Acceleration of the convergence of a non-overlapping domain decomposition method by an approximate deflation technique for high-frequency wave propagation

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Abstract— **The analysis of a non-overlapping domain decomposition method with optimized transmission conditions, applied to a simplified 1-D problem discretized by finite elements, is performed to better understand the spectral properties of the method. An approximate deflation preconditioner is then introduced to modify the spectrum of the iteration operator, and speed up the convergence of the GMRES algorithm used to solve the substructured problem.**

I. Introduction

Domain Decomposition Methods (DDM), combined to classical discretization methods and Krylov solvers, are very powerful for numerically solving PDEs. Though many variations of these methods have been described in the literature, few of them have proved to be effective in dealing with wave propagation, especially at high frequencies [1]. The main reason for the poor convergence of the solvers is to be found in the spectral properties of the associated iteration operator. It is therefore natural to try and get more insights on the spectrum of the operator, improve its spectral properties and speed up convergence.

II. Problem setting and DDM algorithm

As a simplified test problem, we solve the 1-D Helmholtz equation with wavenumber k in an interval $\bar{\Omega} = [0, 1]$, with Dirichlet boundary conditions on one side and Sommerfeld radiation condition to truncate the domain on the other side. The domain is decomposed into N equal-size, non-overlapping subdomains $\Omega_{i,1\leq i\leq N}$, with artificial boundaries $\Sigma_{ij} = \Sigma_{ji}$ between Ω_i and Ω_j. The iterative scheme, detailed in [2], uses impedance-type boundary conditions on Σ_{ij} and recasts the problem in terms of the set of interface data $g = \{g_{ij}, 1 \leq i \neq j \leq N, |i - j| = 1\}$:

$$\begin{aligned} \partial_n u_i^{(m+1)} + \mathcal{S} u_i^{(m+1)} &= -\partial_n u_j^{(m)} + \mathcal{S} u_j^{(m)} \quad \text{on } \Sigma_{ij} \\ &= g_{ij}^{(m)}, \end{aligned}$$

with the update:

$$\begin{aligned} g_{ij}^{(m+1)} &= -\partial_n u_j^{(m+1)} + \mathcal{S} u_j^{(m+1)} \quad \text{on } \Sigma_{ij} \\ &= -g_{ji}^{(m)} + 2\mathcal{S} u_j^{(m+1)}. \end{aligned}$$

We choose the operator $\mathcal{S} = -ik$, i.e. the DtN map for the problem, for optimizing the convergence [3]. This procedure can be rewritten as a fixed point iteration on the unknowns g:

$$Fg = b, \quad \text{with} \quad F = I - A, \tag{1}$$

where applying the operator A amounts to solving the subproblems and updating g. The solution of problem (1) can be accelerated using a Krylov subspace method, e.g. GMRES. We are therefore interested in the spectrum of operator F, that strongly influences the convergence of the solver.

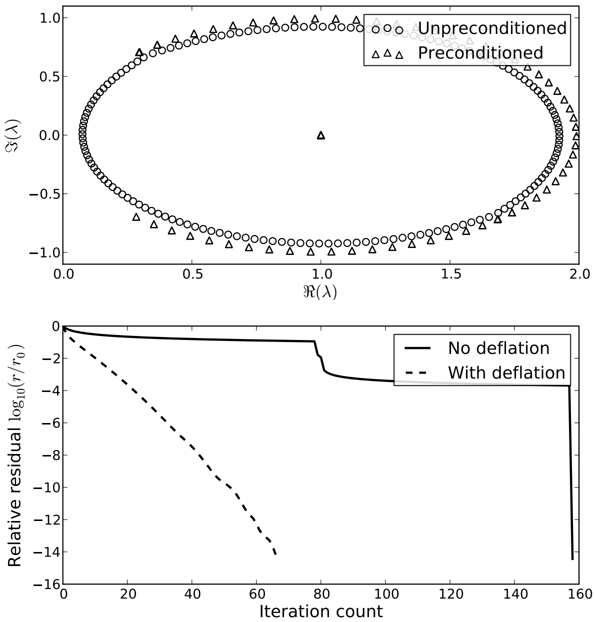

Fig. 1. (a) Eigenvalues distribution and (b) convergence of the GMRES algorithm used to solve the unpreconditioned and deflated systems. A combination of 42 backward and forward plane waves with wavenumbers $k_p'^{\pm} = \pm(k + 5p)_{-10\leq p\leq 10}$ was used as deflation basis. $N = 80$ subdomains ; $k = 188.5$.

III. Spectral analysis and approximate deflation

For the purpose of this analysis, the eigenvalues and eigenvectors of operator F are computed and the eigenvalues are plotted on Fig. 1(a). They are distributed along a circle centered at $(1, 0)$, with some of them approaching $(0, 0)$. These small eigenvalues cause the GMRES algorithm to fail in building a small dimension subspace containing a good approximation of the solution g. Using a (right) preconditioner Q_H inspired by eigenvalues deflation techniques [4], [5] to relocate the smallest eigenvalues at $(1, 0)$, but using plane waves with different wavenumbers k_p' as deflation vectors, produces a modification of the spectrum such that the eigenvalues of FQ_H are arranged on a slightly larger circle, without the smallest ones in magnitude. The convergence of the preconditioned system is smoother and faster, as shown on Fig. 1(b). Further results will be included in the full paper.

References

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