Multi-Dimensional Vector Assignment Problems

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Abstract

We consider a special class of axial multi-dimensional assignment problems called *multi-dimensional vector assignment* (MVA) problems. An instance of the MVA problem is defined by m disjoint sets, each of which contains the same number n of p-dimensional vectors with nonnegative integral components, and a cost function defined on vectors. The cost of an m-tuple of vectors is defined as the cost of their component-wise maximum. The problem is now to partition the m sets of vectors into n m-tuples so that no two vectors from the same set are in the same m-tuple and so that the sum of the costs of the m-tuples is minimized. The main motivation comes from a yield optimization problem in semi-conductor manufacturing. We consider a particular class of polynomial-time heuristics for MVA, namely the *sequential* heuristics, and we study their approximation ratio. In particular, we show that when the cost function is monotone and subadditive, sequential heuristics have a finite approximation ratio for every fixed m. Moreover, we establish smaller approximation ratios when the cost function is submodular and, for a specific sequential heuristic, when the cost function is additive. We provide examples to illustrate the tightness of our analysis. Furthermore, we show that the MVA problem is APX-hard even for the case $m = 3$ and for binary input vectors. Finally, we show that the problem can be solved in polynomial time in the special case of binary vectors with fixed dimension p.

Keywords: multi-dimensional assignment; approximability; worst-case analysis; submodularity; wafer-to-wafer integration.

1. Introduction

1.1. Problem statement

We consider a multi-dimensional assignment problem motivated by an application arising in the semi-conductor industry. Formally, the input of the problem is defined by

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m disjoint sets $V_1, \ldots, V_m$, where each set V_k contains the same number n of p-dimensional vectors with nonnegative integral components, and by a *cost function* $c(u) : \mathbb{Z}_+^p \to \mathbb{R}_+$. Thus, the cost function assigns a nonnegative cost to each p-dimensional vector.

A (*feasible*) *m-tuple* is an mp-dimensional vector $(u^1, u^2, \ldots, u^m) \in V_1 \times V_2 \times \ldots \times V_m$. Define the component-wise maximum operator $\vee$ as follows: for every pair of vectors $u, v \in \mathbb{Z}_+^p$,

$$u \vee v = (\max(u_1, v_1), \max(u_2, v_2), \ldots, \max(u_p, v_p)).$$

We extend the definition of the cost function to k-tuples, for any $k \geq 1$, by setting $cost(u^1, \ldots, u^k) := c(u^1 \vee \ldots \vee u^k)$. More generally, when W is any set of k-tuples, we let $cost(W) = \sum_{a \in W} cost(a)$.

A *feasible assignment* for $V = V_1 \times \ldots \times V_m$ is a set A of n feasible m-tuples such that each element of $V_1 \cup \ldots \cup V_m$ appears in exactly one m-tuple of A. In view of the previous definitions, the cost of a feasible assignment A is the sum of the costs of its m-tuples:

$$cost(A) = \sum_{(u^1, \ldots, u^m) \in A} c(u^1 \vee \ldots \vee u^m). \tag{1.1}$$

With this terminology, the *multi-dimensional vector assignment problem* (MVA-m, or MVA for short) is to find a feasible assignment for V with minimum cost. A case of special interest is the case when all vectors in $V_1 \cup \ldots \cup V_m$ are binary 0–1 vectors; we call this special case *binary* MVA. Finally, the *wafer-to-wafer integration problem* (WWI-m or WWI for short) arises when the cost function of the binary MVA is *additive*, meaning that $c(u) = \sum_{i=1}^p u_i$ for all $u \in \mathbb{Z}_+^p$.

In this paper, we investigate how closely the optimal solution of MVA-m and WWI-m can be approximated by a particular class of approximation algorithms.

Example 1. *An instance of WWI with $m = 3, n = p = 2$ is displayed in Figure 1. The optimal value of the instance is equal to 2: it is achieved by assigning the first vector of V_1, the second vector of V_2, and the first vector of V_3 to the same triple, thus arriving at vector $(1,0)$ with cost $c(1,0) = 1$; the remaining three vectors form a second triple with cost $c(0,1) = 1$.*
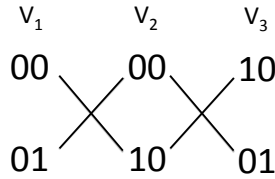

Figure 1: A WWI-3 instance with $m = 3, n = p = 2$ and an illustration of the optimal solution

1.2. Wafer-to-wafer integration and related work

The motivation for studying the WWI problem arises from the optimization of the wafer-to-wafer production process in the electronics industry. We only provide a brief

description of this application; for additional details, we refer to papers by Reda, Smith and Smith [13], Taouil and Hamdioui [19], Taouil et al. [20], and Verbree et al. [21].

For our purpose, a *wafer* can be viewed as a string of elements called *dies*. Each die can be either good (operative) or bad (defective). So, a wafer can be modeled as a binary vector, where each '0' represents a good die and each '1' represents a bad die. There are m lots of wafers, say $V_1, \ldots, V_m$, and each lot contains n wafers. All wafers in a given lot are meant to have identical functionalities, were it not for the occasional occurence of defective dies during the previous production steps. The wafer-to-wafer integration process requires to form *stacks*, where a stack is obtained by "superposing" m wafers chosen from different lots; thus, a stack corresponds to a feasible m-tuple. As a result of integration, each position in the stack gives rise to a *three-dimensional stacked integrated circuit* (3D-SIC) which is 'good' only when the corresponding m entries of the selected wafers are 'good'; otherwise, the 3D-SIC is 'bad'. The yield optimization problem now consists in assigning the available wafers to n stacks so as to minimize the total number of bad 3D-SICs. Thus, the WWI problem provides a model for yield optimization.

The wafer-to-wafer yield optimization problem has recently been the subject of much attention in the engineering literature. One example is the contribution by Reda et al. [13]. These authors formulate WWI as a multi-dimensional assignment problem. A natural formulation of WWI as an integer linear programming problem turns out to be hard to solve to optimality for instances with large values of m (typical dimensions for the instances are: $3 \leq m \leq 10$, $25 \leq n \leq 75$, $500 \leq p \leq 1000$). On the other hand, Reda et al. [13] propose several heuristics and show that they perform well in computational experiments. Some recent work in this direction is also reported in [15, 19, 20, 21].

Our main objective in this paper is twofold: (i) to derive approximation ratios for so-called *sequential heuristics* when they are applied to the MVA problem and to the (more specific) WWI problem, and (ii) to investigate the approximability of the WWI problem. Let us note at this point that the wafer-to-wafer integration problem is usually formulated in the literature as a maximization problem (since one wants to maximize the yield). However, we feel that from the approximation point of view, it is more appropriate to study its cost minimization version. Indeed, in industrial instances, the number of bad dies in each wafer is typically much less than the number of good dies. Therefore, it is more relevant to be able to approximate the (smaller) minimum cost than the (larger) maximum yield.

Since MVA is defined as a multi-dimensional assignment problem with a specific cost structure, our work relates to previous publications on special classes of multi-dimensional assignment problems, such as Bandelt, Crama and Spieksma [1], Burkard, Rudolf and Woeginger [3], Crama and Spieksma [5], Dokka, Kouvela and Spieksma [8], Goossens et al. [9], Spieksma and Woeginger [17], etc. Surveys on multi-dimensional assignment problems can be found in [2] and in [16]. The composition of sum and max operators in the cost function (1.1) is superficially reminiscent of max-algebra formulations of assignment problems, such as those discussed in [2] or [4]. To the best of our knowledge, however, the approximability of MVA has only been previously investigated by Dokka et al. [6], who mostly focused on the case $m = 3$ with additive cost functions. The present paper extends to MVA-m, $m \geq 3$, and considerably strengthens the results presented in [6]. Finally, we point out that the research report [7], on which the present article is based, also analyzes the approximation ratio of another class of algorithms called *hub heuristics*. We refer the reader to the report [7] for more details.

1.3. Contents of the paper

Section 2 contains a formulation of the problem as an integer program (Subsection 2.1), discusses various possible assumptions on the cost function (Subsection 2.2), describes the sequential heuristics (Subsection 2.3), and gives an overview of our results in Subsection 2.4. Section 3 is devoted to the proofs of the approximation ratios for sequential heuristics. In Section 4, we prove that the WWI-m problem is APX-hard even when $m = 3$, all input vectors are binary, and the cost function is additive. Finally, we show in Section 5 that WWI-m can be solved in polynomial time when p is fixed.

2. Problem formulation, properties, heuristics and results

2.1. Problem formulation

Let us provide an integer programming formulation of MVA-m as an m-dimensional axial assignment problem. Recall that $V = V_1 \times V_2 \times \ldots \times V_m$ is the set of feasible m-tuples. For each $a \in V$, let x_a be a binary variable indicating whether m-tuple a is selected ($x_a = 1$) or not ($x_a = 0$) in the optimal assignment. For convenience, we write $u^k \in a$ when $a = (u^1, \ldots, u^m)$ and $1 \le k \le m$. Reda et al. [13] give the following formulation of WWI, which directly extends to MVA:

$$\begin{aligned} \text{minimize} \quad & \sum_{a \in V} c(a) x_a \\ \text{s.t.} \quad & \sum_{a:u \in a} x_a = 1 \qquad \text{for all } u \in \cup_{i=1}^m V_i, \\ & x_a \in \{0,1\} \qquad \text{for all } a \in V. \end{aligned}$$

Notice that this formulation has an exponential number of variables (n^m variables). Other formulations of MVA exist; for instance, Dokka et al. [6] propose an alternative IP-formulation that may be more effective from a computational perspective.

In any application of MVA, the cost function c is likely to have some structure. Indeed, in the WWI-application motivating this study, we have, as mentioned before, an *additive* cost function: $c(u) = \sum_{i=1}^p u_i$. We now list various alternative assumptions on the cost function c.

2.2. Properties of the cost function c

We focus our attention on cost functions $c(u)$ satisfying one or more of the following properties. Notice that, for any pair of vectors $u, v \in \mathbb{Z}_+^p$, we write $u \le v$ when $u_i \le v_i$ for $i = 1, \ldots, p$. Also, $\wedge$ denotes the component-wise minimum operator:

$$u \wedge v = (\min(u_1, v_1), \min(u_2, v_2), \ldots, \min(u_p, v_p)).$$

Let us now define the following properties.

Monotonicity: The cost function c is *monotone* if, for all $u, v \in \mathbb{Z}_+^p$ with $u \le v$, we have $0 \le c(u) \le c(v)$.

Subadditivity: The cost function c is *subadditive* if, for all $u, v \in \mathbb{Z}_+^p$, we have $c(u \vee v) \le c(u) + c(v)$.

Submodularity: The cost function c is *submodular* if, for all $u, v \in \mathbb{Z}_+^p$, we have $c(u \ \vee \ v) + c(u \wedge v) \le c(u) + c(v)$.

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Submodular cost functions frequently appear in the analysis of approximation algorithms for combinatorial optimization problems; for recent illustrations, see for instance [11, 18] and the references therein. The additive cost function of problem WWI actually satisfies a much stronger property than submodularity, namely:

Modularity: The cost function c is *modular* if, for all $u, v \in \mathbb{Z}_+^p$, we have $c(u \vee v) + c(u \wedge v) = c(u) + c(v)$.

It is well-known that c is modular if and only if there exist p functions $f_1(x), \ldots, f_p(x)$ such that $c(u) = \sum_{\ell=1}^p f_\ell(u_\ell)$ (see Theorem 2.3.3 in Simchi-Levy, Chen and Bramel [14]). For the MVA problem, therefore, assuming additivity is essentially equivalent to assuming monotonicity and modularity.

2.3. Sequential heuristics

Consider any heuristic algorithm H for MVA-m. Following standard terminology (see, e.g., Williamson and Shmoys [22]), we say that H is a $\rho^H(m)$-*approximation* algorithm for MVA-m if H runs in polynomial time and if $\rho^H(m)$ is (an upper bound on) the *approximation ratio* of H, in the following sense: for every instance of MVA-m with optimal value c_m^{OPT}, when H returns the assignment A_m, then $cost(A_m) \leq \rho^H(m) c_m^{OPT}$.

Here, we are interested in the behavior of *sequential* heuristics, which rely on the observation that MVA-2 boils down to a classical bipartite assignment (or matching) problem (see, e.g., Bandelt et al. [1] for other examples of assignment-based heuristics).

Consider the *sequential* heuristic H^{seq} described as Algorithm 1: H^{seq} progressively builds a feasible solution H_m by optimally assigning the next set V_i to a partial solution H_{i-1}. We point out that, for WWI-m, Reda et al. [13] proposed an *iterative matching heuristic* which performed very well in their computational experiments. Algorithm 1 is a natural generalization of this iterative matching heuristic. (See also Taouil et al. [20] for a related study where sequential heuristics are called "layer-by-layer" heuristics.)

Algorithm 1 Sequential heuristic H^{seq}

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    let $H_1 := V_1$;
    for $i = 2$ to m **do**
        solve a bipartite assignment problem between H_{i-1} and V_i based on the costs $c(u^1 \vee \ldots \vee u^{i-1} \vee v)$, for all $(u^1, \ldots, u^{i-1}) \in H_{i-1}$ and $v \in V_i$; let H_i be the resulting assignment for $V_1 \times V_2 \times \ldots \times V_i$;
    end for
    output H_m.

Observe that the order of the sets $V_1, \ldots, V_m$ is arbitrary in the sequential heuristic. Let us say that a set V_i ($1 \leq i \leq m$) is *heaviest* when $cost(V_i) \geq cost(V_k)$ for each $k = 1, \ldots, m$. Then, we can obtain a slightly more restrictive heuristic, called the *heaviest-first* heuristic, or H^{heavy}, when we specify that the heaviest set is contained in the first assignment; see Algorithm 2. (An even more specific version, where the sets are ordered by non-increasing weights, was shown by Singh [15] to be computationally effective.)

Clearly, each of these heuristics runs in polynomial time. Both H^{seq} and H^{heavy} solve $O(m)$ bipartite assignment problems; the preprocessing step needed for H^{heavy} does not increase its asymptotic running time.

Algorithm 2 Heaviest-first heuristic H^{heavy}
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   reindex $V_1, \ldots, V_m$ so that V_1 is heaviest;
   apply the sequential heuristic.

2.4. Overview of results

In this section we list the main results proved in our paper. First, in case, c is monotone and subadditive, *no* feasible solution can be arbitrarily far away from the optimum, as expressed by the next theorem.

Theorem 2. *Every heuristic H that returns a feasible solution is an m-approximation algorithm when the cost function c is monotone and subadditive. There exist heuristics H for which the approximation ratio $\rho^H(m) = m$ is tight for all $m \geq 2$, even for WWI-m.*

Next, we establish that both monotonicity and subadditivity must hold for sequential heuristics to have bounded approximation ratio.

Theorem 3. *When the cost function c is not monotone, or when c is not subadditive, even the heaviest-first heuristic H^{heavy} does not have a bounded worst-case ratio for any fixed $m \geq 3$.*

On the other hand, when c is both monotone and submodular, the worst-case performance ratio improves for every sequential heuristic:

Theorem 4. *The sequential heuristic H^{seq} is an $\frac{m}{2}$-approximation algorithm for MVA-m when the cost function c is monotone and submodular, for every order of the sets $V_1, \ldots, V_m$. The approximation ratio $\rho^{seq}(m) = \frac{m}{2}$ is tight for all $m \geq 2$, even for the heaviest-first heuristic and even for binary MVA.*

We prove this result in Subsection 3.2. When c is additive, a better bound can be proved for the heaviest-first heuristic:

Theorem 5. *The heaviest-first heuristic H^{heavy} is a $(\frac{1}{2}(m+1) - \frac{1}{4}\ln(m-1))$-approximation algorithm for MVA-m when the cost function c is additive.*

We prove this result in Subsection 3.3. Although we do not know whether the bound in Theorem 5 is tight, we exhibit in Section 3.4 a family of instances for which H^{heavy} displays the following behavior:

Theorem 6. *There exists an infinite sequence of values of m such that the heaviest-first heuristic produces a feasible assignment with cost larger than $\frac{\sqrt{m}}{2} c_m^{OPT}$ on certain instances of WWI-m.*

This concludes the overview of our results concerning approximation ratios of the heuristics; see also Figure 2.

One might wonder about the precise complexity status of MVA-m and of its special case WWI-m. The following result implies that, when restricting ourselves to polynomial-time algorithms, constant-factor approximation algorithms are the best we can hope for (unless P=NP), even for WWI-3:

Theorem 7. *WWI-3 is APX-hard, even when all vectors in $V_1 \cup V_2 \cup V_3$ are 0–1 vectors with exactly two nonzero entries per vector.*

We prove this in Section 4. Finally, in case the dimension p of the vectors is fixed, we show in Section 5 that binary MVA-m can be solved in polynomial time:

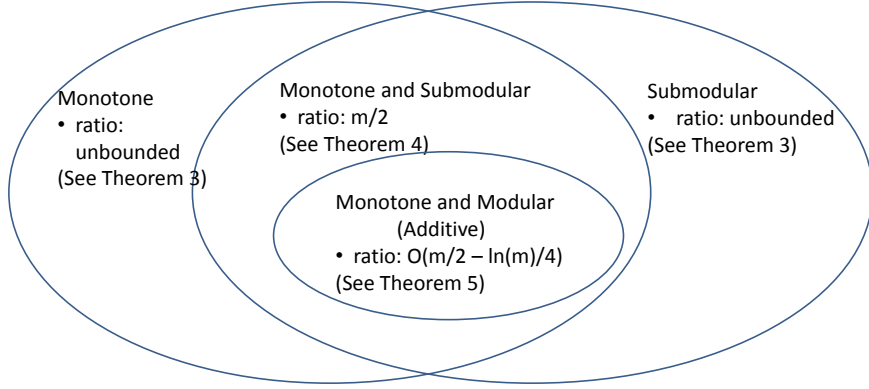**Theorem 8.** *Binary MVA can be solved in polynomial time for each fixed p.*

Figure 2: Overview of approximability results for monotone and submodular cost functions

3. Proofs of approximation ratios

This section is devoted to the proofs of the approximation ratios for sequential heuristics.

3.1. Monotone and subadditive costs: feasible solutions

Here, we first establish some properties of feasible solutions depending on various assumptions on the cost function c. Consider a feasible assignment A_m for $V_1 \times \ldots \times V_m$, and let A_k denote the restriction of this assignment to $V_1 \times \ldots \times V_k$, for all $k \leq m$. Denote by c_k^{OPT} the optimal value of the restricted instance $V_1 \times \ldots \times V_k$.

Lemma 9. *If the cost function c is monotone and if A_m is a feasible assignment, then, for all $i \leq k \leq m$,*
$$cost(V_i) \leq c_k^{OPT} \leq cost(A_k) \leq cost(A_m). \tag{3.2}$$

Proof. Obvious. ∎

Lemma 10. *If the cost function c is subadditive and if A_m is a feasible assignment, then*

$$cost(A_m) \leq cost(A_{m-1}) + cost(V_m). \tag{3.3}$$

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Proof. Assume without loss of generality that the jth m-tuple of A_m is $(u_j^1, \ldots, u_j^m)$ (here, the subscript j runs over the vectors in the set V_i; that is, the jth m-tuple in the assignment contains the jth vector of V_i for each i). Then,

$$
\begin{aligned}
cost(A_m) &= \sum_{j=1}^{n} c(u_j^1 \vee \ldots \vee u_j^m) \\
&\leq \sum_{j=1}^{n} c(u_j^1 \vee \ldots \vee u_j^{m-1}) + \sum_{j=1}^{n} c(u_j^m) \\
&= cost(A_{m-1}) + cost(V_m).
\end{aligned}
$$

∎

These two lemmas allow us to prove:

Theorem 2. *Every heuristic H that returns a feasible solution is an m-approximation algorithm when the cost function c is monotone and subadditive. There exist heuristics H for which the approximation ratio $\rho^H(m) = m$ is tight for all $m \geq 2$, even for WWI-m.*

Proof. The statement holds for $m = 1$. Then, using Eq. (3.3) from Lemma 10, Eq. (3.2) from Lemma 9, and induction on m, we obtain that every feasible solution A_m satisfies

$$
\begin{aligned}
cost(A_m) &\leq cost(A_{m-1}) + cost(V_m) \\
&\leq (m-1)\, c_{m-1}^{OPT} + c_m^{OPT} \\
&\leq m\, c_m^{OPT}.
\end{aligned}
$$

To see that the bound is tight, let $p = 1$, $n = m$, $V_i = \{1, 0, \ldots, 0\}$ for $i = 1, \ldots, m$, and $c(u) = u$ for all $u \in \mathbb{R}$. The cost function is obviously additive, hence this is an instance of WWI-m. A heuristic may produce the worst feasible assignment, namely $\{1, 1, \ldots, 1\}$ with cost m, whereas the optimal assignment has cost 1. ∎

Thus, Theorem 2 implies that every heuristic has bounded worst-case performance (for fixed m) under the assumption that c is monotone and subadditive. On the other hand, if we relax either of the assumptions on c, then even the heaviest-first heuristic does not have a bounded approximation ratio for WWI-3, as shown by the following theorem.

Theorem 3. *When the cost function c is not monotone, or when c is not subadditive, even the heaviest-first heuristic H^{heavy} does not have a constant worst-case ratio for any fixed $m \geq 3$.*

Proof. For any p, we denote by $\mathbf{0}$, $\mathbf{1}$, and e_i, respectively, the all-zero, all-one, and i-th unit vector of $\mathbb{Z}^p$. Let $p = 3$, $V_1 = \{e_1, \mathbf{0}\}$, $V_2 = \{\mathbf{0}, e_2\}$, $V_3 = \{\mathbf{1}, \mathbf{0}\}$, and $c(u) = u_1 + u_2 + u_3 - 3 \min(u_1, u_2, u_3)$. This cost function is nonnegative, subadditive (and even submodular), but not monotone since $1 = c(e_1) \not\leq c(\mathbf{1}) = 0$. The optimal solution for this instance is $\{(e_1, e_2, \mathbf{1}), (\mathbf{0}, \mathbf{0}, \mathbf{0})\}$ with cost 0. Since $cost(V_1) = cost(V_2) > cost(V_3)$, the heaviest-first heuristic could match V_1, V_2 to produce $\{(e_1, \mathbf{0}), (\mathbf{0}, e_2)\}$, then V_3 to produce $\{(e_1, \mathbf{0}, \mathbf{1}), (\mathbf{0}, e_2, \mathbf{0})\}$ with cost 1.

A similar observation applies when c is not subadditive: let $p = 3$, $V_1 = \{e_1, \mathbf{0}\}$, $V_2 = \{\mathbf{0}, e_2\}$, $V_3 = \{e_3, e_3\}$, and $c(u) = u_1 + u_2 + M \min(u_1, u_2, u_3)$ with $M > 0$.

This cost function is nonnegative, monotone (and supermodular), but not subadditive since $c(1,0,1) = c(0,1,1) = 1$ and $c(1,1,1) = M + 2$. The optimal solution is $\{(e_1, \mathbf{0}, e_3), (\mathbf{0}, e_2, e_3)\}$ with cost 2. Note that $cost(V_1) = cost(V_2) = 1$, $cost(V_3) = 0$; hence, heaviest-first could match V_1, V_2 to produce $\{(e_1, e_2), (\mathbf{0}, \mathbf{0})\}$, then V_3 to produce $\{(e_1, e_2, e_3), (\mathbf{0}, \mathbf{0}, e_3)\}$ with cost $M+2$. So, the performance of heaviest-first is unbounded for this instance. ∎

3.2. Monotone and submodular costs: sequential heuristics

Let us now turn to the analysis of sequential heuristics. It follows from the proof of Theorem 2 that the performance ratio of any sequential heuristic is bounded by $m-1$ when the cost function is monotone and subadditive (simply start the induction step with $m = 2$). In this section, we will establish a better bound using the stronger submodularity assumption.

In the sequel, we frequently assume without loss of generality, as in the proof of Lemma 10, that the jth m-tuple of A_m is $(u_j^1, \ldots, u_j^m)$. Under this assumption, we now derive inequalities that are valid for every feasible assignment A_m.

Lemma 11. *If the cost function c is monotone and submodular, and if A_m is a feasible assignment such that the jth m-tuple of A_m is $(u_j^1, \ldots, u_j^m)$, then, for all $k \in \{1, \ldots, m-1\}$,*

$$
\begin{aligned}
cost(A_m) \quad \leq \quad & cost(A_{m-1}) + \sum_{j=1}^{n} c(u_j^k \vee u_j^m) - cost(V_k) && (3.4) \\[1em]
\leq \quad & cost(A_{m-1}) + cost(V_m) - \sum_{j=1}^{n} c(u_j^k \wedge u_j^m) && (3.5) \\[1em]
\leq \quad & cost(A_{m-1}) + cost(V_m). && (3.6)
\end{aligned}
$$

Proof.

$$
\begin{aligned}
cost(A_m) \quad = \quad & \sum_{j=1}^{n} c(u_j^1 \vee \ldots \vee u_j^m) && (3.7) \\[1em]
\leq \quad & \sum_{j=1}^{n} c(u_j^1 \vee \ldots \vee u_j^{m-1}) + \sum_{j=1}^{n} c(u_j^k \vee u_j^m) \\
& - \sum_{j=1}^{n} c((u_j^1 \vee \ldots \vee u_j^{m-1}) \wedge (u_j^k \vee u_j^m)) && (3.8) \\[1em]
\leq \quad & \sum_{j=1}^{n} c(u_j^1 \vee \ldots \vee u_j^{m-1}) + \sum_{j=1}^{n} c(u_j^k \vee u_j^m) - \sum_{j=1}^{n} c(u_j^k) && (3.9) \\[1em]
\leq \quad & \sum_{j=1}^{n} c(u_j^1 \vee \ldots \vee u_j^{m-1}) + \sum_{j=1}^{n} c(u_j^m) - \sum_{j=1}^{n} c(u_j^k \wedge u_j^m) && (3.10) \\[1em]
\leq \quad & \sum_{j=1}^{n} c(u_j^1 \vee \ldots \vee u_j^{m-1}) + \sum_{j=1}^{n} c(u_j^m) && (3.11)
\end{aligned}
$$

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where (3.7) is by definition of the cost function, (3.8) holds by submodularity applied to $u = u_j^1 \vee \ldots \vee u_j^{m-1}$ and $v = u_j^k \vee u_j^m$ for each j, (3.9) follows by monotonicity (since $u_j^k \leq (u_j^1 \vee \ldots \vee u_j^{m-1}) \wedge (u_j^k \vee u_j^m)$), (3.10) by submodularity applied to $u = u_j^k$, $v = u_j^m$, and (3.11) by nonnegativity of c. Inequalities (3.9), (3.10), (3.11) are equivalent to (3.4), (3.5), (3.6), respectively. $\blacksquare$

We can now prove:

Theorem 4. *The sequential heuristic H^{seq} is an $\frac{m}{2}$-approximation algorithm for MVA-m when the cost function c is monotone and submodular, for every order of the sets $V_1, \ldots, V_m$. The approximation ratio $\rho^{seq}(m) = \frac{m}{2}$ is tight for all $m \geq 2$, even for the heaviest-first heuristic and even for binary MVA.*

Proof. Let H_m be a feasible assignment for V found by the sequential heuristic and let H_k be the restriction of this assignment to $V_1 \times \ldots \times V_k$, for all $k \leq m$. We prove the theorem by induction on m. The result is trivial when $m = 2$. For larger values of m, we distinguish among two cases.

Assume first that $cost(V_{m-1}) \leq \frac{1}{2} c_m^{OPT}$. Then, consider the partial assignment $A_{m-2,m}$ that is obtained by assigning optimally V_m to H_{m-2} (independently of V_{m-1}). Let H_m^* be the concatenation of H_{m-1} and $A_{m-2,m}$ (that is, H_m^* assigns V_{m-1} to H_{m-2} as in H_{m-1}, and V_m to H_{m-2} as in $A_{m-2,m}$). Note that $H_{m-1}^* = H_{m-1}$; therefore, $cost(H_m) \leq cost(H_m^*)$ since, by definition, the sequential heuristic assigns V_m optimally to H_{m-1}. Applying (3.6) (Lemma 11) to the assignment H_m^*, we obtain

$$cost(H_m) \leq cost(H_m^*) \leq cost(A_{m-2,m}) + cost(V_{m-1}). \qquad (3.12)$$

Since $A_{m-2,m}$ results from applying the sequential heuristic to $W = V_1 \times \ldots \times V_{m-2} \times V_m$, we have by induction:

$$cost(A_{m-2,m}) \leq \rho^{seq}(m-1)\, c^{OPT}(W) \leq \frac{1}{2}(m-1)\, c_m^{OPT} \qquad (3.13)$$

where $c^{OPT}(W)$ is the cost of an optimal assignment on W.

Finally, using the assumption that $cost(V_{m-1}) \leq \frac{1}{2} c_m^{OPT}$, we conclude from (3.12)–(3.13) that

$$cost(H_m) \leq \left(\frac{m-1}{2} + \frac{1}{2} \right) c_m^{OPT} = \frac{m}{2}\, c_m^{OPT}.$$

Assume now alternatively that $cost(V_{m-1}) \geq \frac{1}{2} c_m^{OPT}$. Let $M_{m-1,m}$ be an optimal matching of V_{m-1} with V_m, and consider the assignment H_m^+ obtained by concatenating H_{m-1} with $M_{m-1,m}$. Assume, without loss of generality, that the jth vector of H_m^+ is $(u_j^1, \ldots, u_j^m)$. Then, by definition of H_m, $cost(H_m) \leq cost(H_m^+)$ and by Eq. (3.4):

$$cost(H_m) \leq cost(H_m^+) \leq cost(H_{m-1}) + \sum_{j=1}^n c(u_j^{m-1} \vee u_j^m) - cost(V_{m-1}). \qquad (3.14)$$

Moreover, $\sum_{j=1}^{n} c(u_j^{m-1} \vee u_j^m) = cost(M_{m-1,m}) \leq c_m^{OPT}$. Thus, we derive

$$
\begin{aligned}
cost(H_m) &\leq cost(H_{m-1}) + cost(M_{m-1,m}) - cost(V_{m-1}) \\
&\leq \rho^{seq}(m-1) c_{m-1}^{OPT} + c_m^{OPT} - \frac{1}{2} c_m^{OPT} \\
&\leq (\frac{m-1}{2} + \frac{1}{2}) c_m^{OPT} \\
&= \frac{m}{2} c_m^{OPT}.
\end{aligned}
$$

This establishes the validity of the approximation ratio $\rho^{seq}(m) = \frac{1}{2}m$.

To prove that the ratio is tight, consider the function $r_2(u) = f(\sum_{i=1}^{p} u_i)$, where $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x$ when $x \leq 2$, and $f(x) = 2$ when $x \geq 2$. Since f is monotone nondecreasing and concave, it follows easily that r_2 is monotone and submodular on $\mathbb{Z}_+^p$ (see, e.g., Theorem 2.3.6 in Simchi-Levy et al. [14]). (When u is a binary vector, $r_2(u)$ is the rank function of the uniform matroid of rank 2.)

Now, let $p = n = m$, $V_i = \{e_i, \mathbf{0}, \dots, \mathbf{0}\}$ for $i = 1, \dots, m$, and $c(u) = r_2(u)$. By symmetry, any of the sets V_i can be chosen as the heaviest set, and the heaviest-first heuristic delivers a solution with the same cost as the sequential heuristic. In particular, it is easy to see that heaviest-first can produce the assignment H_m in which e_i is matched with $m - 1$ zero vectors, for all i. The resulting assignment H_m has cost m, whereas the optimal solution assigns $(e_1, \dots, e_m)$ to the same tuple, and has cost $r_2(e_1 \vee \dots \vee e_m) = 2$. ∎

As a side-remark, the worst-case example used in the proof of Theorem 4 shows that, for monotone submodular instances of binary MVA-m, the same ratio $\frac{m}{2}$ is tight for the (expensive) "multiple-pass" heuristic that results by successively running the sequential heuristic for all possible permutations of the sets $V_1, \dots, V_m$.

Let us also observe that the submodularity assumption is necessary in Theorem 4, as shown by the following example.

Example 12. *Let $m = 3$, $n = 2$, $p = 3$, $V_1 = V_2 = \{e_2, e_3\}$, $V_3 = \{e_1, e_1\}$, and $c(u) = \max(u_1, u_2, u_3) + \min(u_1, u_2, u_3)$. This cost function can be checked to be subadditive, but not submodular. As illustrated by Figure 3, the optimal solution is $\{(e_2, e_2, e_1), (e_3, e_3, e_1)\}$ with cost $c^{OPT} = 2$. However, heaviest-first may find the solution $\{(e_2, e_3, e_1), (e_3, e_2, e_1)\}$ with cost $4 > \frac{m}{2} c^{OPT}$.*
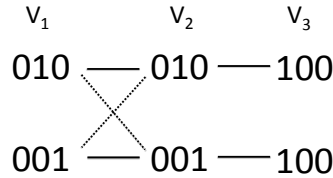

Figure 3: An illustration of optimal (full lines) and heaviest-first (dotted lines) solutions for Example 12.

We return in Section 3.4 to a discussion of the approximation ratio of sequential heuristics for the more restrictive WWI-m problem.

3.3. Additive costs: heaviest-first heuristic

In this section, we explicitly rely on the assumption that the cost function is additive, i.e., $c(u) = \sum_{\ell=1}^{p} u_\ell$, and we derive an improved approximation ratio for the heaviest-first heuristic. We first establish a series of preliminary results.

3.3.1. Preliminary results for additive cost functions

If the jth m-tuple of an arbitrary assignment A_m is $u_j = (u_j^1, \ldots, u_j^m)$, then, for all $j = 1, \ldots, n$

$$c(u_j) = \sum_{\ell=1}^{p} (u_{j\ell}^1 \vee \ldots \vee u_{j\ell}^m).$$

Thus,

$$
\begin{aligned}
cost(A_m) &= \sum_{j=1}^{n} c(u_j) \\
&= cost(A_{m-1}) + cost(V_m) - \sum_{j=1}^{n} \sum_{\ell=1}^{p} \left((u_{j\ell}^1 \vee \ldots \vee u_{j\ell}^{m-1}) \wedge u_{j\ell}^m \right).
\end{aligned}
$$

For each j, ℓ, let $k(j, \ell)$ be an (arbitrary) index $k \in \{1, \ldots, m-1\}$ such that

$$u_{j\ell}^{k(j,\ell)} = u_{j\ell}^1 \vee \ldots \vee u_{j\ell}^{m-1}.$$

For each j, k, let $L(j, k) = \{\ell : k(j, \ell) = k\}$ (roughly speaking, $L(j, k)$ is the set of coordinates ℓ for which the maximum of $u_{j\ell}^1, \ldots, u_{j\ell}^{m-1}$ is attained in set V_k). Then,

$$
\begin{aligned}
cost(A_m) &= cost(A_{m-1}) + cost(V_m) - \sum_{j=1}^{n} \sum_{\ell=1}^{p} \left(u_{j\ell}^{k(j,\ell)} \wedge u_{j\ell}^m \right) \qquad (3.15) \\
&= cost(A_{m-1}) + cost(V_m) - \sum_{j=1}^{n} \sum_{k=1}^{m-1} \sum_{\ell \in L(j,k)} (u_{j\ell}^k \wedge u_{j\ell}^m).
\end{aligned}
$$

Consider now the quantity $Q = \sum_{j=1}^{n} \sum_{k=1}^{m-1} \sum_{\ell \in L(j,k)} (u_{j\ell}^k \wedge u_{j\ell}^m)$. Intuitively, $cost(V_m) - Q$ in Eq. (3.15) represents the amount by which the cost of the partial solution A_{m-1} increases when the set V_m is appended to this partial solution: so, Q can be viewed as the amount of $cost(V_m)$ that is "covered" by $V_1, \ldots, V_{m-1}$.

Clearly, there exists an index $k^* \in \{1, \ldots, m-1\}$ such that

$$\sum_{j=1}^{n} \sum_{\ell \in L(j,k^*)} (u_{j\ell}^{k^*} \wedge u_{j\ell}^m) \geq \frac{1}{m-1} Q$$

(there is a set V_{k^*} that, by itself, covers at least the fraction $\frac{1}{m-1} Q$ of the amount of $cost(V_m)$ that is covered by $V_1, \ldots, V_{m-1}$ together).

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Assume now that A_m is an optimal assignment: $cost(A_m) = c_m^{OPT}$. Denote by H_m the assignment produced by a sequential heuristic which optimally matches the partial assignment H_{m-1} with V_m, and denote by H_{m,k^*} the assignment obtained by concatenating H_{m-1} with the assignment $\{(u_j^{k^*}, u_j^m) : j = 1, \ldots, n\}$ extracted from the optimal solution A_m. Clearly, $cost(H_m) \leq cost(H_{m,k^*})$. Inequality (3.5) implies that

$$
\begin{aligned}
cost(H_m) &\leq cost(H_{m,k^*}) \\
&\leq cost(H_{m-1}) + cost(V_m) - \sum_{j=1}^{n} \sum_{\ell=1}^{p} (u_{j\ell}^{k^*} \wedge u_{j\ell}^{m}) \\
&\leq cost(H_{m-1}) + cost(V_m) - \sum_{j=1}^{n} \sum_{\ell \in L(j,k^*)} (u_{j\ell}^{k^*} \wedge u_{j\ell}^{m}) \\
&\leq cost(H_{m-1}) + cost(V_m) - \frac{1}{m-1} Q.
\end{aligned}
$$

Using the definition (3.15) of Q, we obtain

$$
\begin{aligned}
cost(H_m) &\leq cost(H_{m-1}) + cost(V_m) - \frac{1}{m-1}(cost(A_{m-1}) + cost(V_m) - c_m^{OPT}) \quad (3.16) \\
&= cost(H_{m-1}) + \frac{m-2}{m-1} cost(V_m) + \frac{1}{m-1}(c_m^{OPT} - cost(A_{m-1})). \quad (3.17)
\end{aligned}
$$

Note that the inequality (3.16)-(3.17) is valid for any sequential heuristic. But we are going to apply it next to the analysis of the heaviest-first heuristic.

3.3.2. A bound for the heaviest-first heuristic

We let $\mathcal{H}^{odd}(m) = \sum_{k=1}^{m} \frac{1}{2k-1}$. Then $\mathcal{H}^{odd}(m) = \mathcal{H}(2m-1) - \frac{1}{2}\mathcal{H}(m-1)$, where $\mathcal{H}(m) = \sum_{k=1}^{m} \frac{1}{k}$ is the harmonic function. It is well-known that $\ln(m+1) \leq \mathcal{H}(m) \leq 1 + \ln m$ for all $m \geq 1$. Thus, the function $\mathcal{H}^{odd}$ grows like $\frac{1}{2}\ln(m)$ and $\mathcal{H}^{odd}(m) \geq \frac{1}{2}\ln(m)$.

Theorem 5. *The heaviest-first heuristic H^{heavy} is a $(\frac{1}{2}(m - \mathcal{H}^{odd}(m-1)+1)$-approximation algorithm for MVA-m when the cost function c is additive. Thus, $\rho^{heavy}(m) \leq \frac{1}{2}(m - \mathcal{H}^{odd}(m-1) + 1) \leq \frac{1}{2}(m+1) - \frac{1}{4}\ln(m-1)$.*

Proof. Let H_m be the solution found by the heaviest-first heuristic. The proof proceeds by induction, starting with $m = 2$ and $\rho^{heavy}(2) = 1$.

Consider first the case where $cost(V_1) \leq \frac{m-1}{2m-3} c_m^{OPT}$, with V_1 the heaviest set. By induction,

$$cost(H_{m-1}) \leq \rho^{heavy}(m-1) c_{m-1}^{OPT},$$

where c_{m-1}^{OPT} is the cost of the optimal assignment for $V_1 \times \ldots \times V_{m-1}$. Let A_m again be an optimal assignment for $V_1 \times \ldots \times V_m$. Clearly, $c_{m-1}^{OPT} \leq cost(A_{m-1})$. Using this in

(3.16) together with $cost(V_m) \leq cost(V_1) \leq \frac{m-1}{2m-3} c_m^{OPT}$ yields

$$
\begin{aligned}
cost(H_m) &\leq \rho^{heavy}(m-1)cost(A_{m-1}) + (\frac{m-2}{m-1})(\frac{m-1}{2m-3})c_m^{OPT} + \frac{1}{m-1}(c_m^{OPT} - cost(A_{m-1})) \\
&\leq \rho^{heavy}(m-1)\big(cost(A_{m-1}) + c_m^{OPT} - cost(A_{m-1})\big) + \frac{m-2}{2m-3} c_m^{OPT} \\
&\leq \big(\rho^{heavy}(m-1) + \frac{m-2}{2m-3}\big) c_m^{OPT}.
\end{aligned}
$$

The alternative case is when $cost(V_1) \geq \frac{m-1}{2m-3} c_m^{OPT}$. Repeat the analysis leading to Eq. (3.14) in the second part of the proof of Theorem 4, but this time with V_1 replacing V_{m-1}. From there,

$$
\begin{aligned}
cost(H_m) &\leq cost(H_{m-1}) + cost(M_{1,m}) - cost(V_1) \\
&\leq \rho^{heavy}(m-1) c_{m-1}^{OPT} + c_m^{OPT} - \frac{m-1}{2m-3} c_m^{OPT} \\
&\leq \big(\rho^{heavy}(m-1) + \frac{m-2}{2m-3}\big) c_m^{OPT}.
\end{aligned}
$$

Altogether, we obtain the recurrence equation:

$$
\rho^{heavy}(m) = \rho^{heavy}(m-1) + \frac{m-2}{2m-3}.
$$

To analyze this relation, let $r_m = m - 2\rho^{heavy}(m)$. Then, $r_m - r_{m-1} = \frac{1}{2m-3}$, so that $r_m = r_2 + \sum_{k=2}^{m-1} \frac{1}{2k-1}$. Since $r_2 = 0$, $r_m = \mathcal{H}^{odd}(m-1) - 1$. ∎

The tightness of the bound established in Theorem 5 is discussed in Section 3.4.

3.4. Bad instances for additive cost functions

In this section we complement the previous results by showing that sequential algorithms can perform rather poorly even when the cost function is additive. (Recall that for monotone submodular nonadditive functions, the bound in Theorem 4 was already shown to be tight for all $m \geq 2$, even for the heaviest-first heuristic.)

Let us first consider the case $m = 3$. For MVA-3 with additive costs, Dokka et al. [6] established the validity and the tightness of the bounds established in Theorem 4 and Theorem 5, respectively. To see the former, observe that tightness of the bound $\rho^{seq}(3) = \frac{3}{2}$ follows from the instance depicted in Figure 1: indeed, for this instance, $c^{OPT} = 2$, whereas the sequential heuristic might find a solution with value 3.

To see that $\rho^{heavy}(3) = \frac{4}{3}$, consider the instance with $p = 3$, $V_1 = \{e_1, e_2, \mathbf{0}\}$, $V_2 = \{e_3, e_2, \mathbf{0}\}$, $V_3 = \{e_1, \mathbf{0}, e_3\}$. Its optimal value is $c^{OPT} = 3$, whereas H^{heavy} might produce first $H_2 = \{(e_1, e_3), (e_2, e_2), (\mathbf{0}, \mathbf{0})\}$, then $H_3 = \{(e_1, e_3, e_1), (e_2, e_2, \mathbf{0}), (\mathbf{0}, \mathbf{0}, e_3)\}$, with $cost(H_3) = 4$.

An obvious improvement to heuristics H^{seq} and H^{heavy} would be to run H^{seq} for all possible permutations of the sets $V_1, \ldots, V_m$ in the first step, then to retain the best of the $m!$ feasible solutions found (see Bandelt et al. [1], Crama and Spieksma [5] for related "multiple-pass" heuristics). Interestingly, when $m = 3$, it follows again from the previous example that this multiple-pass sequential heuristic (which involves solving six

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bipartite matching problems) has the same worst-case ratio as H^{heavy} (which only solves two matching problems). This observation also entails that the ratio $\rho(3) = \frac{4}{3}$ is tight for the iterative matching algorithm of Reda et al. [13].

Let us now turn to the general case $m \geq 3$. Dokka et al. [6] observed that the worst-case approximation ratio of the sequential heuristic can grow as fast as $\Omega(\sqrt{m})$ for certain instances with additive cost functions. We now strengthen this result by establishing a lower bound of the same order for the heaviest-first heuristic (and actually, for the multiple-pass heuristic mentioned above).

Theorem 6. *There exists an infinite sequence of values of m such that the heaviest-first heuristic produces a feasible assignment with cost larger than $\frac{\sqrt{m}}{2} c_m^{OPT}$ on certain instances of WWI-m.*

Proof. Fix an arbitrary positive integer r. We are going to describe an instance of WWI-m with $m = r^2 + 1$ and $n = p = 2r$. In order to simplify the description of the instance, we label the input sets from V_0 to V_{r^2}. We write v_{ij} to denote the jth vector of set V_i, $i = 0, \ldots, r^2$, $j = 1, \ldots, 2r$. The construction of the sets $V_0, V_1 \ldots, V_{r^2}$ is as follows. (An instance with $r = 3$ is displayed in Figure 4, and the corresponding heuristic and optimal solutions are illustrated in Figure 5.)

- In V_0, $v_{0j} = e_j$ for $j = 1, \ldots, r$, and $v_{0j} = \mathbf{0}$ for $j = r+1, \ldots, 2r$.

- For $i > 0$, write $i = (k-1)r + \ell$ with $k, \ell \in \{1, \ldots, r\}$. Then, in V_i,
  - for $j = 1, \ldots, r$, $v_{ij} = e_j$ if $j \neq \ell$ and $v_{i\ell} = \mathbf{0}$;
  - for $j = r+1, \ldots, 2r$, $v_{ij} = \mathbf{0}$ if $j \neq r+k$ and $v_{i,r+k} = e_{r+k}$.

For this instance, the optimal cost equals $2r$: for $j = 1, \ldots, 2r$, the jth tuple of the optimal assignment simply collects all vectors e_j (note that there is at most one such e_j in each set V_i).

However, the heaviest-first heuristic may find a solution with cost $r^2 + r$ as follows: First, note that $cost(V_i) = r$ for all i, so that H^{heavy} may consider the sets $V_0, V_1 \ldots, V_{r^2}$ in that order. When matching V_1 to V_0, H^{heavy} may assign the $(r+1)^{st}$ vector of V_1 to the first vector of V_0. In the next $r - 1$ assignment stages, it assigns the $(r+1)^{st}$ vector of V_i ($i = 2, \ldots, r$) to the tuple containing the ith vector of V_0. Then, in the next r assignments, H^{heavy} assigns the $(r+2)^{nd}$ vector of V_i ($i = r+1, \ldots, 2r$) to the tuple $i = 1, \ldots, r$ containing the ith vector of V_0. Proceeding in this way yields a solution with cost $r^2 + r$. ∎

4. WWI-3 is hard to approximate

As mentioned earlier, Reda et al. [13] have observed that WWI-m is NP-hard for all $m \geq 3$. A more detailed proof is found in Dokka et al. [6]. Our objective is now to strengthen this result by showing that WWI-3 does not admit a polynomial-time approximation scheme, unless P=NP.

We shall describe a reduction from 3-*bounded maximum* 3-*dimensional matching* (MAX-3DM-3) to WWI-3. An instance of MAX-3DM-3 consists of three pairwise disjoint sets X, Y, Z such that $|X| = |Y| = |Z| = q$, and of a set of triples $S \subseteq X \times Y \times Z$ such that every element of $X \cup Y \cup Z$ appears in at most three triples of S; let $|S| = s$.

V_0	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9
100000	000000	100000	100000	000000	100000	100000	000000	100000	100000
010000	010000	000000	010000	010000	000000	010000	010000	000000	010000
001000	001000	001000	000000	001000	001000	000000	001000	001000	000000
000000	000100	000100	000100	000000	000000	000000	000000	000000	000000
000000	000000	000000	000000	000010	000010	000010	000000	000000	000000
000000	000000	000000	000000	000000	000000	000000	000001	000001	000001

Figure 4: A bad instance for the heaviest-first heuristic with $r = 3$, $m = 10$

A *matching* in S is a subset $S' \subseteq S$ such that no element of $X \cup Y \cup Z$ appears in two triples of S'. The goal of the MAX-3DM-3 problem is to find a matching of maximum cardinality in S.

Kann [10] showed that MAX-3DM-3 is APX-hard. An instance of MAX-3DM-3 is called a *perfect instance* if its optimal solution consists of q triples that cover all elements of $X \cup Y \cup Z$ (that is, if S contains a feasible assignment). Petrank [12] proved that perfect instances of MAX-3DM-3 are hard to approximate, and that the existence of a polynomial-time approximation scheme for perfect instances would imply P=NP.

Now, consider an arbitrary perfect instance I' of MAX-3DM-3. We build a corresponding instance I of WWI-3 by using the gadget depicted in Figure 6, as explained next.

The instance I consists of three sets V_X, V_Y, V_Z, each of cardinality $q + 3s$. Each element e of each V_k, $k \in \{X, Y, Z\}$, is a 0-1 vector of length $6q + 4s$ containing exactly two nonzero elements. Thus, we have $m = 3$, $n = q + 3s$, and $p = 6q + 4s$. Now, we can view each e as an edge in an undirected graph $G = (U, A)$ where U is a vertex set with cardinality $6q + 4s$ and A can be identified with $V_X \cup V_Y \cup V_Z$. The elements of U are

- x_1, x_2 for each $x \in X$
- y_1, y_2 for each $y \in Y$
- z_1, z_2 for each $z \in Z$
- x_t, y_t, z_t and u_t for each triple $t \in S$

and the edges in $A = V_X \cup V_Y \cup V_Z$ are

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	OPT	H_{heavy}
	100000	100111
	010000	010111
	001000	001111
	000100	000000
	000010	000000
	000001	000000

Figure 5: Cost of optimal solution is 6; cost returned by H^{heavy} is 12

- $(x_1, x_2) \in V_X$ for each $x \in X$ (element edges)

- $(y_1, y_2) \in V_Y$ for each $y \in Y$ (element edges)

- $(z_1, z_2) \in V_Z$ for each $z \in Z$ (element edges)

- $(x_1, x_t) \in V_Y$, $(x_2, x_t) \in V_Z$, $(y_1, y_t) \in V_X$, $(y_2, y_t) \in V_Z$, $(z_1, z_t) \in V_X$, $(z_2, z_t) \in V_Y$, for each $t \in S$ (gadget edges)

- $(x_t, u_t) \in V_X$, $(y_t, u_t) \in V_Y$, $(z_t, u_t) \in V_Z$ for each $t \in S$ (gadget edges).

We say that an element of V_X (V_Y, V_Z) is an X-edge (Y-edge, Z-edge). The subgraph induced by all gadget edges associated with a same triple t is called the *gadget* associated with t and is denoted by $g(t)$. Note that $g(t)$ contains three element edges.

Clearly, a solution to an instance I of WWI-3 consists of a set of $q + 3s$ *vector-triples*, where each vector-triple consists of a vector from V_X, a vector from V_Y, and a vector from V_Z. In the context of the graph $G = (U, A)$, a vector-triple corresponds to a set of three edges, namely an X-edge, a Y-edge, and a Z-edge; we refer to such a triple of edges as a *feasible* triple. Thus, a *feasible triple* for WWI-3 consists of an X-edge, a Y-edge and a Z-edge. A feasible triple of edges $T \subseteq A$ defines (and can be identified with) a subgraph (U_T, T) of G, where U_T is the subset of vertices covered by T. We say that T is connected if (U_T, T) is connected, otherwise it is disconnected. The cost of T is $|U_T|$. Note that a feasible triple is either

- a triangle K_3; its cost equals 3, or

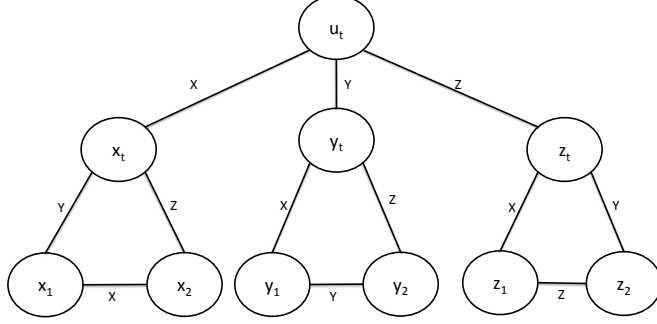- a claw $K_{1,3}$ or a path P_4; its cost equals 4, or

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Figure 6: Gadget $g(t)$ corresponding to a triple $t \in S$

- disconnected; its cost equals either 5 or 6.

A feasible assignment for I is a collection of $q + 3s$ feasible triples covering all edges of G. We now collect some properties of feasible assignments for further reference.

Lemma 13. *Let $M \subseteq V_X \times V_Y \times V_Z$ be a feasible assignment for I, with $|M| = q + 3s$.*
(1) M contains at most $3q$ triangles.
(2) The cost of M (and hence, the optimal value of I) is at least $q + 12s$.
(3) If the cost of M is $q+12s$, then M contains $3q$ triangles, $3s-2q$ additional connected triples, and no disconnected triples.
(4) If the cost of M is equal to $q + 12s + r$ $(r \geq 0)$, then M contains at least $3q - r$ triangles and at most r disconnected triples.

Proof. (1) We say M *covers* A', with $A' \subseteq A$, if all edges in A' are contained in M. Observe that M contains the same number of edges as A (namely, $3q + 9s$ edges), and hence, since M covers A, each edge of A must be covered exactly once. In particular, each element edge can be covered by at most one triangle, which implies that there are at most $3q$ triangles in M.

(2) The cost of M is equal to $3c_3 + 4c_4 + 5c_5 + 6c_6$, where c_k is the number of triples with cost equal to k, and $c_3 + c_4 + c_5 + c_6 = |M|$. There holds:

$$
\begin{aligned}
cost(M) &= 3c_3 + 4c_4 + 5c_5 + 6c_6 & (4.18) \\
&\geq 3c_3 + 4(|M| - c_3 - c_5 - c_6) + 5(c_5 + c_6) & (4.19) \\
&= -c_3 + (c_5 + c_6) + 4|M|. & (4.20)
\end{aligned}
$$

Since $c_3 \leq 3q$ and $c_5 + c_6 \geq 0$, Eq. (4.20) implies that the cost of M is at least $-(3q) + 4(q + 3s) = q + 12s$.

(3) The previous reasoning shows that the cost of $|M|$ can be equal to $q + 12s$ only if $c_3 = 3q$ and $c_5 + c_6 = 0$.

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(4) Intuitively, every missing triangle and every disconnected triple increases the cost of M by at least one unit with respect to the lower bound $q + 12s$, as expressed by the inequality (4.20). More formally, if $c_3 < 3q - r$, then Eq. (4.20) leads to

$$3c_3 + 4c_4 + 5c_5 + 6c_6 > -(3q - r) + 4|M| = q + 12s + r. \qquad (4.21)$$

Similarly, if $c_5 + c_6 > r$, then Eq. (4.20) together with $c_3 \leq 3q$ imply

$$3c_3 + 4c_4 + 5c_5 + 6c_6 > -3q + r + 4|M| = q + 12s + r. \qquad (4.22)$$

∎

We are now ready to establish the relation between the solutions of I and I'

Lemma 14. *If I' is a perfect instance of MAX-3DM-3, then the optimal value of I is $q + 12s$.*

Proof. If $t \in S$ is in the perfect matching, then use three triangles and the claw centered at u_t in the associated gadget $g(t)$. Otherwise, use three claws centered at x_t, y_t and z_t, respectively. Clearly, in the constructed solution for WWI-3 there are only triangles and claws with exactly $3q$ triangles. Hence, by Lemma 13 it follows that the cost of the solution is $q + 12s$. ∎

The converse statement will follow from Lemma 15 hereunder, with $\delta \geq 0$.

Lemma 15. *Let $\delta \geq 0$ be a real number. If instance I has a feasible solution with cost at most $q + 12s + \delta q$, then instance I' possesses a matching with size at least $(1 - 6\delta)q$.*

Proof. Consider a feasible solution M for instance I with cost at most $q + 12s + \delta q$. We call a gadget *damaged* (by M) if :
(Type (g)) at least one of its gadget edges is in a disconnected triple of M, or
(Type (e)) one of its element edges is not included in a triangle of M.
Equivalently, a gadget is *undamaged* if all its gadget edges are in connected triples of M and if all its element edges are in triangles of M.

We call an element edge *damaged* (by M) if it is not included in a triangle of M, or it is in a triangle contained in a damaged gadget. Equivalently, an element edge is *undamaged* if it is in a triangle contained in an undamaged gadget.

It follows from Lemma 13 that M contains at least $3q - \delta q$ triangles. Thus, at most δq element edges are not included in triangles.

Note that if an edge is damaged, then it is contained in a damaged gadget. Since I' is an instance of MAX-3DM-3, each element edge occurs in at most three gadgets. In particular, each damaged element edge can damage at most three gadgets, so that there are at most $3\delta q$ damaged gadgets of type (e).

Furthermore, Lemma 13 also implies that at most δq triples can be disconnected; these triples contain at most $3\delta q$ gadget edges, which can damage at most $3\delta q$ gadgets (damaged gadgets of type (g)).

Since each damaged gadget may yield at most three damaged element edges of type (ii), we find that, altogether there are at most $18\delta q$ damaged element edges, which leaves at least $3(1 - 6\delta)q$ undamaged element edges.
counting of δq potential damaged edges of type (i).)

Now, the main element of the proof of the lemma is the following claim:

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Claim 16. *Every undamaged element edge, say (x_1, x_2), is in a triangle (x_1, x_2, x_t) from some undamaged gadget $g(t)$. We claim that the other two element edges in $g(t)$ are also included in triangles from $g(t)$.*

(Proof of claim.) To see this, consider one of the other element edges in $g(t)$, say (y_1, y_2). Since $g(t)$ is undamaged, (y_1, y_2) must be covered by a triangle contained in a gadget $g(t')$. Assume by contradiction that $t \neq t'$ (otherwise, we are done).

Again because $g(t)$ is undamaged, the X-edge (y_1, y_t) is in a connected triple T, which must necessarily contain the Y-edge (y_t, u_t) (indeed, at vertex y_1, (y_1, y_t) is only incident to X-edges and to the Y-edge (y_1, y_2) which is already covered by a triangle in $g(t')$; so, T must contain either the Y-edge (y_t, u_t) or the Z-edge (y_2, y_t); but the latter case implies the former one).

The previous reasoning applies similarly to (y_2, y_t), so that the claw $\{(y_1, y_t), (y_2, y_t), (y_t, u_t)\}$ must be in M.

This implies, in turn, that (x_t, u_t) and (z_t, u_t) must be in the same triple, which can only contain (z_2, z_t) as Y-edge. Thus, (z_1, z_t) must be in a triple together with (z_1, z_2), contradicting the hypothesis that (z_1, z_2) is undamaged. (End of claim.)

Hence the $3(1 - 6\delta)q$ undamaged element edges can be divided into groups of three that correspond to $(1 - 6\delta)q$ undamaged gadgets. Then the corresponding $(1 - 6\delta)q$ triples in instance I' form a matching. ∎

We are now ready for the main result of this section.

Theorem 7. *WWI-3 is APX-hard even when all vectors in $V_X \cup V_Y \cup V_Z$ are 0–1 vectors with exactly two nonzero entries per vector.*

Proof. When we apply the reduction to a perfect instance I' of MAX-3DM-3, Lemma 14 yields $c^{OPT}(I) = q + 12s$ for the resulting instance I of WWI-3. A $(1+\epsilon)$-approximation algorithm for WWI-3 would imply that we can compute, in polynomial time, a solution of I with objective value at most equal to

$$(1 + \epsilon) \, c^{OPT}(I) \leq q + 12s + 37\epsilon \, q$$

(here we have used $s \leq 3q$). Then Lemma 15 (with $\delta = 37\epsilon$) implies the existence of a matching of size at least $(1 - 222\epsilon)$ for instance I', and this matching can be found in polynomial time. Hence, a PTAS for WWI-3 would imply a PTAS for any perfect instance of 3-bounded MAX-3DM. ∎

5. Binary inputs and fixed p

In this section we consider again the *binary* MVA problem, that is, the special case of MVA where all vectors in $V_1 \cup \ldots \cup V_m$ are binary. We want to argue that the binary MVA problem can be solved in polynomial time when p is fixed.

For an instance of the binary MVA problem, we let, as in Theorem 6, v_{ij} denote the jth vector in set V_i, $j = 1, \ldots, n$, $i = 1, \ldots, m$. Let $b_1, \ldots b_{2^p}$ be all distinct 0-1 vectors of length p, arbitrarily ordered, and consider a feasible m-tuple $(u^1, \ldots, u^m)$. We say that $(u^1, \ldots, u^m)$ is of *type t* if $u^1 \vee \ldots \vee u^m = b_t$.

We construct a mixed integer formulation of MVA featuring variables x_t:

$$x_t = \text{number of } m\text{-tuples of type } t \text{ in the assignment, } t = 1, \ldots, 2^p.$$

We also need assignment variables: for each $i = 1, \ldots, m; j = 1, \ldots, n; t = 1, \ldots, 2^p$,

$$z_{jt}^i \quad = \quad 1 \text{ if } v_{ij} \text{ is assigned to an } m\text{-tuple of type } t.$$

The formulation is now:

$$\min \sum_{t=1}^{2^p} c(b_t) x_t \tag{5.23}$$

$$\sum_{j:\ b_t \geq v_{ij}} z_{jt}^i = x_t \qquad \text{for each } t = 1, \ldots, 2^p, i = 1, \ldots, m, \tag{5.24}$$

$$\sum_{t:\ b_t \geq v_{ij}} z_{jt}^i = 1 \qquad \text{for each } j = 1, \ldots, n, i = 1, \ldots, m, \tag{5.25}$$

$$x_t \quad \text{integer} \qquad \text{for each } t = 1, \ldots, 2^p, \tag{5.26}$$

$$z_{jt}^i \geq 0 \qquad \text{for each } j = 1, \ldots, n, t = 1, \ldots, 2^p, i = 1, \ldots, m. \tag{5.27}$$

The objective function (5.23) minimizes the total cost. Constraints (5.24)-(5.25) are the familiar transportation constraints. Notice further that integrality of x_t implies integrality of z_{jt}^i.

Lemma 17. *Formulation (5.23)-(5.27) is a correct formulation of the binary MVA problem.*

Proof. Consider a feasible solution of the binary MVA problem. This solution prescribes, for each binary vector v_{ij} in each set V_i, whether this vector should be assigned to an m-tuple of type t. This determines the x_t and z_{jt}^i values, which clearly satisfy constraints (5.24)-(5.27).

Conversely, consider x_t, z_{jt}^i values that satisfy (5.24)-(5.27). One can construct a feasible solution of MVA-m as follows: (1) Create a set X containing a copy of vector b_t for each $x_t > 0$. (2) For each $i = 1, \ldots, m$, construct a bipartite graph $G = (V_i \cup X, E)$ where vector v_{ij} of V_i is connected with vector b_t of X if $v_{ij} \leq b_t$. The values x_t, z_{jt}^i define a feasible solution of the transportation problem with supply equal to 1 for each vertex in V_i and demand equal to x_t for vertex t in X. (3) Construct m-tuples of vectors by assigning m vectors – one from each V_i – to the same m-tuple if they all are matched to same vector in X in the solution of the transportation problem (there may be several ways of performing this step; however, any way suffices). This yields a feasible solution of the MVA problem with value at most equal to $\sum_{t=1}^{2^p} c(b_t) x_t$. Hence, the optimal value of (5.23)-(5.27) is equal to the optimal value of the MVA problem. ∎

Theorem 8. *Binary MVA can be solved in polynomial time for each fixed p.*

Proof. Lemma 17 shows that formulation (5.23)-(5.27) is correct. This formulation involves 2^p integer variables x_t, $O(mn2^p)$ continuous variables z_{jt}^i, and $O(m2^p + mn)$ constraints. When we fix p, this results in a fixed number of integer variables x_t, each of which takes at most $n + 1$ distinct values. Therefore, in order to find an optimal solution it is enough to check the feasibility of (5.24)-(5.27) for $O(n^{2^p})$ assignments of values to the x_t variables, and to choose the solution with the minimum cost. ∎

6. Conclusions

In this paper, we have considered the multi-dimensional vector assignment problem MVA-m and we have analyzed the performance of sequential heuristics for this problem in terms of their worst-case approximation ratio. We have also proved that MVA-m is APX-hard, even when $m = 3$. Among the main questions that remain open at this stage, let us mention the following ones:

1. What is the exact approximation ratio of the heaviest-first sequential heuristic in case of additive costs? We know that it lies between $\Omega(\sqrt{m})$ and $O(m - \ln m)$.
2. When the sets V_i are sorted by non-increasing weight, what is the approximation ratio of the sequential heuristic for WWI-m? We know that $\rho(3) = \frac{4}{3}$ is tight when $m = 3$ (see Section 3.4).
3. Does there exist a polynomial-time algorithm with constant (i.e., independent of m) approximation ratio for MVA-m?

Finally, we should mention that the working paper Dokka et al. [7], on which this article is based, also analyzes the performance of a class of so-called *hub heuristics*. For this class, we derive similar approximation ratios as for sequential heuristics; in particular, $\rho^{hub}(m) = \frac{m}{2}$ for certain variants of the heuristics.

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