Identifying codes in vertex-transitive graphs

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Identification in a finite set

How to choose the attributes to **efficiently distinguish** the individuals?
Identification in a finite set

Individuals:

Attributes:

How to choose the attributes to **efficiently distinguish** the individuals?
Identification in a finite set

How to choose the attributes to efficiently distinguish the individuals?
Identification in a graph

- Individuals are vertices of a graph
- Attributes are closed neighbourhoods of the vertices
 Identification in a graph

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- Attributes are **closed neighbourhoods** of the vertices

How many neighbourhoods/vertices to identify these points?
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Formal definition

An identifying code $C$ is a subset of vertices such that

- $\forall u \in V, \ N[u] \cap C \neq \emptyset$ (domination)
- $\forall u, v \in V, \ N[u] \cap C \neq N[v] \cap C$ (separation)

![Graph with vertices and edges]

Given a graph $G$, what is the minimum size $\gamma^{ID}(G)$ of an identifying code of $G$?
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Linear programming formulation

- A variable $x_u$ for each vertex $u$

$x_u \in \{0, 1\} \quad \forall u \in V$
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Minimize $\sum_{u \in V} x_u$

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Linear programming formulation

- A variable $x_u$ for each vertex $u$
- **Goal**: minimize $\sum_{u \in V} x_u$
- **Constraints**: domination and separation

Minimize

\[
\sum_{u \in V} x_u
\]

such that

\[
\sum_{w \in N[u]} x_w \geq 1 \quad \forall u \in V \quad \text{(domination)}
\]

\[
\sum_{w \in N[u] \Delta N[v]} x_w \geq 1 \quad \forall u \neq v \in V \quad \text{(separation)}
\]

$x_u \in \{0, 1\} \quad \forall u \in V$

This problem is NP-complete... but its **fractional** relaxation is not!
Fractional relaxation

Minimize \[ \sum_{u \in V} x_u \]
such that \[ \sum_{w \in N[u]} x_w \geq 1 \quad \forall u \in V \quad \text{(domination)} \]
\[ \sum_{w \in N[u] \Delta N[v]} x_w \geq 1 \quad \forall u \neq v \in V \quad \text{(separation)} \]
\[ x_u \in [0, 1] \quad \forall u \in V \]

Let \( \gamma^f_{ID}(G) \) be the optimal solution of this problem.

\( \gamma^f_{ID}(G) \leq \gamma^ID(G) \)
Vertex-transitive graphs

A graph is **vertex-transitive** if for any two vertices $u$ and $v$, there is an automorphism sending $u$ to $v$.

Examples:

![Hexagonal graph](image1)
![Cube graph](image2)
![Square cubed graph](image3)
Vertex-transitive graphs

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Examples:

![Graph 1](image1)

![Graph 2](image2)

![Graph 3](image3)
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**Examples:**

![Graph 1](image1.png) ![Graph 2](image2.png) ![Graph 3](image3.png)

**Properties:**
- All vertices have the same degree, denoted by $k$.
- There is an optimal solution to the fractional program with all the variables equal.
Fractional value for VT-graphs

There is an optimal solution with $x_u = \lambda$ for all $u \in V$.

Minimize $\sum_{u \in V} x_u$

such that $\sum_{w \in N[u]} x_w \geq 1 \quad \forall u \in V$ (domination)

$\sum_{w \in N[u] \Delta N[v]} x_w \geq 1 \quad \forall u \neq v \in V$ (separation)

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$\Rightarrow \lambda \geq 1/(k + 1)$

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\begin{align*}
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\text{such that} & \quad \sum_{w \in N[u]} x_w \geq 1 \quad \forall u \in V \quad \text{(domination)} \\
& \quad \sum_{w \in N[u] \Delta N[v]} x_w \geq 1 \quad \forall u \neq v \in V \quad \text{(separation)} \\
& \quad x_u \in [0, 1] \quad \forall u \in V \\
\end{align*}
\]

$\Rightarrow \lambda \geq 1/(k + 1)$

$\Rightarrow \lambda \geq 1/d$

where $d$ is the smallest size of sets $N[u] \Delta N[v]$. 

If $G$ is vertex-transitive, $\gamma_{ID}(G) = \lambda \cdot |V| = |V| \min(k + 1, d)$. 

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If $G$ is vertex-transitive, $\gamma_f^{ID}(G) = \lambda \cdot |V| = \frac{|V|}{\min(k+1,d)}$.

**Cycle $C_n$**

- $\gamma_f^{ID}(C_n) = \frac{n}{2}$
- $\gamma^{ID}(C_n) \leq \frac{n+3}{2}$
- $1 \leq \frac{\gamma^{ID}(C_n)}{\gamma_f^{ID}(C_n)} \leq 1 + \frac{3}{n}$

$k = 2, d = 2$
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How big can be the gap between $\gamma_f^{ID}$ and $\gamma^{ID}$?

For any graph $G$, $1 \leq \frac{\gamma^{ID}(G)}{\gamma_f^{ID}(G)} \leq 1 + 2 \ln |V|$
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The upperbound is good if $\gamma_f^{ID}$ is small, i.e. if $k$ and $d$ are large.
Generalized quadrangles

A generalized quadrangle $GQ(s, t)$ is an incidence structure of points and lines such that:

- each line contains $s + 1$ points,
- each point is on $t + 1$ lines,
- if a point $P$ is not on a line $L$, there is a unique line through $P$ intersecting $L$. 

Adjacency graph: points are vertices, lines are clique.

Example: The square grid $n \times n$ or as a graph, the cartesian product $K_n \square K_n$, is a $GQ(n-1, 1)$. 

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Some facts on GQ

Assume $s > 1$, $t > 1$

- A $GQ(s, t)$ is a strongly regular graph with parameters

\[ srg((st + 1)(s + 1), s(t + 1), s - 1, t + 1). \]
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- If it is vertex-transitive,

$$\gamma_f^D(G) = \frac{s^2 t}{st + s + 1} + 1 = \Theta(s).$$

- We have $s \leq t^2$ and $t \leq s^2$. Therefore

$$c_1|V|^{1/4} \leq \gamma_f^D(G) \leq c_2|V|^{2/5}$$
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- The only known values of $(s, t)$ for which there is $GQ(s, t)$ are
  $(q - 1, q + 1), (q + 1, q - 1), (q, q), (q, q^2), (q^2, q), (q^2, q^3), (q^3, q^2)$
  where $q$ is a prime power.
A construction of $GQ(q - 1, q + 1)$ for $q = 2^\ell$

**Step 1:** construction of a hyperconic in the projective plane on $\mathbb{F}_q$.

Projective plane on $\mathbb{F}_q$, $(X_1, X_2, X_3)$
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Conic of equation $X_1X_3 - X_2^2 = 0$
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Hyperconic: $\mathcal{O} = C \cup \{N\}$
- $q + 2$ points
- no 3 collinear points
- a line intersects $\mathcal{O}$ in 0 or 2 points

Nucleus
All the lines through $N$ intersect $C$ in one point
A construction of $GQ(q - 1, q + 1)$ for $q = 2^\ell$

**Step 2:** Construction of $GQ$.

Projective space of dim. 3 on $\mathbb{F}_q$, $(X_0, X_1, X_2, X_3)$

$H_\infty$: projective plane

$X_0 = 0$
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$H_\infty$: projective plane $X_0 = 0$

Points: all except $H_\infty$
Lines: the ones through $\mathcal{O} = \mathcal{C} \cup \{N\}$

- $q^3$ points
- $q$ points by line
- $q + 2$ lines by point
- Unique projection
An identifying code of $GQ(q - 1, q + 1)$

Three non coplanar lines through $N$ form an identifying code.

- Domination: using projection

Domination:

Separation:

Points on $L_i$? adjacent points? non adj. points $P_1$ and $P_2$?

Assume $N[P_1] \cap C = N[P_2] \cap C = \{Q_1, Q_2, Q_3\}$

$\Rightarrow Q_1, Q_2, Q_3$ in a plane containing $N$

$\Rightarrow L_1, L_2, L_3$ are coplanar.
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![Diagram showing three non coplanar lines through N and their domination and separation properties.](image-url)
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  - points on $L_i$ ? ✓
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$\Rightarrow$ $L_1, L_2, L_3$ are coplanar.
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Hence

$$\gamma^{ID}(GQ(q - 1, q + 1)) \leq 3q$$
Bounds for $\gamma^{ID}(GQ(q - 1, q + 1))$

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Hence

$$\gamma^{ID}(GQ(q - 1, q + 1)) \leq 3q - 3$$
Bounds for $\gamma^{ID}(GQ(q - 1, q + 1))$

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Hence

$$\gamma^{ID}(GQ(q - 1, q + 1)) \leq 3q - 3$$

Lower bound?

- Using the fractional value

$$\gamma^{ID}(GQ(q - 1, q + 1)) \geq \frac{q^3}{q^2 + q - 1}$$
Three non coplanar lines through $N$ form an identifying code.

Hence

$$\gamma^{ID}(GQ(q - 1, q + 1)) \leq 3q - 3$$

Lower bound?

- Using the fractional value
  $$\gamma^{ID}(GQ(q - 1, q + 1)) \geq \frac{q^3}{q^2 + q - 1}$$

- With discharging methods (for $q \geq 32$)
  $$\gamma^{ID}(GQ(q - 1, q + 1)) \geq 3q - 7$$

Finally,

$$\gamma^{ID}(GQ(q - 1, q + 1)) \simeq 3q \simeq 3|V|^{1/3}.$$
Let $q$ be a prime power.

- There exists a $GQ(q, q)$ with identifying code of size
  \[ 5q = \Theta(|V|^{1/3}). \]
- There exists a $GQ(q, q^2)$ with identifying code of size
  \[ 5q + 5 = \Theta(|V|^{1/4}). \]
- There exists a $GQ(q^2, q)$ with identifying code of size
  \[ 5q^2 + 3 = \Theta(|V|^{2/5}). \]
Identifying codes in strongly regular graphs

For a \( \text{srg}(n, k, \lambda, \mu) \), we have

\[
d = \min(2(k - 1 - \lambda), 2(k + 1 - \mu)).
\]

Let \( G \) be a primitive strongly regular graph \( \text{srg}(n, k, \lambda, \mu) \), then

\[
k \geq \sqrt{n - 1} \quad \text{and} \quad d \geq \sqrt{n - 3}.
\]

As a consequence:

\[
\gamma^{ID}(G) \leq \frac{n(1 + 2 \ln n)}{\sqrt{n - 3}} = \Theta(\sqrt{n \ln n}).
\]
Another interest of strongly regular graphs

A **resolving set** of a graph is a set of vertices $S$ such that the distances to this set uniquely determine the vertices.
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A **resolving set** of a graph is a set of vertices $S$ such that the
distances to this set uniquely determine the vertices.

Let $G$ be a graph of diameter 2. Let $\text{dim}(G)$ be the size of a
smallest resolving set of $G$.

$$\text{dim}(G) \leq \gamma^{ID}(G) \leq 2\text{dim}(G) + 1$$

Strongly regular graphs have diameter 2.

→ Our constructions give bounds for resolving sets in strongly
regular graphs.

**Example**: For $G$ a $GQ(s, t)$, $c' |V|^{1/4} \leq \text{dim}(G) \leq c_2 |V|^{2/5}$
For any graph $G$, $\gamma_f^{ID}(G) \leq \gamma^{ID}(G) \leq (1 + 2 \ln |V|) \cdot \gamma_f^{ID}(G)$

- New families with $\gamma^{ID}$ and $\gamma_f^{ID}$ of the same order $|V|^\alpha$ with $\alpha \in \{1/3, 1/4, 2/5\}$
- There exists graphs with $\gamma^{ID}$ and $\gamma_f^{ID}$ not of the same order (Paley graphs).
- ... but $\gamma_f^{ID}$ is constant for them!
- Existence of graphs with $\gamma_f^{ID}$ not constant and $\gamma^{ID}$ not of the same order?
- Existence of graphs with order $\gamma^{ID}$ strictly between $\gamma_f^{ID}$ and $\gamma_f^{ID} \cdot \ln |V|$?