A wavelet-based analysis of surface air temperature regularity

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Hölder exponent

Definition

Let f be a signal and x_0 a real number. Then f belongs to the Hölder space $C^\alpha(x_0)$ if there exists a polynomial $P_{x_0,\alpha}$ of degree at most α, a positive constant C and a neighborhood V_{x_0} of x_0 satisfying

$$|f(x) - P_{x_0,\alpha}(x)| \leq C|x - x_0|^\alpha$$

for all $x \in V_{x_0}$.

Hölder exponent

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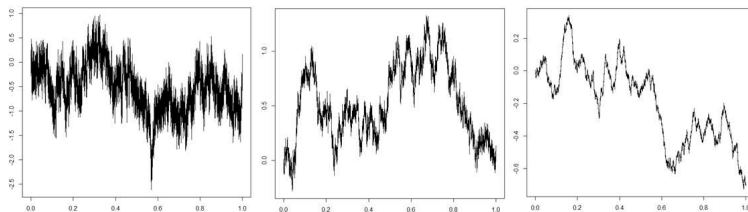
for all $x \in V_{x_0}$.

Definition

The Hölder exponent $h(x_0)$ of f at x_0 is defined as the supremum of the exponents α such that f belongs to $C^\alpha(x_0)$:

$$h(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}.$$

Monofractality

- Hölder exponent changes from point to point : f multifractal
- Constant Hölder exponent : f monofractal, i.e. f is regularly irregular
- Example of a monofractal function : fractional Brownian motion

Fractional Brownian motions with Hölder exponents 0.2, 0.4, 0.6 almost surely.

Spectrum of singularities

How to characterize the global regularity of a signal ?

Definition

The spectrum of singularities of f is the Hausdorff dimension of the set of points sharing the same Hölder exponent :

$$d_f : h \mapsto \dim_{\mathcal{H}}(\{x_0 \in \mathbb{R} : h(x_0) = h\}),$$

where $\dim_{\mathcal{H}}(X)$ denotes the Hausdorff dimension of the set X.

Corollary : f is monofractal if and only if its spectrum of singularities is reduced to a single point.
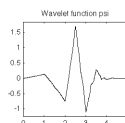
Wavelets

Definition

We say that ψ is a wavelet with $n \in \mathbb{N}$ vanishing moments if $\psi \in L^\infty(\mathbb{R})$, the function $x \mapsto x^k \psi(x)$ belongs to $L^1(\mathbb{R})$ with $\int_\mathbb{R} x^k \psi(x) dx = 0$ for all $k \in \mathbb{N}$ such that $k < n$ and if $\hat{\psi}(0) = 0$ (where $\hat{\psi}$ denotes the Fourier transform of ψ).
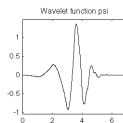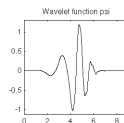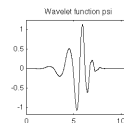
We use Daubechies smooth and compactly supported wavelets.
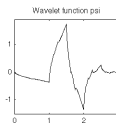
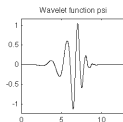db2   db3   db4   db5   db6
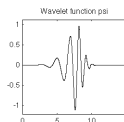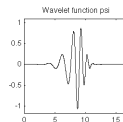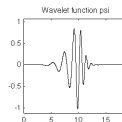

db7   db8   db9   db10

Wavelet leaders method (WLM)

1) Wavelet decomposition of the signal :

$$f(x) = \sum_{j,k \in \mathbb{Z}} c_{j,k} \psi(2^{-j}x - k) = \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda$$

where ψ is a wavelet and $c_{j,k}$ is the wavelet coefficient associated to the dyadic interval λ at scale j and position k :

$$\lambda = \lambda_{j,k} = [2^j k, 2^j(k+1)[$$

and

$$c_{j,k} = 2^{-j} \int_{\mathbb{R}} f(x)\psi(2^{-j}x - k)dx.$$

2) For each λ, compute the wavelet leaders

$$d_\lambda = \sup_{\lambda' \subset \lambda} |c_{\lambda'}|$$

Wavelet leaders method (WLM)

3) Remove the null wavelet leaders and compute

$$S(q,j) = 2^j \sum_{\lambda \in \Lambda_j} d_\lambda^q,$$

where Λ_j is the set of dyadic intervals at scale j.
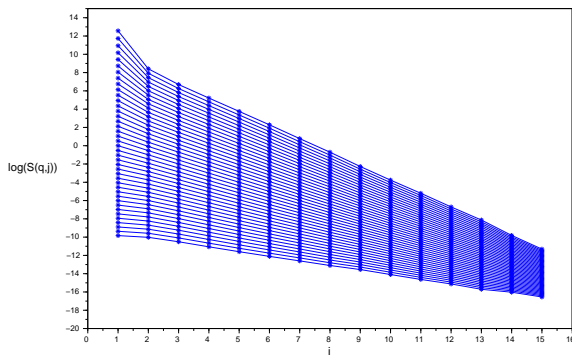
4) Compute the function τ defined as

$$\tau(q) = \varliminf_{j \to -\infty} \frac{\log(S(q,j))}{\log 2^j},$$

which is numerically obtained through the slopes of linear regressions at small scales of $\log(S(q,j))$ seen as a function of j.

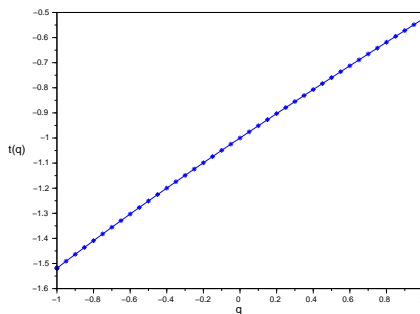5) One can hope to obtain the spectrum of singularities as

$$d(h) = \inf_q \{qh - \tau(q)\} + 1.$$

Wavelet leaders method (WLM)

$\log(S(q,j))$ for a fractional Brownian motion with Hölder exponent 0.5 with q ranging from -1 to 1.

Wavelet leaders method (WLM)

τ function associated to the previous signal. Linear regression gives a slope of 0.494021.

6) Remark : if τ is a straight line, then f is monofractal, in which case the Hölder exponent of f is the slope of τ.
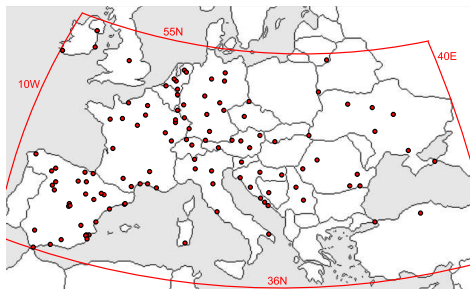
Wavelet leaders method (WLM)

- Remark : if τ is a straight line, then f is monofractal, in which case the Hölder exponent of f is the slope of τ.
- If f is a monofractal signal with Hölder exponent H, then f belongs to the uniform Hölder space C^H, and a "norm" in this space is defined by

$$\|f\|_{C^H} = \sup_{j,k}\{|c_{j,k}|/2^{jH}\} := N$$

- $\Longrightarrow$ Application to surface air temperature signals !

Analyzed data

- Daily mean temperature data from 1951 to 2003, calculated as average of minimum and maximum daily temperatures
- Weather stations located below 1000 meters of altitude
- 115 stations selected
- Missing data up to 7%, less than 1% for 97 stations
- Temperature profile used for more stable numerical results (i.e. x_n replaced by $\sum_{j=1}^{n} x_j$.)

Analyzed data (Rome)
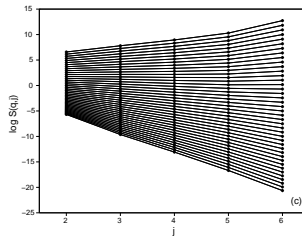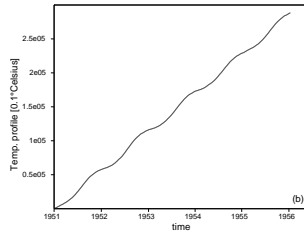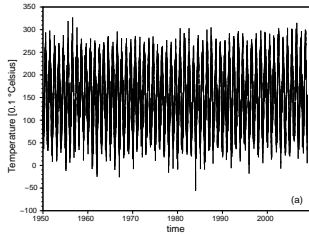
Monofractal nature of the signals
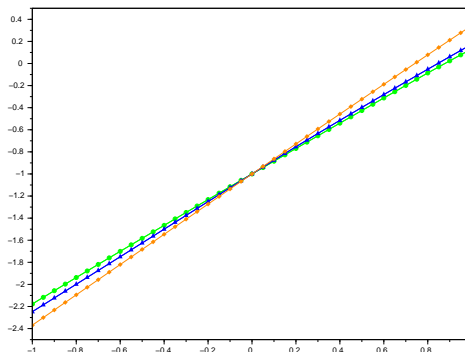
τ functions associated to Aachen (green), Шепетивка (blue) and Rome (orange), with respective slopes 1.156, 1.218, 1.358.

Hölder exponents and norms

- τ linear $\implies$ signals are monofractal
- Mean coefficient of determination : $R^2 = 0.9975 \pm 0.0028$
- Hölder exponents ranging from 1.093 to 1.43
- Norms ranging from 8.23 to 30.45

Comparison with the detrended fluctuation analysis ?

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Description of the method

- Another method to define a notion of regularity.
- Seasonal variation of the temperature signal f have to be removed first.
  If d is a calendar date (e.g. May 22nd), then $\langle f \rangle (d)$ is the average over
  the years of the values $f(t)$ such that t corresponds to the calendar date d
  (May 22nd 1951, May 22nd 1952,...).
  The corresponding trend is

$$\Delta f(t) = f(t) - \langle f \rangle (d).$$

- "Trend profile" used to reduce the noise, i.e. $f(t)$ replaced by $\sum_{u=1}^{t} \Delta f(u)$.

Description of the method

- For a given length l, divide the signal in n segments of length l.
- For each $i = 1, ...n$, determine the best linear fit of segment i and computes the standard deviation $F_i(l)$ of the profile from that straight line.
- Compute the standard deviation of the profile as

$$F(l) = \sqrt{\frac{\sum_{i=1}^{n} F_i(l)}{n}}.$$

- For monofractal signals of exponent γ, we have $F(l) \sim l^{\gamma}$.
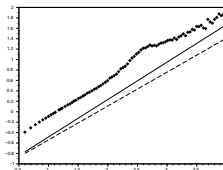
Comparison bewteen WLM and DFA

First simulation : Gaussian noise associated to LRC with index 0.65 + $7\sin\left(\frac{2\pi}{365}t - \frac{\pi}{2}\right)$
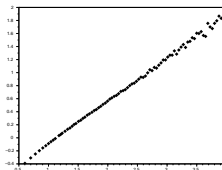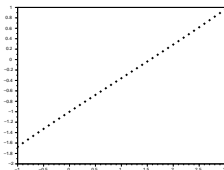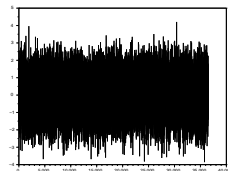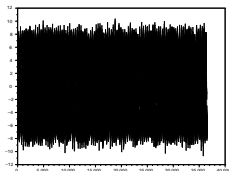
Left : WLM. Slope = 0.651104 . Right : DFA. Slope = 0.657925

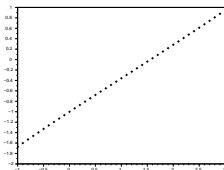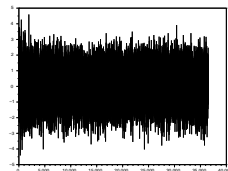## Comparison bewteen WLM and DFA

Second simulation : Gaussian noise associated to LRC with index 0.65 + $7 \sin(\frac{2\pi}{365} t - \frac{\pi}{2} - \frac{1}{20} \log(t+1))$
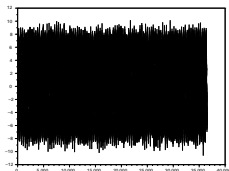
Left : WLM. Slope = 0.647673 . Right : DFA. Slope (straight line) = 0.717090 .

Comparison bewteen WLM and DFA

- On synthetic examples : DFA seems less accurate than WLM.
- On real temperature signals ?
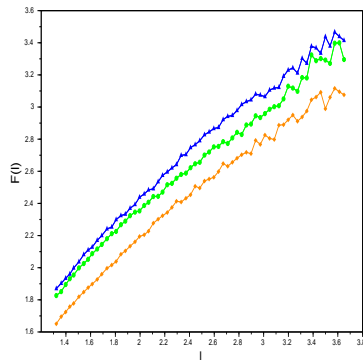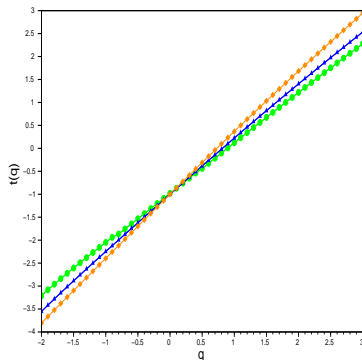
- How to check the "interest" of the results ?
- Idea : temperature variability linked with standard deviation of pressure anomalies.

Relation with pressure anomalies
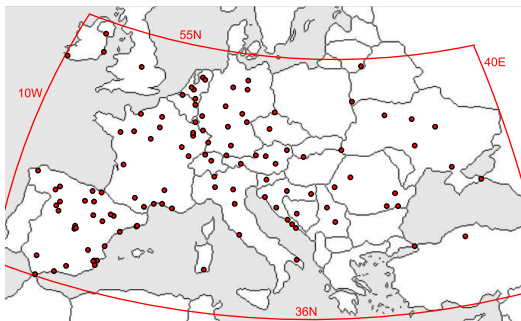
- Normalize the Hölder exponents from WLM between 0 and 1, as well as those from DFA, and standard deviation of pressure anomalies.
- Consider these values as matrices representing Europe (Mw, Md, Mp).

Relation with pressure anomalies

- Normalize the Hölder exponents from WLM between 0 and 1, as well as those from DFA, and standard deviation of pressure anomalies.
- Consider these values as matrices representing Europe (Mw, Md, Mp).
- Compute the Frobenius distance between these matrices :

$$d(M, N) = \sqrt{\sum_{i,j} (M_{i,j} - N_{i,j})^2}.$$

We get $d(Mw, Mp) = 2.68$ and $d(Md, Mp) = 4.67$.

- Are these distances significant ? Confirmation that Mw and Mp are correlated and Md and Mp are not ?

Relation with pressure anomalies

Mw and *Md* shuffled 1000 times and distance with *Mp* measured.
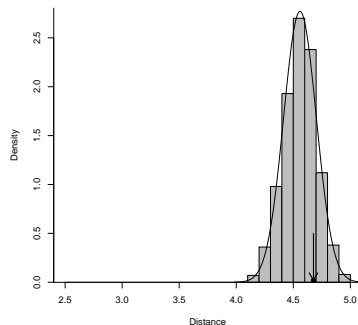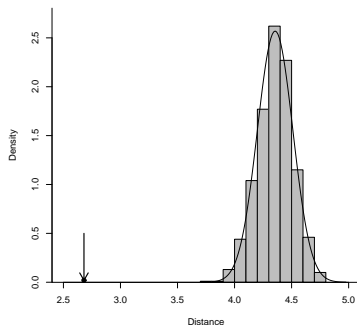
Correlation probable between *Mw* and *Mp* but not between *Md* and *Mp*. Hölder exponents seem to be linked to climate variability.

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   - Spectrum of singularities
   - Wavelet leaders method (WLM)
   - Application to surface air temperature signals

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   - Comparison of the methods
   - Relation with pressure anomalies

3. Link between Hölder exponents and climate types
   - Hölder spaces-based classification and blind test
   - Discussion and conclusions

Adrien DELIÈGE (University of Liège)   Multifractal analysis of temperature signals   ULg, June 2014

Distribution of the exponents and norms
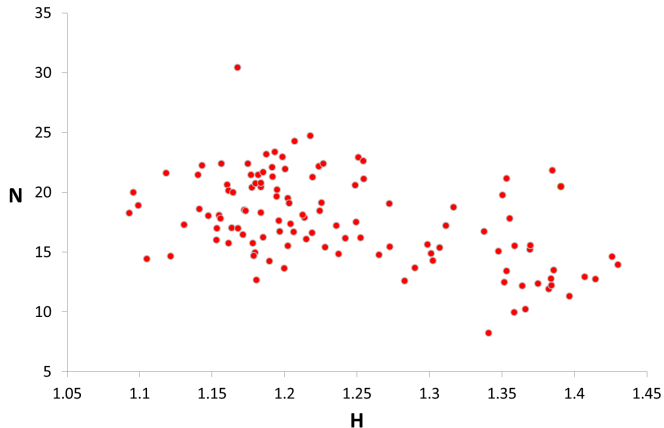
Link with climate types ? What do you think ?

Köppen-Geiger climate classification

Classification based on maximum and minimum monthly mean temperatures (references fixed at 22°C and 0°C). Stations close to 0.5°C of another type of climate were also associated to this second category. Here, precipitations were not taken into account.
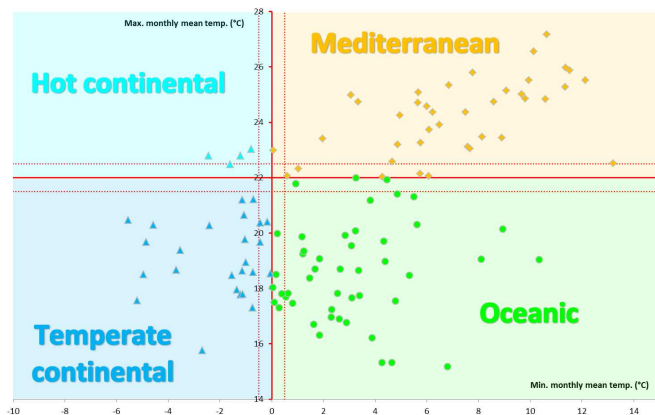
Climate distribution
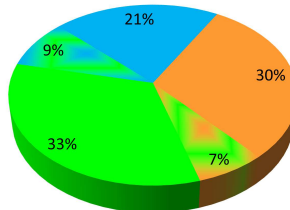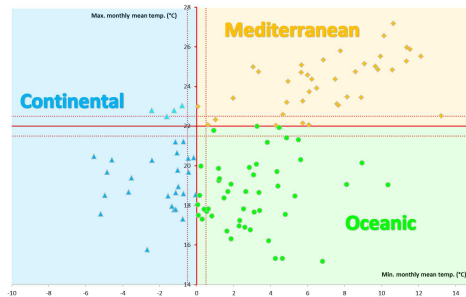
Distribution of the exponents and norms
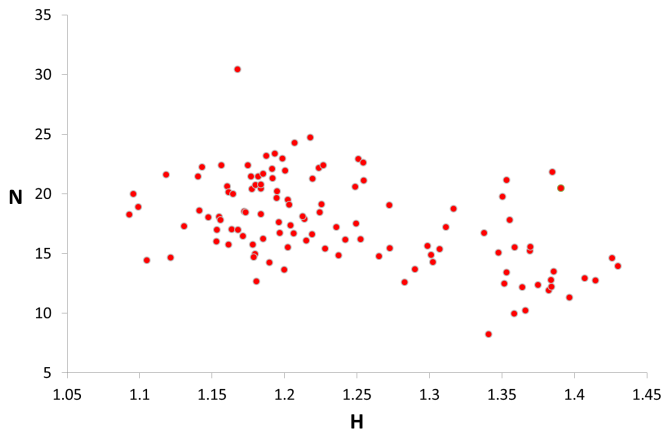
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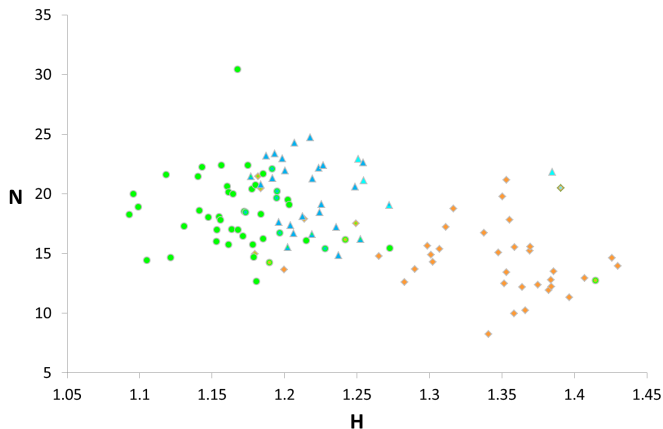
Distribution of the exponents and norms
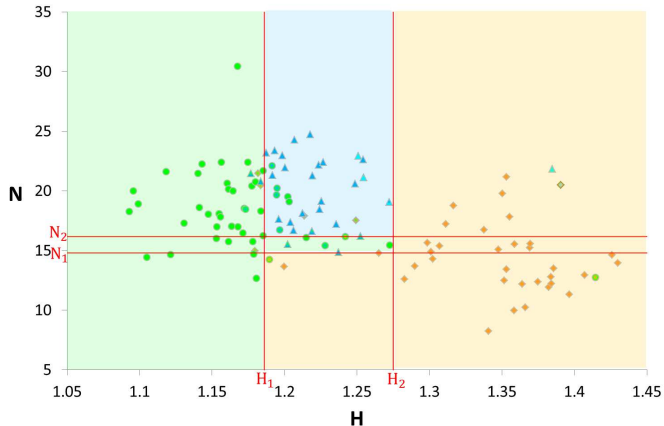
Hölder spaces-based climate classification and results

Maximum matching with Köppen-Geiger classification if

$$
\begin{aligned}
H_1 &= 1.186 \\
H_2 &= 1.275 \\
N_1 &= 14.81 \\
N_2 &= 16.18
\end{aligned}
$$

Result : 93.9% correctly associated

Remark : without the norm, 89.6% correctly associated.

Results on the map
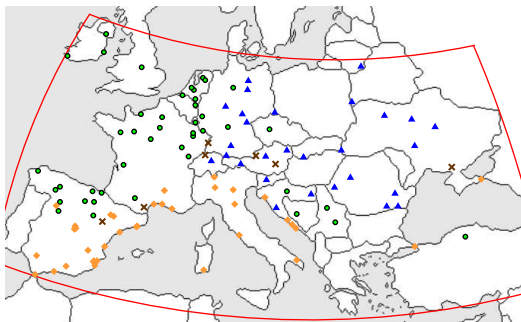
Stations marked with a brown cross are the ones whose type of climate was erroneously predicted. The others were correctly predicted ; green discs stand for Oceanic climate, blue triangles correspond to continental stations and orange orange diamonds are the Mediterranean ones.

Blind test
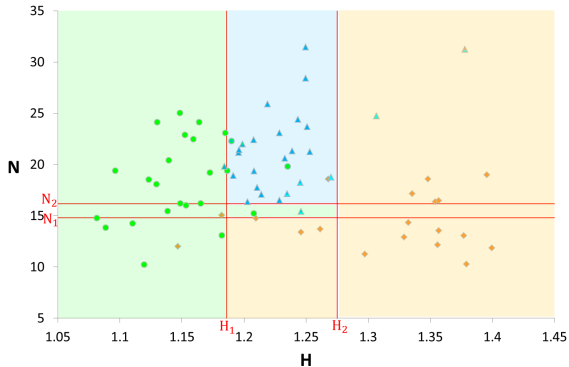
- 69 other stations
- 40 years of data between 1951 and 2003

Blind test
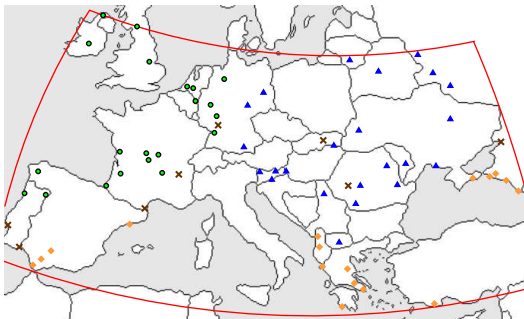
Result : 88.4% correctly associated

Remark : without the norm, 84.1% correctly associated.

Results on the map
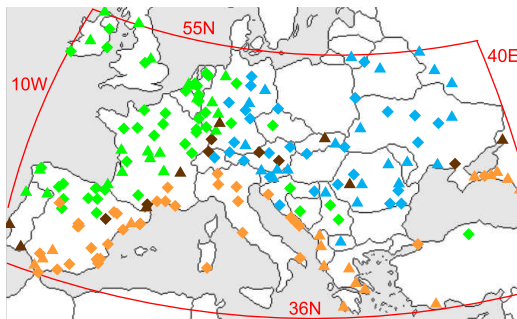
The first 115 weather stations are represented with diamonds, the 69 stations used for the blind test are represented with triangles. Each of them is colored as follows: Oceanic stations (Cb-type) in green, continental stations (D-type) in blue, Mediterranean stations (Ca-type) in orange, and brown diamonds and triangles are used for stations for which the climate type is erroneously predicted by the Hölder-based classification.

Discussion of the results

Results

$$Oceanic \ stations \quad \longleftrightarrow \quad Lowest \ Hölder \ exponents$$
$$Continental \ stations \quad \longleftrightarrow \quad Intermediate \ Hölder \ exponents$$
$$Mediterranean \ stations \quad \longleftrightarrow \quad Largest \ Hölder \ exponents$$

Discussion

- On a daily basis, Oceanic climate is more irregular than the Continental weather, which is less regular than Mediterranean climate.
- Explanation could be the North Atlantic Oscillation (NAO), anticyclonic conditions in Southern Europe, ...

Conclusions and future work

Conclusions

- WLM shows surface air temperatures signals are monofractal signals
- WLM seems to be more accurate than DFA
- Their belonging to functional spaces reflects their temperature-based Köppen-Geiger climate type
- Algorithm and results confirmed through blind tests

Future work

- Checking of the validity of current climatic models
- Analysis of other climate indices (pressure, precipitation,...)
- Generalization to global temperatures

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