Geršgorin Variations III: On a theme of Brualdi and Varga

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Abstract

Brualdi brought to Geršgorin Theory the concept that the digraph $G(A)$ of a matrix A is important in studying whether A is singular. He proved, for example, that if, for every directed cycle of G(A), the product of the diagonal entries exceeds the product of the row sums of the moduli of the off-diagonal entries, then the matrix is nonsingular. We will show how, in polynomial time, that condition can be tested and (if satisfied) produce a diagonal matrix D, with positive diagonal entries, such that AD (where A is any nonnegative matrix satisfying the conditions) is strictly diagonally dominant (and so, A is nonsingular). The same D works for *all* matrices satisfying the conditions. Varga raised the question of whether Brualdi's conditions are sharp. Improving Varga's results, we show, if G is scwaltcy (strongly connected with at least two cycles), and if the Brualdi conditions do not hold, how to construct (again in polynomial time) a complex matrix whose moduli satisfy the given specifications, but is singular.

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1 Introduction

If $M = [m_{ij}]$ is a nonnegative matrix, then $\Omega(M)$ is the set of all complex matrices $A = [a_{ij}]$ such that, $|a_{ij}| = m_{ij}$ for all i and j. If $\mathcal{M}$ is a set of nonnegative matrices,

$$\Omega(\mathcal{M}) = \cup_{M \in \mathcal{M}} \Omega(M).$$

A *transversal* T of a nonnegative matrix of order n is a set of n positive entries of the matrix with no two of the entries in the same row or column; the product of these entries is the *value of the transversal* and is denoted by $v(T)$.

Let $a = (a_1, a_2, \ldots, a_n)$ and $r = (r_1, r_2, \ldots, r_n)$ denote positive vectors of order n, and let G denote a digraph (no loops allowed) with each vertex having positive outdegree. Let $\mathcal{M}(a, r, G)$ be the set of nonnegative matrices $A = [a_{ij}]$ such that $a_{ii} = a_i$ for all i, $a_{ij} > 0$ if and only if (i, j) is an edge of G for all $i \neq j$, and $\sum_{j \neq i} a_{ij} = r_i$ for all i. In particular, for every matrix A in $\mathcal{M}(a, r, G)$, $\prod_i a_i$ is the value of the transversal determined by the diagonal of A.

In this note, we shall prove various theorems related to the question: for given a, r, and G, is there a matrix in $\Omega(\mathcal{M}(a, r, G))$ which is singular? If there is no such matrix, prove it; if there is such a matrix, produce it. We shall decide these questions by calculations that take polynomial time, although we have made no effort to find optimal algorithms.

The work reported in this paper began with an attempt to improve the results of Varga [8] (see also his delightful book [9]) investigating whether the theorems of Brualdi (see [2] and also [3]) were sharp. Brualdi proved that a complex matrix with given positive diagonal and given off-diagonal row sums of moduli, and a given digraph, was nonsingular under certain conditions involving the cycles of the digraph. In this paper (Theorem 1.4), we show that if the diagonal is positive and the digraph is scwaltcy, then the Brualdi's results were exactly sharp; we also show how to construct a singular complex matrix in those cases, and we do it in polynomial time.

Returning to Brualdi's sufficient conditions for nonsingularity, we provide polynomial time algorithms for testing his conditions and (if satisfied) producing a positive vector x such that $a_i x_i > \sum_{j \neq i} |a_{ij}| x_j$ for all i and all complex matrices satisfying those hypotheses. Note that the x is computed *before* a particular A is chosen.

In what follows, a special role is played by the matrix $B(a, r, G) = [b_{ij}]$ defined by:

$$\begin{aligned} b_{ii} &= a_i, \text{ for all } i, \\ b_{ij} &= 0, \text{ if } i \neq j \text{ and } (i, j) \text{ is not an edge of } G, \text{ and} \\ b_{ij} &= r_i, \text{ if } i \neq j \text{ and } (i, j) \text{ is an edge of } G. \end{aligned}$$

Our attention is focussed on the following problem: find a transversal of $B(a, r, G)$ of maximum value. This is a multiplicative version of the well-known assignment problem of linear programming. It is solved by following any classic algorithm for the assignment problem and replacing every appearance of "add" ("subtract") with "multiply" ("divide"). The following results are established:

Theorem 1.1 *If the diagonal of $B(a, r, G)$ is the unique transversal of maximum value, then there is (and we can produce) a positive vector $x = (x_1, x_2, \ldots, x_n)$ such that, for all matrices $A = [a_{ij}]$ in $\mathcal{M}(a, r, G)$,*

$$a_i x_i > \sum_{j \neq i} a_{ij} x_j \quad (i = 1, 2, \ldots, n) \tag{1}$$

Theorem 1.2 *If the diagonal of $B(a, r, G)$ is a transversal of maximum value, but there is at least one transversal of lesser value, then there exists (and we can produce) a positive vector $x = (x_1, x_2, \ldots, x_n)$ such that, for all matrices $A = [a_{ij}]$ in $\mathcal{M}(a, r, G)$,*

$$a_i x_i \geq \sum_{j \neq i} a_{ij} x_j \quad (i = 1, 2, \ldots, n) \tag{2}$$

with strict inequality for at least one i.

Theorem 1.3 *If all transversals of $B(a, r, G)$ have the same value, then there exists (and we can produce) a positive vector x such that, for all matrices $A = [a_{ij}]$ in $\mathcal{M}(a, r, G)$,*

$$a_i x_i = \sum_{j \neq i} a_{ij} x_j \quad (i = 1, 2, \ldots, n). \tag{3}$$

Theorem 1.4 *If there is at least one transversal of $B(a, r, G)$ of greater value than the diagonal, then there exists (and we can produce) a singular complex matrix C in $\Omega(\mathcal{M}(a, r, G))$ and a complex vector z such that*

$$Cz = 0 \tag{4}$$

We prove Theorems 1.1, 1.2, and 1.3 in Section 2. Our principal tool is the duality theorem applied to the assignment problem. The duality theorem has been previously used in [1] in investigating transversals (called diagonal products). We prove Theorem 1.4 in Section 3. Our principal tool there is the Camion-Hoffman theorem [4] (see also [5]). In Section 4, we consider the situation where some of the a_i may be 0, and state and prove Theorem 4.1. In proving Theorems 1.4 and 4.1 we shall restrict ourselves to the nontrivial case where the digraph G is strongly connected with at least two cycles, which we abbreviate as *scwaltcy.*

2 Proofs of Theorems 1.1, 1.2, and 1.3

First, we introduce some additional notation. If A is a nonnegative matrix,

$$v(A) = \max\{v(T) : T \text{ a transversal of } A\}$$

is the *maximum value of a transversal* of A. Also, A_{ij} denoted the submatrix of A obtained by deleting row i and column j, and thus $v(A_{ij})$ is the maximum value of a transversal of A_{ij}. Finally, $v^{ij}(A)$ is the maximum value of a transversal of A that does *not* include the entry from position (i, j).

Lemma 2.1 *Let $A = [a_{ij}]$ be a nonnegative matrix of order n with at least one transversal. Then*

$$v(A) = \min\{y_1 y_2 \cdots y_n z_1 z_2 \cdots z_n : y_i z_j \geq a_{ij} \text{ for all } a_{ij} > 0\}. \tag{5}$$

Further, $v(A)$ and the y's and z's can be calculated in polynomial time.

Proof. Consider the assignment problem of linear programming changing sum to product. The duality statement in the lemma is just a restatement of the duality theorem for the assignment problem. This duality statement can be proved as a consequence of the algorithm, which picks out the optimum transversal and the dual variable y's and z's. This calculation can be done in polynomial time [7]. ♠

Lemma 2.2 . *If the diagonal of $B(a, r, G)$ is a transversal of maximum value, then there exist positive numbers $x_1, x_2, \ldots, x_n$ such that, for all edges (i, j) of G,*

$$a_i x_i \geq r_i x_j. \tag{6}$$

Proof. We apply Lemma 2.1 to $B(a, r, G)$. Then $y_i z_i = a_i$ for all i, because the diagonal is optimal and so $z_i = a_i / y_i$.

Let (i, j) be an edge of G. Then $y_i z_j \geq r_i$, and so $z_j \geq r_i / y_i$. Hence

$$z_j / z_i \geq r_i / a_i. \tag{7}$$

Set $x_i = 1/z_i$ for all i. Then equation (7) becomes

$$a_i x_i \geq r_i x_j \text{ for all edges } (i, j) \text{ in } G \tag{8}$$

which is the assertion of the lemma. ♠

Proofs of Theorems 1.1, 1.2, and 1.3: Suppose that all transversals of $B(a, r, G)$ have the same value. Then $y_i z_j = r_i$ for all edges (i, j) in G. Hence in the proof of Lemma 2.2, we have

$$a_i x_i = r_i x_j \text{ for all } (i, j) \text{ in } G. \tag{9}$$

Equation (9) implies that for each fixed i and all j such that (i, j) is an edge of G, x_j is constant. Since

$$r_i = \sum_{j:(i,j) \text{ is an edge}} a_{ij},$$

we obtain

$$a_i x_i = \sum_{j \neq i} a_{ij} x_j \text{ for all } i \text{ and all matrices } A = [a_{ij}] \text{ in } \mathcal{M}(a, r, G), \tag{10}$$

which is the assertion of Theorem 1.3.

Now suppose that (9) is false. Then

$$a_i x_i \geq r_i x_j \text{ for all edges } (i, j) \text{ of } G \tag{11}$$

with strict inequality for at least one edge $(i, j) = (p, q)$. Since $r_i = \sum_{j \neq i} a_{ij}$ for all i, we have

$$a_i x_i \geq (\sum_{j \neq i} a_{ij}) x_j \geq \sum_{j \neq i} a_{ij} x_j \tag{12}$$

for all i, with strict inequality for $i = p$, and this is the assertion of Theorem 1.2.

Now suppose that $V = \prod_i a_{ii} > v(T)$ for every transversal T of $B(a, r, G)$ different from its diagonal. Then $V > v^{ii}(B(a, r, G))$ for all i. There exists (and we can find) $t > 0$ such that for $a' = a - (t, t, \ldots, t)$, $B(a', r, G)$ satisfies the hypotheses of Theorem 1.2. Thus

$$(a_i - t)x_i \geq \sum_{j \neq i} a_{ij}x_j, \quad (i = 1, 2, \ldots, n)$$

and this implies that the assertion (1) of Theorem 1.1 holds. ♠

3 Proof of Theorem 1.4

In this section, we shall assume that the digraph G is scwaltcy. (All other cases are straighforward). Since G is strongly connected, each nonzero is part of a transvewrsal. Suppose that there is some transversal of $B(a, r, G)$ of greater value than the diagonal. Then, since G is scwaltcy, such a transversal contains a position (i, j) with $i \neq j$ (so (i, j) is an edge in G) such that G also contains a different edge (i, k). Let $A = [a_{ij}]$ be any matrix in $\mathcal{M}(a, r, G)$. If we make all a_{il} very small for $l \neq i, j$ or k then

$$v(A) = \max\{a_{ii}v(A_{ii}), a_{ij}v(A_{ij}), a_{ik}v(A_{ik})\}. \quad (13)$$

Now (13) will hold even as we vary a_{ij} and a_{ik}. Their sum will be close to r_i, so the maximum will be $a_{ij}v(A_{ij})$ when a_{ij} is close to r_i. But as we make a_{ij} smaller and a_{ik} larger by the same amount, at some point the two largest of the expressions on the right of (13) will be the same. Exactly when will involve solving linear equations in one variable. But when those two are the same, that matrix A in $\mathcal{M}(a, r, G)$ has the property that there is a complex matrix, the moduli of whose entries are specified by A, which is singular. The reason comes from the Camion-Hoffman theorem [4, 5]:

N is a nonnegative matrix such that all complex matrices whose entries have moduli specified by N are nonsingular if and only if there is a permuation matrix P and a positive diagonal matrix D such that PND is strictly diagonally domininant.

Since our A has at least two transverals of maximum value, A cannot have the stated property of N in the Camion-Hoffman theorem. Further, reading [4] or [5] will show how to construct the singular complex matrix C with moduli specified by the entries in A.

4 The case when the diagonal is not a transversal

Up until now we have assumed that the vector $a = (a_1, a_2, \ldots, a_n)$ was a positive vector. In this concluding section we only assume that a is a nonnegative vector. Since some of the diagonal entries may be 0, the matrix $B(a, r, G)$, and every matrix in $\mathcal{M}(a, r, G)$, may have positive entries that are not part of any transversal. The replacement of these entries with zeros, equivalently, removing the corresponding edges of G (this takes polynomial time), does not affect the determinant or the nonsingularity of a matrix. The result is a digraph which is a union of strongly connected digraphs. The only new interesting case is that of a scwaltcy digraph where at least one of the a_{ii} equals 0. With one or more zeros on the diagonal, the computations are easier. The proof of the following theorem follows the proof of Theorem 1.4.

Theorem 4.1 *If G is scwaltzy and at least one $a_{ii} = 0$, then there exists (and we can produce) a singular complex matrix C in $\mathcal{M}(a, r, G)$ and a complex vector z such that $Cz = 0$.*

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