Short Prime Quadratizations of Cubic Negative Monomials

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Abstract

Pseudo-Boolean functions naturally model problems in a number of different areas such as computer science, statistics, economics, operations research or computer vision, among others. Pseudo-Boolean optimization (PBO) is \( \mathcal{NP} \)-hard, even for quadratic polynomial objective functions. However, much progress has been done in finding exact and heuristic algorithms for the quadratic case. Quadratizations are techniques aimed at reducing a general PBO problem to a quadratic polynomial one. Quadratizing single monomials is particularly interesting because it allows quadratizing any pseudo-Boolean function by termwise quadratization. A characterization of short quadratizations for negative monomials has been provided. In this report we present a proof of this characterization for the case of cubic monomials, which requires a different analysis than the case of higher degree.

1 Introduction

A pseudo-Boolean function is a mapping \( f : \{0,1\}^n \rightarrow \mathbb{R} \), i.e., a mapping that assigns a real value to each tuple \((x_1,\ldots,x_n)\) of \( n \) binary variables. Every pseudo-Boolean function can be represented by a unique multilinear polynomial, that is, for a function \( f \) on \( \{0,1\}^n \) there exists a unique mapping \( a : 2^{[n]} \rightarrow \mathbb{R} \), which assigns a real value \( a_S \) to every subset \( S \) of the \( n \) variables, such that

\[
f(x_1,x_2,\ldots,x_n) = \sum_{S \subseteq \{1,\ldots,n\}} a_S \prod_{i \in S} x_i.
\]

Pseudo-Boolean optimization (PBO) problems are of the form

\[
\min \{ f(x) : x \in \{0,1\}^n \},
\]

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where \( f(x) \) is a pseudo-Boolean function. Pseudo-Boolean optimization models arise naturally in diverse areas such as computer science, statistics, economics, finance, operations research or computer vision, among others. A detailed list of applications can be found in [2], [3].

Pseudo-Boolean optimization is \( \mathcal{NP} \)-hard, even if the objective function is quadratic. However the quadratic case is particularly interesting; on one hand, because it encompasses relevant problems such as MAX-2-SAT (satisfiability theory) or MAX-CUT (graph theory), and on the other hand, due to much progress that has been done in finding heuristic and exact algorithms for quadratic pseudo-Boolean optimization (QPBO). Therefore, given a pseudo-Boolean function \( f \), we aim to find an equivalent quadratic function \( g \), for which quadratic binary optimization algorithms are applicable.

**Definition 1** Given a pseudo-Boolean function \( f : \{0, 1\}^n \rightarrow \mathbb{R} \), \( g(x, y) \) is a quadratization of \( f \) if \( g(x, y) \) is a quadratic polynomial depending on \( x \) and on \( m \) auxiliary binary variables \( y_1, y_2, \ldots, y_m \), such that

\[
f(x) = \min \{g(x, y) : y \in \{0, 1\}^m\}, \forall x \in \{0, 1\}^n.
\]

Using this definition, \( \min \{f(x) : x \in \{0, 1\}^n\} = \min \{g(x, y) : (x, y) \in \{0, 1\}^{n+m}\} \), reducing a general PBO problem to the quadratic case.

Anthony, Boros, Crama and Gruber have initiated a systematic study of quadratizations of pseudo-Boolean functions [1]. Among other results, they provide a precise characterization of quadratizations for negative monomials. The aim of this report is to provide a proof of this characterization for cubic negative monomials, which is different from the proof for the case of monomials of degree \( \geq 4 \).

2 Negative Monomials

Finding quadratizations for monomials is particularly interesting; if quadratizations for single monomials are known and well-described, it is possible to use termwise quadratization procedures, which are based on the following scheme. For a real number \( c \), let \( \text{sign}(c) = +1 \) (resp., \( -1 \)) if \( c \geq 0 \) (resp., \( c < 0 \)). Then, given \( f \) as in (1),

1. for each \( S \in \mathcal{P}[n] \), let \( g_S(x, y_S) \) be a quadratization of the monomial \( \text{sign}(a_S) \prod_{i \in S} x_i \), where \( (y_S, S \in \mathcal{P}[n]) \) are disjoint vectors of auxiliary variables, one for each \( S \),

2. let \( g(x, y) = \sum_{S \in \mathcal{P}[n]} |a_S| \cdot g_S(x, y_S) \).
Then \( g(x, y) \) is a quadratization of \( f(x) \).

Several quadratizations of monomials have been proposed in the literature (see, e.g., [1]). In this report we describe quadratizations for the case where \( f \) is a negative monomial. We first introduce the notion of prime quadratizations [1], which are interesting because they define ”small” quadratizations, and because our objective is to minimize \( f \). Then, we will prove that there are essentially only two prime quadratizations using a single auxiliary variable for negative cubic monomials.

Definition 2 A quadratization \( g(x, y) \) of \( f \) is prime if there is no quadratization \( h(x, y) \) such that \( h(x, y) \leq g(x, y) \) for all \((x, y) \in \{0, 1\}^{n+m} \) and such that \( h(x^*, y^*) < g(x^*, y^*) \) for at least one point \((x^*, y^*)\).

Definition 3 The standard quadratization of a negative monomial \( M_n = -\prod_{i=1}^{n} x_i \) is the quadratic function

\[
s_n(x, y) = (n - 1)y - \sum_{i=1}^{n} x_i y.
\] (2)

The extended standard quadratization of \( M_n \) is the function

\[
s_n^+(x, y) = (n - 2)x_n y - \sum_{i=1}^{n-1} x_i (y - \bar{x}_n),
\] (3)

where \( \bar{x}_n = 1 - x_n \).

Anthony et al. [1] state the following theorem:

Theorem 1 For \( n \geq 3 \), assume that \( g(x, y) \) is a prime quadratization of \( M_n \) involving a single auxiliary variable \( y \). Then, up to an appropriate permutation of the \( x \)-variables and up to a possible switch of the \( y \)-variable, either \( g(x, y) = s_n \) or \( g(x, y) = s_n^+ \).

The proof in [1] is valid for all \( n \geq 4 \), but the authors skipped the details of the case \( n = 3 \), which requires slightly different arguments. We present next the missing details.

Proof. (case \( n = 3 \)). The proof consists in a case study on the coefficients of the general form of a quadratization with a single auxiliary variable for the cubic negative monomial. Until Claim 2, the proof is identical to the case \( n \geq 4 \) presented in [1].

The general form of a quadratization using a single auxiliary variable is

\[
g(x, y) = ay + \sum_{i=1}^{3} b_i x_i y + \sum_{i=1}^{3} c_i x_i + \sum_{1 \leq i < j \leq 3} p_{ij} x_i x_j.
\] (4)
Notice that there is no constant term because, since we must have $M_3(x) = \min_{y \in [0,1]} g(x, y)$ for all binary vectors $x$, we can assume $g(0, 0) = 0$ after substituting $\bar{y}$ by $y$ if necessary.

For subsets $S \subseteq N = \{1, 2, 3\}$, we write $b(S) = \sum_{i \in S} b_i$, $c(S) = \sum_{i \in S} c_i$, and $p(S) = \sum_{i,j \in S, i < j} p_{ij}$, and we can write

$$g(S, y) = ay + b(S)y + c(S) + p(S).$$

The fact that $g$ is a quadratization of $M_3$ can be written as

$$0 = \min_{y \in [0,1]} (a + b(S))y + c(S) + p(S), \forall S \subset N,$$

$$-1 = \min_{y \in [0,1]} (a + b(N))y + c(N) + p(N).$$

Let us first note that by (6), we have $g(0, 1) \geq 0$, and hence

$$a \geq 0.$$  

(8)

Furthermore, we must have $g((i), 0) \geq 0$ for $i = 1, 2, 3$, implying

$$c_i \geq 0, \text{ for } i = 1, 2, 3.$$  

(9)

Based on (5), we can partition the set of indices as $N = N^0 \cup N^+$, where

$$N^0 = \{u \in N \mid c_u = 0\},$$

$$N^+ = \{i \in N \mid c_i > 0\}.$$  

(10)

(11)

Since $g((i), 0) = c_i$, relation (5) implies

$$g((i), 1) = a + b_i + c_i = 0, \forall i \in N^+, \text{ and}$$

$$g((u), 1) = a + b_u \geq 0, \forall u \in N^0.$$  

(12)

(13)

Let us next write (6) for subsets of size two. Consider first a pair $u, v \in N^0$, $u \neq v$. Since $c_u = c_v = 0$, we get $g((u, v), y) = (a + b_u + b_v)y + p_{uv}$, implying

$$\min(p_{uv}, a + b_u + b_v + p_{uv}) = 0.$$  

(14)

Let us consider next $i, j \in N^+, i \neq j$. Then, by (12) and by the definitions we get $g((i, j), 1) = p_{ij} - a \geq 0$. This, together with (8) implies that $p_{ij} \geq a \geq 0$. Thus, $g((i, j), 0) = c_i + c_j + p_{ij} > 0$ implying that $g((i, j), 1) > 0$, that is

$$p_{ij} = a \geq 0, \forall i, j \in N^+.$$  

(15)

This allows us to establish a property of $N^0$.

**Claim 1** $N^0 \neq \emptyset$. 

4
Proof. If \( N^0 = \emptyset \), then we have \( g(N, y) = (a + b(N^+))y + c(N^+) + \binom{|N^+|}{2}a \) by (15). Since \( |N^+|a + b(N^+) + c(N^+) = 0 \), by (12), we get \( g(N, 1) = \binom{|N^+|}{2}a \geq 0 \) by (8), and \( g(N, 0) = c(N^+) + \binom{|N^+|}{2}a \geq 0 \) by (8) and (9). This contradicts (7) and proves the claim. □

The following two claims distinguish two cases: \( N^+ = \emptyset \), and \( N^+ \neq \emptyset \).

Claim 2 Theorem 1 holds for \( n = 3 \) when \( N^+ = \emptyset \).

1. Case \( p_{12}, p_{13}, p_{23} > 0 \). All quadratizations are of the form:
   \[
   g(x, y) = (2 + p_{12} + p_{13} + p_{23})y \\
   - (1 + p_{12} + p_{13})x_1y - (1 + p_{12} + p_{23})x_2y - (1 + p_{13} + p_{23})x_3y \\
   + p_{12}x_1x_2 + p_{13}x_1x_3 + p_{23}x_2x_3,
   \]
   which is never prime because \( g(x, y) \geq s_3(x, y), \forall (x, y) \in \{0, 1\}^{3+1} \).

2. Case \( p_{12} > 0, p_{13}, p_{23} = 0 \) (w.l.o.g.). All quadratizations are of the form:
   \[
   g(x, y) = (-b_1 - b_2 - p_{12})y + b_1x_1y + b_2x_2y - x_3y + p_{12}x_1x_2,
   \]
   where
   \[
   (2.1) \quad b_2 - p_{12} \geq 1, \\
   (2.2) \quad b_1 - p_{12} \geq 1,
   \]
   which is never prime because \( g(x, y) \geq s_3(x, y), \forall (x, y) \in \{0, 1\}^{3+1} \).

3. Case \( p_{12}, p_{13} > 0, p_{23} = 0 \) (w.l.o.g.). All quadratizations are of the form:
   \[
   g(x, y) = (1 - b_1)y + b_1x_1y - (1 + p_{12})x_2y - (1 + p_{13})x_3y + p_{12}x_1x_2 + p_{13}x_1x_3
   \]
   where
   \[
   (3.1) \quad b_1 - p_{12} \geq 0, \\
   (3.2) \quad b_1 - p_{13} \geq 0, \\
   (3.3) \quad 1 - b_1 - p_{12} - p_{13} \geq 0,
   \]
   which is never prime because \( g(x, y) \geq s_3(x, y), \forall (x, y) \in \{0, 1\}^{3+1} \).

4. Case \( p_{12}, p_{13}, p_{23} = 0 \). All quadratizations are of the form:
   \[
   g(x, y) = (-1 - b_1 - b_2 - b_3)y + b_1x_1y + b_2x_2y + b_3x_3y
   \]
   where
   \[
   (4.1) \quad 1 - b_1 \geq 0, \\
   (4.2) \quad 1 - b_2 \geq 0, \\
   (4.3) \quad 1 - b_3 \geq 0,
   \]
   which is never prime because \( g(x, y) \geq s_3(x, y), \forall (x, y) \in \{0, 1\}^{3+1} \).
Proof.

Since \( N^+ = \emptyset \),
\[
g(x, y) = ay + \sum_{i=1}^{3} b_i x_i y + \sum_{1 \leq i < j \leq 3} p_{ij} x_i x_j. \quad (16)
\]

By (14), \( g(N, 0) = p_{12} + p_{13} + p_{23} \geq 0 \) which implies that \( g(N, 1) = -1 \), or
\[
g(N, 1) = a + b_1 + b_2 + b_3 + p_{12} + p_{13} + p_{23} = -1. \quad (17)
\]

1. Case \( p_{12}, p_{13}, p_{23} > 0 \).

By (14), we have the system of equations
\[
\begin{align*}
    a + b_1 + b_2 + p_{12} &= 0, \\
    a + b_1 + b_3 + p_{13} &= 0, \\
    a + b_2 + b_3 + p_{23} &= 0.
\end{align*}
\]

Considering this system along with equation (17), and solving it as a function of \( p_{12}, p_{13}, p_{23} \), we obtain that the general form (16) of the quadratization in this case is
\[
g(x, y) = (a + b_1 + b_2 + p_{12}) y + b_1 x_1 y + b_2 x_2 y + x_3 y + p_{12} x_1 x_2.
\]

It can be easily checked that \( g(x, y) - s_3(x, y) \geq 0, \forall (x, y) \in \{0, 1\}^{3+1} \), and therefore \( g \) is not prime.

2. Case \( p_{12} > 0, p_{13} = p_{23} = 0 \).

By (14), we have the equation
\[
a + b_1 + b_2 + p_{12} = 0.
\]

Considering this equation along with equation (17), and solving the system as a function of \( b_1, b_2, p_{12} \), we obtain that the general form (16) of the quadratization in this case is
\[
g(x, y) = (-b_1 - b_2 - p_{12}) y + b_1 x_1 y + b_2 x_2 y - x_3 y + p_{12} x_1 x_2.
\]

For \( g \) to be a quadratization we also need
\[
\begin{align*}
    g((1, 3), 1) &= -b_2 - p_{12} - 1 \geq 0, \quad (18) \\
    g((2, 3), 1) &= -b_1 - p_{12} - 1 \geq 0. \quad (19)
\end{align*}
\]

Using conditions (18) and (19), it can be easily checked that \( g(x, y) - s_3(x, y) \geq 0, \forall (x, y) \in \{0, 1\}^{3+1} \), and therefore \( g \) is not prime.
3. Case \( p_{13} \neq 0, p_{23} = 0 \).

By (14), we have the system of equations

\[
\begin{align*}
    a + b_1 + b_2 + p_{12} &= 0, \\
    a + b_1 + b_3 + p_{13} &= 0.
\end{align*}
\]

Considering this system along with equation (17), and solving the system as a function of \( b_1, p_{12}, p_{13} \), we obtain that the general form (16) of the quadratization in this case is

\[
g(x, y) = (1 - b_1)y + b_1x_1y - (1 + p_{12})x_2y - (1 + p_{13})x_3y + p_{12}x_1x_2 + p_{13}x_1x_3.
\]

For \( g \) to be a quadratization we also need

\[
\begin{align*}
    g([2, 3], 1) &= -1 - b_1 - p_{12} - p_{13} \geq 0, \\
    g([2], 1) &= -b_1 - p_{12} \geq 0, \\
    g([3], 1) &= -b_1 - p_{13} \geq 0.
\end{align*}
\]

Using conditions (20), (21), (22), and \( a = 1 - b_1 \geq 0 \), it can be easily checked that \( g(x, y) - s_3(x, y) \geq 0, \forall (x, y) \in \{0, 1\}^{3+1} \), and therefore \( g \) is not prime.

4. Case \( p_{12} = p_{13} = p_{23} = 0 \).

Equation (17) gives

\[
g(N, 1) = a + b_1 + b_2 + b_3 = -1.
\]

Using this equation to express \( a \) in terms of \( b_1, b_2 \) and \( b_3 \) in the general form (16) of the quadratization, we obtain

\[
g(x, y) = (-1 - b_1 - b_2 - b_3)y + b_1x_1y + b_2x_2y + b_3x_3y.
\]

For \( g \) to be a quadratization we also need

\[
\begin{align*}
    g([1, 2], 1) &= -1 - b_3 \geq 0, \\
    g([1, 3], 1) &= -1 - b_2 \geq 0, \\
    g([2, 3], 1) &= -1 - b_1 \geq 0.
\end{align*}
\]

Using conditions (23), (24), (25), it can be easily checked that \( g(x, y) - s_3(x, y) \geq 0, \forall (x, y) \in \{0, 1\}^{3+1} \), and therefore \( g \) is not prime. □

**Claim 3** Theorem 1 holds for \( n = 3 \) when \( N^+ \neq \emptyset \). Since \( N^0 \neq \emptyset \), there are two cases:
1. Case \( c_1, c_2 > 0, c_3 = 0 \) (w.l.o.g.). All quadratizations are of the form:

\[
g(x, y) = ay - (a + c_1)x_1y - (a + c_2)x_2y - (1 + p_{13} + p_{23})x_3y + c_1x_1 + c_2x_2
\]
\[
ax_1x_2 + p_{13}x_1x_3 + p_{23}x_2x_3
\]

where

(5.1) \( c_1 + p_{13} \geq 0 \),
(5.2) \( -1 - p_{13} \geq 0 \),
(5.3) \( c_2 + p_{23} \geq 0 \),
(5.4) \( -1 - p_{23} \geq 0 \),

which is never prime because \( g(x, y) \geq s_2^*(x, y) \forall (x, y) \in [0, 1]^{3+1} \).

2. Case \( c_1 > 0, c_2 = c_3 = 0 \) (w.l.o.g.). Then, any quadratization \( g \) satisfies \( g(x, y) \geq s_3^*(x, \bar{y}) \), \( \forall (x, y) \in [0, 1]^{3+1} \).

Proof.

In this case, the general form of the quadratization is

\[
g(x, y) = ay + \sum_{i=1}^{3} b_i x_i y + \sum_{i=1}^{3} c_i x_i + \sum_{1 \leq i < j \leq 3} p_{ij} x_i x_j, \tag{26}
\]

where \( c_i = 0 \) for at least one \( i \in \{1, 2, 3\} \).

1. Case \( c_1, c_2 > 0, c_3 = 0 \).
   
   By (12) we obtain equations
   
   \[
   a + b_1 + c_1 = 0, \tag{27}
   \]
   \[
   a + b_2 + c_2 = 0. \tag{28}
   \]

   By (15), \( p_{12} = a \geq 0 \).

   For \( g \) to be a quadratization we need
   
   \[
   g([1, 3], 0) = c_1 + p_{13} \geq 0, \tag{29}
   \]
   \[
   g([2, 3], 0) = c_2 + p_{23} \geq 0. \tag{30}
   \]

   Hence,
   
   \[
   g(N, 0) = c_1 + c_2 + a + p_{13} + p_{23} \geq 0. \tag{31}
   \]

   Therefore, for \( g \) to be a quadratization we need \( g(N, 1) = -1 \), i.e.,
   
   \[
   g(N, 1) = a + b_1 + b_2 + b_3 + c_1 + c_2 + a + p_{13} + p_{23} = -1. \tag{32}
   \]
Solving the system given by (27), (28) and (32), as a function of $p_{13}$, $p_{23}$, $a$, $c_1$ and $c_2$, the general form (26) of the quadratization becomes

$$g(x, y) = ay - (a + c_1)x_1y - (a + c_2)x_2y - (1 + p_{13} + p_{23})x_3y + c_1x_1 + c_2x_2 + ax_1x_2 + p_{13}x_1x_3 + p_{23}x_2x_3.$$  

For $g$ to be a quadratization, we also need

$$g([1, 3], 1) = -1 - p_{23} \geq 0, \quad (33)$$

$$g([2, 3], 1) = -1 - p_{13} \geq 0. \quad (34)$$

Using conditions (29), (30), (33) and (34), it can be easily checked that $g(x, y) \geq s_3^\vee(x, y), \forall (x, y) \in \mathbb{R}^3$, therefore $g$ is not prime.

2. Case $c_1 > 0$, $c_2 = c_3 = 0$.

By (12) we obtain equation

$$a + b_1 + c_1 = 0, \quad (35)$$

and by (13),

$$a + b_2 \geq 0, \quad (36)$$

$$a + b_3 \geq 0. \quad (37)$$

Using (35), we obtain the following conditions for $g$ to be a quadratization,

$$g([1, 2], 0) = c_1 + p_{12} \geq 0, \quad (38)$$

$$g([1, 3], 0) = c_1 + p_{13} \geq 0, \quad (39)$$

and

$$g([1, 2], 1) = b_2 + p_{12} \geq 0, \quad (40)$$

$$g([1, 3], 1) = b_3 + p_{13} \geq 0. \quad (41)$$

Equations (38) and (40), (39) and (41), respectively imply

$$\min\{c_1 + p_{12}, b_2 + p_{12}\} = 0, \quad (42)$$

$$\min\{c_1 + p_{13}, b_3 + p_{13}\} = 0. \quad (43)$$

For $i \in \{2, 3\}$, we say that

- $i \in B$ if $b_i + p_{1i} = 0$, and
- $i \in C$ if $c_1 + p_{1i} = 0$, 

...
in equations (42)-(43).

We will now show that \( p_{23} = 0 \).

First, note that by (14),

\[
\min\{p_{23}, a + b_2 + b_3 + p_{23}\} = 0. \tag{44}
\]

Now, (44), (35) and (42)-(43), imply that

\[
g(N, 1) = b_2 + b_3 + p_{12} + p_{13} + p_{23} \geq 0. \tag{45}
\]

Therefore, for \( g \) to be a quadratization we need

\[
g(N, 0) = c_1 + p_{12} + p_{13} + p_{23} = -1. \tag{46}
\]

Assume now that \( p_{23} > 0 \). Then, (44) implies that \( a + b_2 + b_3 + p_{23} = 0 \).

Together with (36) and (37), this implies \( b_2 < 0 \) and \( b_3 < 0 \). From (42), \( p_{12} \geq -b_2 > 0 \) and from (43), \( p_{13} \geq -b_3 > 0 \). Since \( p_{12}, p_{13}, p_{23}, c_1 > 0 \), we have a contradiction with (46).

Therefore, we can assume from now on that \( p_{23} = 0 \). Then, (46) reduces to

\[
g(N, 0) = c_1 + p_{12} + p_{13} = -1. \tag{47}
\]

By (42)-(43), we get \( 2c_1 + p_{12} + p_{13} \geq 0 \) and hence, in view of (47),

\[
c_1 \geq 1. \tag{48}
\]

We distinguish now among several subcases.

- **Case 1.** If \( C = \{2, 3\} \), then (47) implies \( c_1 = 1 \) and \( p_{12} = p_{13} = -1 \).

  With these values, (35) becomes \( b_1 = -1 - a \), and (42)-(43) become \( b_2 \geq 1, b_3 \geq 1 \).

  Moreover, the general form (26) of the quadratization becomes

  \[
g(x, y) = ay + (-1 - a)x_1y + b_2x_2y + b_3x_3y + x_1 - x_1x_2 - x_1x_3. \tag{49}
\]

  Compare this expression with

  \[
s_3^+(x, \bar{y}) = x_1 - x_1y - x_2(x_1 - y) - x_3(x_1 - y),
\]

  (where \( x_1 \) plays the role of \( x_3 \)).

  We obtain

  \[
g(x, y) - s_3^+(x, \bar{y}) = a\bar{x}_1y + (b_2 - 1)x_2y + (b_3 - 1)x_3y \geq 0,
\]

  and \( g \) is not prime.
• **Case 2.** If \( 2 \in B \) and \( 3 \in C \), then by definition, \( p_{12} = -b_2 \) and \( p_{13} = -c_1 \). Then, \((47)\) implies \( p_{12} = -b_2 = -1 \). Let us substitute the values \( p_{12} = -1, b_2 = 1, p_{13} = -c_1 \) and \( b_1 = -c_1 - a \) in the general form \((26)\) of the quadratization,

\[
g(x, y) = ay + (-c_1 - a)x_1y + x_2y + b_3x_3y + c_1x_1 - x_1x_2 - c_1x_1x_3. \quad (50)
\]

When \( y = 0 \), this yields (taking \((48)\) into account),

\[
g(x, 0) = c_1x_1\bar{x}_3 - x_1x_2 \geq x_1\bar{x}_3 - x_1x_2 = s_3^+(x, 1).
\]

When \( y = 1 \),

\[
g(x, 1) = a - ax_1 + x_2 + b_3x_3 - x_1x_2 - c_1x_1x_3
\]

\[
= ax_1 + \bar{x}_1x_2 + (b_3 - c_1)x_3 + c_1\bar{x}_1x_3.
\]

Note that \( a \geq 0, b_3 + p_{13} = b_3 - c_1 \geq 0 \) by \((43)\), and \( c_1 \geq 1 \) by \((48)\). So,

\[
g(x, 1) \geq \bar{x}_1x_2 + \bar{x}_1x_3 = s_3^+(x, 0).
\]

Obtaining that \( g(x, y) \geq s_3^+(x, \bar{y}) \), and \( g \) is not prime.

• **Case 3.** Assume finally that \( B = \{2, 3\} \), meaning that \( p_{12} = -b_2 \) and \( p_{13} = -b_3 \). Substituting in \((47)\) yields \( c_1 - b_2 - b_3 = -1 \), and equations \((42)-(43)\) imply \( c_1 - b_2 \geq 0 \) and \( c_1 - b_3 \geq 0 \). From these relations we deduce

\[
b_2 \geq 1, b_3 \geq 1. \quad (51)
\]

With \( p_{12} = -b_2, p_{13} = -b_3, c_1 = b_2 + b_3 - 1 \) and \( b_1 = -a - c_1 = -a - b_2 - b_3 + 1 \), the general form \((26)\) of the quadratization becomes

\[
g(x, y) = ay + (-a - b_2 - b_3 + 1)x_1y + b_2x_2y + b_3x_3y + (b_2 + b_3 - 1)x_1 - b_2x_1x_2 - b_3x_1x_3.
\]

When \( y = 0 \), and considering \((51)\),

\[
g(x, 0) = (b_2 + b_3 - 1)x_1 - b_2x_1x_2 - b_3x_1x_3
\]

\[
= b_2x_1\bar{x}_2 + b_3x_1\bar{x}_3 - x_1
\]

\[
\geq x_1\bar{x}_2 + x_1\bar{x}_3 - x_1 = s_3^+(x, 1).
\]

When \( y = 1 \),

\[
g(x, 1) = a - ax_1 + b_2x_2 + b_3x_3 - b_2x_1x_2 - b_3x_1x_3
\]

\[
= ax_1 + b_2\bar{x}_1x_2 + b_3\bar{x}_1x_3
\]

\[
\geq \bar{x}_1x_2 + \bar{x}_1x_3 = s_3^+(x, 0).
\]

Obtaining that \( g(x, y) \geq s_3^+(x, \bar{y}) \), and \( g \) is not prime. \( \square \)

We have covered all cases for \( N^+ = \emptyset \) and for \( N^+ \neq \emptyset \). As the theorem states, we have seen that the only possibilities for prime quadratizations using one auxiliary variable of the cubic negative monomial are \( s_3 \) or \( s_3^+ \). \( \square \)
3 Conclusion

Quadratization techniques are aimed at transforming a general pseudo-Boolean function expressed as a multilinear polynomial into a quadratic function, in order to apply quadratic pseudo-Boolean optimization algorithms which have been well-studied in both exact and heuristic approaches. Quadratizations of negative monomials are particularly interesting because they allow using techniques such as termwise quadratization, which can be applied to any pseudo-Boolean function expressed as a multilinear polynomial.

This technical report presented a proof of the theorem of Anthony, Boros, Crama and Gruber [1], characterizing short prime quadratizations for cubic negative monomials. The proof for the cubic case is based on the proof for the general case $n \geq 4$ of the cited article. However the case study is different for $n = 3$, and requires the exhaustive analysis presented in this report.

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References

