Practical information

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- Assistant: Philippe Latour
- Evaluation
  - oral: theory + exercise
- Slides:
  - http://orbi.uliege.be
  - https://orbi.uliege.be/handle/2268/170889
- New from September 2019: manual of exercises:
  - http://hdl.handle.net/2268/239453
In this course:

1. Theory on signals and noise
2. Study of specific aspects of telecommunications systems
3. Towards engineering rules

But why?
Main components:
- signals
- electronics (transmitter, receiver, connectors, repeaters, etc).
- channel: cable (+ adapters), wireless
- propagation issues: antennas, fading

Main constraints:
- shared channel
  - bandwidth
- perturbations due to noise.

Goals of a good design:
- increase the Signal to Noise ratio ($\frac{S}{N}$) to reach the maximal channel capacity (typically for a white Gaussian noise)
- enable a communication with many users (seen as interference for the main link)

- power consumption

Main concerns related to signals

- Signal source handling (preparation of the signal, at the source, in the transmitter):
  - filtering (remove what is useless for communications)
  - analog ↔ digital (digitization)
  - remove the redundancy in the signal: compression

- Signal over the channel:
  - signal shaping to make it suitable for transmission (coding, modulation, multiplexing, etc)
  - signal power versus the noise signal (protect the signal against noise effects)
Outline

1 Reminder
2 Representation of bandpass signals
3 Noise in telecommunications systems
4 Digital modulation
5 Spread spectrum
6 Channels for digital communications and intersymbol interference
7 Navigation systems
8 Multiplexing
9 Telephone traffic engineering
10 Transmission over twisted pairs (fixed telephone network)
11 Radio engineering
- Physical layer (not all the transmission protocols)
- Deterministic - stochastic processes
- A tool for characterizing stochastic processes (including noise) and linear systems: the power spectrum or power spectral density
- Properties of the power spectrum
- Gaussian process
- Noise
- Modulation
- Digital communications
Deterministic vs stochastic tools

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Power Spectral Density (PSD)

$$E \left\{ X^2(t) \right\} = \int_{-\infty}^{+\infty} \gamma_X(f)df$$
**Deterministic vs stochastic tools**

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**More about the Power Spectral Density (PSD)**

Let $X(t)$ be a (wide-sense) *stationary* stochastic process (note that we use a "capital" letter $X$ for stochastic processes).

**Definition (Auto-correlation function)**

$$\Gamma_{XX}(\tau) = E\{X(t+\tau)X(t)\} \quad \forall t \quad (1)$$

Extremely useful because it expresses the average power (when $\tau = 0$):

$$\Gamma_{XX}(\tau = 0) = E\left\{X^2(t)\right\} \quad (2)$$

In practice, we have that the power $P_X$ of a stochastic process is given by:

$$P_X = E\left\{X^2(t)\right\} \quad (3)$$
More about the Power Spectral Density (PSD) II

Definition (Power spectrum or power spectral density of a stationary process PSD)

\[
\gamma_X(f) = \int_{-\infty}^{+\infty} \Gamma_{XX}(\tau) e^{-2\pi j f \tau} d\tau
\] (4)

Summary:

\[
P_X = E\{X^2(t)\} = E\{X(t+0)X(t)\} = \int_{-\infty}^{+\infty} \gamma_X(f)e^{2\pi j f 0} df
\] (6)

\[
= \int_{-\infty}^{+\infty} \gamma_X(f) df
\] (7)

Therefore, \(\gamma_X(f)\) expresses the distribution of power for all the frequencies.

Example of a PSD

Let us consider a signal with a random phase \(\theta\), uniformly distributed over \([-\pi, +\pi]\) (or, likewise, \([0, 2\pi]\))

\[
X(t) = A_c \cos (2\pi f_c t + \Theta)
\] (8)

\[ Mean of X(t)?\]

\[
\mu_X(t) = E\{X(t)\} = \int_{-\pi}^{+\pi} A_c \cos (2\pi f_c t + \theta) \frac{1}{2\pi} d\theta = 0
\] (9)

\[ Auto-correlation?\]

\[
\Gamma_{XX}(t_1, t_2) = E\{X(t_1)X(t_2)\}
\] (10)

\[ = \int_{-\pi}^{+\pi} A_c \cos (2\pi f_c t_1 + \theta) A_c \cos (2\pi f_c t_2 + \theta) \frac{1}{2\pi} d\theta
\]

\[ = \frac{A_c^2}{2} \cos [2\pi f_c (t_2 - t_1)] = \frac{A_c^2}{2} \cos [2\pi f_c \tau]
\] (11)

The signal is stationary. So,

\[ Power spectral density?\]

\[
\gamma_X(f) = \frac{A_c^2}{4} [\delta(f - f_c) + \delta(f + f_c)]
\] (12)
Practical link between $x(t)$ and $X(t)$? I

Let:
- $X(t)$ be the stochastic (unknown) process behind the scenes,
- $x(t)$ be the observation. This is what you measure/observe.

You can estimate the PSD of $X(t)$ by taking the Fourier transform of the observation $x(t)$.

**Proof.**
Assume $x(t)$ is deterministic and has finite energy, that is
\[
\int_{-\infty}^{+\infty} |x(t)|^2 \, dt \quad (13)
\]

Let us define:
- a “pseudo” auto-correlation function by
  \[
  \Gamma_{xx}(\tau) = \int_{-\infty}^{+\infty} x(t)x(t + \tau) \, dt \quad (14)
  \]

Practical link between $x(t)$ and $X(t)$? II

- and a “pseudo” PSD, which is an estimate of the true PSD, as
  \[
  \gamma_x(f) = \int_{-\infty}^{+\infty} \Gamma_{xx}(\tau) e^{-2\pi j f \tau} \, d\tau \quad (15)
  \]

Some calculations lead to:
\[
\gamma_x(f) = \int_{-\infty}^{+\infty} \Gamma_{xx}(\tau) e^{-2\pi j f \tau} \, d\tau \quad (16)
\]
\[
= \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} x(t)x(t + \tau) \, dt \right) e^{-2\pi j f \tau} \, d\tau \quad (17)
\]
\[
= \int_{-\infty}^{+\infty} x(t) \left( \int_{-\infty}^{+\infty} x(t + \tau) e^{-2\pi j f \tau} \, d\tau \right) \, dt \quad (18)
\]
\[
= \int_{-\infty}^{+\infty} x(t) \left( \mathcal{X}(f) e^{2\pi j f t} \right) \, dt \quad (19)
\]
\[
= \mathcal{X}(f) \int_{-\infty}^{+\infty} x(t) e^{2\pi j f t} \, dt \quad (20)
\]
\[
= \mathcal{X}(f) \mathcal{X}^*(f) \quad (21)
\]
\[
= \|\mathcal{X}(f)\|^2 \quad (22)
\]
In decibels (remember that $v \leftrightarrow 10 \log_{10}(v) [dB]$),

$$\gamma_x(f) [dB] = 20 \log_{10} \|\mathcal{X}(f)\| \quad (23)$$

So, $\gamma_x(f) [dB]$ can be estimated with the help of $\|\mathcal{X}(f)\|$.

However,

- $\gamma_x(f)$ is not based on a single observation and therefore $\gamma_x(f)$ is not equal to $\gamma_X(f)$.

<table>
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<tr>
<th>$x , [W]$</th>
<th>$10 \log_{10}(x) , [dBW]$</th>
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<tr>
<td>1 [W]</td>
<td>0 [dBW]</td>
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<tr>
<td>2 [W]</td>
<td>3 [dBW]</td>
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<tr>
<td>0.5 [W]</td>
<td>−3 [dBW]</td>
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<tr>
<td>5 [W]</td>
<td>7 [dBW]</td>
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<tr>
<td>$10^n$ [W]</td>
<td>$10 \times n$ [dBW]</td>
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What if a process is not stationary?

Stationary (in the wide sense) means

- the mean of $X(t)$ is time independent:

$$\mu_X(t) = \mu_x \quad \forall t \quad (24)$$

- the autocorrelation only depends on time differences (and not the origin of time):

$$\Gamma_{XX}(t_1, t_2) = \Gamma_{XX}(t_2 - t_1) \quad \forall t_1, t_2 \quad (25)$$

If a process $X(t)$ is not stationary, then

- you cannot define a PSD for a non-stationary process.
- But, you can sometimes make it stationary.

2 classical ways worth trying:

1. inject a random (independent) phase, suited for modulated signals: $S(t) = X(t) \cos(2\pi f_c t + \Phi)$ with $\text{pdf}_\Phi(\phi) = \frac{1}{2\pi}$ for $\phi \in [0, 2\pi]$.
2. inject a random (independent) time shift (≡jitter), suited for digital signals: $X(t + T_0)$ with $\text{pdf}_{T_0}(t_0) = \frac{1}{T}$ for $t_0 \in [0, T]$. 
Consider a stationary process $X(t)$, a linear system whose transfer function is given by $H(f)$, and $Y(t)$ the output process.

**Theorem (Mean of a filtered stochastic process)**

Mean of $Y(t)$:

$$\mu_Y = \mu_X H(0) \quad (26)$$

**Theorem (Wiener-Kintchine)**

Power spectrum of a filtered stochastic process $Y(t)$:

$$\gamma_Y(f) = \|H(f)\|^2 \gamma_X(f) \quad (27)$$

**Useful considerations about the PSDs**

- **Sum of (stationary) stochastic processes:**
  
  $$Y(t) = K(t) + N(t) \quad (28)$$

  If both signals are uncorrelated (which they are if they are independent), then

  $$\gamma_{YY}(f) = \gamma_{KK}(f) + \gamma_{NN}(f) \quad (29)$$
What does it mean that a process $X(t)$ is “Gaussian”?

- the probability density function (pdf) of its voltage/current is Gaussian distributed:

$$pdf_X = f_X(x) = \frac{1}{\sigma_X \sqrt{2\pi}} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$$  \hspace{1cm} (30)

- the mean and variance of $X$ suffice to characterize it.

- it is a good approximation for the sum of a number of independent random variables with arbitrary one-dimensional pdfs.

Useful properties:

- If the *input* of a linear system is a Gaussian stochastic process, then the *output* is also a Gaussian process.

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**Definition (White noise)**

A white noise is defined as a stochastic process whose power spectral density is constant for each frequency

$$\gamma_N(f) = \frac{N_0}{2} \left[ \frac{W}{Hz} \right]$$  \hspace{1cm} (31)

In practice, there is no “pure” white noise, but it does not matter as long as the PSD is constant inside the useful bandwidth.

A common signal in telecommunications is a *wide-sense stationary zero-mean white Gaussian noise*:

- the probability density function of the voltage of the noise is a Gaussian.

- the observed mean voltage has a zero mean.

- its power spectrum is constant for each frequency.
Fact

Power of a white noise (for a $B$ large bandwidth)

$$P_N = N = \int_{-\infty}^{+\infty} \gamma_N(f) \, df = 2 \int_{f_c - \frac{B}{2}}^{f_c + \frac{B}{2}} \frac{N_0}{2} \, df = 2 \times B \times \frac{N_0}{2} = B \, N_0$$

(32)

Modulation

Principle: modulation is all about using of a carrier $f_c$ for transmitting information

$$s(t) = A(t) \cos (2\pi f(t) t + \phi(t))$$

Frequency modulation [FM]

Amplitude modulation [AM]

Phase modulation [PM]
Consequences of modulation

- On analog signals:
  - frequency band is shifted towards the carrier frequency ($\Rightarrow f_c$)
  - bandwidth modification

- On digital signals:
  - power spectral density is shifted

\[
\gamma_S(f) = \frac{1}{4} \left[ \gamma_M(f - f_c) + \gamma_M(f + f_c) \right]
\] (33)

- shape of the power spectral density may be modified

Rice’s decomposition

An expression such as

\[
s(t) = A(t) \cos(2\pi f_c t + \phi(t))
\] (34)

may also be written as

\[
s(t) = A(t) \cos(2\pi f_c t + \phi(t)) = s_I(t) \cos(2\pi f_c t) - s_Q(t) \sin(2\pi f_c t)
\] (35)

such that

\[
s_I(t) = A(t) \cos \phi(t)
\] (36)

\[
s_Q(t) = A(t) \sin \phi(t)
\] (37)

Rice’s decomposition is essential because, as seen later, it means that any modulated signal can be decomposed into two amplitude modulated signals.

Many receivers in telecommunications use this principle!
Information theory and channel capacity: there is maximum bit rate! I

**Theorem (Shannon-Hartley)**

*Channel capacity* $C$ *(conditions for the error rate $P_e \to 0$)*

$$C [b/s] = B \log_2 \left( 1 + \frac{S}{N} \right)$$  \hspace{5mm} (38)

where

- $B$ is the channel bandwidth in Hz
- $\frac{S}{N}$ the signal-to-noise ratio (in watts/watts, not in dB).

---

**On the importance of the \( \frac{E_b}{N_0} \) ratio for digital transmissions**

Assume infinite bandwidth and a Gaussian white channel,

$$C = \lim_{B \to \infty} \left\{ B \log_2 \left( 1 + \frac{S}{N} \right) \right\}$$  \hspace{5mm} (39)

As

- $S = E_b R_b$ ($E_b$ is the energy of one bit and $R_b = \frac{1}{T_b}$ is the bitrate)
- $N = B N_0$

Therefore,

$$C = \lim_{B \to \infty} \left\{ B \log_2 \left( 1 + \frac{E_b R_b}{B N_0} \right) \right\} = \lim_{x \to 0} \left\{ \frac{\log_2 \left( 1 + x \frac{E_b R_b}{N_0} \right)}{x} \right\}$$

$$H = \log_2 e \lim_{x \to 0} \left\{ \frac{1}{1 + x \frac{E_b R_b}{N_0}} \right\} = \frac{1}{\ln 2} \frac{E_b R_b}{N_0}$$  \hspace{5mm} (40)

At maximum capacity: $C = R_b$, so that $\frac{E_b}{N_0} = \ln 2 \equiv -1.59 \ [dB]$ is the absolute minimum.
Motivations:

- it is important for linear systems, and most communication systems are linear (channel, filter, etc).
- channels can be shared if signals are “frequency”-friendly (multiplexing).

What type of frequency signals do we have?

- in its original form, a signal is in its baseband (voice is in $[300, 3400]$ Hz).
- bandwidth $\equiv$ band occupied by the signal.
- for digital signals, sampling $\rightarrow$ sampling frequency (driven by Nyquist’s criterion and relating to the highest frequency, thus the frequency content).
- spectrum
  - $\mathcal{V}(f)$ for deterministic signals
  - $\gamma_{\mathcal{V}}(f)$ for stochastic processes.
Purposes of dealing with bandpass signals specifically

1. Lower the sampling frequency (not twice the highest frequency!)
2. There are many more convenient representations of such a signal (analytic signal, complex envelope, baseband equivalent, quadrature components, etc):
   1. They are equivalent, but not equal.
   2. Which one is the most appropriate depends on the context (hard to foresee).
   3. See these representations as tools!
   4. The many representations are convenient theoretical and practical ways to process signals.
   5. Almost every receiver uses the underlying theory of bandpass signals.

Representation of bandpass signals: outline

- Sampling of band-limited signals
- Representations of deterministic band-limited (bandpass) signals:
  - bandpass equivalent,
  - analytic signal,
  - complex envelope,
  - and more!
- Representation of bandpass systems
- Representation of bandpass stochastic processes
Theorem (“Revised” sampling theorem)

Assume that \( v(t) \) is a deterministic, energy limited signal, with a \( W \) large bandwidth, whose Fourier \( \mathcal{V}(f) \) transform is upper bounded by \( f_u \) (that is \( \mathcal{V}(f) = 0 \) for \( f > f_u \)). Then, it is possible to characterize this signal with samples \( v[nT_s] \), \( n \in \{-\infty, +\infty\} \), taken at a \( f_s \) sampling frequency if this frequency is equal to \( f_s = \frac{2f_u}{k} \), where \( k \) is the largest integer strictly smaller than \( \frac{f_u}{W} \).

It should be noted that not all sampling frequencies are valid (in order to reconstruct \( v(t) \) perfectly), except for all those that are strictly larger than \( 2f_u \).

Important practical consequences:

1. sampling frequency: from \( f_s = 2f_u \) to \( f_s = \frac{2f_u}{k} \). \( k \) times less!

2. there are other ways to look at bandpass signals: new representations!

Definitions:

A bandpass signal \( v(t) \) is a signal if there exist two values \( W \) and \( f_0 \), such that \( W \ll f_0 \), and

\[
\forall f \notin \left[ f_0 - \frac{W}{2}, f_0 + \frac{W}{2} \right], \; \mathcal{V}(f) = 0 \tag{41}
\]

Assume a deterministic bandpass signal \( v(t) \). \( \overline{v}(t) \) is an equivalent baseband of \( v(t) \) if there exists a frequency \( f_0 \), comprised inside the frequency band of \( v(t) \), such that

\[
v(t) = \text{Re} \left( \overline{v}(t)e^{2\pijf_0t+j\varphi_0} \right) \tag{42}
\]

Note that \( \overline{v} \) in fact covers a family of equivalent baseband signals \( (\overline{v}(t) = (v(t) + jz(t))e^{-2\pijf_0t-j\varphi_0} \) are all valid candidates).
**Definition (Bandpass)**

A *bandpass* signal \( v(t) \) is a signal if there exist two values \( W \) and \( f_0 \), such that \( W \ll f_0 \), and

\[
\forall f \not\in \left[ f_0 - \frac{W}{2}, f_0 + \frac{W}{2} \right], \quad \mathcal{V}(f) = 0 \tag{41}
\]

**Definition (Equivalent baseband)**

Assume a deterministic bandpass signal \( v(t) \). \( \mathcal{V}(t) \) is an *equivalent baseband* of \( v(t) \) if there exists a frequency \( f_0 \), comprised inside the frequency band of \( v(t) \), such that

\[
v(t) = \text{Re} \left( \mathcal{V}(t) e^{2\pi jf_0 t + j\phi_0} \right) \tag{42}
\]

Note that \( \mathcal{V} \) in fact covers a family of equivalent baseband signals \((\mathcal{V}(t) = (v(t) + jz(t))e^{-2\pi jf_0 t - j\phi_0} \) are all valid candidates).

**Summary of some representations of a bandpass signal**

\[
v(t) \quad \leftarrow \quad \text{Re}(.) \quad \otimes \left( \delta(t) + \frac{j}{\pi t} \right) \quad \otimes e^{-2\pi jf_0 t} \quad \otimes e^{+2\pi jf_0 t} \quad \text{Re}(.) \quad \otimes \left( \delta(t) + \frac{j}{\pi t} \right) \quad \otimes e^{-2\pi jf_0 t} \quad \otimes e^{+2\pi jf_0 t} \quad \equiv \quad v_I(t) \cos(.) - v_Q(t) \sin(.) \quad \equiv \quad v_I(t) + jv_Q(t) \quad \equiv \quad v_a(t) \quad \otimes e^{-2\pi jf_0 t} \quad \otimes e^{+2\pi jf_0 t} \quad \equiv \quad e_v(t) \quad \equiv \quad a_v(t)e^{j\phi_v(t)}
\]
Why do we double the height of $\mathcal{V}(f)$ to define $\mathcal{V}_a(f)$?
Let us consider the following filter that removes the negative frequency components of a signal with the (Heaviside) step function:

\[
\mathcal{H}(f) = \begin{cases} 
0 & \text{if } f < 0 \\
2 & \text{if } f \geq 0
\end{cases} = 1 + \text{sign}(f) \tag{43}
\]

whose impulse response is

\[
h(t) = \delta(t) + \frac{j}{\pi t} \tag{44}
\]

**Definition (Analytic signal)**

A signal which has no negative-frequency components is called an analytic signal. In the time domain, it is obtained as

\[
\nu_a(t) = \nu(t) \otimes \left( \delta(t) + \frac{j}{\pi t} \right) \tag{45}
\]

\[
= \nu(t) + j\nu(t) \otimes \frac{1}{\pi t} \tag{46}
\]

The norm of the analytic signal is named “envelope”.
Hilbert transform

Definition (Hilbert transform)

The Hilbert transform of a signal \( v(t) \), denoted as \( \tilde{v}(t) \), is defined by

\[
\tilde{v}(t) = v(t) \otimes \frac{1}{\pi t}
\]  

(47)

With this definition,

\[
v_a(t) = v(t) + j\tilde{v}(t)
\]

(48)

Properties of the Hilbert transform

- The energy (or power) of a signal and that of its Hilbert transform are equal.

- [Hilbert transform of a modulated signal] Assume that \( v(t) \) is a baseband signal, then

\[
v(t) \cos(2\pi f_c t) = v(t) \sin(2\pi f_c t)
\]

(49)
### Definition (Hilbert transform)

The **Hilbert transform** of a signal $v(t)$, denoted as $\tilde{v}(t)$, is defined by

$$
\tilde{v}(t) = v(t) \otimes \frac{1}{\pi t}
$$

(47)

With this definition,

$$
v_a(t) = v(t) + j\tilde{v}(t)
$$

(48)

### Properties of the Hilbert transform

- The energy (or power) of a signal and that of its Hilbert transform are equal.
- **[Hilbert transform of a modulated signal]** Assume that $v(t)$ is a baseband signal, then

$$
v(t) \cos(2\pi f_c t) = v(t) \sin(2\pi f_c t)
$$

(49)

---

### Going backwards

$$
v_a(t) = v(t) + j\tilde{v}(t)
$$

(50)

So,

$$
v(t) = \text{Re} (v_a(t))
$$

(51)
Properties of the analytic signal

- it has no negative frequency
- it carries the same power as the original signal
- \( v(t) \) can be reconstructed from \( v_a(t) \)
- \( v_a(t) \) is not a real signal. Is that a problem?

**Why would we want to use** \( v_a(t) \)?

Baseband representation derived from the analytic signal I

\[
E_v(f) = V_a(f + f_0) \quad (52)
\]
**Definition (Complex envelope of a signal)**

The signal that results from a right-to-left shift of the analytic signal in the frequency domain is named the **complex envelope** of the signal. It is denoted as \( e_v(t) \).

Mathematically, the complex envelope and its spectrum are related to the analytic signal as follows:

\[
e_v(t) = v_a(t)e^{-2\pi j f_0 t} \quad (53)
\]

\[
E_v(f) = V_a(f + f_0) \quad (54)
\]

It remains to find a good practical method to determine the complex envelope!

---

**Question**

What sampling frequency to choose for the complex envelope?
In-phase component: \( v_I(t) \)

\[
\begin{align*}
 v_I(t) & = \text{Re} \left( e_v(t) \right) \\
        & = \text{Re} \left( v_a(t) e^{-2\pi jf_0 t} \right) \\
        & = \text{Re} \left( (v(t) + j\tilde{v}(t)) e^{-2\pi jf_0 t} \right) \\
        & = v(t) \cos(2\pi f_0 t) + \tilde{v}(t) \sin(2\pi f_0 t)
\end{align*}
\] (59)

Quadrature component: \( v_Q(t) \)

\[
\begin{align*}
 v_Q(t) & = \text{Im} \left( e_v(t) \right) \\
        & = \text{Im} \left( v_a(t) e^{-2\pi jf_0 t} \right) \\
        & = -v(t) \sin(2\pi f_0 t) + \tilde{v}(t) \cos(2\pi f_0 t)
\end{align*}
\] (62)

Therefore,
Theoretical determination of Rice components II

\[ v(t) = \text{Re} \left( e_v(t) e^{2\pi j f_0 t} \right) \]  \hspace{1cm} (63)

\[ = \text{Re} \left( (v_I(t) + jv_Q(t)) e^{2\pi j f_0 t} \right) \]  \hspace{1cm} (64)

\[ = v_I(t) \cos(2\pi f_0 t) - v_Q(t) \sin(2\pi f_0 t) \]  \hspace{1cm} (65)

\[ v(t) \xleftarrow{v_I(t) \cos(\cdot) - v_Q(t) \sin(\cdot)} v_I(t) + jv_Q(t) \]

\[ \text{Re}(\cdot) \xleftarrow{\otimes (\delta(t) + j \pi t) \times e^{-2\pi j f_0 t}} e_v(t) \xrightarrow{\equiv} a_v(t)e^{j \phi_v(t)} \]

Theoretical determination of Rice components III

Figure: Recover \( v(t) \) starting from the Rice components.

\[ v_I(t) \xrightarrow{\times} \cos(2\pi f_c t) \]

\[ \frac{\pi}{2} \]

\[ \sin(2\pi f_c t) \]

\[ v_Q(t) \xrightarrow{\times} \]

\[ + \]

\[ v(t) \]

\[ + \]

\[ v(t) \]

Figure: Recover \( v(t) \) starting from the Rice components.
Any bandpass signal can be seen as the sum of two modulated signals by lowpass signals.

Theoretically, the components of a bandpass signal can be computed using the Hilbert transform. The diagram illustrates this process, where $v(t)$ is multiplied by the Hilbert transform of $v(t)$, resulting in $v_I(t)$.
\[ v(t) \times 2 \cos(2\pi f_0 t) = 2 \left[ v_I(t) \cos(2\pi f_0 t) - v_Q(t) \sin(2\pi f_0 t) \right] \cos(2\pi f_0 t) \]
\[ = 2 \left[ v_I(t) \cos^2(2\pi f_0 t) - v_Q(t) \sin(2\pi f_0 t) \cos(2\pi f_0 t) \right] \]
\[ = v_I(t) + v_I(t) \cos(4\pi f_0 t) - v_Q(t) \sin(4\pi f_0 t) \]

And after a low-pass filter,

\[ v(t) \times 2 \cos(2\pi f_0 t) \rightarrow \text{lowpass filter} \rightarrow v_I(t) \quad (68) \]

Likewise,

\[ v(t) \times 2 \sin(2\pi f_0 t) \rightarrow \text{lowpass filter} \rightarrow -v_Q(t) \quad (69) \]
Amplitude/phase representation of the complex envelope I

\[ e_v(t) = v_I(t) + jv_Q(t) = a_v(t) e^{j\phi_v(t)} \quad (70) \]

Amplitude/phase representation of the complex envelope II

By substituting \( e_v(t) \) by its amplitude + phase description, we have

\[ v(t) = v_I(t) \cos(.) - v_Q(t) \sin(.) \]

\[ \equiv \Re(__.) \]

\[ \times e^{-2\pi j f_0 t} \]

\[ \otimes \left( \delta(t) + \frac{j}{\pi t} \right) \]

\[ \times e^{+2\pi j f_0 t} \]

\[ v_a(t) \]

\[ \equiv \]

\[ e_v(t) \]

\[ a_v(t)e^{j\phi_v(t)} \]

with, by definition,

\[ a_v(t) = \sqrt{v_I^2(t) + v_Q^2(t)} \quad (74) \]

\[ \phi_v(t) = \tan^{-1} \frac{v_Q(t)}{v_I(t)} \quad (75) \]
Conclusions for $v(t) = a_v(t) \cos(2\pi f_0 t + \phi_v(t))$

1. the amplitude $a_v(t)$ of the complex envelope is the envelope of the original signal $v(t)$.
2. the instantaneous phase of $v(t)$ is given by the phase of the complex envelope.

Any bandpass signal can be seen as a signal modulated both in amplitude and in phase.

Linear bandpass systems

Assume a bandpass filter, around $f_0$, whose response filter is

$$h(t) = \text{Re} \left( e_h(t)e^{2\pi j f_0 t} \right)$$

(76)

The filtered signal is given by

$$y(t) = v(t) \otimes h(t)$$

(77)

$$= \int_{-\infty}^{+\infty} h(\lambda)v(t - \lambda)d\lambda$$

(78)

Thesis:

$$e_y(t) = \frac{1}{2} e_h(t) \otimes e_v(t)$$

(79)
Assume a bandpass filter, around $f_0$, whose response filter is

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**Thesis:**

$$e_y(t) = \frac{1}{2} e_h(t) \otimes e_v(t) \quad (79)$$

**Proof I**

$$y(t) = v(t) \otimes h(t) = \int_{-\infty}^{+\infty} h(\lambda)v(t - \lambda)d\lambda \quad (80)$$

To get rid of $\text{Re}()$, we use

$$\text{Re}(a + jb) = \frac{a + jb}{2} + \frac{a - jb}{2} = \frac{a + jb}{2} + (\frac{a + jb}{2})^* \quad (81)$$

So, $v(t - \lambda) = \frac{1}{2} \left( e_v(t - \lambda)e^{2\pi jf_0 t}e^{-2\pi jf_0 \lambda} + e_v^*(t - \lambda)e^{-2\pi jf_0 t}e^{2\pi jf_0 \lambda} \right)$ and

$h(\lambda) = \frac{1}{2} \left( e_h(\lambda)e^{2\pi jf_0 \lambda} + e_h^*(\lambda)e^{-2\pi jf_0 \lambda} \right)$

Therefore,

$$y(t) = \frac{1}{4} e^{2\pi jf_0 t} \int_{-\infty}^{+\infty} e_h(\lambda)e_v(t - \lambda)d\lambda + \frac{1}{4} e^{-2\pi jf_0 t} \int_{-\infty}^{+\infty} e_h^*(\lambda)e_v^*(t - \lambda)d\lambda$$

$$+ \frac{1}{4} e^{-2\pi jf_0 t} \int_{-\infty}^{+\infty} e_h(\lambda)e_v^*(t - \lambda)e^{4\pi jf_0 \lambda}d\lambda$$

$$+ \frac{1}{4} e^{2\pi jf_0 t} \int_{-\infty}^{+\infty} e_h^*(\lambda)e_v(t - \lambda)e^{-4\pi jf_0 \lambda}d\lambda$$
\[ y(t) = \frac{1}{2} \text{Re} \left( \int_{-\infty}^{+\infty} e_h(\lambda) e_v(t - \lambda) d\lambda e^{2\pi j f_0 t} \right) \quad (82) \]

\[ + \frac{1}{2} \text{Re} \left( \int_{-\infty}^{+\infty} e_h^*(\lambda) e_v(t - \lambda) e^{-4\pi j f_0 \lambda} d\lambda e^{2\pi j f_0 t} \right) \quad (83) \]

But, \( e_h^*(\lambda) e_v(t - \lambda) \) is low frequency and \( e^{-4\pi j f_0 \lambda} \) is high frequency. Therefore,

\[ \int_{-\infty}^{+\infty} e_h^*(\lambda) e_v(t - \lambda) e^{-4\pi j f_0 \lambda} d\lambda \approx 0 \quad (84) \]

Finally,

\[ y(t) = \frac{1}{2} \text{Re} \left( (e_h(t) \otimes e_v(t)) e^{2\pi j f_0 t} \right) \quad (85) \]

or

\[ e_y(t) = \frac{1}{2} e_h(t) \otimes e_v(t) \quad (86) \]

**Bandpass filtering**

*Use of \( e_y(t) = \frac{1}{2} e_h(t) \otimes e_v(t) \)*

- the output is a bandpass signal too (\( \Rightarrow \) it has a baseband equivalent).
- we have a way to filter a bandpass signal by its baseband equivalent. This is really helpful when we work with digital signals (because the sampling frequency is much lower).
- why do we have the \( \frac{1}{2} \) factor?
Two signals can be considered:

1. the **stochastic process**: $X(t)$
2. its **auto-correlation function** $\Gamma_{XX}(\tau)$ or its **power spectrum** $\gamma_X(f)$

The theory of equivalent baseband signals applies both to $X(t)$ and to $\Gamma_{XX}(\tau)$, but they are very different concepts.

Remember that when a stochastic process $X(t)$ passes through a (linear) filter, its power spectrum is multiplied by $\|H(f)\|^2$.

---

**Link between a stochastic process and its complex envelope**

$X(t)$ is related to its complex envelope by

$$X(t) = \text{Re} \left( e_X(t) e^{2\pi f_0 t} \right) \quad (87)$$

$X(t)$ being a stochastic process, its complex envelope $e_X(t)$ is also a stochastic process.

In general, $X(t)$ is *not stationary* because its mean is time dependent.

**Solution:** introduction of a random phase $\Theta$ uniformly distributed over $[0, 2\pi[$

$$X(t) = \text{Re} \left( e_X(t) e^{j(2\pi f_0 t + \Theta)} \right) \quad (88)$$
Likewise to developments for deterministic signals, the *in-phase* and *quadrature* components of a stochastic process can be expressed as

\[ e_X(t) = X_I(t) + jX_Q(t) \]  

(89)

Rice components of the \( X(t) \) stochastic process are then obtained by

\[ X(t) = \text{Re} \left( e_X(t) e^{2\pi j f_0 t} \right) \]  

(90)

\[ = \text{Re} \left( (X_I(t) + jX_Q(t)) e^{2\pi j f_0 t} \right) \]  

(91)

\[ = X_I(t) \cos(2\pi f_0 t) - X_Q(t) \sin(2\pi f_0 t) \]  

(92)

By analogy with the developments for deterministic signals, we build the analytic signal by filtering \( X(t) \) with the following filter \( H(f) \) that removes all components for negative frequencies:

\[ H(f) = \begin{cases} 
0 & \text{if } f < 0 \\
2 & \text{if } f \geq 0 
\end{cases} \]  

(93)

By applying Wiener-Kintchine, the power spectral density of the analytic signal is given by

\[ \gamma_{X_a}(f) = \|H(f)\|^2 \gamma_X(f) \]  

(94)

\[ = \begin{cases} 
4\gamma_X(f) & \text{if } f \geq 0 \\
0 & \text{if } f < 0 
\end{cases} \]  

(95)

Note that it can also be written as \( \gamma_{X_a}(f) = 2H(f) \gamma_X(f) \).
\[ \gamma_{eX}(f) = \gamma_{Xa}(f + f_0) \]  

(96)

\[ \Gamma_{XX}(\tau) = \frac{1}{2} \text{Re} \left( \Gamma_{XaXa}(\tau) \right) \]  

(97)

\[ = \frac{1}{2} \text{Re} \left( \Gamma_{eXeX}(\tau) e^{2\pi j f_0 \tau} \right) \]  

(98)

It is shown in a later chapter that, after stationarization, we get

\[ \gamma_X(f) = \frac{\gamma_{eX}(f - f_0) + \gamma_{eX}^*(-f - f_0)}{4} \]  

(99)

Important practical result: \( \gamma_X(f) \) can be derived from \( \gamma_{eX}(\cdot) \).
Noise in telecommunications systems I

Noise is a recurrent issue/problem in telecommunications systems. Remember for example the following theorem:

**Theorem (Shannon-Hartley)**

\[
C \left[ \frac{b}{s} \right] = B \log_2 \left( 1 + \frac{S}{N} \right)
\]

(100)

where

- \( B \) is the channel bandwidth in Hz
- \( \frac{S}{N} \) the signal-to-noise ratio (in watts/watts, not in dB).

Therefore, we need to understand how to deal with noise.

Noise in telecommunications systems II

There are several physical sources of noise:

- *thermal noise*
- shot noise
- ...

There are also “system” sources of noise

- quantization noise
- intermodulation noise
- crosstalk
- interference noise
- ...

Approach to deal with noise in telecommunications systems?

1. model the most common noise: *thermal noise* for electronic circuits
2. the principles of thermal noise can also be used to model noises that are not of thermal origin (such as white noises).

Modeling noise in electronic circuits

1. Characterization of a one-port circuit (source)
   - Available noise power
   - Noise temperature of a one-port circuit
   - Signal to noise ratio

2. Characterization of a two-port circuit (channel, amplifiers, filters, etc)
   - Gain
   - Noise factor
   - Figure of merit
   - Effective noise temperature
   - Particular case: attenuator

3. Cascade of two-port circuits (a complete chain)
Thermal noise: a measuring experiment

A conductive element with two terminals (≡ one-port circuit) may be characterized by:

- its resistance $R$.
- Free electrons have some random motion depending on the temperature $T \Rightarrow$ noise voltage source $E(t)$.

The noise voltage source, $E(t)$, is a stochastic process.

Thermal noise: towards a model for $E(t)$

A natural source of noise is thermal noise, caused by the omnipresent motion of free electrons in conducting material.

Because a voltage that would be measured at the output of a resistance $R$ is produced by many free electrons, by the central limit theorem, the amplitude statistics are modeled by a Gaussian.

- The thermal noise has a zero mean (can be shown analytically and experimentally).
- It can be shown that the autocorrelation function of thermal noise is well modeled by

$$\Gamma_{EE}(\tau) = k_B T R e^{-|\tau|/t_0}$$

where

- $k_B = 1.38 \times 10^{-23}$ [J/K] is Boltzmann’s constant.
- $T = 273.15 + C$ is the absolute temperature of the resistor (in Kelvin); $C$ is the temperature in Celsius.
- $t_0 = 10^{-12}$ [s] is the average time between collisions of electrons.
As the autocorrelation function is

$$\Gamma_{EE}(\tau) = k_B TR \frac{e^{-|\tau|/t_0}}{t_0} \tag{102}$$

the PSD is

$$\gamma_E(f) = \frac{2k_B TR}{1 + (2\pi ft_0)^2} \tag{103}$$

Empirically, at room temperature, for low frequencies ($< 1000 \text{ [GHz]}$), the noise PSD is almost flat so that we may take

$$\gamma_E(f) \simeq 2k_B TR \tag{104}$$

$\gamma_E(f)$ corresponds to the power of $E$ (remember that the power is in the form of $E^2(t)$).

Note: for a bandwidth, a resistor in a short circuit dissipates a noise power of (non sinusoidal signal):

$$P = \int_{-\infty}^{\infty} \gamma_E(f)df / R = 2 \int_{0}^{\infty} \gamma_E(f)df / R = 4k_B T B \tag{105}$$

Consider a thermal noise with a noise temperature of $T = 290 \text{ [K]}$ and $R = 1 \text{ [\Omega]}$, then we have

$$\gamma_E(f) = 2 \times 1.38 \times 10^{-23} \times 290 \times 1 = 8 \times 10^{-21} \frac{\text{W}}{\text{Hz}} \tag{106}$$

Note that $\gamma_E(f)$ is the power due to thermal noise; it is not the power available at the output!
Characterization of a single-port circuit (dipole)

**Source impedance:**

\[ Z_s(f) = R_s(f) + jX_s(f) \]  \hspace{1cm} (107)

**Load impedance:**

\[ Z_L(f) \]  \hspace{1cm} (108)

\[ Z_s \]

\[ Z_L \]

**Figure:** Thermal source with a load \( Z_L \).

---

**Available power**

**Definition (Available power)**

The *available power* is the maximum power that can be drawn from a source.

**Theorem (Maximum power transfer)**

*The maximum power transfer occurs when the load impedance is equal to the conjugate of the source impedance (matched impedances):*

\[ Z_L(f) = Z_S^*(f) \]  \hspace{1cm} (109)

If impedances are almost purely resistive, then

\[ R_L = R_S \]  \hspace{1cm} (110)
The case of sinusoidal signals I

When the signals are sinusoidal, the power provided by a source $S$ at the output of the dipole, $P_{pS}$, is given by

$$P_{pS} = \lim_{T \to \infty} \frac{1}{T} \int_0^T v(t)i(t)dt$$

(111)

$$= \frac{1}{2} \text{Re} \left( \hat{V}\hat{I}^* \right)$$

(112)

where $\hat{V}$ and $\hat{I}$ are phasors are defined with *peak values*, instead of *root mean square (rms) values*, and therefore there is a $\frac{1}{2}$ factor.

The case of sinusoidal signals II

In a load $Z_L$, we have (voltage divider):

$$\hat{V} = \frac{Z_L}{Z_s + Z_L} \hat{E}$$

and

$$\hat{I} = \frac{\hat{E}}{Z_s + Z_L}$$

Therefore,

$$P_{pS} = \frac{1}{2} \text{Re} \left( \hat{V}\hat{I}^* \right) = \frac{1}{2} \text{Re} \left( \frac{Z_L\hat{E}}{Z_s + Z_L} \frac{\hat{E}^*}{(Z_s + Z_L)^*} \right) = \frac{\text{Re} (Z_L) \hat{E}^2}{2 \|Z_s + Z_L\|^2}$$

(113)
So, the available power (that is when $Z_L(f) = Z_S^*(f)$, so that $Z_s + Z_L = 2 \Re (Z_s)$) from the Source is

$$P_{aS} = \frac{\Re (Z_L) \hat{E}^2}{2 \|Z_s + Z_L\|^2} = \frac{\Re (Z_L) \hat{E}^2}{8 \Re (Z_s)^2} = \frac{\hat{E}^2}{8 \Re (Z_s)} \quad (114)$$

### The case of sinusoidal signals IV

**Summary**

The available power in the load is

$$P_{aS} = \frac{\hat{E}^2}{8 \Re (Z_s)} \quad (115)$$

By definition, the effective power produced by the source is (for an open circuit, matched impedance, so that $Z_s + Z_L = 2 \times \Re (Z_s)$)

$$P_S = \frac{1}{2} \hat{E} \hat{l}^* = \frac{1}{2} \hat{E} \frac{\hat{E}^*}{(Z_s + Z_L)^*} = \frac{\hat{E}^2}{4 \Re (Z_s)} \quad (116)$$

It follows then that the power delivered by the Thévenin generator and the power dissipated in the generator’s Thévenin resistance are the same.
Available noise power (matched impedance)

For an arbitrary random noise source, the provided noise power is

\[ P_{pN} = \lim_{T \to +\infty} \frac{1}{T} \int_0^T V(t)I(t)dt \]  \hspace{1cm} (117)

For the particular case of thermal noise

The available power spectral density is, by applying Wiener-Kintchine:

\[ \gamma_{aN}(f) = \|H(f)\|^2 \gamma_E(f) = \left( \frac{R}{R+R} \right)^2 2k_B T = \frac{k_B T}{2} \]  \hspace{1cm} (118)

Remember that this assumes the adaptation of the impedance \((Z_s = Z_L^*)\); otherwise, the result would depend on the impedances!

Noise at the output of an antenna pointing towards the sky

Sky temperature \(T_a\)

Noiseless resistor \(R\)

\[ \gamma_E(f) = 2k_B T_a \]

Example

Contributions to \(T_a\):

- the sky contributes to 10 [K] (3 [K] of residual temperature of the Big Bang + 7 [K] due to atmospheric absorption).
- the ground contribution is typically 0.1 of 290 [K] (contributions of secondary lobes looking at the ground)

Thus,

\[ T_a = 0.9 \times 10 + 0.1 \times 290 = 38 \text{ [K]} \]  \hspace{1cm} (119)

The noise PSD is then

\[ \gamma_{aN}(f) = \frac{k_B T}{2} = 2.6 \times 10^{-22} \left[ \frac{W}{\text{Hz}} \right] \]  \hspace{1cm} (120)
There exist two ways to match loads:

1. $Z_S^* = Z_L$ (conjugate matching). This ensures the maximum transfer of power (that is the “available power”).

2. $Z_c = Z_L$, where $Z_c$ denotes the characteristic impedance of a transmission line. $Z_c$ is the ratio of the amplitudes of voltage and current of a single wave propagating along the line. When $Z_c = Z_L$, there is no reflection.

In practice, what should we do when we connect a circuit to a line?

- $Z_S^* = Z_L$ is mandatory. It is the matching condition for this chapter.
- Luckily, $Z_c = R_c + jL_c$ and $L_c \ll R_c$, so that $Z_c \approx R_c$. We can almost fulfill both conditions simultaneously.

### Summary I

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<th>Matched load</th>
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<td>$P_{pS} = \frac{1}{2} \text{Re} \left( \hat{V}^* \right) = \frac{E^2 \text{Re}(Z_L)}{2 |Z_s + Z_L|^2}$</td>
<td>$P_{aS} = \frac{E^2}{8 \text{Re}(z_s)}$</td>
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<tr>
<td>Stochastic processes</td>
<td>$P_{pN} = \lim_{T \to +\infty} \frac{1}{T} \int_0^T V(t)I(t)dt$</td>
<td>$\gamma_{aN}(f) = \frac{\gamma_E(f)}{4 \text{Re}(z_s)}$</td>
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<tr>
<td>Thermal noise</td>
<td></td>
<td>$\gamma_{aN}(f) = \frac{k_B T}{2}$</td>
</tr>
</tbody>
</table>

Table: Power provided by a one-port circuit.

### Theorem (Available power)

The available power from a thermal source in a bandwidth of $B$ is

$$p_N = \int_{-\infty}^{+\infty} \gamma_{aN}(f) df = 2 \times \int_{f_0 - \frac{B}{2}}^{f_0 + \frac{B}{2}} \frac{k_B T}{2} df = k_B TB \quad (121)$$
Conclusions:
We have a model for the power of (thermal) noise

\[ P_N = k_B T B \]  \hspace{1cm} (122)

It does not depend on the impedance (because we assume impedance matching). It depends only on
\begin{itemize}
  \item the temperature \( T \)
  \item the bandwidth \( B \)
\end{itemize}

Noise power
Consider a thermal noise with a noise temperature of \( T = 290 \) [K] and \( B = 10 \) [MHz], then we have

\[ P_N(f) = 1.38 \times 10^{-23} \times 290 \times 10^7 = 40 \times 10^{-15} \] [W] \hspace{1cm} (123)

Where does noise matter? [transmitter? channel? receiver?]

Summary III

\[ P_N = k_B T B \]  \hspace{1cm} (124)

So the noise power increases with the bandwidth \( \Rightarrow \) we may want to reduce the bandwidth
However, the capacity is given by

\[ C = B \log_2 \left( 1 + \frac{S}{N} \right) \]  \hspace{1cm} (125)

If \( S \) is fixed and we only have a white noise, then

\[ C = B \log_2 \left( 1 + \frac{S}{k_B T B} \right) \]  \hspace{1cm} (126)

To increase the capacity \( C \), there is no alternative other than to increase the bandwidth \( \Rightarrow \) trade-off!
Definition (Noise temperature [at a given frequency])

The *noise temperature at a given frequency* is the absolute temperature that an impedance should have to produce, by thermal effect, for a given frequency, a noise power spectral density equal to that of the circuit.

By definition thus,

\[ \gamma_{aN}(f) = \frac{k_B T(f)}{2} \]  \hfill (127)

Definition (Frequency average for \( T(f) \) and bandwidth)

The maximum temperature, denoted \( T \), the noise temperature of the dipole and the bandwidth are defined such that

\[ P_{aN} = k_B T B \]  \hfill (128)
**Definition**

The Signal to Noise ratio \( S/N \) of a dipole is defined as the ratio between the available power of the signal and the noise power

\[
\frac{S}{N} = \frac{P_{aS}}{P_{aN}}
\]  

(129)

**Warning:** in the following \( S \) denotes the Signal and not the noise source!

By convention, if the signal is modulated, the power of the useful signal is defined as follows (it is just a reference to compare techniques):

- for amplitude or angular modulations, we consider the *power of the carrier* (it is then a carrier to noise ratio \( \frac{C}{N} \)),
- for suppressed carrier amplitude modulation techniques, we take the *mean power of the modulating signal*, and
- for impulse coded modulation techniques, it is the *peak power*.

**Characterization of a two-port circuit (quadripole) I**

![Scheme of a two-port circuit](image)

*Figure:* Scheme of a two-port circuit.
Characterization of a two-port circuit (quadripole) II

Steps:

▶ Notion of gain?
▶ Characterization of the amount of internal noise of the quadripole by means of the normalized notion of noise figure
▶ Equivalent circuits
▶ Figure of merit (not normalized)
▶ Effective noise temperature
▶ Special case: purely resistive attenuator

Hypothesis: loads are matched (complex conjugate) at the input and at the output

Noise figure of a two-port circuit

Definition (Noise Figure (NF), $F_0$)

Assuming fixed internal impedances, the spot noise figure of a two-port circuit, for a given frequency $f$, denoted $F_0(f)$, is the ratio between

(1) the noise power spectral density at the output of the quadripole, for the appropriate frequency $f$, when the noise temperature of the generating one-port put at the input is normalized to be $T_0 = 290\ [K]$, and

(2) the contribution of the generating input source, at a frequency $f$, to the noise power spectral density at the output.
Property: $F_0 > 1$

So, by definition,

$$F_0(f) = \frac{\gamma_{aN_2}(f)}{G(f)\gamma_{aN_1}(f)} = \frac{G(f)\gamma_{aN_1}(f) + \gamma_{aNq}(f)}{G(f)\gamma_{aN_1}(f)} > 1 \quad (130)$$

**Equivalent circuit**

$$F_0(f) = \frac{G(f)\gamma_{aN_1}(f) + \gamma_{aNq}(f)}{G(f)\gamma_{aN_1}(f)} \iff \gamma_{aNq}(f) = [(F_0(f) - 1)\gamma_{aN_1}(f)] G(f)$$

Figure: Scheme of a noisy quadripole and an equivalent circuit (we model the internal noise by $(F_0(f) - 1)\gamma_{aN_1}(f)$ at the entrance)
The input signal to noise ratio is given by
\[
\left( \frac{S}{N} \right)_{\text{in}} = \frac{\gamma_{\text{in}}(f)}{\gamma_{aN_1}(f)}
\]

At the output of the two-port circuit, we have
\[
\left( \frac{S}{N} \right)_{\text{out}} = \frac{\gamma_{\text{out}}(f)}{\gamma_{aN_2}(f)}
\]

Therefore,
\[
\frac{\left( \frac{S}{N} \right)_{\text{in}}}{\left( \frac{S}{N} \right)_{\text{out}}} = \frac{\gamma_{\text{in}}(f)}{\gamma_{\text{out}}(f)} \cdot \frac{\gamma_{aN_2}(f)}{\gamma_{aN_1}(f)}
\]

As \( \gamma_{\text{out}}(f) = G(f)\gamma_{\text{in}}(f) \), this ratio becomes
\[
\frac{\left( \frac{S}{N} \right)_{\text{in}}}{\left( \frac{S}{N} \right)_{\text{out}}} = \frac{\gamma_{aN_2}(f)}{\gamma_{aN_1}(f) G(f)} \cdot \frac{G(f) F_0(f)\gamma_{aN_1}(f)}{\gamma_{aN_1}(f) G(f)} = F_0(f)
\]

\( F_0 \) therefore expresses the signal to noise ratio degradation.

Noise figures for un-normalized temperatures (\( T_s \neq T_0 \))

What if the input temperature \( T_s \neq T_0 \)?
This leads to define a different notion \( \Rightarrow \) the figure of merit \( F \).

Link between \( F \) and \( F_0 \) (\( F_0 \) is provided by the manufacturer)?
Remember that, considering \( \gamma_{aN_1}(f) \big|_{T=T_0} = \frac{1}{2} k_B T_0 \):
\[
\gamma_{aNq}(f) = (F_0 - 1) \frac{1}{2} k_B T_0 G(f)
\]

The internal noise of a two-port circuit is independent of the input temperature (the last is just a convention). Therefore, another temperature \( T_s \) then leads to another figure of merit \( F \). It is derived as follows
\[
\gamma_{aNq}(f) = (F_0 - 1) \frac{1}{2} k_B T_0 G(f) = (F - 1) \frac{1}{2} k_B T_s G(f)
\]

and, finally,
\[
F = 1 + \frac{T_0}{T_s} (F_0 - 1)
\]
**Effective noise temperature of a two-port circuit I**

\[ \gamma_{an_2}(f) = G(f)\gamma_{an_1}(f) + \gamma_{an_q}(f) \]

\[ \gamma_{an_1}(f) \]

\[ (F_0(f) - 1)\gamma_{an_1}(f) \]

\[ \gamma_{an_2}(f) = G(f)F_0(f)\gamma_{an_1}(f) \]

\[ \gamma_{an_2}(f) = \frac{1}{2}k_B T_0 G(f) + \gamma_{an_q}(f) = \frac{1}{2}k_B [T_0 + (F_0 - 1)T_0] G(f) \]

**Effective noise temperature of a two-port circuit II**

**Definition (Effective (input-)noise temperature)**

\[ T_e = (F_0 - 1)T_0 \]  \hspace{1cm} (138)

It is the additional temperature required for an input source to produce the same available power at the output.

Note that:

\[ T_e = (F_0 - 1)T_0 \iff F_0 = 1 + \frac{T_e}{T_0} \]  \hspace{1cm} (139)

**Noisy two-port circuit**

Consider an effective noise temperature \( T_e = 120 \) [K], then

\[ F_0 = 1 + \frac{120}{290} = 1.41 = 1.5 \) [dB]  \hspace{1cm} (140)
Let us take an **attenuator** (for example a purely resistive circuit or a lossy transmission line) at temperature $T_0$.

(1) Assuming matched input and output impedances, the *available noise power at the output* is

$$\gamma_{aN_2}(f) = \frac{1}{2} k_B T_0$$  \hspace{1cm} (141)

(2) But that the attenuator is characterized by its effective noise temperature $T_e$, then the output noise power is

$$\gamma_{aN_2}(f) = \frac{1}{2} k_B \left( T_0 + T_e \right) \frac{1}{L}$$  \hspace{1cm} (142)

By combining these two expressions: $T_e = (L - 1) T_0$. So that,

$$F_0 = 1 + \frac{(L - 1) T_0}{T_0} = L$$  \hspace{1cm} (143)

**Conclusion**: attenuators have a noise figure $F_0$ equal to their attenuation ratio $L$ when their physical temperature equals $T_0$. 

---

Attenuator with a “Gain” $G = 1/L$ or Loss $L$

Let us take an **attenuator** (for example a purely resistive circuit or a lossy transmission line) at temperature $T_0$.

(1) Assuming matched input and output impedances, the *available noise power at the output* is

$$\gamma_{aN_2}(f) = \frac{1}{2} k_B T_0$$  \hspace{1cm} (141)

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By combining these two expressions: $T_e = (L - 1) T_0$. So that,

$$F_0 = 1 + \frac{(L - 1) T_0}{T_0} = L$$  \hspace{1cm} (143)

**Conclusion**: attenuators have a noise figure $F_0$ equal to their attenuation ratio $L$ when their physical temperature equals $T_0$. 
For a two-port network with 2 stages,

\[
F_0 = \frac{\gamma_{aN_1}(f) G_1 G_2 + (F_{01} - 1) \gamma_{aN_1}(f) G_1 G_2 + (F_{02} - 1) \gamma_{aN_1}(f) G_2}{\gamma_{aN_1}(f) G_1 G_2} \\
= 1 + (F_{01} - 1) + \frac{(F_{02} - 1)}{G_1} \\
= F_{01} + \frac{(F_{02} - 1)}{G_1}
\]

Likewise,

\[
T_e = T_{e1} + \frac{T_{e2}}{G_1} + \frac{T_{e3}}{G_1 G_2} + \cdots = T_{e1} + \sum_{i=2}^{n} \frac{T_{ei}}{\prod_{j=1}^{i-1} G_j}
\]
Low-noise amplifier for receivers

\[ F_0 = F_{01} + \frac{F_{02} - 1}{G_1} + \frac{F_{03} - 1}{G_1 G_2} + \cdots \]  

(148)

Two consequences:

1. The noise figure always increases with an additional stage.
2. The overall noise figure of a receiver is primarily set by the noise figure of its first amplifying stage.

Therefore, the first stage amplifier is often a *Low-Noise Amplifier* (LNA). Then, the overall receiver noise figure is

\[ F_{\text{receiver}} \approx F_{\text{LNA}} + \frac{F_{\text{others}} - 1}{G_{\text{LNA}}} \]  

(149)

Outline

1. Reminder
2. Representation of bandpass signals
3. Noise in telecommunications systems
4. Digital modulation
5. Spread spectrum
6. Channels for digital communications and intersymbol interference
7. Navigation systems
8. Multiplexing
9. Telephone traffic engineering
10. Transmission over twisted pairs (fixed telephone network)
11. Radio engineering
Criteria ruling the selection of a modulation scheme

Definition and typology of digital modulations

Classic linear modulations
- Description
- Determination of the power spectrum
- Amplitude modulation (ASK)
- Phase modulation (PSK)
- Quadrature modulation (QPSK)

Offset modulations
- Description
- Determination of the power spectrum
- Offset quadrature modulation (OQPSK)
- Minimum shift modulation techniques (MSK)

Introduction to digital modulation
Criteria ruling the selection of a modulation scheme

1. Resistance to distortion and perturbation. That includes:
   - resistance to additive white Gaussian noise. Noise usually results in a bit error rate $P_e$, expressed in terms of the $E_b/N_0$ ratio.
   - sensitivity to interference (multipath, other users, etc).
   - sensitivity to imperfect filters. This is associated to the phenomenon of intersymbol interference.
   - sensitivity to non-linearities.

2. Spectral occupancy:
   - spectral efficiency $\eta$, expressed in bit per second per Hertz $\left[\frac{b}{s} \cdot \frac{1}{Hz}\right]$, which measures the bit rate that can be transmitted per unit of frequency bandwidth for a given modulation.
   - asymptotic behavior, defined by the values of the spectral density for frequencies relatively distant from the carrier frequency.

3. Simplicity of implementation.

Spectral efficiency of linear modulation techniques

$$C = B \log_2 \left(1 + \frac{E_b R_b}{B N_0}\right) \quad \text{max: } R_b \to C \quad \eta \approx \frac{R_b}{B} = \log_2 \left(1 + \eta \frac{E_b}{N_0}\right)$$
We will make an intensive use of alternative representations.

\[ v(t) = \Re(v_I(t) \cos(.)) - \Re(v_Q(t) \sin(.)) \]

\[ e_v(t) = a_v(t)e^{j\phi_v(t)} \]

Reminder: building a digital signal \( \sum_{k=-\infty}^{+\infty} A_k p(t - kT) \)

Main characteristics of digital signals:
- With digital signals, the fundamental unit is a time slot \( T \).
- One information symbol \( A_k \) per time slot \( T \) (no overlap).
- Same pulse shape for each time slot (eases the task of the receiver).
Reminder: building a digital signal \( \sum_{k=-\infty}^{+\infty} A_k p(t-kT) \)

<table>
<thead>
<tr>
<th>( A_0 )</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

Information: \( A_k \)

Pulse shape: \( p(t-kT) \)

Combining modulated pulse shapes

Result: \( \sum_{k=-\infty}^{+\infty} A_k p(t-kT) \)

Main characteristics of digital signals:
- With digital signals, the fundamental unit is a time slot \( T \).
- One information symbol \( A_k \) per time slot \( T \) (no overlap).
- Same pulse shape for each time slot (eases the task of the receiver).

Definition and typology of digital modulations I

General expression for digital modulations (based on the complex envelope)

\[
s(t) = \text{Re} \left( \psi[m(t)] e^{j(2\pi f_c t + \varphi_c)} \right) \tag{150}
\]

The complex function \( \psi[m(t)] \), which is related to the modulating waveform \( m(t) \), defines the type of modulation. It is also the complex envelope \( e_s(t) \) of the modulated signal \( s(t) \).

Depending on the form of \( \psi(.) = \psi_I(.) + j \psi_Q(.) \), we generally distinguish:
- **linear modulations** for which \( \psi[m(t)] \) is a linear function of \( m(t) \).
- **angular modulations** for which \( \psi[m(t)] \) has the form of

\[
\psi[m(t)] = e^{j\varphi[m(t)]} \tag{151}
\]

where \( \varphi[m(t)] \) is a linear function of \( m(t) \).
The modulated signal can also be expressed as
\[
s(t) = \psi_I [m(t)] \cos (2\pi f_c t + \varphi_c) - \psi_Q [m(t)] \sin (2\pi f_c t + \varphi_c)
\] (152)
and
\[
s(t) = \|\psi [m(t)]\| \cos (2\pi f_c t + \varphi_c + \arg \psi [m(t)])
\] (153)

In the following, we will concentrate on modulations that can be written as
\[
s(t) = \text{Re} \left( \sum_{k=-\infty}^{+\infty} d_k(t) e^{i(\theta_k - 2\pi f_c kT)} \right) e^{i(2\pi f_c t + \varphi_c)}
\] (154)

Two types of linear modulations will be studied:
- **classic modulations**, for which \( \theta_k = 2\pi f_c kT \)
- **offset modulations**, for which \( \theta_k = 2\pi f_c kT + k \frac{\pi}{2} \)

### Classic linear modulations I

**Description**
Classic linear modulations are such that \( \theta_k = 2\pi f_c kT \). Therefore, we have that
\[
s(t) = \text{Re} \left( e_s(t) e^{i(2\pi f_c t + \varphi_c)} \right)
\] (155)

The complex envelope takes the form
\[
e_s(t) = \sum_{k=-\infty}^{+\infty} d_k(t)
\] (156)
\[
= \sum_{k=-\infty}^{+\infty} D_k p_k(t - kT)
\] (157)

where \( D_k = A_k + jB_k \) and \( A_k, B_k \) are two real random variables. In most cases, the pulse shape of \( p_k(t - kT) \) is the same for each symbol \( k \). Therefore, \( p_k(.) \) becomes \( p(.) \).
The complex envelope can also be written as $e_s(t) = s_I(t) + js_Q(t)$, so that

$$s_I(t) = \sum_{k=-\infty}^{+\infty} A_k p(t - kT)$$  \hspace{1cm} (158)

$$s_Q(t) = \sum_{k=-\infty}^{+\infty} B_k p(t - kT)$$  \hspace{1cm} (159)

resulting in

$$s(t) = s_I(t) \cos (2\pi f_c t + \varphi_c) - s_Q(t) \sin (2\pi f_c t + \varphi_c)$$ \hspace{1cm} (160)

and, by replacing $s_I$ and $s_Q$ by their value,

$$s(t) = \left[ \sum_{k=-\infty}^{+\infty} A_k p(t - kT) \right] \cos (2\pi f_c t + \varphi_c) - \left[ \sum_{k=-\infty}^{+\infty} B_k p(t - kT) \right] \sin (2\pi f_c t + \varphi_c)$$

---

**Power spectral density of a modulated signal?**

The modulated signal is a stochastic process $S(t)$ that can be written, taking $\varphi_c = 0$, as

$$S(t) = \text{Re} \left( M(t) e^{j2\pi f_c t} \right)$$ \hspace{1cm} (161)

where $M(t)$ is a complex stochastic process (such as the complex envelope $e_s(t)$ in our case).

**But is $S(t)$ stationary?**

Obviously, we take $M(t)$ stationary. However, even then

$$\mu_S = E \{ S(t) \} = \mu_M e^{j2\pi f_c t}$$ \hspace{1cm} (162)

is time-dependent (not constant).

Therefore, as such, we cannot calculate $\gamma_S(f)$. 
Stationarization

The $S(t)$ process is not stationary because its mean is time-dependent. We have to stationarize the signal.

For that, we add the random phase $\Theta$ whose probability density function (pdf) is uniformly distributed over $[0, 2\pi]$ (in other words, pdf$_{\Theta}(\theta) = \frac{1}{2\pi}$ for $\theta \in [0, 2\pi]$ and 0 outside):

$$S(t) = \text{Re} \left( M(t) e^{j(2\pi f_c t + \Theta)} \right)$$

Mean of $S(t)$?

$M(t)$ and $\Theta$ being independent,

$$\mu_S = E \{ S(t) \} = 0$$
**Derivation of the power spectral density IV**

**Autocorrelation function of** $S(t)$?

\[ \Gamma_{SS}(t, t - \tau) = E \{S(t) S(t - \tau)\} \]  

(165)

As

\[ S(t) = \text{Re} \left( M(t) e^{j(2\pi f_c t + \Theta)} \right) \]  

(166)

\[ = \frac{1}{2} \left[ M(t) e^{j(2\pi f_c t + \Theta)} + M^*(t) e^{-j(2\pi f_c t + \Theta)} \right] \]  

(167)

we have

\[ S(t) S(t - \tau) = \frac{1}{2} \left[ M(t) e^{j(2\pi f_c t + \Theta)} + M^*(t) e^{-j(2\pi f_c t + \Theta)} \right] \]  

(168)

\[ \times \frac{1}{2} \left[ M(t - \tau) e^{j(2\pi f_c (t-\tau) + \Theta)} + M^*(t - \tau) e^{-j(2\pi f_c (t-\tau) + \Theta)} \right] \]

---

**Derivation of the power spectral density V**

\[ S(t) S(t - \tau) = \frac{1}{4} M(t) e^{j(2\pi f_c t + \Theta)} M(t - \tau) e^{j(2\pi f_c (t-\tau) + \Theta)} \]  

(169)

\[ + \frac{1}{4} M(t) e^{j(2\pi f_c t + \Theta)} M^*(t - \tau) e^{-j(2\pi f_c (t-\tau) + \Theta)} \]  

(170)

\[ + \frac{1}{4} M^*(t) e^{-j(2\pi f_c t + \Theta)} M(t - \tau) e^{j(2\pi f_c (t-\tau) + \Theta)} \]  

(171)

\[ + \frac{1}{4} M^*(t) e^{-j(2\pi f_c t + \Theta)} M^*(t - \tau) e^{-j(2\pi f_c (t-\tau) + \Theta)} \]

\[ S(t) S(t - \tau) = \frac{1}{4} M(t) M(t - \tau) e^{j(2\pi f_c (2t-\tau) + 2\Theta)} \]  

(172)

\[ + \frac{1}{4} M(t) M^*(t - \tau) e^{j2\pi f_c \tau} \]  

(173)

\[ + \frac{1}{4} M^*(t) M(t - \tau) e^{-j2\pi f_c \tau} \]  

(174)

\[ + \frac{1}{4} M^*(t) M^*(t - \tau) e^{-j(2\pi f_c (2t-\tau) + 2\Theta)} \]  

(175)
Because, $\Gamma_{SS}(t, t - \tau) = E\{S(t)S(t - \tau)\}$, the terms of $E\{.\}$ containing $2\Theta$ are null. Therefore, we are left with

$$\Gamma_{SS}(t, t - \tau) = \frac{1}{4} E \{ M(t) M^*(t - \tau) e^{j2\pi f_c \tau} + M^*(t) M(t - \tau) e^{-j2\pi f_c \tau} \}$$

$$= \frac{1}{4} E \{ 2 \text{Re} \left( M(t) M^*(t - \tau) e^{j2\pi f_c \tau} \right) \}$$

$$= \frac{1}{2} \text{Re} \left( E \{ M(t) M^*(t - \tau) e^{j2\pi f_c \tau} \} \right)$$

$$= \frac{1}{2} \text{Re} \left( \Gamma_{MM}(t, t - \tau) e^{j2\pi f_c \tau} \right)$$  \hspace{1cm} (176)

Finally, because

$$\Gamma_{SS}(\tau) = \frac{1}{4} [\Gamma_{MM}(\tau) e^{j2\pi f_c \tau} + \Gamma_{MM}(\tau)^* e^{-j2\pi f_c \tau}]$$  \hspace{1cm} (179)

and $x^*(t) \leftrightarrow X^*(-f)$, we have that

$$\gamma_S(f) = \frac{\gamma_M(f - f_c) + \gamma_M^*(-f - f_c)}{4}$$  \hspace{1cm} (180)

where $\gamma_M(f)$ is the power spectral density of $M(t)$.

---

**Power spectral density of a modulation signal for classic linear modulation techniques**

If $M(t)$ is real:

$$\gamma_S(f) = \frac{\gamma_M(f - f_c) + \gamma_M(f + f_c)}{4}$$  \hspace{1cm} (181)

If $M(t)$ is complex:

$$\gamma_S(f) = \frac{\gamma_M(f - f_c) + \gamma_M^*(-f - f_c)}{4}$$  \hspace{1cm} (182)
Calculation of the power spectrum of the complex envelope (for classic linear modulation techniques)

The complex envelope of the modulated signal is \( e_s(t) = M(t) \)

\[
M(t) = \sum_{k=-\infty}^{+\infty} D_k p(t - kT)
\]  \hspace{1cm} (183)

The sequence of complex random variables \( D_k \) is characterized by

- mean: \( \mu_D = E\{D_k\} \)
- variance: \( \sigma_D^2 = E\{(D_k - \mu_D)(D_k - \mu_D)^*\} \)
- autocorrelation function: \( \Gamma_{DD}(k, k-l) = E\{D_k D_{k-l}^*\} \)
- covariance function: \( C_{DD}(k, k-l) = E\{(D_k - \mu_D)(D_{k-l} - \mu_D)^*\} \)

After the stationarization of \( D_k \), the PSD of the baseband complex envelope is (cf. first course in telecommunications)

\[
\gamma_M(f) = \frac{||P(f)||^2}{T} \left[ \sigma_D^2 + ||\mu_D||^2 \sum_{m=-\infty}^{+\infty} \frac{1}{T} \delta\left(f - \frac{m}{T}\right) \right]
\]  \hspace{1cm} (184)

In conclusion (\( \gamma_M(f) \) being real in this case):

\[
\gamma_S(f) = \frac{\gamma_M(f - f_c) + \gamma_M(f + f_c)}{4}
\]  \hspace{1cm} (185)

Amplitude Shift Keying (amplitude modulation) I

Complex envelope of the Amplitude Shift Keying (ASK)

\[
e_s(t) = \sum_{k=-\infty}^{+\infty} A_k p(t - kT)
\]  \hspace{1cm} (186)

Choice for shaping the pulse function over \( T \): \( p(t) = \text{rect}_{[0,T]}(t) \)

Goal: we need to find the envelope \( a(t) \) and the phase \( \varphi(t) \) of the modulated signal. These two signals can be derived from

\[
e_s(t) = a(t) e^{j\varphi(t)}
\]  \hspace{1cm} (187)
Amplitude Shift Keying (amplitude modulation) II

**Trick:** $A_k$ can be expressed as $A_k = \|A_k\|e^{\left(1-\text{sign}(A_k)\right)\frac{\pi}{2}j}$

Let us verify:

- $A_k$ is positive ($A_k = \|A_k\|$):
  
  $$A_k = \|A_k\|e^{\left(1-\text{sign}(A_k)\right)\frac{\pi}{2}j} = \|A_k\|e^{(1-1)\frac{\pi}{2}j} = \|A_k\|e^{0\frac{\pi}{2}j} = \|A_k\|$$ (188)

- $A_k$ is negative ($A_k = -\|A_k\|$):
  
  $$A_k = \|A_k\|e^{(1-(-1))\frac{\pi}{2}j} = \|A_k\|e^{2\frac{\pi}{2}j} = \|A_k\|e^{\pi j} = -\|A_k\|$$ (189)

So, we have

$$e_s(t) = \sum_{k=-\infty}^{+\infty} A_k p(t - kT)$$ (190)

$$= \sum_{k=-\infty}^{+\infty} \|A_k\|e^{\left(1-\text{sign}(A_k)\right)\frac{\pi}{2}j} p(t - kT)$$ (191)

Amplitude Shift Keying (amplitude modulation) III

Therefore, the envelope $a(t)$ and the phase $\varphi(t)$ of the modulated signal are given by

$$a(t) = \sum_{k=-\infty}^{+\infty} \|A_k\| \text{rect}_{[0,T]}(t - kT)$$ (192)

$$\varphi(t) = \sum_{k=-\infty}^{+\infty} \frac{\pi}{2} \left(1 - \text{sign}(A_k)\right) \text{rect}_{[0,T]}(t - kT)$$ (193)

Note the presence of the time window $\text{rect}_{[0,T]}(t - kT)$ in these expressions. **Why?**
Hypothesis: both signals ±A have an equal probability.

- The mean $\mu_A$ of the random variable $A_k$ is equal to 0.
- The variance is given by $\sigma_A^2 = E\{A_k^2\} = A^2$.
- $p(t) = \text{rect}_{[0,T]}(t)$. Thus, its Fourier transform is

$$P(f) = e^{-j2\pi f \frac{T}{2}} \frac{T}{2} \text{sinc}(fT)$$  \hspace{1cm} (194)

Therefore, the power spectrum of the complex envelop is

$$\gamma_{es}(f) = \frac{\|P(f)\|^2}{T} \left[\sigma_A^2 + \|\mu_A\|^2 \sum_{m=-\infty}^{+\infty} \frac{1}{T} \delta (f - \frac{m}{T})\right]$$

$$\gamma_{es}(f) = A^2 \frac{T}{4} \text{sinc}^2 (fT)$$  \hspace{1cm} (195)

and that of the ASK-2 modulated signal is

$$\gamma_s(f) = A^2 \frac{T}{4} \left\{\text{sinc}^2 [(f - f_c) T] + \text{sinc}^2 [(f + f_c) T]\right\}$$  \hspace{1cm} (196)

### ASK-2: bandwidth and spectral efficiency

The PSD of ASK-2 is

$$\gamma_s(f) = A^2 \frac{T}{4} \left\{\text{sinc}^2 [(f - f_c) T] + \text{sinc}^2 [(f + f_c) T]\right\}$$  \hspace{1cm} (197)

At fixed bitrate $R_b$:

<table>
<thead>
<tr>
<th>Technique</th>
<th>bandwidth</th>
<th>spectral efficiency $\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseband (NRZ)</td>
<td>$W = 0.6 R_b$</td>
<td>$\eta = \frac{R_b}{0.6 R_b} \simeq 1.6$</td>
</tr>
<tr>
<td>ASK-2</td>
<td>$B = 2 \times 0.6 R_b = 1.2 R_b$</td>
<td>$\eta = \frac{R_b}{1.2 R_b} \simeq 0.8$</td>
</tr>
</tbody>
</table>
We have:

\[ s(t) = s_I(t) \cos(2\pi f_c t) - s_Q(t) \sin(2\pi f_c t) \]  \hspace{1cm} (198)

and, for the complex envelope:

\[ e_s(t) = s_I(t) + j s_Q(t) \]  \hspace{1cm} (199)

The alternative is

\[ s(t) = a_s(t) \cos(2\pi f_c t + \phi_s(t)) \]  \hspace{1cm} (200)

with

\[ a_s(t) = \sqrt{s_I^2(t) + s_Q^2(t)} \]  \hspace{1cm} (201)

and

\[ \phi_s(t) = \tan^{-1}\frac{s_Q(t)}{s_I(t)} \]  \hspace{1cm} (202)

**Definition (Constellation diagram)**

The plot of \( e_s(t) \) in a diagram whose axis units are \((\cos(2\pi f_c t), -\sin(2\pi f_c t))\) defines the **constellation diagram**.

---

**Constellation or state diagram: definition II**

Constellation diagram of a ASK-2 or BPSK modulation

The presence of \( p(t) \) reinforces the idea that there is only one symbol per time slot, but it is often dropped. Likewise, \( \varphi_c \) is often taken equal to 0.
Constellation diagram: purposes

- Representation of the possible states in the complex plane.
- See how the state diagram is used (here, we immediately see that the sin() axis is not used).
- The distance between states is essential for finding the $P_e$. Closer states mean less resistance to noise.
- See the paths from one state to another (trajectories).

Infrared signals are usually sent using On-Off Shift Keying (because it is hard to determine the phase of an infrared signal).
OOK, ASK-2 ≡ BPSK ≡ PSK-2, and QPSK.

16-QAM constellation diagram

Q

0000 0100 1100 1000

0001 0101 1101 1001

0011 0111 1111 1011

0010 0110 1110 1010
Example of a noisy (real) constellation diagram (ASK-2)

16-PSK constellation diagrams at different stages

Constellation diagrams and trajectories (transitions between states)
Phase Shift Keying modulation (PSK) I

Description [Phase Shift Keying modulation (PSK)]

\[ s(t) = A \sum_{k=-\infty}^{+\infty} \text{rect}[0,T] (t - kT) \cos (2\pi f_c t + \varphi_c + \psi_k) \] (203)

where \( \psi_k \) is a random variable that:

1. remains constant over the \([kT, (k + 1)T]\) interval,
2. takes a value among \( N \) possible values:

\[ \psi_k \in \left\{ \psi \mid \psi = \varphi_0 + \frac{2\pi i}{N}, \; i = 0, \ldots, N - 1 \right\} \] (204)
Phase Shift Keying modulation (PSK) II

The modulated signal is (using \( \cos(a + b) = \cos a \cos b - \sin a \sin b \))

\[
s(t) = A \sum_{k=-\infty}^{+\infty} \text{rect}[0,T](t - kT) [\cos(2\pi f_c t + \varphi_c) \cos \psi_k - \sin(2\pi f_c t + \varphi_c) \sin \psi_k]
\]

\[
= \sum_{k=-\infty}^{+\infty} \left[ A \cos \psi_k \text{rect}[0,T](t - kT) \right] \cos(2\pi f_c t + \varphi_c) - A \sin \psi_k \text{rect}[0,T](t - kT) \sin(2\pi f_c t + \varphi_c)
\]

We then derive its complex envelope

\[
es(t) = s_I(t) + j s_Q(t) \quad (208)
\]

\[
= A \sum_{k=-\infty}^{+\infty} \text{rect}[0,T](t - kT) (\cos \psi_k + j \sin \psi_k) \quad (209)
\]

Phase Shift Keying modulation (PSK) III

\[
es(t) = A \sum_{k=-\infty}^{+\infty} \text{rect}[0,T](t - kT) (\cos \psi_k + j \sin \psi_k) \quad (210)
\]

\[
= A \sum_{k=-\infty}^{+\infty} \text{rect}[0,T](t - kT) e^{j \psi_k} \quad (211)
\]

Conclusions for a PSK modulation:

Therefore, the envelope and phase of the modulated signal \( s(t) \) are:

\[
a(t) = A \sum_{k=-\infty}^{+\infty} \text{rect}[0,T](t - kT) = A \quad (212)
\]

\[
\varphi(t) = \sum_{k=-\infty}^{+\infty} \psi_k \text{rect}[0,T](t - kT) \quad (213)
\]

The envelope of a PSK modulated signal is constant.
The PSD of ASK-2 is
\[ \gamma_s(f) = \frac{A^2 T \{ \text{sinc}^2 [(f - f_c) T] + \text{sinc}^2 [(f + f_c) T] \}}{4} \] (214)

At fixed bitrate \( R_b \):

<table>
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<tr>
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Criteria ruling the selection of a modulation scheme

Definition and typology of digital modulations

Classic linear modulations
- Description
- Determination of the power spectrum
- Amplitude modulation (ASK)
- Phase modulation (PSK)
- Quadrature modulation (QPSK)

Offset modulations
- Description
- Determination of the power spectrum
- Offset quadrature modulation (OQPSK)
- Minimum shift modulation techniques (MSK)

Quadrature Phase Shift Keying modulation (QPSK) I

Why doing quadrature modulation?
- Starting point: for quadrature modulation, we would like to use the \( \cos(2\pi f_c t) \) axis AND the \( \sin(2\pi f_c t) \) axis.
- There exist techniques capable to retrieve \( m_1(t) \) and \( m_2(t) \) separately from a modulated signal such as

\[
s(t) = m_1(t) \cos(2\pi f_c t) + m_2(t) \sin(2\pi f_c t) \quad (215)
\]

- Benefit: reduce the used bandwidth.

Description [QPSK]
A typical Quadrature Phase Shift Keying (QPSK) typically uses different phases (4 in this particular example):

\[
D_k \in \left\{ Ae^{-j\frac{3\pi}{4}}, Ae^{-j\frac{\pi}{4}}, Ae^{j\frac{\pi}{4}}, Ae^{j\frac{3\pi}{4}} \right\} \quad (216)
\]
Quadrature Phase Shift Keying modulation (QPSK) II

This corresponds to the following constellation diagram:

\[-p(t) \sin (2\pi f_c t + \varphi_c)\]

\[(+\frac{A}{\sqrt{2}}, +\frac{A}{\sqrt{2}})\]

\[01\]

\[(0, 0)\]

\[11\]

\[(+\frac{A}{\sqrt{2}}, -\frac{A}{\sqrt{2}})\]

\[10\]

\[(-\frac{A}{\sqrt{2}}, -\frac{A}{\sqrt{2}})\]

\[00\]

\[p(t) \cos (2\pi f_c t + \varphi_c)\]

Side note: this diagram is equivalent to a constellation diagram whose states are located on the axes (diagram rotated by $\pi/4$)!

Quadrature Phase Shift Keying modulation (QPSK) III

For the development, we introduce a new binary random sequence:

Let us consider a binary source generated by a regularly time-spaced series of Dirac delta functions:

\[I(t) = \sum_{k=-\infty}^{+\infty} I_k \delta(t - kT_b)\]  \hspace{1cm} (217)

where $I_k = -1$ or $+1$ corresponds to a binary symbol 0 or 1, respectively. It is a sequence whose symbols are normalized to an amplitude of 1.
Starting from $I(t)$, we split the original sequence into two “slower” sequences (even $l_{2k}$ and odd $l_{2k+1}$ bits):

$$s_I(t) = A \sqrt{2} \sum_{k=-\infty}^{+\infty} l_{2k} p(t - k T) = \sum_{k=-\infty}^{+\infty} A_k p(t - k T)$$

$$s_Q(t) = A \sqrt{2} \sum_{k=-\infty}^{+\infty} l_{2k+1} p(t - k T) = \sum_{k=-\infty}^{+\infty} B_k p(t - k T)$$

where:

- $T = 2T_b$. $T$ is twice longer than $T_b$!
- $p(t)$ is a $T$-long rectangular pulse: $p(t) = \text{rect}_{[0,T]}(t)$
- $A_k = l_{2k} A \sqrt{2}$ (even index bits)
- $B_k = l_{2k+1} A \sqrt{2}$ (odd index bits)

### Figure: Formation of in-phase and quadrature signals of a QPSK.
\[ e_s(t) = s_I(t) + j s_Q(t) = \sum_{k=-\infty}^{+\infty} (A_k + j B_k) \text{rect}_{[0,T]}(t - kT) \]

\[ = \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} (l_{2k} + j l_{2k+1}) \text{rect}_{[0,T]}(t - kT) \tag{219} \]

\[ a(t) = \sqrt{s_I^2(t) + s_Q^2(t)} \tag{220} \]

\[ = \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} \sqrt{l_{2k}^2 + l_{2k+1}^2} \text{rect}_{[0,T]}(t - kT) \tag{221} \]

\[ = A \sum_{k=-\infty}^{+\infty} \text{rect}_{[0,T]}(t - kT) = A \tag{222} \]

\[ \Rightarrow \text{The amplitude of a QPSK is constant.} \]

\[ \varphi(t) = \tan^{-1}\left(\frac{s_Q(t)}{s_I(t)}\right) \tag{223} \]

\[ = \sum_{k=-\infty}^{+\infty} \text{rect}_{[0,T]}(t - kT) \tan^{-1}\left(\frac{l_{2k+1}}{l_{2k}}\right) \tag{224} \]

**Conclusions about the phase of a QPSK modulated signal:**

- knowing that \( l_{2k}, l_{2k+1} = \pm 1 \), we have \( \tan^{-1}\left(\frac{l_{2k+1}}{l_{2k}}\right) = \pm \frac{\pi}{4} \).
- but there is an ambiguity between \( \frac{1}{1} \) or \( -\frac{1}{-1} \), and \( -\frac{1}{1} \) or \( \frac{1}{-1} \).
Figure: Illustration for the QPSK modulation: (a) binary input sequence $I(t)$, (b) in-phase signal $s_I(t)$, (c) quadrature signal $s_Q(t)$, (d) $s_I(t) \cos(2\pi f_c t)$, (e) $s_Q(t) \sin(2\pi f_c t)$, and (f) modulated signal $s(t)$. 

QPSK modulator
QPSK demodulator

\[ s(t) \]

\[ \cos(2\pi f_c t) \]

\[ \sin(2\pi f_c t) \]

\[ \frac{1}{2} s_i(t) \]

\[ \frac{1}{2} s_o(t) \]

Matched filter and decision

parallel

serial

\[ \frac{1}{2} I(t) \]

\[ \frac{1}{2} Q(t) \]

Power spectral density

\[
\gamma_s(f) = \frac{A^2 T_b}{2} \left\{ \text{sinc}^2 [(f - f_c) 2 T_b] + \text{sinc}^2 [(f + f_c) 2 T_b] \right\} \quad (225)
\]

For the same bitrate, it needs half the bandwidth of that of a PSK.

Discussion on the spectral efficiency and resilience to noise

At fixed bitrate \( R_b \):

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</tr>
<tr>
<td>ASK-2(^N) or PSK-2(^N)</td>
<td>( \frac{1.2}{N T_b} = \frac{1.2}{N} R_b )</td>
<td>( \eta = \frac{R_b}{1.2 R_b} N \approx 0.8 N )</td>
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Concatenating \( N \) bits in larger symbols has the following consequences:

- (+) it increases the spectral efficiency.
- (–) for the same power level, it decreases the average distance between symbols, which leads to an increase in the Bit Error Rate (BER or \( P_e \)).

- Solution: use pre-coding to change the input sequence. Probabilities to move between neighboring states should be less than that of distant states in the constellation diagram.
Possible transitions: self, close neighbors, opposite state (transition crosses the origin)

Time rhythm between state changes: $2T_b$

Offset modulation techniques I

General form for linear modulation techniques:

$$s(t) = \text{Re} \left( e^{j(2\pi f_c t + \varphi_c)} \sum_{k=-\infty}^{+\infty} d_k(t) e^{j(\theta_k - 2\pi f_c k T_b)} \right)$$  \hspace{1cm} (226)

**Offset modulations** have $\theta_k = 2\pi f_c k T_b + k \frac{\pi}{2}$.

Description

The complex envelope of offset modulation techniques is:

$$e_s(t) = \sum_{k=-\infty}^{+\infty} A_k p(t - k T_b) e^{j k \frac{\pi}{2}}$$  \hspace{1cm} (227)

We rotate the constellation diagram by $\frac{\pi}{2}$ after each symbol!
Offset modulation techniques II

\[ s(t) = \sum_{k=-\infty}^{+\infty} A_k p(t - kT_b) \cos \left( 2\pi f_c t + \varphi_c + k \frac{\pi}{2} \right) \]  
\[ = \left[ \sum_{k=-\infty}^{+\infty} A_k p(t - kT_b) \cos \left( k \frac{\pi}{2} \right) \right] \cos (2\pi f_c t + \varphi_c) \]
\[ - \left[ \sum_{k=-\infty}^{+\infty} A_k p(t - kT_b) \sin \left( k \frac{\pi}{2} \right) \right] \sin (2\pi f_c t + \varphi_c) \]

- For \( k \) odd (1, 3, 5, ...) \( \Rightarrow \cos \left( k \frac{\pi}{2} \right) = 0. \)
One out of two cosine terms is null; remaining terms: \( k \) even.
- For \( k \) even (0, 2, 4, ...) \( \Rightarrow \sin \left( k \frac{\pi}{2} \right) = 0. \)
One out of two sine terms is null; remaining terms: \( k \) odd.

Offset modulation techniques III

Considering that \( \cos \left( k \pi/2 \right) = 0 \) for \( k \) odd and that \( \sin \left( k \pi/2 \right) = 0 \) for \( k \) even, in-phase \( s_I(t) \) and quadrature \( s_Q(t) \) are respectively:

\[ s_I(t) = \sum_{k=-\infty}^{+\infty} A_k p(t - kT_b) \cos \left( k \frac{\pi}{2} \right) \]  
\[ = \sum_{k'=-\infty}^{+\infty} A_{2k'} p \left( t - 2k'T_b \right) \cos \left( 2k' \frac{\pi}{2} \right) \]  
\[ = \sum_{k'=-\infty}^{+\infty} A_{2k'} (-1)^{k'} p \left( t - 2k'T_b \right) \]  

and

\[ s_Q(t) = \sum_{k=-\infty}^{+\infty} A_k p(t - kT_b) \sin \left( k \frac{\pi}{2} \right) \]  
\[ = \sum_{k'=-\infty}^{+\infty} A_{2k'+1} (-1)^{k'} p \left( t - (2k' + 1) T_b \right) \]
Building an Offset Quadrature Phase Shift Keying signal
Consider a binary source generated by a regularly time-spaced series of Dirac delta functions:

\[ I(t) = \sum_{k=-\infty}^{+\infty} I_k \delta(t - kT_b) \]  

(234)

We construct two signals with the odd and even bits of the input sequence:

\[ s_i(t) = \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} (-1)^k l_{2k} p(t - 2kT_b) = \sum_{k=-\infty}^{+\infty} A_{2k} p(t - kT) \]  

(235)

\[ s_Q(t) = \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} (-1)^k l_{2k+1} p(t - (2k + 1)T_b) = \sum_{k=-\infty}^{+\infty} A_{2k+1} p(t - kT - T_b) \]

where \( p(t) \) is a unit rectangular pulse over \([0, 2T_b] = [0, \ T] \), \( A_{2k} = \frac{A}{\sqrt{2}} (-1)^k l_{2k} \) and \( A_{2k+1} = \frac{A}{\sqrt{2}} (-1)^k l_{2k+1}, \ \forall k \).

Figure: Formation of in-phase and quadrature signals of a OQPSK (for this drawing, we assume that the constellation diagram does not turn by \( \pi/2 \) for each bit).

Important note: the signal changes every \( T_b \) instead of every \( 2T_b \)!
Offset Quadrature Phase Shift Keying (OQPSK) III

Figure: Constellation diagram of an OQPSK (note: we have frozen the $\frac{\pi}{2}$ rotation factor between consecutive symbols).

Offset Quadrature Phase Shift Keying (OQPSK) IV

Figure: Illustration for the OQPSK modulation: (a) binary input sequence $I(t)$, (b) in-phase signal $s_I(t)$, (c) quadrature signal $s_Q(t)$, (d) $s_I(t) \cos(2\pi f_c t)$, (e) $s_Q(t) \sin(2\pi f_c t)$, and (f) modulated signal $s(t)$. 
First guess?

- construction of two independent sequences.
- one sequence modulates the \( \cos(.) \) function, the other the \( \sin(.) \) function.
- the power spectrum of a sum is the sum of the power spectrum densities if the sequences are independent.

⇒ why should the spectral occupancy of an OQPSK be different to that of a QPSK?

---

Towards an analytic expression of the power spectral density of an offset modulation.

**Trick:** we modify the expression of the pulse function by adding the \( e^{-j2\pi \frac{t}{4T_b}} \) term (later compensated in the general expression by \( e^{j2\pi \frac{t}{4T_b}} \)), to get:

\[
p(t - kT_b) e^{jk \frac{\pi}{2}} e^{-j2\pi \frac{t}{4T_b}} = p(t - kT_b) e^{-j \frac{\pi}{2T_b} (t - kT_b)} \tag{236}
\]

The introduction of a phase factor in the envelope has to be compensated by introducing a new carrier frequency \( f'_{c} \):

\[
s(t) = \text{Re} \left( e_s(t) e^{-j2\pi \frac{t}{4T_b}} e^{j(2\pi f_c t + \varphi_c)} e^{j2\pi \frac{t}{4T_b}} \right) \tag{237}
\]

\[
= \text{Re} \left( e_s(t) e^{-j2\pi \frac{t}{4T_b}} e^{j \left( 2\pi \left( f_c + \frac{1}{4T_b} \right) t + \varphi_c \right)} \right) \tag{238}
\]
So,

\[ s(t) = \text{Re} \left( e_s(t) e^{-j2\pi \frac{t}{4T_b}} e^{j \left( 2\pi \left( f_c + \frac{1}{4T_b} \right) t + \phi_c \right)} \right) \]

We introduce \( z(t) = e_s(t) e^{-j2\pi \frac{t}{4T_b}} \) and \( f'_c = f_c + \frac{1}{4T_b} \)

\[ s(t) = \text{Re} \left( z(t) e^{j (2\pi f'_c t + \phi_c)} \right) \quad (239) \]

If the power spectral density of the complex signal \( z(t) \) was known, then

\[ \gamma_s(f) = \frac{\gamma_z(f - f'_c) + \gamma^*_z(-f - f'_c)}{4} \quad (240) \]

\[ = \frac{\gamma_z(f - f_c - \frac{1}{4T_b}) + \gamma^*_z(-f - f_c - \frac{1}{4T_b})}{4} \quad (241) \]

It remains to calculate \( \gamma_z(f) \):

\[ z(t) = \sum_{k=-\infty}^{+\infty} A_k p(t - kT_b) e^{j k \frac{\pi}{2}} e^{-j2\pi \frac{t}{4T_b}} \quad (242) \]

\[ = \sum_{k=-\infty}^{+\infty} A_k p(t - kT_b) e^{-j \frac{\pi}{2T_b} (t - kT_b)} \quad (243) \]

\[ = \sum_{k=-\infty}^{+\infty} A_k h(t - kT_b) \quad (244) \]

where we have defined the new function \( h(x) \) by

\[ h(x) = p(x) e^{-j \frac{\pi x}{2T_b}} = p(x) e^{2\pi j (\frac{x}{4T_b})} \quad (245) \]

\( h(t) \) is in fact a new “shaping” function whose Fourier transform is derived from that of \( p(t) \) by

\[ H(f) = \mathcal{P} \left( f - \left( -\frac{1}{4T_b} \right) \right) = \mathcal{P} \left( f + \frac{1}{4T_b} \right) \quad (246) \]
Therefore, we only need to adapt the power spectral density of $z(t)$ by considering $\mathcal{H}(f)$ in replacement of $P(f)$

$$\gamma_z(f) = \left\| P \left( f + \frac{1}{4T_b} \right) \right\|^2 \left[ \sigma^2_A + \mu^2_A \sum_{m=-\infty}^{+\infty} \frac{1}{T_b} \delta \left( f - \frac{m}{T_b} \right) \right]$$

(247)

in

$$\gamma_s(f) = \frac{\gamma_z \left( f - f_c - \frac{1}{4T_b} \right) + \gamma_z^* \left( -f - f_c - \frac{1}{4T_b} \right)}{4}$$

(248)

Obviously, the $\frac{1}{4T_b}$ component of the filter annihilates the factor $-\frac{1}{4T_b}$ in the expression of $\gamma_s(f)$ (except for the series of $\delta()$)

**In conclusion,** the OQPSK and the QPSK have the same power spectral density (except for the locations of $\delta()$ in some cases).

**Calculation of the power spectral density of OQPSK-4**

The complex envelope of the modulated signal is given by

$$e_s(t) = \sum_{k=-\infty}^{+\infty} A_k p(t - kT_b) e^{jk \frac{\pi}{2}}$$

(249)

where $p(t)$ is the shaping function and $A_k$ is the random variable containing the information, respectively given by

$$p(t) = \text{rect}_{[0, \, 2T_b]}(t)$$

(250)

$$A_k \in \left\{ +\frac{A}{\sqrt{2}}, -\frac{A}{\sqrt{2}} \right\}$$

(251)

The power spectral density of $z(t)$ is calculated as follows:

$$\gamma_z(f) = 2A^2 T_b \text{sinc}^2 \left[ \left( f + \frac{1}{4T_b} \right) 2T_b \right]$$

(252)

Therefore, the power spectral density of the OQPSK modulation is

$$\gamma_s(f) = \frac{A^2 T_b}{2} \left\{ \text{sinc}^2 [(f - f_c) 2T_b] + \text{sinc}^2 [(f + f_c) 2T_b] \right\}$$

(253)

which is also that of the QPSK.
Minimum Shift Keying modulation (MSK)

Description

For defining the Minimum Shift Keying modulation (MSK) modulation, we use the principle of the OQPSK and a non-rectangular pulse shape. More precisely, we take

\[ p(t) = \text{rect}_{[0, 2T_b]}(t) \sin \left( \frac{\pi t}{2T_b} \right) \]

(254)

\( s_I(t) \) and \( s_Q(t) \) are built exactly as for the OQPSK. \( s_I(t) \) is calculated as

\[
s_I(t) = \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} l_{2k} \text{rect}_{[0, 2T_b]}(t - 2kT_b) \sin \left[ \frac{\pi (t - 2kT_b)}{2T_b} \right]
\]

\[
= \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} l_{2k} \text{rect}_{[0, 2T_b]}(t - 2kT_b) \sin \left[ \frac{\pi t}{2T_b} - k\pi \right]
\]
Minimum Shift Keying modulation (MSK) II

Because \( \sin(a - b) = \sin a \cos b - \cos a \sin b \),

\[
s_I(t) = \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} I_{2k} \text{rect}_{[0, 2T_b]}(t - 2kT_b) \sin \left( \frac{\pi t}{2T_b} - k\pi \right)
\]

(255)

\[
s_I(t) = \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} I_{2k} \text{rect}_{[0, 2T_b]}(t - 2kT_b) \left[ \sin \left( \frac{\pi t}{2T_b} \right) \cos(k\pi) - \cos \left( \frac{\pi t}{2T_b} \right) \sin(k\pi) \right]
\]

(256)

\[
s_I(t) = \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} I_{2k} \text{rect}_{[0, 2T_b]}(t - 2kT_b) \sin \left( \frac{\pi t}{2T_b} \right) \cos(k\pi)
\]

(257)

\[
s_I(t) = \cos \left( \frac{\pi}{2} - \frac{\pi t}{2T_b} \right) \sum_{k=-\infty}^{+\infty} \frac{A}{\sqrt{2}} I_{2k} (-1)^k \text{rect}_{[0, 2T_b]}(t - 2kT_b)
\]

(258)

\[
s_I(t) = \cos \left( \frac{\pi t}{2T_b} - \frac{\pi}{2} \right) \sum_{k=-\infty}^{+\infty} \frac{A}{\sqrt{2}} I_{2k} (-1)^k \text{rect}_{[0, 2T_b]}(t - 2kT_b)
\]

(259)

Minimum Shift Keying modulation (MSK) III

Likewise,

\[
s_Q(t) = \sin \left( \frac{\pi t}{2T_b} - \frac{\pi}{2} \right) \sum_{k=-\infty}^{+\infty} \frac{A}{\sqrt{2}} I_{2k+1} (-1)^k \text{rect}_{[0, 2T_b]}(t - (2k + 1)T_b)
\]

The instantaneous envelope of the modulated signal is then

\[
a(t) = \sqrt{s_I^2(t) + s_Q^2(t)}
\]

(260)

\[
a(t) = \sqrt{\left( \frac{A}{\sqrt{2}} \right)^2 \cos^2 \left( \frac{\pi t}{2T_b} - \frac{\pi}{2} \right) + \left( \frac{A}{\sqrt{2}} \right)^2 \sin^2 \left( \frac{\pi t}{2T_b} - \frac{\pi}{2} \right)}
\]

(261)

\[
a(t) = \sqrt{\left( \frac{A}{\sqrt{2}} \right)^2} = \frac{A}{\sqrt{2}}
\]

(262)

The instantaneous phase is

\[
\varphi(t) = \tan^{-1} \left[ \frac{s_Q(t)}{s_I(t)} \right]
\]

(263)

\[
\varphi(t) = \tan^{-1} \left[ \tan \left( \frac{\pi t}{2T_b} - \frac{\pi}{2} \right) \frac{\sum_{k=-\infty}^{+\infty} \frac{A}{\sqrt{2}} I_{2k+1} (-1)^k \text{rect}_{[0, 2T_b]}(t - (2k + 1)T_b)}{\sum_{k=-\infty}^{+\infty} \frac{A}{\sqrt{2}} I_{2k} (-1)^k \text{rect}_{[0, 2T_b]}(t - 2kT_b)} \right]
\]
Minimum Shift Keying modulation (MSK) IV

$\varphi(t) = \tan^{-1} \left[ \tan \left( \frac{\pi t}{2T_b} - \frac{\pi}{2} \right) \frac{\sum_{k=-\infty}^{+\infty} \frac{A}{\sqrt{2}} l_{2k+1} (-1)^k \text{rect}[0,2T_b](t - (2k+1)T_b)}{\sum_{k=-\infty}^{+\infty} \frac{A}{\sqrt{2}} l_{2k} (-1)^k \text{rect}[0,2T_b](t - 2kT_b)} \right]$

Figure: Constellation diagram of the MSK modulation.

Understanding the phase of an MSK

▶ there is an ambiguity between $\frac{1}{1}$ or $\frac{-1}{-1}$, and $\frac{-1}{1}$ or $\frac{1}{-1}$.
▶ the phase evolves linearly with time:

$$\varphi(t) = \tan^{-1} \left[ \tan \left( \frac{\pi t}{2T_b} - \frac{\pi}{2} \right) \times (\pm 1) \right] \quad (264)$$

$$= \pm \left( \frac{\pi t}{2T_b} - \frac{\pi}{2} \right) \quad (265)$$

It is continuous!
▶ transition rate given by the duration $T_b$.
   - the time slope is $\frac{\pi}{2T_b}$ or $-\frac{\pi}{2T_b}$.
   - the phase increment after a $T_b$ period is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$. 
Minimum Shift Keying modulation (MSK) VI

\[ \Delta \varphi(t) \]

\[ 3\pi/2 \]

\[ \pi \]

\[ \pi/2 \]

\[ -\pi/2 \]

\[ -\pi \]

\[ -3\pi/2 \]

\( T_b \)

Figure: Phase trellis diagram of the MSK modulation.

Minimum Shift Keying modulation (MSK) VII

\[ s_I(t) \cos(2\pi f_c t) \]

\[ s_Q(t) \sin(2\pi f_c t) \]

\[ s(t) \]

Figure: Illustration for the MSK modulation: (a) binary input sequence \( I(t) \), (b) in-phase signal \( s_I(t) \), (c) quadrature signal \( s_Q(t) \), (d) \( s_I(t) \cos(2\pi f_c t) \), (e) \( s_Q(t) \sin(2\pi f_c t) \), and (f) modulated signal \( s(t) \).
Which family of modulation techniques does the MSK belong to?

Because the envelope is constant and the phase evolves as (over a period of $T_b$)

$$\Delta \varphi(t) = \pm \frac{\pi t}{2T_b}$$

we can write that

$$s(t) = A \cos \left( 2\pi f_c t \pm \frac{\pi t}{2T_b} \right)$$

$$= A \cos \left[ 2\pi \left( f_c \pm \frac{1}{4T_b} \right) t \right]$$

(268)

This illustrates that, in fact, the MSK modulation is a “pure” frequency modulation (whose frequency excursion is limited to $\Delta f = \frac{1}{4T_b}$).

Power spectral density of the MSK I

The complex envelope of the modulated signal is again given by

$$e_s(t) = \sum_{k=-\infty}^{+\infty} A_k p(t - kT_b) e^{jk \frac{\pi}{2}}$$

(269)

where $p(t)$ is the shaping function and $A_k$ is the random variable containing the information, respectively given by

$$p(t) = \text{rect}_{[0, 2T_b]}(t) \sin \left( \frac{\pi t}{2T_b} \right)$$

(270)

$$A_k \in \{ +A, -A \}$$

(271)

After considering the Fourier transform of $\mathcal{H}(f) = \mathcal{P} \left( f + \frac{1}{4T_b} \right)$,

$$\gamma_z(f) = \frac{16 A^2 T_b}{\pi^2} \left\{ \frac{\cos \left[ 2\pi \left( f + \frac{1}{4T_b} \right) T_b \right]}{1 - 16 \left( f + \frac{1}{4T_b} \right)^2 T_b^2} \right\}^2$$

(272)
This yields the power spectral density of the modulation:

\[
\gamma_s(f) = \frac{4 A^2 T_b}{\pi} \left\{ \left( \frac{\cos [2\pi (f - f_c) T_b]}{1 - 16 (f - f_c)^2 T_b^2} \right)^2 + \left( \frac{\cos [2\pi (f + f_c) T_b]}{1 - 16 (f + f_c)^2 T_b^2} \right)^2 \right\}
\]

(273)
Spread spectrum

- Introduction
  - Utilization
  - Techniques for spread spectrum

- Direct sequence spreading
  - Principles
  - Generation of pseudo-random sequences
  - Principles of baseband transmission
  - BPSK modulated spreading

- Performance study
  - Error probability
  - Interference margin
  - Capacity
Spread-spectrum involves the transmission of a signal in a frequency bandwidth substantially greater than the information bandwidth to achieve a particular operational advantage.

<table>
<thead>
<tr>
<th>Purposes</th>
<th>Military domain</th>
<th>Commercial domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anti-jamming</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Multiple access</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Detection harvesting</td>
<td>✓</td>
<td></td>
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<tr>
<td>Message privacy</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Selective calling</td>
<td>✓</td>
<td>✓</td>
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<tr>
<td>Identification</td>
<td>✓</td>
<td>✓</td>
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<tr>
<td>Navigation</td>
<td>✓</td>
<td>✓</td>
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<tr>
<td>Multipath protection</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Low radiated flux density</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

There exist two main spread spectrum families:

1. **Frequency Hopping**.
2. **Direct Sequence Spread Spectrum (DSSS or DS)**. This technique leads to **Code-Division Multiplexing (CDM)**.

**Frequency Hopping Spread Spectrum I**

**Definition**

**Frequency Hopping** Spread Spectrum (FHSS) is a method of transmitting radio signals by rapidly switching a carrier among many frequency channels, using a pseudo-random sequence known to both transmitter and receiver.

**Bandwidth.** (−) The overall bandwidth required for frequency hopping is much wider than that required to transmit the same information using only one carrier frequency.

**Challenge.** (−) Need to synchronize the hopping sequence between the transmitter and the receiver.

**Advantage.** (+) Possibility to avoid being permanently in a “bad” frequency channel.
**Bluetooth**

- total bandwidth of 79 [MHz]
- width of individual signals: 1 [MHz]
- 1600 changes per second (hopping time of 625 [$\mu$s])

---

**Direct Sequence spreading: principles I**

**Definition**

*Direct Sequence Spread Spectrum (DSSS or DS)* transmissions multiply the data being transmitted by a “noise/carrier” signal. This noise/carrier signal is a pseudo-random NRZ-like sequence of +1 and −1 values, also known as *spreading sequence*, at a bitrate much higher than that of the original signal.

For a *Direct Sequence spreading (DS) system*, there are two signals:

1. a binary baseband waveform $b(t)$, whose symbol rate is $R_b = 1/T_b$. This is the original sequence.
2. a pseudo-random binary spreading waveform $c(t)$, whose “chip” rate $R_c = 1/T_c$ is much faster than the symbol rate $R_b$ ($T_c \ll T_b$). This waveform is the *spreading sequence*. 

---

![Frequency Hopping Spread Spectrum II](image_url)
Note that the two signals are aligned (synchronized). In the following, we will consider:

\[ R_c = NR_b \iff T_b = NT_c \quad (274) \]

\( N \) is the spreading factor.

Questions:

- effects of spreading (bandwidth, Bit Error Rate \( P_e \), etc)?
- how do we build an appropriate spreading waveform? How should a spreading sequence look like?

Effects of direct sequence spreading

Compare the power of the original signal and the power of the spread signal.

What is the bandwidth of a spread waveform? (in baseband or modulated form)
Bandwidth? Summary

- The bandwidth of $b(t)$ is related to $1/T_b$ [Hz]
- The bandwidth of $c(t)$ is related to $1/T_c$ [Hz]

Because $T_c \ll T_b$, $c(t)$ is a wideband signal.

The resultant spectrum of the product $c(t)b(t)$ is the convolution of two spectra.

Essentially, it will occupy a bandwidth that is practically the same as that of $c(t)$.

The bandwidth of a spread waveform is about $N$ times larger.
**Generation of pseudo-random spreading sequences I**

How can we construct a pseudo-random spreading sequence?

There are digital circuits to generate sequences that “look like” random sequences.

![Feedback shift register diagram](image)

**Figure:** Feedback shift register.

Why should the spreading sequences be “almost” random?

- random sequences have a flat power spectral density.

**Generation of pseudo-random spreading sequences II**

Example

![Example of a “linear” feedback shift register diagram](image)

**Figure:** Example of a “linear” feedback shift register.

**Questions:**

- how does it work?
- what are the properties of the generated sequence? For example, what is the spreading factor that can be achieved with the circuits?
Consider an example I

(a) \([5,2]\)

Figure: A valid configuration of feedback shift register of length \(R = 5\).

- Generation of a sequence \([5,2]\)
  - 5 registers.
  - The contents of registers 2 and 5 are XORed (eXclusive OR) to feed register 1.

- We are interested in understanding what is outputted by the circuit.

Consider an example II

<table>
<thead>
<tr>
<th>2 XOR 5</th>
<th>Register</th>
<th>Feedback bit</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>Output bit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>1</td>
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<td>0 →</td>
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<td>0 →</td>
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<td>1 →</td>
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<td></td>
<td>0 →</td>
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<tr>
<td></td>
<td>0 →</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table: State of the registers over time (one row per clock pulse).
Consider an example III

- The length of the outputted sequence is related to the number of possible states stored in the registers.
  - Maximum of $2^5 = 32$ possible states.
- One state is forbidden.
- Finally, we have $2^5 - 1 = 31$ possible states.

Conclusions

- The generated (pseudo-random) sequence is periodic with a period of $2^R - 1$. Such sequences are named of "maximum-length".
- The length of the period is odd.

Choosing a maximum-length sequence I

![Diagram of two different configurations of feedback shift register of length $R = 5$.](image)

Figure: Two different configurations of feedback shift register of length $R = 5$. 
<table>
<thead>
<tr>
<th>Shift register length $R$</th>
<th>Possible configurations for the feedback taps</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>[2,1]</td>
</tr>
<tr>
<td>3</td>
<td>[3,1]</td>
</tr>
<tr>
<td>4</td>
<td>[4,1]</td>
</tr>
<tr>
<td>5</td>
<td>[5,2], [5,4,3,2], [5,4,2,1]</td>
</tr>
<tr>
<td>6</td>
<td>[6,1], [6,5,2,1], [6,5,3,2]</td>
</tr>
<tr>
<td>7</td>
<td>[7,1], [7,3], [7,3,2,1], [7,4,3,2], [7,6,4,2], [7,6,3,1], [7,6,5,2], [7,6,5,4,2,1], [7,5,4,3,2,1]</td>
</tr>
<tr>
<td>8</td>
<td>[8,4,3,2], [8,6,5,3], [8,6,5,2], [8,5,3,1], [8,6,5,1], [8,7,6,1], [8,7,6,5,2,1], [8,6,4,3,2,1]</td>
</tr>
</tbody>
</table>

Figure: Maximum-length sequences of shift register lengths 2 through 8.

Properties of maximum-length sequences I

Maximal length
The all-zeros 0000... state is forbidden. So, for $R$ flip-flops, the maximum length is exactly $N = 2^R - 1$.

Balance property
The number of 1s is always one more than the number of 0s.

Therefore, the mean of a pseudo-random sequence $c(t)$ is

$$E \{c(t)\} = \frac{1}{N}$$

(275)

It is not equal to 0 as for a real random sequence.
The autocorrelation function of a maximum-length sequence is periodic.

**[Definition]** The autocorrelation of a periodic sequence (new definition because otherwise the autocorrelation function would be infinite) is defined as

\[
\Gamma_{cc}(\tau) = \frac{1}{T_b} \int_0^{T_b} E \{c(t)c(t-\tau)\} \, dt \tag{276}
\]

**[Auto-correlation function]** The autocorrelation function of a maximum-length sequence:

\[
\Gamma_{cc}(\tau) = \begin{cases} 
1 - \left(1 + \frac{1}{N}\right) \frac{|\tau|}{T_c}, & |\tau| \leq T_c \\
-\frac{1}{N}, & \text{elsewhere}
\end{cases} \tag{277}
\]

**[Periodicity of the autocorrelation function]**
The auto-correlation function of a maximum-length sequence is periodic (with period \(T_b = NT_c\)).
Power spectral density of a spread signal I

\[ \begin{align*}
\gamma_c(f) &= \frac{1}{N^2} \delta(f) + \frac{1 + N}{N^2} \sum_{n=\infty}^{+\infty} \left( \frac{\sin(\pi \frac{n}{N})}{\pi \frac{n}{N}} \right)^2 \delta \left( f - \frac{n}{NT_c} \right) \\
\end{align*} \]

Let us compare this expression (for a maximum-length sequence) with that of a random sequence:

- for a purely random sequence (which is not periodic), the autocorrelation function would be

\[ \begin{align*}
\gamma_c(\tau) &= \begin{cases} 
1 - \frac{|\tau|}{T_c} & |\tau| \leq T_c \\
0 & |\tau| > T_c
\end{cases} 
\end{align*} \]

(278)

The waveforms have the same envelope \( \text{sinc}^2() \), for their power spectral densities.

- the main difference is that, for a maximum-length sequence, it consists of delta functions spaced \( 1/NT_c \) [Hz] apart.
Let \( \{b_k\} \) denote a binary data sequence, and \( b(t) \) be its polar \( \pm 1 \) NRZ representation.

For a baseband transmission, the transmitted signal is the product \( y(t) \) obtained by
\[
y(t) = c(t) b(t)
\] (279)
where \( c(t) \) is the spreading sequence.

In the Fourier domain,
\[
\mathcal{Y}(f) = \mathcal{B}(f) \otimes \mathcal{C}(f) = \int_{-\infty}^{+\infty} \mathcal{C}(\tau) \mathcal{B}(f - \tau) \, d\tau
\] (280)
If all signals were deterministic, the bandwidth of \( \mathcal{Y}(f) \) would be almost equal to that of \( \mathcal{C}(f) \).
Input of the receiver before despreading

The received signal $r(t)$ consists of the transmitted signal $y(t)$ plus an additive interference, denoted by $i(t)$,

$$r(t) = y(t) + i(t)$$  \hspace{1cm} (281)

$$= c(t)b(t) + i(t)$$  \hspace{1cm} (282)

$i(t)$ is the signal due to other users. It has the form of

$$i(t) = \sum_k c_k(t)b_k(t)$$

At the receiver after despreading

Assuming a perfect synchronization between the transmitter and the receiver, the input of the receiver is built as (the synchronized sequence $c(t)$ also serves to despread the signal):

$$z(t) = c(t)r(t)$$  \hspace{1cm} (283)

$$= c^2(t)b(t) + c(t)i(t)$$  \hspace{1cm} (284)

The $c(t)$ sequence alternates between $-1$ and $+1$. Therefore, $c^2(t)$ is equal to $+1$, for all $t$. This leads to

$$z(t) = b(t) + c(t)i(t)$$  \hspace{1cm} (285)

At the multiplier output of the receiver, we have

- $b(t)$, the original decoded signal.
- an interference signal spread by $c(t)$. 


The receiver gets
\[ z(t) = c^2(t)b(t) + c(t)i(t) = b(t) + c(t)i(t) \] \hspace{1cm} (286)

If \( i(t) \) originates from another user, it is of the same type as of \( b(t) \). Let assume that:
\[ i(t) = c'(t)b'(t) \] \hspace{1cm} (287)

Then, it is important to compare:
- \( c^2(t) \), whose average value is the value of the autocorrelation function for \( \tau = 0 \), and
- \( c(t)c'(t) \), whose average is given by the cross-correlation function \( \Gamma_{cc'}(\tau) \).

More about \( c(t)c'(t) \)

In fact, there are two possibilities for building the signal \( i(t) \) of another user:
1. use a different spreading sequence \( c'(t) \). Then, because we don’t know the synchronization time of \( i(t) \), we need to upper bound \( \Gamma_{cc'}(\tau) \) to a much lower value than \( \Gamma_{cc}(0) \):
\[ \Gamma_{cc'}(\tau) \ll \Gamma_{cc}(0) \] \hspace{1cm} (288)
2. use the same spreading sequence, but delayed by some known time shift value \( \Delta T \). Then, the condition becomes
\[ |\Gamma_{cc}(\Delta T)| \ll \Gamma_{cc}(0) \] \hspace{1cm} (289)

Many systems, such as the GPS system, are designed to work with a unique sequence \( \Rightarrow \) hardware simplification.
For systems that use different spreading sequences

Figure: Cross-correlation functions.

Direct sequence spread BPSK modulation I

Figure: Direct sequence spread modulation: general overview.
Figure: Direct sequence spread coherent phase-shift keying.

**Figure**: Spreading + BPSK modulation.
From a conceptual point of view, it is possible to interchange the order of spectrum spreading and phase modulation. The interference, modeled by $j(t)$, limits the performance.

**Figure:** Model of direct sequence spread BPSK system.

**Sources of noise in the channel of a spread spectrum system**

In a channel, we have:

1. *additive white Gaussian noise*, that models a sum of independent noise signals.

2. *small bandwidth noises* (“colored” noises), that are very localized in the spectrum. Note that despreading a useful signal will spread these noises.

3. *BPSK/CDMA signals of other users.*

   If $M$ represents the number of users that share the same bandwidth, the level of noise power is estimated as $M - 1$ times the nominal power of one user.

It is important to understand what happens after despreading.
Sources of noise in the channel of a spread spectrum system II

![Diagram of a spread spectrum system]

**Figure:** A small bandwidth noise (drawn in red) in the channel is spread by the despreading sequence of the user (drawn in blue).

Sources of noise in the channel of a spread spectrum system III

<table>
<thead>
<tr>
<th>in the channel</th>
<th>after despreading at the transmitter</th>
</tr>
</thead>
<tbody>
<tr>
<td>additive white noise</td>
<td>remains a white noise at the same level</td>
</tr>
<tr>
<td>small bandwidth noises (“colored” noise)</td>
<td>is spread by the specific spreading sequence of the user. The total amount of power is spread but remains unchanged.</td>
</tr>
<tr>
<td>signals of other users</td>
<td>are spread by the specific spreading sequences of the users. The total amount of power is spread but remains unchanged.</td>
</tr>
</tbody>
</table>
A thorough analysis of the performance leads to (no proof provided)

\[ 10 \log \left( \frac{S}{N} \right)_{\text{OUT}} = 10 \log \left( \frac{S}{N} \right)_{\text{IN}} + 3 + 10 \log \left( \frac{T_b}{T_c} \right) \quad (290) \]

- The 3 [dB] term originates from the coherent demodulation mechanism.
- There is a gain that is proportional to

\[ \frac{T_b}{T_c} = N \quad (291) \]

\( N \) is named the processing gain.

---

**Theorem (Bit Error Rate for a DS/BPSK modulation system)**

\[ P_e \approx \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{J T_c}} \right) \quad (292) \]

where \( J \) is the average interference power: \( J = \frac{1}{T_b} \int_0^{T_b} j^2(t) \, dt. \)

In other terms, if \( N_J = \int_0^{T_b} j^2(t) \, dt \), then

\[ P_e \approx \frac{1}{2} \text{erfc} \left( \sqrt{\frac{T_b}{T_c} \frac{E_b}{N_J}} \right) = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{N E_b}{N_J}} \right) \quad (293) \]

The \( N \) factor is obtained because:

- noise is unaffected by the despreading operation
- the useful signal is concentrated in a \( N \) times smaller bandwidth after despreading.
Performance: interference margin I

Definitions and notations:

- $S [W]$ received power for the desired signal. It is equal to the energy per bit \times the bitrate ($S = E_b \times R_b$),
- $E_b [J = W \times s]$ received energy per bit for the desired signal,
- $R_b = \frac{1}{T_b} [Hz]$ data bitrate,
- $B [Hz]$ spread bandwidth in Hz,
- $J [W]$ received power for the jamming signals,
- $N_0 [W/Hz]$ equivalent noise power spectral density.

Performance: interference margin II

Case 1: absence of spreading

The signal to noise (jamming) ratio is given by

$$ \frac{S}{J} = \frac{E_b R_b}{N_0 B} $$

For a system that is not spread in bandwidth: $B \approx R_b$. 

$$ \left( \frac{E_b}{N_0} \right) $$ is then equal to the signal to noise ratio:

$$ \frac{S}{J} = \frac{E_b}{N_0} $$

Equivalently,

$$ \frac{J}{S} = \frac{1}{E_b/N_0} $$

and the minimal $\frac{S}{J}$ is set by the required $\frac{E_b}{N_0}$:

$$ \frac{J}{S} [\text{dB}] = - \left( \frac{E_b}{N_0} \right)_{\text{req}} [\text{dB}] $$
Case 2: with spreading

The ratio of the equivalent “noise” power \( J \) to \( S \) is given by

\[
\frac{J}{S} = \frac{N_0 B}{E_b R_b} = \frac{B}{E_b N_0}
\]

When the value of \( \frac{E_b}{N_0} \) is set to that required for acceptable performance of the communication system \( \left( \frac{E_b}{N_0} \right)_{\text{req}} \), then the ratio \( \frac{J}{S} \) bears the interpretation of a jamming margin:

\[
\frac{J}{S} \text{ [dB]} = \text{margin [dB]} = \frac{B}{R_b} \text{ [dB]} - \left( \frac{E_b}{N_0} \right)_{\text{req}} \text{ [dB]}
\]

The quantity \( \frac{B}{R_b} \) is called the spread-spectrum processing gain. It is equal to \( (B \approx R_c) \)

\[
\frac{B}{R_b} \approx \frac{R_c}{R_b} = \frac{T_b}{T_c} = N
\]

Capacity of a spread spectrum system

**Theorem**

The capacity (=number of users) of a multi-user spread spectrum system is the ratio of the spread bandwidth to the data rate:

\[
M = \alpha \frac{B}{R_b} = \alpha N
\]

To calculate its value, we consider a unique cell.

- The carrier power is

\[
C = S = R_b E_b
\]

- Likewise, the interference power \( I \), at the base station receiver, is

\[
I = B N_0
\]

where \( B \) is the transmission bandwidth.
Thus, the carrier to interference power ratio for a particular mobile user at the base station is given by

$$\frac{C}{I} = \frac{R_b E_b}{BN_0} = \frac{E_b}{N_0} \frac{B}{R_b}$$ (304)

Let $M$ denote the number of users. Assuming that all users have the same power level at the base station, then the total interference power for one user is cause by $M - 1$ interferers

$$I = C(M - 1)$$ (305)

which leads to

$$\frac{C}{I} = \frac{1}{M - 1}$$ (306)

By combining the expressions (304) and (306) for $\frac{C}{I}$, we derive that

$$M = \frac{B}{R_b} \frac{1}{E_b/N_0} + 1 \simeq \frac{B}{R_b} \frac{1}{E_b/N_0} = N \frac{1}{E_b/N_0}$$ (307)

**Additional factors I**

- **Duty-cycle** $\alpha$ of a full-duplex voice conversations is 35%. In practice, we take 50%.
  **Consequence:**
  - the power sent by each user drops to $\alpha R_b E_b$

  $$M \simeq \frac{B}{\alpha R_b} \frac{1}{E_b/N_0} = \frac{B}{R} \frac{1}{E_b/N_0} \frac{1}{\alpha}$$ (308)

- **Antenna directivity.** Cells are divided into 3 sectors (with 15% overlap) and they provide a “gain”.
  **Consequence:**
  - The number of possible users is multiplied by the number of available frequencies (3).
  - Because a user in between two cells must chose one single frequency, we have 15% loss. Therefore, we have a “gain” of $G = 3 \times 0.85 = 2.55$:

  $$M \simeq \frac{B}{R_b} \frac{1}{E_b/N_0} \frac{1}{\alpha} G$$ (309)
The re-usability factor of frequencies \( F_e \) is less than 1 (\( F_e \approx 0.65 \) for the IS-95 mobile system):

\[
F_e = \frac{\text{Effective surface covered by a frequency}}{\text{Total geographic surface}}
\]  

(310)

**Final estimation of users per cell**

\[
M \simeq \frac{B}{R_b} \frac{1}{E_b/N_0} \frac{1}{\alpha} G F_e
\]  

(311)

Example

For the IS-95 radio mobile system: \( \frac{B}{R_b} = 128 \), \( \frac{E_b}{N_0} = 7 \) [dB] = 5, \( \alpha = 0.5 \), \( G = 2.55 \), and \( F_e = 0.65 \).

These parameters yield a capacity of

\[
M = 128 \times \frac{1}{5} \times \frac{1}{0.5} \times 2.55 \times 0.65 = 85
\]  

(312)

Outline

1. Reminder
2. Representation of bandpass signals
3. Noise in telecommunications systems
4. Digital modulation
5. Spread spectrum
6. Channels for digital communications and intersymbol interference
7. Navigation systems
8. Multiplexing
9. Telephone traffic engineering
10. Transmission over twisted pairs (fixed telephone network)
11. Radio engineering
Introduction I

In a communication chain, we have:

- a transmitting filter (why?)
- a channel that, if linear, is described by a transfer function
- a receiver filter (why?)

One should “optimize” the shape of the transmitting and the receiver filters with respect to the channel characteristics.
The pulse of each may be smeared into adjacent time slots: this causes InterSymbol Interference (ISI).

**Theorem (Interpolation formula of Whittaker)**

Let \( y(t) \) be a signal band limited to \([-W, W]\). Take the \( \{y[nT_b]\} \) set of samples regularly spaced by \( T_b = 1/f_b \). Then the \( y(t) \) function can be obtained by (with \( f_b = 2W \))

\[
y(t) = \sum_{n=-\infty}^{+\infty} y[nT_b] \text{sinc} \left( \frac{t - nT_b}{T_b} \right)
\]

(313)
Interpretation of Whittaker’s formula

- $y(t)$: is an analog signal (theoretically, perfectly reconstructed)
- $y[nT_b]$: are samples taken every $T_b$
- $\text{sinc}\left(\frac{t-nT_b}{T_b}\right) = p(t)$ is a real shape (pulse) whose spectrum is strictly limited to $[-W, W]$, which is unfeasible in practice.

What about imperfections in the channel?
- Noise? ⇒ matched “filter”
- Pulse shape? Some pulse shapes are impossible to build.
- Multi-path, obstacles, etc.
  This is a problem from an operational point of view.
  ⇒ multiple time-shifted versions of the signal.
  ⇒ need to “compensate” the effects of the channel.

Choice for a “better” pulse shape $p(t)$ than $\text{sinc}\left(\frac{t-nT_b}{T_b}\right)$

Constraints:

1. A pulse $p(t)$ should be time limited. We expect that $p(t) = 0$ for $t < -\tau$.
2. $P(f)$ should be a bandpass signal, whose content is mainly concentrated inside of $[-1/2T_b, 1/2T_b]$.
3. When there is no noise, we should be able to reconstruct the signal perfectly.

Note that 1 and 2 are incompatible.

Therefore, we are looking for a trade-off:

- $p(t)$ should be close to 0 outside its main interval, and
- $P(f)$ should be almost 0 outside its main bandwidth.
At the output of the complete chain, we can write that
\[
y(t) = \sum_{k=\infty}^{\infty} A_k \delta(t - kT_b) \otimes g_T(t) \otimes h(t) \otimes g_R(t) + w(t) \otimes g_R(t) 
\]  
(314)

where \( \otimes \) denotes the convolution.

We adopt the following notations (we take \( \mu \) such that \( p(0) = 1 \)):
\[
\begin{align*}
\mu p(t) &= g_T(t) \otimes h(t) \otimes g_R(t) \\
n(t) &= w(t) \otimes g_R(t) 
\end{align*}
\]  
(315) \hspace{0.5cm} (316)

In the frequency domain,
\[
\mu \mathcal{P}(f) = \mathcal{G}_T(f) \mathcal{H}(f) \mathcal{G}_R(f) 
\]  
(317)
As \( w(t) \) is modeled as an additive zero mean white Gaussian noise, \( w(t) \otimes g_R(t) \) is of the same type. It is denoted \( n(t) \) hereafter.

\[
y(t) = \sum_{k=\infty}^{+\infty} A_k \delta(t - kT_b) \otimes \mu p(t) + w(t) \otimes g_R(t) \quad (318)
\]

\[
y(t) = \mu \sum_{k=\infty}^{+\infty} A_k p(t - kT_b) + n(t) \quad (319)
\]

Once sampled at \( t_i = iT_b \), the output signal \( y(t) \) becomes

\[
y[t_i] = \mu \sum_{k=\infty}^{+\infty} A_k p[(i - k)T_b] + n[t_i] \quad (320)
\]

\[
y[t_i] = \mu A_i + \sum_{k=\infty}^{+\infty} A_k p[(i - k)T_b] + n[t_i] \quad (321)
\]

Analysis of

\[
y[t_i] = \mu A_i + \sum_{k=-\infty}^{+\infty} A_k p[(i - k)T_b] + n[t_i]
\]

- Useful signal: \( \mu A_i \)
- Noise:
  - \( \sum_{k=-\infty}^{+\infty} A_k p[(i - k)T_b] \) represents the residues of all other symbols \( k \neq i \) on the \( i \)-iest bit. It is at the origin of the phenomenon of intersymbol interference. This is an artificial noise that should be minimized or even nullified.
  - the noise effect due to \( n[t_i] \) is “reduced” via the matching filter.

Minimizing the intersymbol interference increases the signal to noise ratio (and decreases the bit error rate \( P_e \)).
Nyquist’s criterion

Goal: chose the best pulse shape $p(t)$ to minimize, or suppress, intersymbol interferences.

This is achievable if $p(t)$ verifies (noise-free channel)

$$p[(i - k)T_b] = \begin{cases} 
1 & \text{if } i = k \\
0 & \text{if } i \neq k 
\end{cases} \quad (322)$$

where $p[0] = 1$ (by normalization).

Let us determine the best pulse shape $p(t)$. To compute it, we inject the constraints on $p[(i - k)T_b]$ in the Fourier transform of a sampled version of $p(t)$. 
Towards Nyquist’s criterion I

**Sampled version of \( p(t) \)**

Consider \( p(t) \), represented by a series of its samples \( \{ p[mT_b] \} \), for \( m = 0, \pm 1, \pm 2, \ldots \). Then, the signal

\[
p_s(t) = \sum_{m=-\infty}^{+\infty} p[mT_b] \delta(t - mT_b) \tag{323}
\]

is a sampled version of \( p(t) \) (by definition of the sampling process).

Its Fourier transform is given by

\[
\mathcal{P}_s(f) = f_b \sum_{m=-\infty}^{+\infty} \mathcal{P}(f - mf_b) \tag{324}
\]

where \( f_b = 1/T_b \) is the bitrate (or rhythm) expressed in \([b/s]\).

Towards Nyquist’s criterion II

We have two expressions for the Fourier transform \( \mathcal{P}_s(f) \)

1. **Expression [1]**: \( \mathcal{P}_s(f) = f_b \sum_{m=-\infty}^{+\infty} \mathcal{P}(f - mf_b) \)

2. **Expression [2]**: based on the definition, it can be written as

\[
\mathcal{P}_s(f) = \int_{-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} [p[mT_b] \delta(t - mT_b)] e^{-2\pi jft} \, dt \tag{325}
\]

but, as the sum reduces that to the \( m = 0 \) term (according to the \( p[(i - k)T_b] = 0 \) if \( i \neq k \) constraint), we may write

\[
\mathcal{P}_s(f) = \int_{-\infty}^{+\infty} p[0] \delta(t) e^{-2\pi jft} \, dt = e^{-2\pi j0f} = 1 \tag{326}
\]

By combining [1] and [2]:

\[
\sum_{m=-\infty}^{+\infty} \mathcal{P}(f - mf_b) = \frac{1}{f_b} = T_b \tag{327}
\]
This leads to Nyquist’s criterion for an ideal, noiseless, baseband transmission.

**Theorem (Nyquist’s criterion)**

The Fourier transform $\mathcal{P}(f)$ of the pulse shaping function $p(t)$ removes all intersymbol interferences due to other samples taken every $T_b$ seconds if

$$
\sum_{m=-\infty}^{+\infty} \mathcal{P}(f - mf_b) = T_b \tag{328}
$$

Note that the expression $\mathcal{P}(f)$ encompasses the transmitting filter, the channel, and the receiver filter.

"Ideal" channel

The easiest way to satisfy Nyquist’s criterion consists to take a rectangular shape for $\mathcal{P}(f)$ (in the spectral domain):

$$
\mathcal{P}(f) = \begin{cases} 
\frac{1}{2W} & -W < f < W \\
0 & |f| > W 
\end{cases} \tag{329}
$$

$$
= \frac{1}{2W} \text{rect}[-W,+W](f) \tag{330}
$$

where the bandwidth $W$ is

$$
W = \frac{f_b}{2} = \frac{1}{2T_b} \tag{331}
$$

![Diagram showing rectangular pulse shape in the frequency domain](image)
Alternative pulse shapes I

There are several reasons that prevent us to use a time sinc filter (or rectangular frequency filter):

1. such a filter requires $\mathcal{P}(f)$ to be constant over $[-W, +W]$ and 0 outside.
2. the function $p(t)$ decreases as $1/|t|$ for large $|t|$ values; it decreases slowly, which means that many samples interfere with each others.

Solution: raised cosine-pulse, whose spectrum is given by

$$
\mathcal{P}(f) = \begin{cases} 
\frac{1}{4W} \left\{ 1 - \sin \left[ \frac{\pi|f|}{2W} \right] \right\} & 0 \leq |f| < f_1 \\
0 & f_1 \leq |f| < 2W - f_1 \\
0 & |f| \geq 2W - f_1
\end{cases}
$$  (332)

The constant $f_1$ and bandwidth parameter $W$ are related by ($\alpha$ is the roll-off factor)

$$
\alpha = 1 - \frac{f_1}{W}
$$  (333)

Alternative pulse shapes II

The bandwidth $B_T$ is therefore given by: $B_T = 2W - f_1 = W(1 + \alpha)$
Practical remarks

\[ \mu P(f) = G_T(f)H(f)G_R(f) \]  \hspace{1cm} (334)

It is the complete chain that needs to be optimized, but \( H(f) \) is unknown!

**Solution:**

1. estimate \( H(f) \): \( \Rightarrow \hat{H}(f) \).
2. compensate for the presence \( H(f) \) by taking the inverse of \( \hat{H}(f) \).

In conclusion, after compensation, we should have

\[ \mu P(f) = G_T(f)G_R(f) \]  \hspace{1cm} (335)

But how to chose the transmitter and the receiver filters?

In a real communication environment:

- we need a **transmitter filter**, whose purpose consists to avoid the appearance of a spectral content outside the useful bandwidth.
- there must be a **receiver filter**. It eliminates out-of-band noise.
- \( P(f) \) is therefore constrained to be partly implemented at the transmitter and at the receiver.

\[ \Rightarrow \text{we can equally divide } P(f) \text{ over the transmitter and the receiver (such filters are named } \text{half-Nyquist}) \]

Link with the notion of ideal channel for the transmission of an analog signal in a channel (delay and no amplitude distortion):

\[ H(f) = Ae^{-2\pi jf\tau} \]  \hspace{1cm} (336)
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Fundamentals of the Global Positioning System (GPS)

- Introduction
- Principles of localization/positioning
- Signals
- Improvements
Definition and objectives

GPS = Global Positioning System

Definition
Network of satellites that sent signals permanently to allow the positioning on the earth surface by \textit{trilateration} measurements (sometimes, incorrectly, named as \textit{triangulation}).

- Started by the U.S. Department of Defense
- First operational satellite in 1978
- Full constellation since 1994
- Precision of 6 to 12 [m] since May 2000

Constraints:
* need to be able to receive the transmitted signals.
* does not work inside of buildings or underground.
3 segments:

- **Space segment**
  - comprising a constellation of 24 satellites (with 21 in service)
- **Control segment**
  - comprising ground stations
  - goal: correct the position of satellites
- **User segment**

**Satellite trajectories**

- Located on high orbits (but sub-geostationary)
- Revolution period of 12 hours
- Transmitting power of 20 to 50 [W]
- 6 planes with a 55° angle with the equator, spaced by 60° and with 4 satellites per plane (24 satellites in total)
Definition (Geoid)
The geoid is essentially the figure of the earth abstracted from its topographical features. It is an idealized equilibrium surface of sea water, the mean sea level surface in the absence of currents, air pressure variations, etc, and continued under the continental masses.

- Because of the non-uniform repartition of masses, the earth is not a sphere.
- The geoid is close to an ellipsoid whose long axis is equal to 6400 [km].

Principles of localization/positioning

1. A distance $d$ is related to a propagation time by $t$

$$d = c \times t$$

(337)

where $c = 3 \times 10^8 \text{[m/s]}$ is the speed of light.

2. **Trilateration** (or even multilateration).

Figure: Positioning by means of intersecting the loci of 2 or 3 satellites.
Basic equations for finding user position

\[ \rho_1 = \sqrt{(x_1 - x_u)^2 + (y_1 - y_u)^2 + (z_1 - z_u)^2} \quad (338) \]
\[ \rho_2 = \sqrt{(x_2 - x_u)^2 + (y_2 - y_u)^2 + (z_2 - z_u)^2} \quad (339) \]
\[ \rho_3 = \sqrt{(x_3 - x_u)^2 + (y_3 - y_u)^2 + (z_3 - z_u)^2} \quad (340) \]

- Second order equations → difficulties
- Resolution with linearization and iterative approach

In principle, 3 distances \( \rho_i \) suffice. But, to minimize errors due to inaccuracies, 4 distances are needed (and thus 4 satellites in line of sight).

Measurement of pseudo-range

Every satellite \( i \) sends a signal at a certain time \( t_{si} \). It is received by the user at time \( t_u \). The distance between the user and the satellite is

\[ \rho_i T = c \times (t_u - t_{si}) \quad (341) \]

where \( c \) is the speed of light.

In practice, both times are biased \( t_u \) (inaccuracies)

\[ t'_u = t_u + b_{ut} \quad (342) \]

and, for \( t_{si} \),

\[ t'_i = t_{si} + \Delta b_i \quad (343) \]

But there are other sources of errors (\( \Delta D_i = \) satellite position error on range, \( \Delta T_i = \) tropospheric delay error, \( v_i = \) receiver measurement noise error, \( \triangle v_i = \) relativistic time correction), leading to

\[ \rho_i = \rho_i T + \Delta D_i - c \times (\Delta b_i - b_{ut}) + c \times (\Delta T_i + \Delta l_i + v_i + \triangle v_i) \quad (344) \]
A GPS signal comprises 2 spectral components: L1 (Link 1), L2

- $L1 = 1575.42 \text{ [MHz]} = 154 \times 10.23 \text{ [MHz]}$ is used for consumer market receivers.
- $L2 = 1227.6 \text{ [MHz]} = 120 \times 10.23 \text{ [MHz]}$ reserved for military applications.

The frequencies are very accurate as their reference is an atomic frequency standard.

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>C/A</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>$-133 \text{ [dBm]}$</td>
<td>$-130 \text{ [dBm]}$</td>
</tr>
<tr>
<td>L2</td>
<td>$-136 \text{ [dBm]}$</td>
<td>$-136 \text{ [dBm]}$</td>
</tr>
</tbody>
</table>

Table: Minimal required power levels at the receiver.
Each satellite sends three types of information:

1. **data**
   - type of satellite
   - maintenance data
   - precise calculus of the orbit (precision < 1 m), etc

2. A **precision code P**: this allows a precise calculus of delays. Civilian users have access to a degraded version of P.

3. A **coarse (or clear) acquisition code C/A**: allows an approximated calculus of delays. Usually acquired first.

---

**Signal sent by a satellite**

![Diagram of signal sent by a satellite]

*Figure:* Building the signal sent by GPS satellites.
- Bi-phase modulating signal → BPSK modulation
- Bitrate of 10.23 [Mb/s] → bandwidth as defined by the main lobe size: 20.46 [MHz].
- $T_c = 97.8$ [ns]
- Generated from two pseudorandom noise codes
  - sequence 1: 15, 345, 000 chips, 1.5 [s] long period
  - sequence 2: 15, 345, 037 chips
- As these numbers are relative prime, they have no common factors between them. Therefore, the code length generated by these codes is

$$1.5 \times 15,345,037 = 23,017,555.5 \text{ [s]}$$

which is slightly longer than 38 weeks.

However, the actual code length is 1 week as the code is reset every week. This code can be divided into 37 different $P$ codes and each satellite can use a different portion of the code (identification).

**Code generation**
C/A code

- Bi-phase NRZ code → modulation BPSK
- Bitrate of 1.023 [Mb/s] → width of the main lobe 2.046 [MHz].
  - → the bandwidth is thus fixed by the P code, not the C/A code.
- $T_c = 977.5$ [ns]
- Generated by means of a pseudorandom sequence of 1023 chips → period of 1 [ms]
Better precision with the help of ground stations:
Differential GPS

Figure: Working of the DGPS.

- FM band
- Precision up to 1 to 5 [m]

Galileo

- Orbital altitude: 23,222 [km] (MEO - Medium Earth Orbit)
- 3 orbital planes, 56° inclination, separated by 120° longitude
- Constellation of 30 satellites (with working 24 [3x8] satellites and 6 [3x2] spares)
Deployment of Galileo

- First launches: 2 satellites in October 2011, 2 satellites in October 2012. These were test satellites.
- First *Full Operational Capability* satellite launched in November 2013.
- August 2014, two more satellites (but ... injected on a wrong orbit).
- November 2019: 23 satellites fully operational, 3 for testing or not available.

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Multiplexing and multiple access

- Multiplexing: principles and access modes
- Specific techniques for multiplexing:
  - Frequency Division Multiplexing (FDM)
  - Time Division Multiplexing (TDM)
    - Example: signals between telephone exchanges
  - Code Division Multiplexing (CDM)

Historically, there are two major techniques:

1. **Frequency Division Multiplexing (FDM)**. This technique allocates a dedicated frequency band to each signal (user).
2. **Time Division Multiplexing (TDM)**. This technique monitors the time allocated to each user. Two users cannot use the channel simultaneously.

More recent technique: *multiplexing by spread spectrum* → **Code Division Multiplexing (CDM)**

Variants for optical fibers:

- **Wave(length) Division Multiplexing (WDM)**: it is equivalent to FDM (but in terms of wavelengths)
- some variants especially targeting high speed links: **Dense Wave Division Multiplexing (DWDM)**.
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---

**Multiple access**

Multiplexing is introduced for *sharing* resources. Therefore, a multiplexing technique should propose and implement:

- **techniques to access individual resources.** We refer to the techniques and protocols to access a resource by the letter $A$, denoting *Access*: FDMA, TDMA, CDMA, ...

- **strategies to allocate resources** to users. These strategies, which are specific to a network operator, are based on performance targets: high bitrate, low delay, low congestion rate, ... These strategies are studied in other courses.
Principles of Frequency Division Multiplexing (FDM) I

Multiplexed signal

Figure: Frequency demultiplexing.

Principles of Frequency Division Multiplexing (FDM) II

Multiplexed signal

Figure: Frequency demultiplexing.
Bandwidth and access to the multiplex

\[ W > \sum_i W_i \quad (346) \]

**Figure:** Frequency Division Multiple Access (FDMA): multiplexing mechanism.

**Illustration: Plain Old Telephone Service (POTS)**

- Wired network
- Trunks are shared
- Frequency band per user: [300 Hz, 3400 Hz]

Voice channel: [0 → 4 kHz]
Figure: Construction of a basic group and a super group.

Figure: Building mechanism for super groups.
The time slots for users, plus the information necessary for the multiplex itself, constitute a frame.

**Bandwidth**

The total number of samples is \( n_s = \sum_i 2W_i T \)

If \( W \) is the bandwidth, we need \( 2W \) samples. For an interval \( T \), we need

\[
ns = 2WT = \sum_i 2W_i T \quad (347)
\]

So that,

\[
W = \sum_i W_i \quad (348)
\]
Public switch telephone network in Europe: 30 digital voice channels (E1)

**Characteristics**

The European standard TDM frame for 30(+2) digital voice channels (E1):

- \( W = \text{bandwidth of the signal} = 300 - 3400 \text{ [Hz]} \) and \( f_s = 8 \text{ [k sample/s]} \).

- PCM encoding with the A (\( \approx 87,6 \)) quantization law, compression with 13 segments on 8 bits, \( R_v = \text{bitrate for one digital voice channel} = 64 \text{ [kb/s]} \).

- total bitrate \( = 32 \times 64 \text{ [kb/s]} = 2.048 \text{ [Mb/s]} \).

This is the minimal bitrate for communicating with a digital public network in Europe.
Public switch telephone network in Europe: 30 digital voice channels (E1) II

- Time multiplexing with 32 time slots comprising:
  - TS0 channel: framing, synchronization, alarms, CRC, etc.
  - TS1-TS15 + TS17-T31: 30 voice channels
  - TS16: signaling

**Frame structure I**

- All time slots contain 8 bit words
- 1 frame = 256 bits
- 3.9 µs
- 488 ns

Figure: Frame structure at the bit level.
In the network, we need higher bit rates than E1 bit rates. **Synchronous Digital Hierarchy** (SDH) is a standardized protocol that transfer multiple digital bit streams synchronously over optical fiber.

<table>
<thead>
<tr>
<th>SDH level and frame format</th>
<th>Payload bandwidth [kb/s]</th>
<th>Line rate [kb/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>STM-1</td>
<td>150 336</td>
<td>155 520</td>
</tr>
<tr>
<td>STM-4</td>
<td>601 344</td>
<td>622 080</td>
</tr>
<tr>
<td>STM-16</td>
<td>2 405 376</td>
<td>2 488 320</td>
</tr>
<tr>
<td>STM-64</td>
<td>9 621 504</td>
<td>9 953 280</td>
</tr>
<tr>
<td>STM-256</td>
<td>38 486 016</td>
<td>39 813 120</td>
</tr>
</tbody>
</table>
SDH frame structure

1. **RSOH** = regenerator section overhead
2. Pointers are used to address a **Virtual Container** in the payload.
3. **MSOH** = multiplex section overhead

**Typical structure of a SDH network**

- Presence of Add and Drop Multiplexers (ADM)
- Structure of a ring → redundancy in the paths
Main multiplexing techniques:

- **FDMA**: use *some* of bandwidth *all* of the time
- **TDMA**: *all* of the bandwidth *some* of the time
  and ...
- **CDMA**: *all* of the bandwidth *all* of the time!

**Figure**: Scheme for the study the cross-correlation phenomenon.
The interference (only) due to user $i$ evaluated at the input of the decision device in the receiver of user $j$ is given by

$$v_j(\tau)\bigg|_{b_j(t)=0} = \int_0^{T_b} b_i(t-\tau) c_i(t-\tau) c_j(t)dt \quad (349)$$

$$= \pm \int_0^{T_b} c_j(t)c_i(t-\tau)dt \quad (350)$$

We may rewrite this in the following form

$$v_j(\tau)\bigg|_{b_j(t)=0} = \pm T_b \Gamma_{ji}(\tau) \quad (351)$$

where

$$\Gamma_{ji}(\tau) = \frac{1}{T_b} \int_0^{T_b} c_j(t)c_i(t-\tau)dt \quad (352)$$

This time-averaged quantity is called the *partial cross-correlation function* of two pseudo-random sequences.

*Figure*: Cross-correlation of two sequences of length $N = 63$ (generated by [6, 1] and [6, 5, 2, 1]).
There are two techniques for multiplexing with Direct Sequence Spread Spectrum (DSSS):

- **one identical spreading sequences** for all the users. As long as synchronization time are kept separated by a time larger to one chip, this is not problematic
  - → the auto-correlation function is important to measure the performance of the system.

- **different spreading sequences** for all the users.
  - → the cross-correlation function is important to measure the performance.

**Figure:** Auto-correlation and cross-correlation ($R_{4,1}(\tau)$ and $R_{4,6}(\tau)$).
It is important to find an upper bound for the cross-correlation function. There are such results:

**Theorem**

*It can be shown that the Gold sequences, under certain generation conditions, satisfy the following cross-correlation function values (where \( n \) is the number of registers):*

\[
|R_{1,t}(k)| \leq \begin{cases} 
2^{\frac{n+1}{2}} + 1 & \text{if } n \text{ is odd} \\
2^{\frac{n+2}{2}} + 1 & \text{if } n \text{ is even but } n \neq 0 \text{mod 4} 
\end{cases} \tag{353}
\]
Combination of multiplexing techniques

**Figure:** Code Division Multiple Access (CDMA).

**Figure:** Resource sharing by combining time and frequency multiplexing (TDMA/FDMA).
Overview of *multiplexing/modulation* techniques (+ *diversity* techniques)

**Space-time transmission**

**Figure:** Space diversity: (a) one transmit and two receive antennas and (b) two transmit and one receive antennas.

*Figure:* Space diversity: (a) one transmit and two receive antennas and (b) two transmit and one receive antennas.
Objectives:
1. improve the link reliability (reduces fading)
2. increase the spectral efficiency (more bits per second per hertz of bandwidth)

One possibility:

\[
\begin{align*}
time & \rightarrow \quad [(k - 1)T_b, kT_b] \quad [kT_b, (k + 1)T_b] \\
\text{antenna 1} & \quad A_k \sqrt{E_b/T_b} \cos(2\pi f_c t) \quad -A_{k+1} \sqrt{E_b/T_b} \cos(2\pi f_c t) \\
\text{antenna 2} & \quad A_{k+1} \sqrt{E_b/T_b} \cos(2\pi f_c t) \quad A_k \sqrt{E_b/T_b} \cos(2\pi f_c t)
\end{align*}
\]
Ethernet protocol: Carrier Sense Multiple Access with Collision Detection (CSMA-CD)

- It is a protocol (not a multiplexing technique)
- Relates to the Access
- Complex state diagram

Comparison of mobile standards

<table>
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<td>Access mode</td>
<td>TDMA/FDMA</td>
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<td>TDMA/FDMA</td>
<td>CDMA/FDMA</td>
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<td>1805-1880</td>
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<td>Distance between channels</td>
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<td>30</td>
<td>1250</td>
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<td>π/4 DQPSK</td>
<td>BPSK/QPSK</td>
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<td>Characteristics for the mobile device</td>
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<td>Max./Mean</td>
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<td>RPE-LTP</td>
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<td>VSELP</td>
<td>QCELP</td>
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<td>Voice rate (kb/s)</td>
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<td>13</td>
<td>7,95</td>
<td>8 (var.)</td>
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<tr>
<td>Channel bitrate</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upwards (kb/s)</td>
<td>270,833</td>
<td>270,833</td>
<td>48,6</td>
<td></td>
</tr>
<tr>
<td>Downwards (kb/s)</td>
<td>270,833</td>
<td>270,833</td>
<td>48,6</td>
<td></td>
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<tr>
<td>Frame (ms)</td>
<td>4,615</td>
<td>4,615</td>
<td>40</td>
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</tr>
</tbody>
</table>
Telephone traffic engineering

- Characterization of a telephone network
  - Intensity
  - Carried traffic load
  - Reference load
  - Offered load

- Probabilistic analysis of calls
  - Counting process
  - Poisson distribution: definition and properties

- Probabilistic law for telephone traffic
  - Memoryless model: Erlang B statistic
  - Sizing of a trunk
  - Other models
For economic reasons, there are fewer lines available than the number of telephones or potential callers (the cost is proportional to the number of lines).

There is a probability that someone willing to make a call find that all the lines are in use. The call is then “blocked”.

Goal: establish the relationship between the number of lines $N$ of a trunk and the blocking probability.

Intensity, traffic and load: which measure(s)?

There are two types of links in a switched telephone network:

- links that carry telephone signals or data
- links dedicated to the transmission of signaling data

Figure: A trunk between two switches.
We need a measure for the level of activity!

**Intensity measures II**

**Definition (Intensity of the carried traffic)**

The intensity \( I \) of the carried traffic is defined as the ratio between the observed traffic volume divided by the observation time \( T \):

\[
I = \frac{\int_0^T \sum_{i=1}^N 1_i(t) \, dt}{T} = \frac{\sum_{i=1}^N \int_0^T 1_i(t) \, dt}{T}
\]

where \( 1_i(t) \) is the *indicator function* of a link (line \( i \)).

The intensity represents the average carried traffic during a given amount of time. It has no unit; however it is given a unit name and is expressed in **Erlang**, denoted \([E]\).
In practice, the traffic is characterized by two important parameters:

1. the **average rate of carried calls** \( \lambda_c \), measured in [call/s],
2. the **average call duration** \( t_m \), in [s/call].

If \( \#_T \) represents the number of calls done during a time \( T \), then the average duration is given by

\[
t_m = \frac{\sum_{i=1}^{N} \int_0^T 1_i(t)dt}{\#_T}
\]  

(355)

### Notions of load(s)

**Definition (Carried traffic load)**

The **carried load**, expressed in Erlang, is the product of the entering call rate (thus carried) by the average duration of a call. It is denoted \( A_c \).

\[
A_c = \lambda_c t_m
\]

(356)

**Reference load?**

The maximum load on a \( N \)-lines trunk is theoretically \( N \). But in practice, not all the lines are used permanently and we distinguish between two types of loads:

1. the **offered load**; it is the load that would be carried if the network could carry all calls, without any limit.
2. the **carried load**; it is the load really measured in the network.

By definition, the **carried load** is always lower than the **offered load**: \( A_c \leq A \).
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By definition, the **carried load** is always lower than the **offered load**: \( A_c \leq A \).

Offered load

Definition
The **offered load** \( A \) is defined by

\[
A = \lambda t_m
\]  

(357)

where \( \lambda \) is the average rate of **call trials**.

How do we determine the load when we have no possibility to measure it? ⇒ rules of **good practice**

It is common to choose a **practical reference load** (capacity) that represents the load to be allocated to a user during the **peak hours**.

Typically, the network could be sized to allocate a capacity \( A_0 \) comprised between 0.02 and 0.1 [E] per user (and per type of activity: internal/in/out)

A good network should be sized such that the carried load is (remember that \( A_c \leq A \)):

- as close as possible to the offered load during the peak hours.
- equal to the offered load outside the peak hours.
A telephone switch is usually able to collect the following statistics, for a given period $T$ (typically a quarter of an hour):

- the average call duration $t_m$. Note that certain calls might have started before the observation time and finish later.
- the number of call trials [attempts] (counter), $\lambda$.
- the number of carried calls (counter).
- the number of terminated calls (counter).

To establish our statistical analysis of telephone calls, we will use $T$ and $\lambda$.

**Probabilistic analysis of telephone calls I**

**Counting process**

We start by establishing the law of the random process $D(t)$ that counts the number of calls initiated after $t = 0$.

![Figure: Integrating the number of calls over time (realization of $D(t)$).]
Elements of our model:

1. **[Time discretization]** We divide the observation time $T$ into $m$ sub-intervals $\Delta T$ such that $T = m\Delta T$. Important note: we take $\Delta T$ so small that only one occurrence per interval is possible.

2. **[Probability]** Let $p$ be the probability (supposed to be stationary) of a call trial to occur during $\Delta T$. Because $\Delta T$ is fixed, we can define $\lambda$ such that $p = \alpha \Delta T$, where $\alpha$ is a constant. Later, it appears that in fact $\alpha = \lambda$, so that we take $p = \lambda \Delta T$ directly.

Let consider one random variable per sub-interval $\Delta T$: $D_1, D_2, \ldots$ (there are $m$ of them). Each variable assigns 1 if a trial has occurred, 0 otherwise.

By accumulating the values of all the $D_1, D_2, \ldots$ variables, we get the expression of $D(t)$:

- the probability of having $n$ call trials during $T$ is ($n$ successes for a binomial law):
  \[
  (\lambda \Delta T)^n = \left( \frac{\lambda T}{m} \right)^n 
  \]
  (358)

- the probability of having $m - n$ sub-intervals $\Delta T$ with no trial is ($m - n$ failures for a binomial law)
  \[
  (1 - \lambda \Delta T)^{m-n} = \left( 1 - \frac{\lambda T}{m} \right)^{m-n} 
  \]
  (359)

- we need to consider all the possible permutations: $C_m^n$
The number of occurrences (call trials) \( D_m \) during \( T = m \Delta T \) is therefore given by the following binomial probability density function (pdf)

\[
f_{D_m}(n) = p(D_m = n) = \begin{cases} \binom{n}{m} \left( \frac{\lambda T}{m} \right)^n \left( 1 - \frac{\lambda T}{m} \right)^{m-n} & n = 0, 1, \ldots, m \\ 0 & n \neq 0, 1, \ldots, m \end{cases}
\]

Its expectation (average/mean) is given by the number of trials \( T \) times the probability to succeed: \( m \left( \frac{\lambda T}{m} \right) = \lambda T \) (property of binomial law).

\[\lambda\] can thus be interpreted as the average number of trials during the time period \( T \).

Poisson distribution: definition and properties I

The previous expression can be rewritten as

\[
f_{D_m}(n) = \binom{n}{m} \left( \frac{\lambda T}{m} \right)^n \left( 1 - \frac{\lambda T}{m} \right)^{m-n}
\]

Then we take the limit \( \Delta T \to 0 \quad (\equiv m \to +\infty) \) of it:

\[
\lim_{m \to +\infty} f_{D_m}(n) = \lim_{m \to +\infty} \frac{m(m - 1) \ldots (m - n + 1)}{n!} \left( \frac{\lambda T}{m} \right)^n \left( 1 - \frac{\lambda T}{m} \right)^{m-n}
\]

\[
= \lim_{m \to +\infty} 1 \times \left( \frac{\lambda T}{m} \right)^n \left( 1 - \frac{\lambda T}{m} \right)^{m-n}
\]

\[
= \frac{(\lambda T)^n}{n!} \lim_{m \to +\infty} \left( 1 - \frac{\lambda T}{m} \right)^{m-n}
\]
If $x$ is small, then $e^{-x} = 1 - x + \frac{x^2}{2!} - \ldots$, so that

$$\lim_{m \to +\infty} \left(1 - \frac{\lambda T}{m}\right)^{m-n} = \lim_{m \to +\infty} \left(1 - \frac{\lambda T}{m}\right)^m = e^{-\lambda T}$$

Therefore, for $m \to +\infty$,

$$f_D(n) = \begin{cases} \frac{(\lambda T)^n}{n!} e^{-\lambda T} & n = 0, 1, \ldots \\ 0 & n \neq 0, 1, \ldots \end{cases}$$

This is a Poisson probability distribution function of parameter $\lambda T$.

**The mean and variance of a Poisson distribution are respectively**

$$\mu_D = \lambda T \quad \sigma^2_D = \lambda T$$
Proof.

The expectation (mean) is defined as

$$
\mu_D = \sum_{i=0}^{\infty} \text{id}(i) = \sum_{i=0}^{\infty} \frac{(\lambda T)^i}{i!} e^{-\lambda T} \tag{373}
$$

$$
= 0 \frac{\lambda T}{0!} e^{-\lambda T} + \sum_{i=1}^{\infty} \frac{i (\lambda T)^i}{i!} e^{-\lambda T} = \sum_{i=1}^{\infty} \frac{i (\lambda T)^i}{i!} e^{-\lambda T} \tag{374}
$$

Then we take $j = i - 1$ (variable change):

$$
\mu_D = \sum_{j=0}^{\infty} (j + 1) \frac{(\lambda T)^{j+1}}{(j + 1)!} e^{-\lambda T} = \sum_{j=0}^{\infty} \frac{(\lambda T)^{j+1}}{j!} e^{-\lambda T} \tag{375}
$$

$$
= \lambda T \sum_{j=0}^{\infty} \frac{(\lambda T)^j}{j!} e^{-\lambda T} = \lambda T \times 1 = \lambda T \tag{376}
$$

Remember that $\lambda$ is given/measured by the switch and $T$ is chosen!

Sizing a trunk

- We can measure some statistics ($\lambda$, $t_m$, etc). If they are not available, we take values given by good practice (reference load depending of the users profile).
- We know the law for call trials. A similar law counts the call releases; it is also a Poisson probability distribution function (although with a different parameter value).
- We need to establish the probability $P_k$ to have $k$ busy lines out of $N$ lines of the trunk. Then, $P_N$ is the value we are looking for (for sizing the trunk at peak hours).

Approach:

1. find the expression of transition probabilities from $p(k; t)$ to $p(k; t + dt)$.
2. add some assumptions to reduce the number of unknown values.
3. derive the expression of $P_N$. 
**Number of arriving calls**
In a time interval $\triangle t$, the number of *arriving* calls, $N_A$ is a Poisson process given by

$$p(N_A = n) = \frac{(\lambda \triangle t)^n}{n!} e^{-\lambda \triangle t}, \quad n = 0, 1, \ldots$$  \hspace{1cm} (377)

where $\lambda$ is the average *arrival rate* (for the trunk).

**Example**
The probability of exactly one arrival in $\triangle t$ is $\lambda \triangle t e^{-\lambda \triangle t}$.

**Number of calls that departure (releases)**
Likewise, the number $N_D$ of calls that departure is a Poisson process with average *departure rate* $\eta$:

$$p(N_D = n) = \frac{(\eta \triangle t)^n}{n!} e^{-\eta \triangle t}, \quad n = 0, 1, \ldots$$  \hspace{1cm} (378)
Load analysis I

At any time, the load fluctuates between 0 and \( N \) ongoing calls.

For a very brief time interval \( dt \), considering one line only,

- the probability that exactly one call will arrive \((n = 1)\) is
  \[
  \lambda dte^{-\lambda dt} \simeq \lambda dt
  \]
  (379)

- the probability that exactly one call will depart from an occupied line is
  \[
  \eta dte^{-\eta dt} \simeq \eta dt
  \]
  (380)

Thus, given that \( k \) lines (of the \( N \) lines) are occupied at time \( t \), we need to calculate the following transition probabilities, from time \( t \) to time \( t + dt \):

- \( P_{+1} \) the probability that one call will arrive in the interval
- \( P_{-1} \) the probability that one call will depart in the interval
- \( P_{=} \) the probability that neither an arrival nor a departure occurs in the interval

Load analysis II

Calculation of \( P_{+1} \), \( P_{-1} \) and \( P_{=} \):

- \( P_{+1} \): the probability of a call arriving is independent of the number of lines. Therefore:
  \[
  P_{+1} = \lambda dt
  \]
  (381)

- \( P_{-1} \): if one of the \( k \) lines is released, then the \( k - 1 \) other lines remain occupied.
  Taking all permutations into account:
  \[
  P_{-1} = C_k^1(\eta dt)^1(1 - \eta dt)^{k-1} \simeq k\eta dt
  \]
  (382)

- \( P_{=} \): unchanged situation \( \equiv \) no entering call, no leaving call
  (don’t forget that \( k \) lines are occupied)
  \[
  P_{=} = (1 - \lambda dt)(1 - k\eta dt) \simeq 1 - \lambda dt - k\eta dt
  \]
  (383)
Establishing the transition equations:
Let \( p(k; t + dt) \) be the probability to have \( k \) busy lines at time \( t + dt \). It is then possible to determine \( p(k; t + dt) \) based on \( P_{+1}, P_{-1}, P_{=} \), and probabilities at time \( t \):

\[
p(k; t + dt) = P_{=} p(k; t) + P_{+1} p(k - 1; t) + P_{-1} p(k + 1; t) \quad (384)
\]

\[
(1 - \lambda dt - k\eta dt) p(k; t) \quad (385)
\]

\[
\lambda dt p(k - 1; t) \quad (386)
\]

\[
(k + 1)\eta dt p(k + 1; t) \quad (387)
\]

With are two particular cases:

- \( k = 0 \) (no line occupied)
  
  \[
p(0; t + dt) = (1 - \lambda dt) p(0; t) + \eta dt p(1; t) \quad (388)
  \]

- \( k = N \) (all lines are occupied)
  
  \[
p(N; t + dt) = (1 - \lambda dt - N\eta dt) p(N; t) + \lambda dt p(N - 1; t) \quad (389)
  \]

### At steady state

**Summary:**

- we have \( N + 1 \) transition equations of type
  \[
p(k; t+dt) = (1-\lambda dt-k\eta dt) p(k; t)+\lambda dt p(k-1; t)+(k+1)\eta dt p(k+1; t)
  \]

- \( p(k; dt) \) and \( p(k; t + dt) \) probabilities are unknown: \( 2(N+1) \)
  unknown values

\( \Rightarrow \) we need to reduce the number of unknown values.

**At steady state**, it is assumed that the *probabilities are not function of time* (reasonable assumption of [strict!] time stationarity):

\[
p(k; t + dt) = p(k; t) = P_k, \quad k = 0, 1, 2, ..., N \quad (390)
\]

The transition equations then become

\[
P_k = (1 - \lambda dt - k\eta dt) P_k + \lambda dt P_{k-1} + (k + 1)\eta dt P_{k+1} \quad (391)
\]

\[
0 = [(-\lambda - k\eta) P_k + \lambda P_{k-1} + (k + 1)\eta P_{k+1}] dt \quad (392)
\]

\[
(\lambda + k\eta) P_k = \lambda P_{k-1} + (k + 1)\eta P_{k+1}, \quad 0 < k < N \quad (393)
\]
Likewise, we have, for the two particular cases:

\[
\lambda P_0 = \eta P_1, \quad k = 0 \quad (394)
\]
\[
(\lambda + N\eta) P_N = \lambda P_{N-1}, \quad k = N \quad (395)
\]

**Intermediate conclusion:** we have \(N + 1\) unknown probabilities and \(N + 1\) equations. But these are relative equations (all probabilities can be derived up to a scale factor).

**[Scaling condition]** Because the number of occupied lines is restricted to \(0 \leq k \leq N\), the probabilities \(P_k\) also must satisfy

\[
P_0 + P_1 + \ldots + P_N = 1 \quad (396)
\]

The form of the probability \(P_k\) that satisfies all the conditions is (for all \(k\))

\[
P_k = \frac{\left(\frac{\lambda}{\eta}\right)^k}{k!} \sum_{i=0}^{N} \frac{\left(\frac{\lambda}{\eta}\right)^i}{i!} \quad (397)
\]

This is the probability to have \(k\) occupied lines.

---

**Erlang B formula I**

**Erlang B formula**

The state of having all lines occupied is referred to as congestion. If calls are rejected when all \(N\) lines are occupied, then \(P_N (k = N)\) is the probability that a call is rejected or blocked:

\[
B = P_N = \frac{\left(\frac{\lambda}{\eta}\right)^N}{N!} \sum_{i=0}^{N} \frac{\left(\frac{\lambda}{\eta}\right)^i}{i!} \quad (398)
\]

This is known as the Erlang B formula.

If (1) \(K\) is a random variable representing the number of occupied lines in a \(N\)-lines trunk, and (2) \(P_k\) is the probability to have \(k\) occupied lines of the trunk, then the carried load \(A_c\) is given by the expectation of \(K\):

\[
A_c = E\{K\} = \sum_{k=0}^{N} k P_k \quad (399)
\]
**Erlang B formula II**

**Theorem (Expectation of \( K \))**

The expectation (mean/average) of the variable \( K \) provides the average number of occupied lines; this is the carried load \( A_c \). It is equal to

\[
A_c = E\{K\} = A(1 - B) \tag{400}
\]

where \( A \) is the offered load.

**Erlang B formula III**

**Proof.**

Let \( \beta = \lambda/\eta \). The expectation (mean) is defined as

\[
E\{K\} = \sum_{k=0}^{N} k P_k = \sum_{k=0}^{N} k \frac{\beta^k}{k!} = \sum_{k=1}^{N} k \frac{\beta^k}{k!} \sum_{i=0}^{N} \frac{\beta^i}{i!} \tag{401}
\]

\[
= \frac{\sum_{k=1}^{N} k \beta^k}{\sum_{i=0}^{N} \beta^i} \frac{\sum_{i=0}^{N} \beta^i}{\sum_{i=0}^{N} \beta^i} \tag{402}
\]

\[
= \beta \sum_{l=0}^{N-1} \frac{\beta^l}{\sum_{i=0}^{N} \beta^i} \tag{403}
\]

Therefore also, \( \frac{\lambda}{\eta} \) defines the offered load \( A \):

\[
A = \frac{\lambda}{\eta} \quad \text{and} \quad A_c = A(1 - B) \tag{404}
\]
In the expression of the Erlang B statistic,

\[ B = P_N = \frac{A^N}{N!} \sum_{i=0}^{N} \frac{A^i}{i!} \]  

we have three parameters:

- the **blocking probability** \( B \)
- the **size of the trunk** \( N \)
- the **offered load** \( A \)

\( \Rightarrow \) if we set two parameters, we can calculate the third one. But this expression is not easily inverted.

In practice, engineers use tables with values or graphics.

---

**Interpreting Erlang B probability law I**

Blocking probability \( B \) as a function of the offered load per line \( (A/N) \)
Interpreting Erlang B probability law II

Table: Example: proportionality of $A/N$ to $N$ for $B = 0.01$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>4.5</td>
<td>12.0</td>
<td>20.3</td>
<td>29.0</td>
<td>37.9</td>
<td>46.9</td>
<td>56.1</td>
<td>65.4</td>
<td>74.7</td>
<td>84.1</td>
</tr>
<tr>
<td>$A/N$</td>
<td>0.45</td>
<td>0.60</td>
<td>0.68</td>
<td>0.73</td>
<td>0.76</td>
<td>0.78</td>
<td>0.80</td>
<td>0.82</td>
<td>0.83</td>
<td>0.84</td>
</tr>
</tbody>
</table>

Sizing the capacity

Table: Erlang B: offered load, given $B$ and $N$.
Alternative models I

Assumptions for building the Erlang B model

- stationarity.
- rejected calls are not re-submitted.
- the blocking probability is given by $P_N$.

Types of loads:

1. carried load $A_c = E\{k\} = A(1 - B)$
2. offered load $A = \frac{\lambda}{\eta}$
3. what if rejected calls are returning? Then the offered load $A'$ is larger than $A$, because users return until they are served. Therefore,

$$A' \geq A \geq A_c \quad (406)$$

How do we determine $A'$?

Alternative models II

Trials and calls returning.

Until now, we have ignored what happens when a call has been blocked.

Let $A'$ be the estimated offered load that considers the effects of blocked calls returning and being accepted:

- $A$ is the original load (1st attempt)
- $AB$ is the load that returns to the trunk after calls have been blocked (2nd attempt)
- $AB^2$ results from 3rd attempts
- ...

Therefore, $A'$ can be expressed as

$$A' = A + AB + (AB)B + (AB^2)B + ... = \frac{A}{1 - B} \quad (407)$$

Note that $A' > A$. 
### Alternative blocking formulas

#### Blocking analysis

#### Treatment of lost calls

#### Blocking formula

### Formulas for a large (infinite) number of traffic sources

<table>
<thead>
<tr>
<th>Formula</th>
<th>Calls “cleared”; no recur</th>
<th>( B_1 = \sum_{k=0}^{\infty} P_k ) with ( P_k = \frac{A^k}{k!} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Erlang B</td>
<td>Calls reenter until served</td>
<td>Given ( B ), effective load is ( A' = A/(1 - B) )</td>
</tr>
<tr>
<td>Lost calls return</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Molina</td>
<td>Same as Erlang C</td>
<td>( B = 1 - e^{-A} \sum_{k=0}^{N-1} P_k = e^{-A} \sum_{k=N}^{\infty} P_k )</td>
</tr>
</tbody>
</table>

### Formulas for finite number of traffic sources, \( M \)

<table>
<thead>
<tr>
<th>Formula</th>
<th>Lost calls are cleared</th>
<th>( B_2(\rho) = \sum_{k=0}^{\infty} p_k ), ( p_k = \binom{M}{k} \rho^k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Engest</td>
<td></td>
<td>and ( A(\rho) \approx \frac{M \rho}{1 + \rho B_2(\rho)} )</td>
</tr>
<tr>
<td>Bernoulli</td>
<td>Lost calls held</td>
<td>( B = \sum_{k=N}^{M} \left( \binom{M}{k} \left( \frac{A}{M} \right) \right)^k \left( \frac{1}{M} \right)^{M-k} )</td>
</tr>
</tbody>
</table>

### Comparison of blocking probability formulas

![Comparison of blocking probability formulas](image-url)
Transmission over twisted pairs ("fixed" telephone network)

- Introduction
  - bit/packet error rate
- Transmission over copper lines
  - Electrical properties
  - Model
  - Spectral study
  - Examples of lines
- Telephone network
- Crosstalk and high bitrate transmission
  - Principles
  - Crosstalk study
  - NEXT, FEXT, signal to noise ratio
- Estimation of the channel capacity
  - Information, uncertainty and entropy
  - Memoryless discrete channel
  - Mutual information
  - Channel capacity
How will we proceed?

1. Understand the **principles** of transmission over copper lines
2. Establish a **model** for dealing with **disturbers** (other users on neighboring copper wires)
3. Estimate the **efficiency** of transmission over copper lines ⇒ notion of **channel capacity**

---

**Introduction**

![Figure: Comparison of different transmission media.](image-url)
Several phenomenons affect the transmission over copper lines:

**Signal [S] attenuation** Attenuation results in a decrease of the power along the copper line. It is commonly expressed \([dB]\) per kilometer. It will be expressed by the \(\alpha\) line attenuation constant.

**Noise [N]** Noise originates from electromagnetic disturbances: environment + other users (named *disturbers*).

**Non-linearities** Here, we will assume that the channel (made of copper wires) is linear to a large extend.

---

**Cables**

There are many types of cables, but basically two families:
- coaxial cables (mainly used by cable TV operators)
- cables of twisted pairs (used for network and telephone lines)
Cables of twisted pairs

Why do we twist pairs? I

A uniform (but time-varying) magnetic field induces a current in a loop of two wires
Conclusion: twisting pairs make them more resilient to electro-magnetic noise.

Cables might be organized in binder groups.
At the other side:
Local Distribution Center (LDC)

Pulling a cable through a conduit
Electrical model of a line and primary line constants (reminder)

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Units</th>
<th>Unit symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>loop resistance</td>
<td>$R$</td>
<td>ohms per meter</td>
<td>$\Omega/m$</td>
</tr>
<tr>
<td>loop inductance</td>
<td>$L$</td>
<td>henries per meter</td>
<td>$H/m$</td>
</tr>
<tr>
<td>insulator capacitance</td>
<td>$C$</td>
<td>farads per meter</td>
<td>$F/m$</td>
</tr>
<tr>
<td>insulator conductance</td>
<td>$G \approx 0$</td>
<td>siemens per meter</td>
<td>$S/m$</td>
</tr>
</tbody>
</table>

Figure: Infinitesimal section of a copper line.

Equivalent circuit representation of a transmission line using distributed elements and (telegrapher’s) equations

By taking a small section:

$$\frac{\partial V}{\partial x} = -RI - L \frac{\partial I}{\partial t} \quad (408)$$

$$-\frac{\partial I}{\partial x} = GV + C \frac{\partial V}{\partial t} \quad (409)$$
Solution to these equations

\[
\frac{\partial^2 V}{\partial x^2} = RGV + (RC + LG) \frac{\partial V}{\partial t} + LC \frac{\partial V^2}{\partial t^2} \quad (410)
\]

At steady state
At steady state, we express voltages and currents with phasors \( V(x, t) = V(x)e^{j\omega t} \). Then, the equation becomes

\[
\frac{\partial^2 V}{\partial x^2} = (R + jL\omega)(G + jC\omega)V(x) = \gamma^2 V(x) \quad (411)
\]

with \( \gamma = \alpha + j\beta \).

We then obtain

\[
V(x) = V_F e^{-\gamma x} + V_B e^{\gamma x} \quad (412)
\]

- \( V_F e^{-\gamma x} \): forwards propagating wave
- \( V_B e^{\gamma x} \): backwards propagating wave

Secondary line constants

Primary line constants aim at describing the line behavior at the “microscopic” scale. Lines are easier to characterize by their secondary line constants:

- the characteristic impedance \( Z_c \).
  It is defined as the impedance looking into an infinitely long line. Such a line will never return a reflection since the incident wave will never reach the end to be reflected.

- the propagation constant, \( \gamma \), whose real and imaginary parts are the attenuation constant, \( \alpha \), and phase change constant, \( \beta \), respectively:
  \[
  \gamma = \alpha + j\beta \quad (413)
  \]
Primary and secondary line constants are related by the following equations:

\[ Z_c = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \]  
\[ \gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \]

If we admit that \( G \approx 0 \),

\[ Z_c \approx \sqrt{\frac{R + j\omega L}{j\omega C}} \]
\[ \gamma \approx \sqrt{(R + j\omega L)j\omega C} \]

Spectral dependence of parameters

At high frequencies \( (\omega = 2\pi f) \) and taking \( \sqrt{1 + \alpha} \approx 1 + \frac{1}{2} \alpha \),

\[ \gamma \approx \sqrt{(R + j\omega L)j\omega C} \]
\[ \approx \sqrt{j\omega C} \sqrt{j\omega L} \sqrt{1 + \frac{R}{j\omega L}} \]
\[ \approx \sqrt{j\omega C} \sqrt{j\omega L} \left(1 + \frac{R}{j2\omega L}\right) \]
\[ \approx \frac{1}{2} R \sqrt{\frac{C}{L}} + j\omega \sqrt{LC} = \alpha + j\beta \]

In the following, the take the following approximations:

- \( C \) and \( L \) are independent of the frequency.
- due to the skin effect,
  - \( R \) is proportional to \( \sqrt{f} \): \( R = R_0 \sqrt{f} \)
  - and consequently \( \alpha \) is also proportional to \( \sqrt{f} \): \( \alpha = \alpha_0 \sqrt{f} \)
If $V_{in}(f, 0)$ is the input voltage ($x = 0$), then

$$V(f, L) = V_{in}(f, 0)e^{-\gamma(f)L} \quad (422)$$

is the voltage at location $L$.

**Definition**

The attenuation is defined as

$$A = \frac{|V_{in}(f, 0)|}{|V(f, L)|} = |e^{\gamma(f)L}| = e^{\alpha(f)L} \quad (423)$$

In decibels and considering that $\alpha = \alpha_0\sqrt{f}$, we have

$$A \text{[dB]} = 20 \log \left[ e^{\alpha_0\sqrt{f}L} \right] = \frac{20 \ln \left[ e^{\alpha_0\sqrt{f}L} \right]}{\ln 10} = A_0\sqrt{f}L \text{[dB]} \quad (424)$$

This means that doubling the length doubles the losses in decibels.

**Examples of line constants**

**Twisted pair**

| Frequency     | $R \text{[Ω/km]}$ | $L \text{[µH/km]}$ | $|Z_c| \text{[Ω]}$ | $\alpha \text{[mNP/km]}$ |
|---------------|-------------------|-------------------|-------------------|-------------------|
| 10 [kHz]      | 52,3              | 766               | 188               | 151               |
| 120 [kHz]     | 98,7              | 67,5              | 156               | 363               |

**Table**: Sample values of lines encountered in a telephone network (note that a neper [$\text{Np}$] is approximately 8.7 [dB])
Structure of a telephone network

**Figure:** Transmission of analog voice signals over the telephone network (so called “analog” lines).

Bandwidth for voice communications: [300 Hz, 3400 Hz]

Digital transmission by using modems over a telephone network

**Figure:** Digital information is modulated and sent transparently in the [300 Hz, 3400 Hz] bandwidth.
The entry level interface to ISDN is the Basic(s) Rate Interface (BRI), a $2 \times 64 \text{[kb/s]}$ service delivered over a pair of standard telephone copper wires (these channels are identical to the E1 channels).

**Figure:** Integrated Services Digital Network (ISDN).

Digital Subscriber Line (DSL) technologies and transmission

**Figure:** Configuration of an Asymmetric DSL (ADSL) line.
Cutoff filter (in the splitter)

Spectral occupancy

Voice signal: 0 – 4kHz  
downstream: 100kHz – 1MHz
upstream: 25 – 100kHz

Figure: Spectral occupancy of an ADSL line.

- Lower frequencies are dedicated to the voice and upstream bitrate.
Technologies for asymmetric bit streams (ADSL):

<table>
<thead>
<tr>
<th>Version</th>
<th>Standard name</th>
<th>Common name</th>
<th>Downstream</th>
<th>Upstream</th>
<th>Approved in</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADSL</td>
<td>ITU G.992.1</td>
<td>ADSL (G.dmt)</td>
<td>8 [Mb/s]</td>
<td>1.3 [Mb/s]</td>
<td>1999-07</td>
</tr>
<tr>
<td>ADSL2</td>
<td>ITU G.992.3</td>
<td>ADSL2</td>
<td>12 [Mb/s]</td>
<td>1.3 [Mb/s]</td>
<td>2002-07</td>
</tr>
</tbody>
</table>

Other DSL technologies:

- **High-bit-rate digital subscriber line (HDSL)** is a telecommunications protocol standardized in 1994. It was the first digital subscriber line (DSL) technology to use a higher frequency spectrum of copper, twisted pair cables.

- **Very-high-bit-rate digital subscriber line (VDSL or VHDSL):** up to 52 [Mb/s] downstream and 16 [Mb/s] upstream using the frequency band from 25 [kHz] to 12 [MHz]. **VDSL2** is even faster.
Real cables I

Real cables II
Crosstalk and high speed transmission

**Principle**

Figure: Crosstalk originates from the proximity of neighboring copper wires.
Towards an analysis of crosstalk effects

**Each input** signal generates **3 output** signals

Types of crosstalk in telephone networks I

1. **NEXT (Near-End Crosstalk):** disturbing pair’s source is at the same size
2. **FEXT (Far-End Crosstalk)**

NEXT is usually more damaging than FEXT.

**Figure:** Two types of crosstalk. We examine NEXT and FEXT at one side.
For two neighboring twisted pairs, capacitive and inductive components exist between each of the four wires and also between each wire and ground. Therefore, we consider two models:

1. **capacitive unbalance model**
2. **inductive unbalance model**

We will see that they produce similar results.

### Capacitive unbalance model I

**Framework:**

- we consider 2 pairs, or 4 wires
- there are 8 unknown quantities: \( V_1, V_2, V_3, V_4, I_1, I_2, I_3, I_4 \) (voltages are defined with respect to the ground reference)
- how many circuits do we have?
- how many circuits do we use in practice?
  - **differential (metallic) modes** are easier to use. They are defined by \( V_1 - V_2, V_3 - V_4, I_1 - I_2, \) and \( I_3 - I_4 \).
  - so we use 2 circuits only.
Capacitive unbalance model II

\[
\begin{align*}
V_1(x + \Delta x) &= V_1(x) - I_1(x)Z_1\Delta x \\
V_2(x + \Delta x) &= V_2(x) - I_2(x)Z_2\Delta x \\
V_3(x + \Delta x) &= V_3(x) - I_3(x)Z_3\Delta x \\
V_4(x + \Delta x) &= V_4(x) - I_4(x)Z_4\Delta x 
\end{align*}
\]

Figure: Capacitive model of a short section of two twisted pairs.

Capacitive unbalance model III

For convenience:

- we write all capacitances as admittances: \( Y = j\omega C \) (we assume that \( G = 0 \)).
- All voltages are defined with respect to the ground reference.

According to Kirchhoff’s mesh rule (on voltages):

\[
\begin{align*}
V_1(x + \Delta x) &= V_1(x) - I_1(x)Z_1\Delta x \\
V_2(x + \Delta x) &= V_2(x) - I_2(x)Z_2\Delta x \\
V_3(x + \Delta x) &= V_3(x) - I_3(x)Z_3\Delta x \\
V_4(x + \Delta x) &= V_4(x) - I_4(x)Z_4\Delta x 
\end{align*}
\]
Capacitive unbalance model IV

Likewise, the current nodes law provides:

\[ I_1(x + \Delta x) = I_1(x) - V_1(x + \Delta x)Y_{1G}\Delta x - [V_1(x + \Delta x) - V_2(x + \Delta x)]Y_{12}\Delta x \]

\[ -[V_1(x + \Delta x) - V_3(x + \Delta x)]Y_{13}\Delta x - [V_1(x + \Delta x) - V_4(x + \Delta x)]Y_{14}\Delta x \]

\[ I_2(x + \Delta x) = I_2(x) - V_2(x + \Delta x)Y_{2G}\Delta x - [V_2(x + \Delta x) - V_1(x + \Delta x)]Y_{12}\Delta x \]

\[ -[V_2(x + \Delta x) - V_3(x + \Delta x)]Y_{23}\Delta x - [V_2(x + \Delta x) - V_4(x + \Delta x)]Y_{24}\Delta x \]

\[ I_3(x + \Delta x) = I_3(x) - V_3(x + \Delta x)Y_{3G}\Delta x - [V_3(x + \Delta x) - V_1(x + \Delta x)]Y_{13}\Delta x \]

\[ -[V_3(x + \Delta x) - V_2(x + \Delta x)]Y_{23}\Delta x - [V_3(x + \Delta x) - V_4(x + \Delta x)]Y_{34}\Delta x \]

\[ I_4(x + \Delta x) = I_4(x) - V_4(x + \Delta x)Y_{4G}\Delta x - [V_4(x + \Delta x) - V_1(x + \Delta x)]Y_{13}\Delta x \]

\[ -[V_4(x + \Delta x) - V_2(x + \Delta x)]Y_{24}\Delta x - [V_4(x + \Delta x) - V_3(x + \Delta x)]Y_{34}\Delta x \]

We have:

- 8 equations
- but 16 unknown values?!

\[ V_j(x), \ V_j(x + \Delta x), \ I_j(x), \ I_j(x + \Delta x), \text{ with } j \in \{1, 2, 3, 4\}. \]

Therefore, in the equations, we divide both members by \( \Delta x \) and take the limit for \( \Delta x \rightarrow 0 \).

Capacitive unbalance model V

For example,

\[ V_1(x + \Delta x) = V_1(x) - I_1(x)Z_1\Delta x \]

becomes

\[ \lim_{\Delta x \rightarrow 0} \frac{V_1(x + \Delta x) - V_1(x)}{\Delta x} = -Z_1I_1(x) \]

\[ \frac{d}{dx} V_1(x) = -Z_1I_1(x) \]

Note that \( V_1(x) \) depends on a current. Likewise it appears that currents depend on voltages. \( \Rightarrow \) equations are coupled.
In a matrix form, the 8 equations can be expressed as

\[
\begin{bmatrix}
    \frac{d}{dx} & V_1 \\
    V_2 \\
    V_3 \\
    V_4 \\
    l_1 \\
    l_2 \\
    l_3 \\
    l_4
\end{bmatrix}
= \begin{bmatrix}
    0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    V_1 \\
    V_2 \\
    V_3 \\
    V_4 \\
    l_1 \\
    l_2 \\
    l_3 \\
    l_4
\end{bmatrix}
= \begin{bmatrix}
    V_1 \\
    V_2 \\
    V_3 \\
    V_4 \\
    l_1 \\
    l_2 \\
    l_3 \\
    l_4
\end{bmatrix}
\]

with

\[
A_1 = -(Y_{1G} + Y_{12} + Y_{13} + Y_{14})
\]

\[
A_2 = -(Y_{2G} + Y_{12} + Y_{23} + Y_{24})
\]

\[
A_3 = -(Y_{3G} + Y_{13} + Y_{23} + Y_{34})
\]

\[
A_4 = -(Y_{4G} + Y_{14} + Y_{24} + Y_{34})
\]

Capacitive unbalance model VII

Change of variables (we transform the voltages and currents on the four lines to metallic \(\equiv\) differential and longitudinal voltages and currents):

\[
V_{1M} = V_1 - V_2
\]

\[
V_{2M} = V_3 - V_4
\]

\[
V_{1L} = \frac{V_1 + V_2}{2}
\]

\[
V_{2L} = \frac{V_3 + V_4}{2}
\]

\[
l_{1M} = \frac{l_1 - l_2}{2}
\]

\[
l_{2M} = \frac{l_3 - l_4}{2}
\]

\[
l_{1L} = l_1 + l_2
\]

\[
l_{2L} = l_3 + l_4
\]
Capacitive unbalance model VIII

In a matrix form:

\[ \vec{\mathbf{S}} = T \begin{bmatrix} V_{1M} \\ V_{3L} \\ V_{2M} \\ V_{2L} \\ I_{1M} \\ I_{1L} \\ I_{2M} \\ I_{2L} \end{bmatrix} = T \vec{\mathbf{S'}} \quad (445) \]

where \( T \) is a transformation matrix:

\[
T = \begin{bmatrix}
\frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{bmatrix} \quad (446)\]

So that,

\[ \vec{\mathbf{S'}} = T^{-1} \vec{\mathbf{S}} \quad (447) \]

Capacitive unbalance model IX

This can be written as

\[
\frac{d}{dx} \vec{\mathbf{S}} = A \vec{\mathbf{S}} \quad (448)
\]

\[
\frac{d}{dx} T^{-1} \vec{\mathbf{S}} = T^{-1} A \vec{\mathbf{S}} \quad (449)
\]

\[
\frac{d}{dx} \vec{\mathbf{S'}} = T^{-1} A T \vec{\mathbf{S'}} \quad (450)
\]

This results in currents depending only on the metallic and longitudinal voltages.
First solution: the capacitive model I

By replacing \( Y \) by \( j\omega C \) (only capacitive effects are considered),

\[
\frac{d}{dx} \begin{bmatrix} I_{1M} \\ I_{1L} \\ I_{2M} \\ I_{2L} \end{bmatrix} = -\frac{j\omega}{4} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} V_{1M} \\ V_{1L} \\ V_{2M} \\ V_{2L} \end{bmatrix}
\]

(451)

with

\[
\begin{align*}
a_{11} &= C_1G + C_2G + 4C_{12} + C_{13} + C_{14} + C_{23} + C_{24} \\
a_{21} &= a_{12} = 2C_1G - 2C_2G + 2C_{13} + 2C_{14} - 2C_{23} - 2C_{24} \\
a_{31} &= a_{13} = -C_{13} + C_{14} + C_{23} - C_{24} \\
a_{41} &= a_{14} = -2C_{13} - 2C_{14} + 2C_{23} + 2C_{34} \\
a_{22} &= 4C_1G + 4C_2G + 4C_{13} + 4C_{14} + 4C_{23} + 4C_{24} \\
a_{23} &= a_{32} = -2C_{13} + 2C_{14} - 2C_{23} + 2C_{34} \\
a_{24} &= a_{42} = -4C_{13} - 4C_{14} - 4C_{23} - 4C_{24} \\
a_{33} &= C_3G + C_4G + C_{13} + C_{14} + C_{23} + C_{24} + 4C_{34} \\
a_{34} &= a_{43} = 2C_{3G} - 2C_4G + 2C_{13} - 2C_{14} + 2C_{23} - 2C_{24} \\
a_{44} &= 4C_3G + 4C_4G + 4C_{13} + 4C_{14} + 4C_{23} + 4C_{24}
\end{align*}
\]

(452-461)

First solution: the capacitive model II

The parameter \( a_{31} \) defines the the coupling between the metallic (differential) voltage in the disturbing pair \( V_{1M} \) to the metallic (differential) current in the disturbed pair \( I_{2M} \); it is referred to as capacitance unbalance of a twisted pair:

\[
\frac{d}{dx} I_{2M} = -\frac{j\omega}{4} C_{M1M2} V_{1M}
\]

(462)

where \( C_{M1M2} \) is equal to \( a_{31} \) (or \( a_{13} \)).
A detailed analysis leads to

\[
\frac{d}{dx} I_{2M} \simeq \frac{j\omega M}{4Z_c^2} V_{1M} \tag{463}
\]

where \( M \) equals \( M_1 + M_2 + M_3 + M_4 \).
Both models provide expressions that can be combined to
\[
\frac{d}{dx} I_{2M} = \left( \frac{j\omega M}{4Z_c^2} - \frac{j\omega}{4} C_{M_1M_2} \right) V_{1M}
\]  
(464)
that can be grouped together to form a new unbalanced constant
\[
\frac{d}{dx} I_{2M}(x) = j\omega Q_{M_1M_2}(x) V_{1M}(x)
\]  
(465)
where

- \(Q_{M_1M_2}(x)\) takes into account capacitance and inductive effects.
- \((x)\) emphasizes the dependence with the location along the line.

---

**Near-end crosstalk (NEXT) calculation I**

\[j\omega Q_{M_1M_2}(l)V_0(f)e^{-2\gamma(f)l}\]
\[j\omega Q_{M_1M_2}(l)V_0(f)e^{-\gamma(f)l}\]

Figure: NEXT at location \(x = l\).
Near-end crosstalk (NEXT) calculation II

1. At the input of the disturbing line, the voltage is $V(f, x = 0) = V_0(f)$.

2. At location $x = l$, the voltage along the disturbing pair is equal to ($\rightarrow$ propagation)

$$V_1(f, x = l) = V_0(f) e^{-\gamma(f)l} \quad (466)$$

3. At location $x = l$, the induced current on the disturbed pair is

$$\frac{d}{dx} I_2(f, x = l) = j\omega Q_{M_1 M_2}(l) V_1(f, x = l) = j\omega Q_{M_1 M_2}(l) V_0(f) e^{-\gamma(f)l} \quad (467)$$

4. This current propagates back to the input ($x = 0$) of the disturbed line ($\leftarrow$ propagation):

$$I_2(f, x = 0) = I_2(f, x = l) e^{\gamma(f)(-l)} = I_2(f, x = l) e^{-\gamma(f)l} \quad (468)$$

so that

$$\frac{d}{dx} I_2(f, x = 0) = j\omega Q_{M_1 M_2}(l) V_0(f) e^{-2\gamma(f)l} \quad (469)$$

Derivation of the NEXT power transfer function: integration along the line

We have

$$I_2(f) = \int_0^L j\omega Q_{M_1 M_2}(l) V_0(f) e^{-2\gamma(f)l} dl \quad (470)$$

Power?

The power due to the disturbing pair is given by

$$P_2(f) = V_2(f) I_2^*(f) = Z_L I_2(f) I_2^*(f) \quad (471)$$

$Z_L$ is chosen to:

- maximize the transfer of power (conjugate matching: $Z_L^* = Z_c$) and
- to avoid any reflection ($Z_L = Z_c$), where $Z_c$ is the characteristic impedance of a transmission line.

So that, $Z_L = R_L$. Consequently,

$$P_2(f) = R_L I_2(f) I_2^*(f) \quad (472)$$
Derivation of the NEXT power transfer function: integration along the line II

The problem is that $Q_{M_1M_2}$ is unknown and it depends on the location ⇒ we adopt a probabilistic approach for evaluating the power

$$E\{P_2(f)\} = E\left\{ R_L V_0^2(f) \int_0^L j\omega Q_{M_1M_2}(x) e^{-2\gamma(f)x} \, dx \int_0^L -j\omega Q^*_{M_1M_2}(y) e^{-2\gamma^*(f)y} \, dy \right\}$$

$$= R_L \omega^2 V_0^2(f) E\left\{ \int_0^L \int_0^L Q_{M_1M_2}(x) Q^*_{M_1M_2}(y) e^{-2\gamma(f)x} e^{-2\gamma^*(f)y} \, dx \, dy \right\} \quad (473)$$

$$= R_L \omega^2 V_0^2(f) \int_0^L \int_0^L E\left\{ Q_{M_1M_2}(x) Q^*_{M_1M_2}(y) \right\} e^{-2\gamma(f)x} e^{-2\gamma^*(f)y} \, dx \, dy \quad (474)$$

How do we estimate $E\{ Q_{M_1M_2}(x) Q^*_{M_1M_2}(y) \}$?

Derivation of the NEXT power transfer function I

We assume that $Q_{M_1M_2}(x)$ and $Q^*_{M_1M_2}(y)$ (at two different locations) are uncorrelated, which means that

$$E\{ Q_{M_1M_2}(x) Q^*_{M_1M_2}(y) \} = k \delta(x - y) \quad (475)$$

According to this assumption:

$$E\{P_2(f)\} = R_L \omega^2 V_0^2(f) \int_0^L \int_0^L k \delta(x - y) e^{-2\gamma(f)x} e^{-2\gamma^*(f)y} \, dy \, dx$$

Knowing that there is no contribution to crosstalk outside of $[0, L]$ and that $\int_{-\infty}^{+\infty} \delta(x - y)f(y)dy = f(x)$:

$$\int_0^L k \delta(x - y) e^{-2\gamma(f)x} e^{-2\gamma^*(f)y} \, dy = \int_{-\infty}^{+\infty} k \delta(x - y) e^{-2\gamma(f)x} e^{-2\gamma^*(f)y} \, dy$$

$$= k e^{-2\gamma(f)x} e^{-2\gamma^*(f)x}$$

$$= e^{-2(\alpha(f)+j\beta(f))x} e^{-2(\alpha(f)-j\beta(f))x}$$

$$= e^{-4\alpha(f)x}$$
Therefore,
\[
E \{ P_2(f) \} = R_L \omega^2 V_0^2(f) \int_0^L \left[ \int_0^L k \delta(x - y) e^{-2\gamma(f)x} e^{-2\gamma^*(f)y} dy \right] dx
\]
\[
= R_L \omega^2 V_0^2(f) k \int_0^L e^{-4\alpha(f)x} dx
\]
\[
= R_L \omega^2 V_0^2(f) k \left[ \frac{e^{-4\alpha(f)x}}{-4\alpha(f)} \right]_0^L
\]
\[
= \frac{R_L \omega^2 V_0^2(f) k}{-4\alpha(f)} \left( e^{-4\alpha(f)L} - 1 \right)
\]
\[
(476)
\]
\[
(477)
\]
\[
(478)
\]
For long lines, \( e^{-4\alpha(f)L} \ll 1 \) and, consequently,
\[
E \{ P_2(f) \} \approx \frac{R_L \omega^2 V_0^2(f) k}{-4\alpha(f)} (-1) = \frac{R_L \omega^2 V_0^2(f) k}{4\alpha(f)}
\]
\[
(479)
\]
At high frequencies (the ones used for ADSL transmission), \( \alpha(f) \) is proportional to \( \sqrt{f} \), so that
\[
E \{ P_2(f) \} = \frac{R_L (2\pi f)^2 V_0^2(f) k}{4\alpha_0 \sqrt{f}} = \frac{R_L V_0^2(f) \pi^2 k}{\alpha_0} f^{\frac{3}{2}}
\]
\[
(480)
\]
Thus, for a disturber whose power is given by \( V_0^2(f)/R_L \), we have the NEXT power transfer function between the disturbing transmitter and the disturbed receiver:
\[
H_{\text{NEXT}}(f) = \frac{R_L^2 \pi^2 k}{\alpha_0} f^{\frac{3}{2}} = K_{\text{NEXT}} f^{\frac{3}{2}}
\]
\[
(481)
\]
Interpretation:
- the NEXT is proportional to \( f^{\frac{3}{2}} \).
- it does not depend on the length. Why? What does it mean?
Measured crosstalk values I

Setup:

Measured crosstalk values II

Result: the received power decreases as 15 [dB] per frequency decade
Calculation of the FEXT I

At the **input** of the disturbing line, the voltage is $V_0(f)$.

At location $x = l$, the voltage on the disturbing pair is equal to ($\rightarrow$ propagation)

$$V_1(f, x = l) = V_0(f) e^{-\gamma(f)l}$$  \hfill (482)

At location $x = l$, the induced current on the disturbed pair is

$$\frac{d}{dx} I_2(f, x = l) = j\omega Q_{M_1M_2}(l) V_1(f, x = l) = j\omega Q_{M_1M_2}(l) V_0(f) e^{-\gamma(f)l}$$

This current propagates to the **output** ($x = L$) of the disturbed line ($\rightarrow$ propagation):

$$I_2(f, x = L) = I_2(f, x = l) e^{-\gamma(f)(L-l)}$$  \hfill (483)

so that

$$\frac{d}{dx} I_2(f, x = L) = j\omega Q_{M_1M_2}(l) V_0(f) e^{-\gamma(f)L} e^{-\gamma(f)(L-l)}$$

$$= j\omega Q_{M_1M_2}(l) V_0(f) e^{-\gamma(f)L} $$  \hfill (484)

Figure: Far-end crosstalk due to only one unbalance point $x = l$. 

Calculation of the FEXT II

Multi-pair cable

At the **input** of the disturbing line, the voltage is $V_0(f)$.

At location $x = l$, the voltage on the disturbing pair is equal to ($\rightarrow$ propagation)

$$V_1(f, x = l) = V_0(f) e^{-\gamma(f)l}$$  \hfill (482)

At location $x = l$, the induced current on the disturbed pair is

$$\frac{d}{dx} I_2(f, x = l) = j\omega Q_{M_1M_2}(l) V_1(f, x = l) = j\omega Q_{M_1M_2}(l) V_0(f) e^{-\gamma(f)l}$$

This current propagates to the **output** ($x = L$) of the disturbed line ($\rightarrow$ propagation):

$$I_2(f, x = L) = I_2(f, x = l) e^{-\gamma(f)(L-l)}$$  \hfill (483)

so that

$$\frac{d}{dx} I_2(f, x = L) = j\omega Q_{M_1M_2}(l) V_0(f) e^{-\gamma(f)L} e^{-\gamma(f)(L-l)}$$

$$= j\omega Q_{M_1M_2}(l) V_0(f) e^{-\gamma(f)L} $$  \hfill (484)
Derivation of the FEXT power transfer function:
integration along the line I

We have, at the output,

$$l_2(f) = \int_0^L j\omega Q_{M_1M_2}(x)V_0(f)e^{-\gamma(f)L}dx$$  \hspace{1cm} (485)

Power?

The **power** due to the disturbing pair is given by

$$P_2(f) = V_2(f)l_2^* = RLl_2(f)l_2^*(f)$$  \hspace{1cm} (486)

Derivation of the FEXT power transfer function:
integration along the line II

By a similar reasoning to that of the NEXT,

$$E\{P_2(f)\} = RLV_0^2(f)E\left\{\int_0^L j\omega Q_{M_1M_2}(x)e^{-\gamma(f)L}dx\int_0^L -j\omega Q_{M_1M_2}^*(y)e^{-\gamma^*(f)L}dy\right\}$$

$$= RL\omega^2V_0^2(f)e^{-2\alpha(f)L}\int_0^L \int_0^L k\delta(x-y)dxdy$$  \hspace{1cm} (487)

$$= RL\omega^2V_0^2(f)e^{-2\alpha(f)L}\int_0^L kdy$$  \hspace{1cm} (488)

$$= RL(2\pi f)^2V_0^2(f)e^{-2\alpha(f)L}kL$$  \hspace{1cm} (489)

So, we get the **FEXT power transfer function**

$$H_{FEXT}(f) = k_{FEXT}f^2e^{-2\alpha(f)L}L$$  \hspace{1cm} (490)

where \( L \) is the length of the cable with some FEXT effects.
Derivation of the FEXT power transfer function:
integration along the line III

Interpretation:

\[ H_{FEXT}(f) = k_{FEXT} f^2 e^{-2\alpha(f)L} \]  \hspace{0.5cm} (491)

- it depends on the length \( L \).
- it increases with the frequency (similar effect to that of NEXT).
- \( e^{-2\alpha(f)L} = \left(e^{-\alpha(f)L}\right)^2 \) is the power attenuation for any signal along the line. It is the usual input-output power transfer function or channel power transfer function \( \|H_c(f)\|^2 \).

Calculation of the FEXT Signal to Noise ratio I

\[ \gamma_R(f) = \gamma_T(f) \|H_c(f)\|^2 + \gamma_D(f) H_{FEXT}(f) \]  \hspace{0.5cm} (492)

Figure: Calculation of the power spectral density at the receiver with the presence of FEXT.
This leads to the following signal to noise ratio:

\[
\frac{S(f)}{N(f)} = \frac{\gamma_T(f) \| \mathcal{H}_c(f) \|^2}{\gamma_D(f) H_{FEXT}(f)}
\] (493)

\[
= \frac{\gamma_T(f) \| \mathcal{H}_c(f) \|^2}{\gamma_D(f) Lf^2 k_{FEXT} \| \mathcal{H}_c(f) \|^2}
\] (494)

If we have \( \gamma_T(f) \simeq \gamma_D(f) \), which means that both signals are of the same type, then

\[
\frac{S(f)}{N(f)} = \frac{\gamma_T(f) \| \mathcal{H}_c(f) \|^2}{\gamma_D(f) Lf^2 k_{FEXT} \| \mathcal{H}_c(f) \|^2}
\] (495)

\[
= \frac{k'}{L f^2}
\] (496)

Note that the transfer function of the channel does not appear in this expression.

**Influence of the number of disturbers**

Let consider a cable with \( N \) disturbing lines. A simple idea consists to sum up the individual effects of the \( N \) disturbing lines:

\[
\gamma_{out}(f) = \sum_{i=1}^{N} \gamma_{in_i}(f) H_{FEXT/\text{NEXT}}(f)
\] (497)

If all the signals have the same power spectrum,

\[
\gamma_{out}(f) = N \gamma_{in}(f) H_{FEXT/\text{NEXT}}(f)
\] (498)

However, this formula overestimates the real disturbing power because only neighboring lines interfere with the considered line.

**Unger’s formula (empirical formula)**

\[
\gamma_{out}(f) = N^{0.6} \gamma_{in}(f) H_{FEXT/\text{NEXT}}(f)
\] (499)
Cable building considerations I

In the field
Impact of an error during the transmission: bit/packet error rate (for packets of length $N$)

Assume a packet of size $N$ and let $P_e$ be the probability error on one bit.

The probability for the packet to be correct is

$$\left(1 - P_e\right)^N$$

(500)

Therefore the packet error rate is

$$P_P = 1 - \left(1 - P_e\right)^N.$$  

(501)

For large packets and small $P_e$, this becomes

$$P_P \simeq 1 - (1 - NP_e) = N \times P_e.$$  

(502)

Example

With $N = 10^5$ bits and a bit error rate of $P_e = 10^{-7}$, $P_P \simeq 10^{-2}$.

We thus need to lower $P_e \Rightarrow$ understand how disturbance occurs.

Estimation of the channel capacity I

Approach:

1. Define the notion of information
2. Provide a model for the channel
3. Establish the channel capacity

Example

Two types of channels for carrying information:

- A, C, .. $\rightarrow$ noise-free channel $\rightarrow$ A, C, ..
- A, C, .. $\rightarrow$ noisy channel $\rightarrow$ D, B, ..
Notion of information

Definition (Information of a symbol)

Let $s_k$ be an event with a probability of $p_k$ of a source $S$, we quantify the information provided by its observation by

$$i(s_k) = \log_2 \left( \frac{1}{p_k} \right) = -\log_2 p_k \quad (\geq 0) \quad (503)$$

- **Rare event**: small $p_k \rightarrow i(s_k) = -\log_2 p_k$ tends towards $+\infty$
  $\rightarrow$ a lot of information
- $p_k$ close to 1 $\rightarrow i(s_k) = -\log_2 p_k$ tends towards 0
  $\rightarrow$ no information

The unit of information is the **bit of information**.

Note that there is a difference between a **bit** (which measures the number of sent binary signals) and a **bit of information** (which measures the rate of information).

Notion of entropy I

Definition (Entropy of a source $S$ comprising $K$ symbols)

The average information, per symbol, provided by a source is the entropy. It is denoted by $H(S)$ and is defined as

$$H(S) = \sum_{k=0}^{K-1} p_k \log_2 \left( \frac{1}{p_k} \right) = \sum_{k=0}^{K-1} p_k i(s_k) \quad (504)$$

Figure: Entropy of a binary source ($p$ denotes the probability of one symbol).
So, $H(S) = p \log_2 p + (1-p) \log_2(1-p)$. 
Binary source with $p(0) = p(1) = \frac{1}{2}$.

In that case, we have

$$H(S) = \frac{1}{2} \log_2 2 + \frac{1}{2} \log_2 2 = \log_2 2 = 1 \text{ [bit of information/symbol]}$$

For all other values of $p(0)$ and $p(1)$, $H(S) < 1$.

Source with 4 symbols, such that $p(0) = p(1) = p(2) = p(3) = \frac{1}{4}$.

In that case, we have

$$H(S) = 4 \times \frac{1}{4} \log_2 4 = \log_2 4 = 2 \text{ [bit of information/symbol]}$$

Question:
Do we want a source with a minimal or maximal amount of information? Why?

Memoryless discrete channel

$$
\begin{array}{c}
X \\
\hline
x_0 \\
x_1 \\
\vdots \\
x_{J-1} \\
\end{array}
\quad
\begin{array}{c}
Y \\
\hline
y_0 \\
y_1 \\
\vdots \\
y_{K-1} \\
\end{array}
\quad
\text{Channel: } p(y_k|x_j)
$$

Figure: Model of a discrete memoryless channel.

Ideally,

$$p(y_k|x_j) = 0 \text{ if } k \neq j$$

when there is no noise (and no pre-encoding). In practice however, this is not the case, because there is always noise on the channel.
For the noisy channel, the question consists to evaluate the amount of uncertainty relative to the input $X$ when we observe $Y = y_k$ at the output. This leads to the definition of *conditional entropy*.

The uncertainty on $X$, given $Y = y_k$, is

$$H(X|Y = y_k) = E \{- \log_2 p(X|y_k)\} = \sum_{j=0}^{J-1} p(x_j|y_k) \log_2 \left( \frac{1}{p(x_j|y_k)} \right)$$

**Definition (Conditional entropy)**

Average of this information ⇒ *conditional entropy* $H(X|Y)$

$$H(X|Y) = \sum_{k=0}^{K-1} H(X|Y = y_k)p(y_k)$$

$$= \sum_{k=0}^{K-1} \sum_{j=0}^{J-1} p(x_j|y_k)p(y_k) \log_2 \left( \frac{1}{p(x_j|y_k)} \right)$$

**Mutual information $I$**

The uncertainty about $X$ must be larger before observing $Y$ than after. The difference is a measure of the information passing through the channel, on average.

**Definition (Mutual information)**

We define

$$I(X; Y) = H(X) - H(X|Y)$$

as the average *mutual information*.

**Theorem**

*The mutual information is symmetric* $I(X; Y) = I(Y; X)$

Therefore:

$$I(X; Y) = H(X) - H(X|Y)$$

$$I(Y; X) = H(Y) - H(Y|X)$$
Questions:

1. Is it intuitive to have that the mutual information is symmetric?

2. Is it better to have a channel with the lowest or highest mutual information \( I(X, Y) = H(X) - H(X|Y) \)? Why?

Channel capacity

Definition

The channel capacity per symbol is defined as the maximum information conveyed over all possible input probability distributions

\[
C_s = \max_{\{p(x_j)\}} I(X; Y) \tag{513}
\]

It corresponds to the best possible usage of the channel.

This capacity is in bit/symbol, where a symbol is one sample of \( X \)!

If the channel is used \( s \) times per second, then the channel capacity is bits per second is

\[
C = s \, C_s \tag{514}
\]
The capacity for one symbol is given by

\[ C_s = \max_{\{p(x_j)\}} I(X; Y) \]  \hspace{1cm} (515)

\[ = \max_{\{p(x_j)\}} [H(X) - H(X|Y)] \]  \hspace{1cm} (516)

\[ = \max_{\{p(x_j)\}} [H(Y) - H(Y|X)] \]  \hspace{1cm} (517)

We have that

\[ H(Y|X) = \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} p(y_k|x_j)p(x_j) \log_2 \left( \frac{1}{p(y_k|x_j)} \right) \]  \hspace{1cm} (518)

So, for the input-output pair \((x_0, y_0)\), the conditional entropy is

\[ H(Y|X)_{(x_0,y_0)} = -p(x_0)p(y_0|x_0) \log_2 (p(y_0|x_0)) \]  \hspace{1cm} (519)

Given that \(p(x_0) = \alpha\) and that \(p(y_0|x_0) = p (= 1 - q)\), we have

\[ H(Y|X)_{(x_0,y_0)} = -\alpha p \log_2 p \]  \hspace{1cm} (520)

Likewise:

\[ H(Y|X)_{(x_0,y_1)} = -\alpha q \log_2 q \]  \hspace{1cm} (521)

\[ H(Y|X)_{(x_1,y_0)} = -(1 - \alpha)q \log_2 q = -q \log_2 q + \alpha q \log_2 q \]

\[ H(Y|X)_{(x_1,y_1)} = -(1 - \alpha)p \log_2 p = -p \log_2 p + \alpha p \log_2 p \]
Calculation of the capacity of a binary symmetric channel

By summing all these contributions,

\[
H(Y|X) = \sum_{i=0}^{1} \sum_{j=0}^{1} H(Y|X)_{(x_i,y_j)} \quad (522)
\]

\[
= -p \log_2 p - q \log_2 q \quad (523)
\]

We observe that \( H(Y|X) \) is \((\geq 0)\) and that it only depends on the channel characteristics.

Finally, we get

\[
I(X; Y) = H(Y) + p \log_2 p + q \log_2 q \quad (524)
\]

that we want to maximize to find the capacity.

Calculation of the capacity of a binary symmetric channel

Maximum of ?

\[
I(X; Y) = H(Y) + p \log_2 p + q \log_2 q \quad (525)
\]

Note that \( p \log_2 p + q \log_2 q = -H_c \leq 0 \) and that it depends on the channel. Therefore, we maximize \( I(X; Y) \) by maximizing \( H(Y) \), whose maximum is 1 for a binary output alphabet.

In conclusion, (remember that \( q = 1 - p \))

\[
C_S = 1 + p \log_2 p + q \log_2 q = 1 - H_c < 1 \quad (526)
\]

In the case of a two-state technique (such as the NRZ or the ASK-2), the bit error rate \( q = p_e \) depends on the \( \frac{E_b}{N_0} \) ratio and is given by

\[
q = p_e = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_0}} \right) \quad (527)
\]
Influence of the type of modulation

For a four-state modulation technique, $C_s$ has the same form and is given by

$$C_s = H(Y) - H_c < 2$$ \hfill (528)

where that maximum of $H(Y)$ is 2.

Discrete inputs and continuous output: Shannon’s second law

In an additive noise channel, the output is

$$Y = X + N(0, \sigma_N^2)$$ \hfill (529)

where $X$ is a discrete random input to the channel and $N(0, \sigma_N^2)$ is a continuous noise variable (taken to be a Gaussian here).

Theorem (Shannon’s second law)

*For an additive Gaussian noise channel (zero-mean, variance $\sigma_N^2$), the capacity per symbol is given by*

$$C_s = \frac{1}{2} \log_2 \left(1 + \frac{\sigma_X^2}{\sigma_N^2}\right) \quad \text{[bit/symbol]}$$ \hfill (530)

where $\frac{\sigma_X^2}{\sigma_N^2}$ represents the signal to noise ratio.
Illustration of the second theorem of Shannon

\[ C_s (\text{bit/symbol}) \]

Shannon’s limit

\[ p_e = 10^{-5} \]

According to Rauschmayer (page 144):

<table>
<thead>
<tr>
<th>Disturbers</th>
<th>Upstream (kb/s)</th>
<th>Downstream (Mb/s)</th>
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<td>AWGN only</td>
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<td>24 ISDN NEXT</td>
<td>1485</td>
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<td>24 ADSL NEXT</td>
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<td>24 ADSL FEXT</td>
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<td>Dowstream ADSL NEXT</td>
<td>1109</td>
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</tr>
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Outline

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2 Representation of bandpass signals
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4 Digital modulation
5 Spread spectrum
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7 Navigation systems
8 Multiplexing
9 Telephone traffic engineering
10 Transmission over twisted pairs (fixed telephone network)
11 Radio engineering

Engineering of mobile radio communication systems

- Introduction
  - Propagation and fading
  - Mobile sensitivity
- Propagation model
  - Propagation loss
  - Shadowing
    - Lognormal distribution
  - Fading
    - Rayleigh and Rician fading laws
- Empirical models
  - Types of environment
  - Influence of ground and antenna heights
  - Macrocellular model
  - Indoor propagation
At the receiver, the power $P_R$, expressed in [dB], is given by

$$P_R = P_T - L_T + G_T - A_e + G_R - L_R$$  \hspace{1cm} (531)

### Fading

**Definition (Fading)**

Fading is a deviation of the attenuation affecting a signal over certain propagation media.

The fading may vary with time, location or radio frequency. Different classifications for fading effects exist:

- slow/fast fading. It relates to time and delays considerations.
- flat/frequency selective fading.
- large-scale/small-scale fading.
- ...

**Mitigation techniques:**

- Rake receiver (estimation of the channel impulse response)
- Convolutional encoding and Viterbi decoding (pre-encoding)
- Diversity (MIMO, ...)
- Margins
Fading channel manifestations

Key manifestations that a fading channel can exhibit:

- **large-scale fading**, which represents the average signal-power attenuation or the path loss due to motion over large areas ⇒ “slow” fading.

- **small-scale fading**. It refers to the dramatic changes in signal amplitude and phase that can be experienced as a result of small changes (as small as a half wavelength) in the spatial positioning between a receiver and a transmitter ⇒ “fast” fading.

We will study examples of both types of fading and develop *probabilistic models* for them.

---

Mobile sensitivity and quality of service I

For a “good” radio communication, the system needs to fulfill *two conditions*:

1. The level of power received by the mobile device must be larger than the mobile/device sensitivity level, that is the minimal amount of power that the device can interpret for a given level of noise at the input.

2. The channel should not distort the signal and the level of noise should be acceptable. This relates to $E_b/N_0$.

Calculation: link the $\frac{E_b}{N_0}$ ratio to the mobile sensitivity (taking a spectral efficiency of $1 \Rightarrow R_b \approx W$)

$$\frac{C}{N} = \frac{E_b R_b}{N_0 W} \approx \frac{E_b W}{N_0 W} = \frac{E_b}{N_0}$$

(532)

This leads to

$$C = \frac{E_b}{N_0} |_{\text{threshold}} + N$$

(533)
Calculation of the device sensitivity $C$, at the decoding point in the receiver

For the GSM:

- $\frac{E_b}{N_0}\bigg|_{\text{threshold}} = 8 \text{ [dB]}$ for a channel with fading (6 [dB] for devices with higher power levels)

- The bandwidth of a GSM channel is $W = 271 \text{ [kHz]}$. Therefore, for a temperature of $290 \text{ [K]}$, the amount of noise due to the channel is

  $$N = k_B T W = 1,1 \times 10^{-12} \text{ [mW]} \equiv -120 \text{ [dBm]}$$  

- As the typical input amplifier gain is $10 \text{ [dB]}$, the amplified amount of noise at the entrance of the decoder, $N'$, is

  $$N' = -110 \text{ [dBm]}$$

$$C = \frac{E_b}{N_0}\bigg|_{\text{threshold}} + N' \quad (536)$$

<table>
<thead>
<tr>
<th>Receiver type</th>
<th>Device sensitivity $C$ in [dBm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base station device</td>
<td>$-104$</td>
</tr>
<tr>
<td>8 [W] mobile device</td>
<td>$-104$</td>
</tr>
<tr>
<td>2 [W] mobile device (GSM 900)</td>
<td>$-102$</td>
</tr>
<tr>
<td>Dual-band mobile device (GSM)</td>
<td>$-102$</td>
</tr>
</tbody>
</table>
**General propagation model I**

(a) Lateral view

(b) Top view

*Figure:* Views of a transmission between a mobile and a base station.

**General propagation model II**

(a) First measurement

(b) Second measurement

*Figure:* Power measurements as a function of the distance along a same path. There are differences between the two observations.
Fading is often computed as a combination of three terms:

1. a *median attenuation* due to *distance,*
2. a random component that considers large-scale fading effects. We first consider *shadowing* effects.
3. a random component that takes into account small-scale fading effects. Later, we study *multipath* effects.

---

**Shadowing effects: towards a probabilistic model I**

*Shadowing* is due to obstacles (hills, building, etc). To model shadowing effects, we assume that the total attenuation is given by (multiplicative model)

\[ A = A_1 \times A_2 \times \ldots \times A_N \] (537)

with \( A_i \geq 1. \)

When there is shadowing, the received power \( P_R \) is

\[ P_R = \frac{P_T}{(A_{PL} \times A_1 \times A_2 \times \ldots \times A_N)} \] (538)

where \( A_{PL} \) is the propagation loss.

In decibels, the total attenuation loss is the sum of these terms:

\[ L_{total} = L_{PL} + L = L_{PL} + L_1 + L_2 + \ldots + L_N \] (539)
More specifically, the loss due to shadowing $L$ is the sum of $N$ contributions

$$L = L_1 + L_2 + \ldots + L_N$$  \hspace{1cm} (540)

If all the contributions are random variables with an identical mean and variance, $L$ is a Gaussian random variable:

$$L \, [\text{dB}] = L_{50\%} \, [\text{dB}] + \sigma_s \, [\text{dB}] \times N(0, 1) = N(L_{50\%}, \sigma_s^2) \, [\text{dB}]$$  \hspace{1cm} (541)

where $L_{50\%}$ is the median value of the distribution of the propagation loss and $\sigma_s \, [\text{dB}]$ is its standard deviation (which can vary over the range of 8 to 10 [dB]).

If we consider $L_{\text{total}} = L_{PL} + L$, then

$$L_{\text{total}} \, [\text{dB}] = N(L_{PL} + L_{50\%}, \sigma_s^2) \, [\text{dB}]$$  \hspace{1cm} (542)

In absolute numbers (natural units, instead of [dB]) and denoting the reduced Gaussian by $N(0, 1) = X$, the loss is the variable $L_a$

$$L_a = 10^{L_{[\text{dB}]/10}} = 10^{(L_{50\% \, [\text{dB}]} + \sigma_s \, [\text{dB}] \times X)/10}$$

$$= 10^{L_{50\% \, [\text{dB}]/10}} \times 10^{\sigma_s \, [\text{dB}] \times X/10} = L_o \, V$$  \hspace{1cm} (543)

where $L_o \triangleq 10^{L_{50\% \, [\text{dB}]/10}}$ is a constant, and

$$V = 10^{\sigma_s \, [\text{dB}] \times X/10}$$  \hspace{1cm} (544)

$V$, and subsequently the loss $L_a = L_o \, V$, are random variables with a lognormal probability density function.
Let us find the expression of the lognormal probability density function of \( V \)

The probability density function (pdf) of the zero-mean, unit-variance Gaussian variable \( X \) is, for \( x \in [-\infty, +\infty] \),

\[
f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{546}
\]

The lognormal variable \( V \) is given by (changing the base: \( b^z = (e^{\ln b})^z = (e^{z \ln b}) \))

\[
V = 10^{\frac{\sigma_s X}{10}} = e^{\beta \sigma_s X} \tag{547}
\]

where

\[
\beta = \frac{\ln(10)}{10} = 0.23 \tag{548}
\]

The pdf is obtained by taking the formula of variable changes applicable to pdfs:

\[
f_V(v) = \left. \frac{f_X(x)}{|\partial V/\partial x|} \right|_{x=\frac{\ln v}{\beta \sigma_s}} = \left. \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right|_{x=\frac{\ln v}{\beta \sigma_s}} e^{\beta \sigma_s x} \tag{549}
\]

\[
= \begin{cases} 
\frac{1}{\beta \sigma_s v} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\ln v)^2}{2\beta^2 \sigma_s^2}} & \text{if } v \geq 0 \\
0 & \text{if } v < 0 
\end{cases} \tag{550}
\]
Median of $L_a = L_o V$?

- The median of $V$, denoted by $v_m$, is obtained for $x = 0$, that is
  
  $$v_m = 10^{\frac{\sigma_s x}{10}} = 10^0 = 1$$

  (551)

  It does not depend on the variance $\sigma_s$.

- Because $V$ has a median of 1, the median of the loss $L_a = L_o V$ is given by

  $$L_o \triangleq 10^{L_{50\%}[dB]/10}$$

  (552)
Impact of shadowing on the coverage zone at the border of a mobile cell

We have seen that the loss due to shadowing can be modeled as
\[ L[\text{dB}] = L_{50\%}[\text{dB}] + \sigma_s[\text{dB}] \times N(0, 1) = N(L_{50\%}, \sigma_s^2)[\text{dB}] \] (553)

where
- \( L_{50\%} \) is the median loss, as given by an empirical model (such models are obtained from measurements and they include \( L_{PL} \); see page 472).
- \( N \) is a Gaussian variable.

If the observed loss, \( L_{\text{observed}} \), is such that
- \( L_{\text{observed}} \leq L_{50\%}[\text{dB}] \), then we have a favorable case.
- \( L_{\text{observed}} \geq L_{50\%}[\text{dB}] \), then there is a risk for a connection failure.

\[ \Rightarrow \text{the probability to be in the favorable case or in the unfavorable case depends on a threshold value } L_{50\%}[\text{dB}] \]

Lognormal margin

If we tailor the system in such a way that the mobile sensitivity is reached for the median value of the attenuation, then the connection will fail for 50% of time.

To decrease the risk (%) of failure and mitigate the effects of shadowing, we add a power margin.

Towards the computation of a power margin \( m_s[\text{dB}] \)

\[ p(l_s < m_s) = \int_{-\infty}^{m_s} \frac{1}{\sigma_s \sqrt{2\pi}} e^{-\frac{l_s^2}{2\sigma_s^2}} dl_s \] (554)

\[ = \frac{1}{2} + \int_{0}^{m_s} \frac{1}{\sigma_s \sqrt{2\pi}} e^{-\frac{l_s^2}{2\sigma_s^2}} dl_s \] (555)

\[ = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{m_s}{\sqrt{2\sigma_s}} \right) \] (556)

\( \text{erf}(x) \) is the error function defined as

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt \] (557)
Power margin

So, there are two terms:

1. \( \frac{1}{2} \rightarrow \text{level of power to get 50\% of favorable cases}. \)

2. \( \frac{1}{2} \text{erf} \left( \frac{m_s}{\sqrt{2}\sigma_s} \right) \rightarrow \text{leads to a power margin } m_s \text{ if we want more favorable cases (in excess of 50\% thus)}. \)

Definition (Power margin)

The power margin is the additional amount of power sent by the transmitter to obtain more favorable cases at the receiver. The power margin “compensates” for possibly higher fading losses.

Lognormal margin III

Relationship between a \( \% \) of coverage and a power margin (at the border of a cell)

![Graph showing the relationship between coverage probability (in %) and margin in dB](image-url)
Key manifestations that a fading channel can exhibit:

- **large-scale fading**, which represents the average signal-power *attenuation* or the path loss due to motion over large areas ⇒ “slow” fading.

- **small-scale fading**. It refers to the dramatic changes in *signal amplitude* and *phase* that can be experienced as a result of small changes (as small as a half wavelength) in the spatial positioning between a receiver and a transmitter ⇒ “*fast*” fading.

We now study the effect of fast fading.

**Fast fading I**

Consider the transmission over a *scattering* or *multipath channel*
The transmitter emits a clean sinusoidal wave of the form \( A \cos(2\pi f_0 t) \). At the receiver, we get

\[
X(t) = \sum_i C_i \cos(2\pi f_0 t + \theta_i) \tag{558}
\]

This expression can be rewritten to express the in-phase and quadrature components (\( \cos(a + b) = \cos a \cos b - \sin a \sin b \)):

\[
X(t) = \sum_i [C_i \cos(2\pi f_0 t) \cos \theta_i - C_i \sin(2\pi f_0 t) \sin \theta_i] \tag{559}
\]

\[
= \sum_i A_i \cos(2\pi f_0 t) - \sum_i B_i \sin(2\pi f_0 t) \tag{560}
\]

\[
= X_I \cos(2\pi f_0 t) - X_Q \sin(2\pi f_0 t) \tag{561}
\]

where we have defined

\[
X_I = \sum_i A_i = \sum_i C_i \cos \theta_i \quad \text{and} \quad X_Q = \sum_i B_i = \sum_i C_i \sin \theta_i \tag{562}
\]

**Fast fading III**

What are \( X_I = \sum_i A_i = \sum_i C_i \cos \theta_i \) and \( X_Q = \sum_i B_i = \sum_i C_i \sin \theta_i \)?

- we can consider \( X_I \) and \( X_Q \) as two random variables.
- they are sums of terms that are identically distributed.

\( \Rightarrow \) \( X_I \) and \( X_Q \) are two Gaussian random variables with:

1. a zero mean and,
2. the same variance \( \sigma_X^2 \).

**Calculation of the mean:**

\[
E\{X_I\} = E\left\{ \sum_i C_i \cos \theta_i \right\} = \sum_i E\{C_i \cos \theta_i\} \tag{563}
\]

\[
= \sum_i E\{C_i\} E\{\cos \theta_i\} = \sum_i E\{C_i\} \times 0 \tag{564}
\]

\[
= 0 \tag{565}
\]
Calculation of the variance:

\[
\sigma_{XI}^2 = E \{(XI - \mu_I)^2\} = E \{(XI - 0)^2\} 
\]  
(566)

\[
\sigma_{XI}^2 = E \left\{ \left( \sum_i C_i \cos \theta_i \right)^2 \right\} 
\]  
(567)

\[
\sigma_{XI}^2 = E \left\{ \sum_i C_i^2 \cos^2 \theta_i + \sum_{i,j, i \neq j} C_i \cos \theta_i C_j \cos \theta_j \right\} 
\]  
(568)

\[
\sigma_{XI}^2 = \sum_i E \{ C_i^2 \cos^2 \theta_i \} + \sum_{i,j, i \neq j} E \{ C_i \cos \theta_i C_j \cos \theta_j \} 
\]  
(569)

\[
\sigma_{XI}^2 = \sum_i E \{ C_i^2 \} E \{ \cos^2 \theta_i \} + \sum_{i,j, i \neq j} E \{ C_i \} E \{ \cos \theta_i \} E \{ C_j \} E \{ \cos \theta_j \} 
\]  
(570)

\[
\sigma_{XI}^2 = \frac{1}{2} \alpha = \sigma_X^2 
\]  
(571)

Fast fading V

Probability density functions of \(X_I\) and \(X_Q\)

In conclusion, we have

\[
f_{X_I}(x_I) = \frac{1}{2\pi\sigma_X^2} e^{-\frac{x_I^2}{2\sigma_X^2}} 
\]  
(572)

and

\[
f_{X_Q}(x_Q) = \frac{1}{2\pi\sigma_X^2} e^{-\frac{x_Q^2}{2\sigma_X^2}} 
\]  
(573)

Note also that \(X_I\) and \(X_Q\) are the Rice components of the stochastic process \(X(t)\):

\[
X(t) = X_I \cos(2\pi f_o t) - X_Q \sin(2\pi f_o t) 
\]  
(574)
To determine the amplitude of the electrical field at the receiver, we need to analyze the amplitude of $X(t)$. It is given by

$$ R = \sqrt{X_I^2 + X_Q^2}, \quad R \geq 0 $$  \hspace{1cm} (575)

while the phase is

$$ \Phi = \tan^{-1} \frac{X_Q}{X_I}, \quad \Phi \in [0, 2\pi] $$  \hspace{1cm} (576)

Towards the probability density function of $R$ and $\Phi$

Both $R$ and $\Phi$ are obtained from $X_I$ and $X_Q$. So, we have to find the joint $f_{R,\Phi}(r, \phi)$ from that of $f_{X_I, X_Q}(x_I, x_Q)$.

The joint pdf $f_{X_I, X_Q}(x_I, x_Q)$ is, by definition,

$$ f_{X_I, X_Q}(x_I, x_Q) = \frac{1}{2\pi \sigma_X^2} e^{-\frac{x_I^2 + x_Q^2}{2\sigma_X^2}} $$  \hspace{1cm} (577)

From equations (575) and (576), we have that $x_I = r \cos \phi$, $x_Q = r \sin \phi$. Therefore,

$$ f_{R,\Phi}(r, \phi) = \begin{vmatrix} \cos \phi & \sin \phi \\ -r \sin \phi & r \cos \phi \end{vmatrix} f_{X_I, X_Q}(r \cos \phi, r \sin \phi) $$

$$ = \frac{r}{2\pi \sigma_X^2} e^{-\frac{r^2 \cos^2 \phi + r^2 \sin^2 \phi}{2\sigma_X^2}} $$  \hspace{1cm} (579)

$$ = \begin{cases} \frac{r}{2\pi \sigma_X^2} e^{-\frac{r^2}{2\sigma_X^2}}, & r \geq 0, \phi \in [0, 2\pi] \\ 0, & r < 0 \end{cases} $$  \hspace{1cm} (580)
By integrating over $\Phi$, we obtain the marginal probability density function of the envelope:

$$f_R(r) = \begin{cases} \frac{r}{\sigma_X^2} e^{-\frac{r^2}{2\sigma_X^2}}, & r \geq 0 \\ 0 & r < 0 \end{cases}$$  \hspace{1cm} (581)$$

This is a Rayleigh distribution.

![Rayleigh probability density function](image)

**Figure**: Rayleigh probability density function.

### Probability density function of the phase

To obtain $f_\Phi(\phi)$, we integrate on the $r$ parameter from 0 to $+\infty$, to get:

$$f_\Phi(\phi) = \frac{1}{2\pi}, \quad \phi \in [0, 2\pi]$$  \hspace{1cm} (582)$$

$\Phi$ is thus the probability density function of a uniform random variable, which is independent of $R$.

In conclusion,

$$f_{R,\Phi}(r, \phi) = f_R(r)f_\Phi(\phi)$$  \hspace{1cm} (583)$$
Experimental data

Rayleigh distribution (experimental and theoretical)

Number of occurrences (frequencies)

Signal level [dBm]

Figure: Histogram of real measured power levels.

Fading in the presence of a direct path I

If there is a direct path to the receiver, the signal at the receiver is

$$Z(t) = A \cos(2\pi f_0 t + \theta) + X(t)$$

(584)

Assume that $Z(t)$ is a bandpass signal, we can take its Rice decomposition

$$Z(t) = Z_I \cos(2\pi f_0 t) - Z_Q \sin(2\pi f_0 t)$$

(585)

where

$$Z_I = A \cos \theta + X_I$$

(586)

$$Z_Q = A \sin \theta + X_Q$$

(587)

In terms of amplitude and phase, this leads to

$$Z = R \cos(2\pi f_0 t + \Phi)$$

(588)

with

$$R = \sqrt{Z_I^2 + Z_Q^2}, \quad R \geq 0$$

(589)
and
\[ \Phi = \tan^{-1} \frac{X_Q}{X_I}, \quad \Phi \in [0, 2\pi] \] (590)

Therefore,
\[ Z_I = R \cos \Phi \quad (591) \]
\[ Z_Q = R \sin \Phi \quad (592) \]

For a given \( \theta \), the \( X_I \) and \( X_Q \) are independent variables. This remains true, so that
\[ Z_I = N(A \cos \theta, \sigma^2_X) \quad (593) \]
\[ Z_Q = N(A \sin \theta, \sigma^2_X) \quad (594) \]

The joint pdf, given \( \theta_o \), is then
\[ f_{Z_I Z_Q}(z_I, z_Q | \theta = \theta_o) = \frac{1}{2\pi \sigma^2_X} e^{-\frac{(z_I - A \cos \theta_o)^2 + (z_Q - A \sin \theta_o)^2}{2\sigma^2_X}} \quad (595) \]

**Probability density function**

\[ f_{R, \Phi}(r, \phi | \theta_o) = \begin{vmatrix} \cos \phi & \sin \phi \\ -r \sin \phi & r \cos \phi \end{vmatrix} f_{Z_I Z_Q}(r \cos \phi, r \sin \phi | \theta_o) \]
\[ = \frac{r}{2\pi \sigma^2_X} e^{-\frac{(r \cos \phi - A \cos \theta_o)^2 + (r \sin \phi - A \sin \theta_o)^2}{2\sigma^2_X}} \quad (596) \]
\[ = \frac{r}{2\pi \sigma^2_X} e^{-\frac{r^2 + A^2 - 2rA \cos(\theta_o - \phi)}{2\sigma^2_X}} \quad (597) \]
When integrating over $\Phi$, we obtain the Rician pdf

$$f_R(r) = \begin{cases} \frac{r}{\sigma_X^2} e^{-\frac{r^2 + A^2}{2\sigma_X^2}} I_0 \left( \frac{Ar}{\sigma_X^2} \right), & r \geq 0 \\ 0, & r < 0 \end{cases} \quad (598)$$

where $I_0(x)$ is the modified Bessel function of order 0.

**Figure:** Modified Bessel function of order 0.

**Figure:** Rician pdf (for different values of $a = \frac{A}{\sigma_X}$).
Fast fading: conclusions

1. The receiver gets $X(t) = \sum_i C_i \cos(2\pi f_0 t + \theta_i)$ (no direct path):
   - the amplitude of $X(t)$ follows a Rayleigh distribution
   - the phase of $X(t)$ follows a uniform distribution

2. The receiver gets $Z(t) = A \cos(2\pi f_0 t + \theta) + X(t)$ (there is a direct path):
   - the amplitude of $Z(t)$ follows a Rician distribution
   - the phase of $Z(t)$ is not uniformly distributed

Influence of ground and antenna heights on the radio budget link I

Figure: Influence of a reflection on the ground.
Influence of ground and antenna heights on the radio budget link II

Analysis:
▶ direct path → \( Ae^{-jd_1} \),
▶ reflected signal → \(-Ae^{-jd_2}\),
▶ the received signal is \( Ae^{-jd_1} \left( 1 - e^{j\beta(d_2 - d_1)} \right) \).

So, the reflection adds the following factor:

\[
\Gamma = 1 - e^{j\beta(d_2 - d_1)} 
\] (599)

As (for right-angled triangles)

\[
d_1 = \sqrt{d^2 + (h_b - h_m)^2} \quad (600)
\]

\[
d_2 = \sqrt{d^2 + (h_b + h_m)^2} \quad (601)
\]

Influence of ground and antenna heights on the radio budget link III

we have that

\[
d_2 - d_1 = \sqrt{d^2 + (h_b + h_m)^2} - \sqrt{d^2 + (h_b - h_m)^2} 
\] (602)

\[
= d \sqrt{1 + \left( \frac{h_b + h_m}{d} \right)^2} - d \sqrt{1 + \left( \frac{h_b - h_m}{d} \right)^2} 
\] (603)

\[
\simeq \frac{4h_b h_m}{d} 
\] (608)
Factor modifying the power

To deal with the power, we take the square of the electrical field modifying factor \( \Gamma^2 \)

\[
\Gamma^2 = \left| 1 - e^{-j\beta(d_2-d_1)} \right|^2 
\]

\[
= \left| e^{-j\beta(d_2-d_1)/2} \right|^2 \left| e^{j\beta(d_2-d_1)/2} - e^{-j\beta(d_2-d_1)/2} \right|^2 
\]

\[
= \left| e^{-j\beta(d_2-d_1)/2} \right|^2 \left| 2 \sin \left( \beta \frac{d_2 - d_1}{2} \right) \right|^2 
\]

\[
= 1 \times \left( 2 \sin \left( \frac{h_b h_m}{d} \right) \right)^2 
\]

\[
= 4 \sin^2 \left( \frac{h_b h_m}{d} \right) 
\]

\[
\approx 4 \left( \frac{h_b h_m}{d} \right)^2 = 4 \left( \frac{2\pi h_b h_m}{\lambda d} \right)^2 = \left( \frac{4\pi h_b h_m}{\lambda d} \right)^2 
\]

Receiver power \( P_R \)

Finally, the received power \( P_R \) is given by

\[
P_R \approx P_E G_E G_R \left( \frac{\lambda}{4\pi d} \right)^2 \left( \frac{4\pi h_b h_m}{d} \right)^2 
\]

\[
\approx P_E G_E G_R \frac{h_b^2 h_m^2}{d^4} 
\]

or, in decibels,

\[
P_R [\text{dB}] = 10 \log_{10}(P_E G_E G_R) + 20 \log_{10} h_b + 20 \log_{10} h_m - 40 \log_{10} d 
\]
Different types of environments

- **rural.** The topographic relief is the major factor.
- **suburban** (small towns)
- **urban** (large cities)

<table>
<thead>
<tr>
<th>Environment</th>
<th>exponent</th>
</tr>
</thead>
<tbody>
<tr>
<td>rural</td>
<td>3.2</td>
</tr>
<tr>
<td>suburban</td>
<td>3.5</td>
</tr>
<tr>
<td>dense urban</td>
<td>3.8</td>
</tr>
</tbody>
</table>

*Table: Exponent for the distance dependency of the attenuation loss.*

Model for macrocells

**COST 231-Hata model**

In a urban environment, the median attenuation $L_u$ is given, in [dB], by

$$L_u = 46.33 + 33.9 \log(f) - 13.82 \log(h_b) - a(h_m) + [44.9 - 6.55 \log(h_b)] \log d + C_m$$  \hspace{1cm} (6.18)

where

- $f$ is the frequency, $d$ the distance, $h_b$, $h_m$, heights; all these values are expressed in, respectively, [MHz], [km] and [m].

- $a(h_m) = (1.1 \log(f) - 0.7) h_m - (1.56 \log(f) - 0.8)$ for middle-sized towns; this correction factor depends on the mobile height but also from the type of environment.

- $C_m = 0$ [dB] middle-sized towns and suburbs, and $C_m = 3$ [dB] for large cities.
Figure: Transmission, reflection and diffraction.

Figure: Wave guide effects.
“Outdoor-indoor” propagation

- we can define two types of situations:
  - *soft indoor*, representing the attenuation close to the front of a building, and
  - *deep indoor*, representing the attenuation deeper inside the building.

- Typical values are 10 [dB] *soft indoor* and 20 [dB] for *deep indoor* at 900 [MHz].

“Indoor-indoor” propagation”