Identifying codes in vertex-transitive graphs

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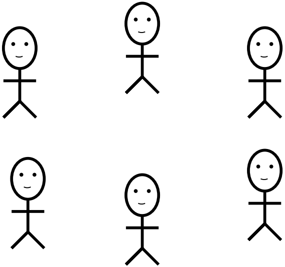
INSTITUT FOURIER
maths à modeler
Université de Liège

Identification in a finite set



- Individuals : 
- Attributes :

How to choose the attributes to efficiently distinguish the individuals?

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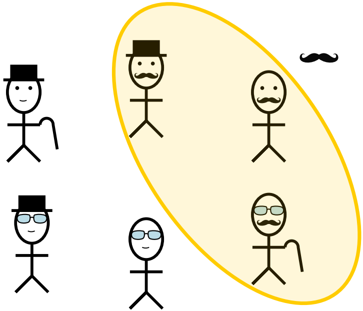

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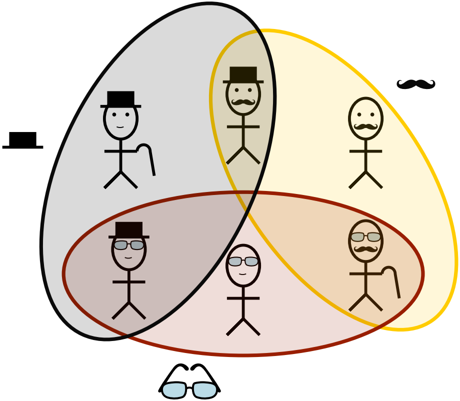

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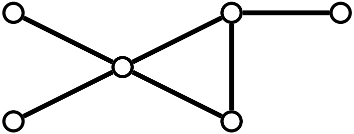
Identification in a graph

- Individuals are vertices of a graph
- Attributes are closed neighbourhoods of the vertices



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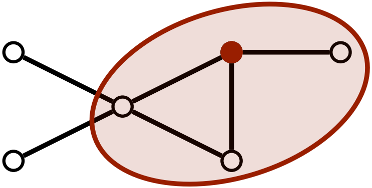

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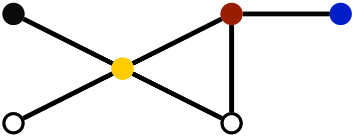

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Formal definition

An identifying code C is a subset of vertices such that

- $\forall u \in V, \quad N[u] \cap C \neq \emptyset$ (domination)
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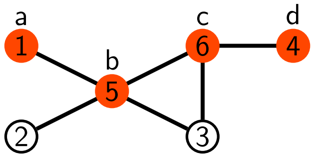

V \ C	a	b	c	d
1	•	•	-	-
2	-	•	-	-
3	-	•	•	-
4	-	-	•	•
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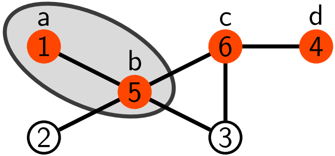

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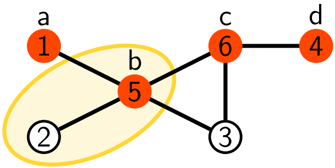

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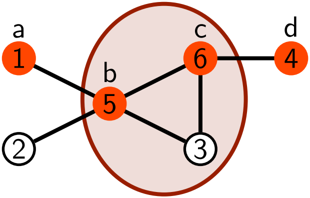

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Linear programming formulation

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- Constraints: domination and separation

$$\begin{array}{llllll}
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This problem is NP-complete... but its fractional relaxation is not !

Fractional relaxation

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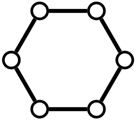
Let $\gamma_f^{ID}(G)$ be the optimal solution of this problem.

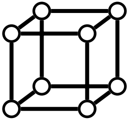
$$\gamma_f^{ID}(G) \leq \gamma^{ID}(G)$$

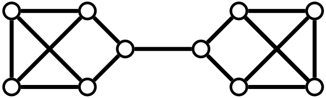
Vertex-transitive graphs

A graph is vertex-transitive if for any two vertices u and v, there is an automorphism sending u to v.

Examples :







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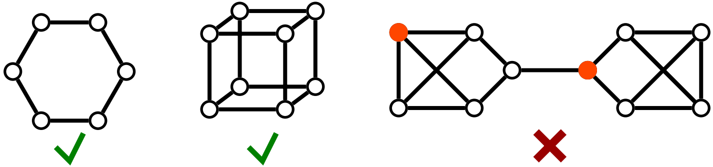

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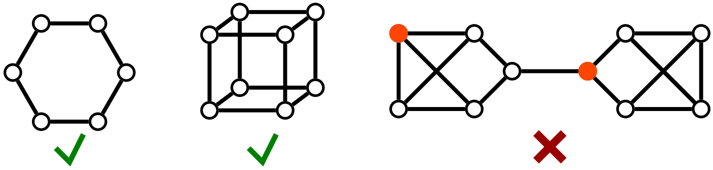

Properties:

- All vertices have the same degree, denoted by k.
- There is an optimal solution to the fractional program with all the variables equal.

Fractional value for VT-graphs

There is an optimal solution with $x_u = \lambda$ for all $u \in V$.

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If G is vertex-transitive, $\gamma_f^{ID}(G) = \lambda \cdot |V| = \frac{|V|}{\min(k+1,d)}$.

Example

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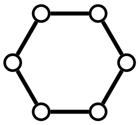
Cycle $\mathcal{C}_n$



$k = 2, d = 2$

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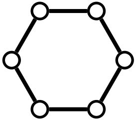

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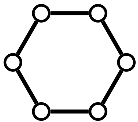

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The upperbound is good if γ_f^{ID} is *small*, i.e. if k and d are *large*.

Generalized quadrangles

A generalized quadrangle $GQ(s, t)$ is an incidence structure of points and lines such that:

- each line contains $s + 1$ points,
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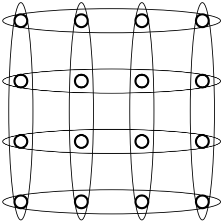
Example:

The square grid $n \times n$ or as a graph, the cartesian product $K_n \Box K_n$, is a $GQ(n - 1, 1)$.



Some facts on GQ

Assume $s > 1, t > 1$

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$$srg((st+1)(s+1), s(t+1), s-1, t+1).$$

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$$\gamma_f^{ID}(G) = \frac{s^2 t}{st+s+1} + 1 = \Theta(s).$$

- We have $s \leq t^2$ and $t \leq s^2$. Therefore

$$c_1 |V|^{1/4} \leq \gamma_f^{ID}(G) \leq c_2 |V|^{2/5}$$

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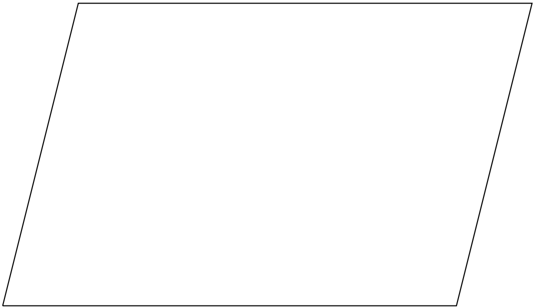
- The only known values of (s,t) for which there is $GQ(s,t)$ are $(q-1, q+1), (q+1, q-1), (q,q), (q,q^2), (q^2,q), (q^2,q^3), (q^3,q^2)$ where q is a prime power.

A construction of $GQ(q-1, q+1)$ for $q = 2^\ell$

Step 1: construction of a hyperconic in the projective plane on $\mathbb{F}_q$.

Projective plane on $\mathbb{F}_q$, (X_1, X_2, X_3)



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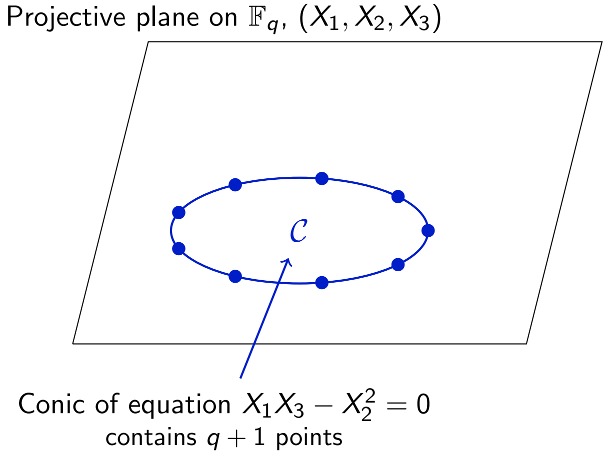

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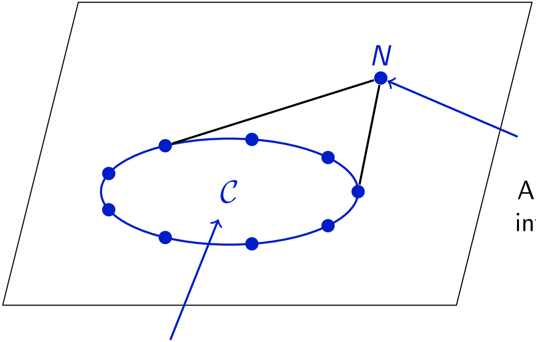

Nucleus
All the lines through N
intersect $\mathcal{C}$ in one point

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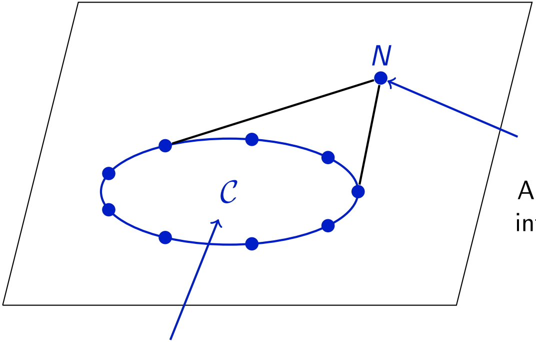

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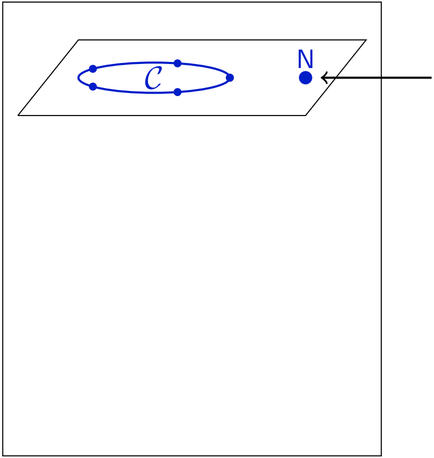
Hyperconic: $\mathcal{O} = \mathcal{C} \cup \{N\}$
$q+2$ points
no 3 collinear points
a line intersects $\mathcal{O}$ in 0 or 2 points

A construction of $GQ(q-1, q+1)$ for $q = 2^\ell$

Step 2: Construction of GQ.

Projective space of dim. 3 on $\mathbb{F}_q$, (X_0, X_1, X_2, X_3)



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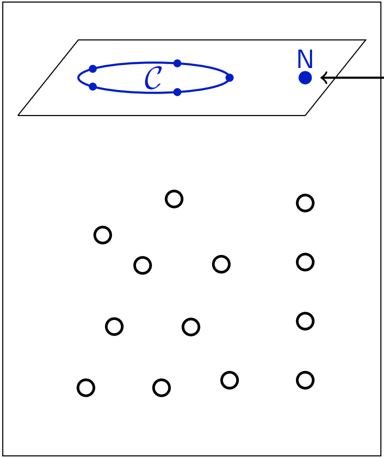

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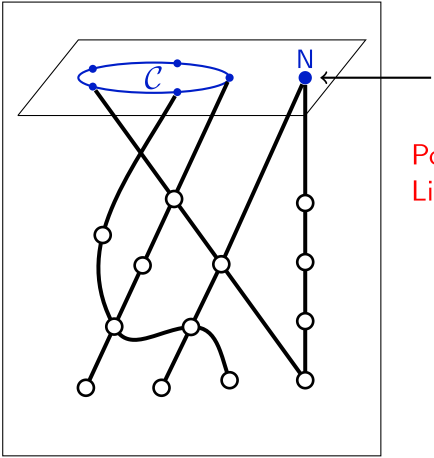

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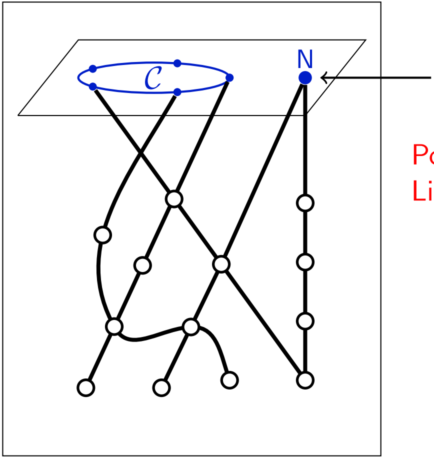

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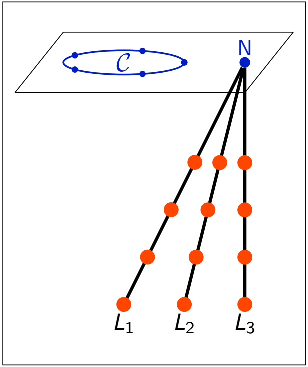
- q^3 points
- q points by line
- $q+2$ lines by point
- unique projection

An identifying code of $GQ(q-1, q+1)$

Three non coplanar lines through N form an identifying code.



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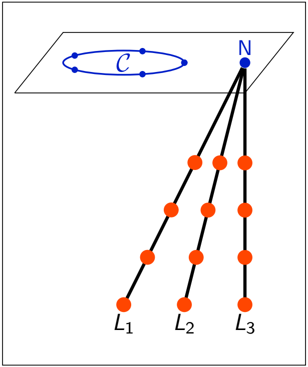

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- Separation:

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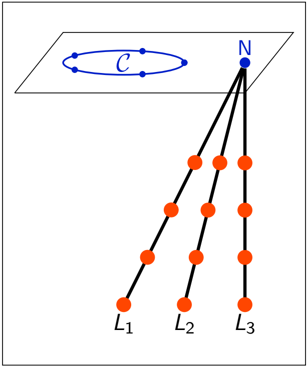

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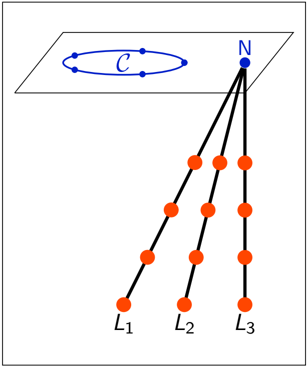

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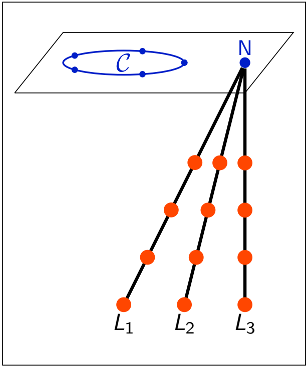

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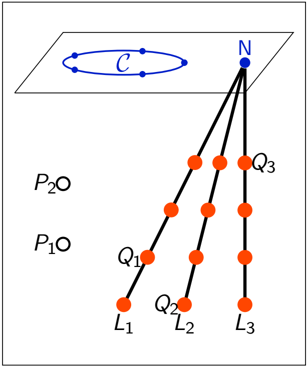

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$$N[P_1] \cap C = N[P_2] \cap C = \{Q_1, Q_2, Q_3\}$$

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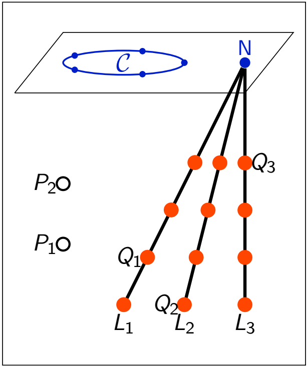

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Assume

$$N[P_1] \cap C = N[P_2] \cap C = \{Q_1, Q_2, Q_3\}$$

$\Rightarrow Q_1, Q_2, Q_3$ in a plane containing N

$\Rightarrow L_1,\ L_2,\ L_3$ are coplanar.

Bounds for $\gamma^{ID}(GQ(q-1, q+1))$

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Hence

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- Using the fractional value

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- With discharging methods (for $q \geq 32$)

$$\gamma^{ID}(GQ(q-1,q+1)) \geq 3q-7$$

Finally,

$$\gamma^{ID}(GQ(q-1,q+1)) \simeq 3q \simeq 3|V|^{1/3}.$$

Other results

Let q be a prime power.

- There exists a $GQ(q,q)$ with identifying code of size
$$5q = \Theta(|V|^{1/3}).$$
- There exists a $GQ(q,q^2)$ with identifying code of size
$$5q+5 = \Theta(|V|^{1/4}).$$
- There exists a $GQ(q^2,q)$ with identifying code of size
$$5q^2+3 = \Theta(|V|^{2/5}).$$

Identifying codes in strongly regular graphs

For a $\mathrm{srg}(n, k, \lambda, \mu)$, we have

$$d = \min(2(k - 1 - \lambda), 2(k + 1 - \mu)).$$

Let G be a primitive strongly regular graph $\mathrm{srg}(n, k, \lambda, \mu)$, then

$$k \geq \sqrt{n-1} \text{ and } d \geq \sqrt{n} - 3.$$

As a consequence:

$$\gamma^{ID}(G) \leq \frac{n(1 + 2\ln n)}{\sqrt{n} - 3} = \Theta(\sqrt{n} \ln n).$$

Another interest of strongly regular graphs

A resolving set of a graph is a set of vertices S such that the distances to this set uniquely determine the vertices.

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A resolving set of a graph is a set of vertices S such that the distances to this set uniquely determine the vertices.

Let G be a graph of diameter 2. Let $dim(G)$ be the size of a smallest resolving set of G.

$$dim(G) \leq \gamma^{ID}(G) \leq 2dim(G) + 1$$

Strongly regular graphs have diameter 2.
$\rightarrow$ Our constructions give bounds for resolving sets in strongly regular graphs.
Example : For G a $GQ(s, t)$, $c'|V|^{1/4} \leq dim(G) \leq c_2|V|^{2/5}$

Conclusion

For any graph G, $\gamma_f^{ID}(G) \leq \gamma^{ID}(G) \leq (1 + 2\ln|V|) \cdot \gamma_f^{ID}(G)$

- New families with γ^{ID} and γ_f^{ID} of the same order $|V|^\alpha$ with $\alpha \in \{1/3, 1/4, 2/5\}$
- There exists graphs with γ^{ID} and γ_f^{ID} not of the same order (Paley graphs)..
- ... but γ_f^{ID} is constant for them !
- Existence of graphs with γ_f^{ID} not constant and γ^{ID} not of the same order ?
- Existence of graphs with order γ^{ID} strictly between γ_f^{ID} and $\gamma_f^{ID} \cdot \ln|V|$?