Calculation of third order joint acceptance function for line-like structures

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1 Introduction

A horizontal line-like structure is exposed to a random stationary 1-direction wind flow with mean velocity $U(\xi)$ and a fluctuating gaussian turbulence component $u(\xi,t)$ with standard deviation $\sigma(\xi)$, see Fig. 1. The wind velocities are assumed to be perfectly correlated in the vertical direction $d(\xi)$ of the line-like structure. Turbulence intensity is defined as $I_u(\xi) = \sigma(\xi)/U(\xi)$.

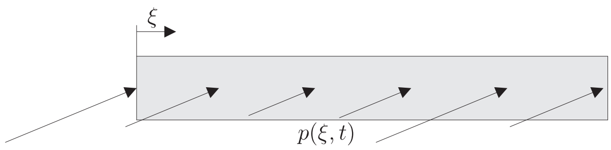

Figure 1. A line-like structure immersed in a wind velocity field. The dimensionless curvilinear abscissa is ξ.

The covariance between two wind velocities at two positions is defined by

$$\kappa_{2u}(\xi_i, \xi_j) \quad = \quad \sigma(\xi_i)\sigma(\xi_j)\rho\left(s_{ij}\right) \tag{1}$$

where the correlation function $\rho\left(s_{ij}\right)$ is assumed to be a function of only the absolute value of the spatial distance $s_{ij} = |\xi_i - \xi_j|$ between the two positions. Vertical line-like structures, such as buildings, do not meet this assumption because the correlation function is also function of the two distinct positions, i.e. $\rho\left(\xi_i, \xi_j, s_{ij}\right)$.

Assuming quasi-steady aerodynamics, neglecting aerodynamic damping and assuming that the velocity of the structural displacement aligned with the wind is low compared to the wind velocity, the total non-gaussian aerodynamic pressure $p(\xi,t)$ on the structure is expressed by

$$p = a + bu + cu^2 \tag{2}$$

where $a(\xi) = \gamma(\xi)U(\xi)^2$; $b(\xi) = 2\gamma(\xi)U(\xi)$; $c(\xi) = \gamma(\xi)$ with $\gamma(\xi) = \frac{1}{2}\rho(\xi)d(\xi)C(\xi)$ where $\rho(\xi)$ is the air density and $C(\xi)$ is the aerodynamic coefficient. The cross cumulants of order 2 $\kappa_{2p}(\xi_i, \xi_j)$ and order 3 $\kappa_{3p}(\xi_i, \xi_j, \xi_k)$ between aerodynamic pressures are respectively given by

$$\kappa_{2p_{ij}} = b_i b_j \rho_{ij}\sigma_i\sigma_j + 2c_i c_j \rho_{ij}^2 \sigma_i^2 \sigma_j^2 \tag{3}$$

and

$$\kappa_{3p_{ijk}} = 2\sigma_i\sigma_j\sigma_k\left(c_i b_j b_k \rho_{ij}\rho_{ik}\sigma_i + b_i c_j b_k \rho_{ij}\rho_{jk}\sigma_j + b_i b_j c_k \rho_{ik}\rho_{jk}\sigma_k\right) + 8c_i c_j c_k \rho_{ij}\rho_{ik}\rho_{jk}\sigma_i^2\sigma_j^2\sigma_k^2. \tag{4}$$

2 Background turbulent response

In its continuous form, the background structural response, $R(t)$, is derived from integration of the aerodynamic pressures field $p(\xi,t)$ multiplied by its response-influence function $I(\xi)$ over the line-like structure

$$R(t) = \int_0^1 I(\xi)p(\xi,t)\mathrm{d}\xi \tag{5}$$

and its second cumulant is obtained as

$$\kappa_{2R} = \iint_0^1 I(\xi_i)I(\xi_j)\kappa_{2p}(|\xi_i-\xi_j|)\mathrm{d}\xi_i\mathrm{d}\xi_j \simeq \iint_0^1 g_1(\xi_i)g_1(\xi_j)\rho\left(|\xi_i-\xi_j|\right)\mathrm{d}\xi_i\mathrm{d}\xi_j \tag{6}$$

where $g_1(\xi) = b(\xi)\sigma_u(\xi)I(\xi)$ and neglecting the second term in Eq. (3), which is marginal. Closed-form expressions of the double integrals in Eq. (6) are attractive in order to avoid numerical integrations and the consideration of numerical admittance (Denoël and Maquoi, 2012). This has been achieved by (Dyrbye and Hansen, 1988) who simplified this double integral thanks to the change of variable $s_{ij} = |\xi_i-\xi_j|$ and interchange of the order of integration, which finally yields

$$\kappa_{2R} = \int_0^1 k(s)\rho\left(s\right)\mathrm{d}s \tag{7}$$

with an influence function defined as

$$k(s) = 2\int_0^{1-s} g_1(\xi)g_1(\xi+s)\mathrm{d}\xi. \tag{8}$$

In the case of integrable expressions for $g_1(\xi)$, analytical formulations for the influence function $k(s)$ are derived and even if Eq. (6) has no analytical solution, Eq. (7) has the advantage to reduce the double integration to a single one which is also easier to treat numerically if it had to. Nonetheless, for specific $\rho(s)$ such as a decaying exponential function

$$\rho(s) = e^{-\phi s} \tag{9}$$

with parameter ϕ, analytical solutions of Eq. (7) can be derived. We must also emphasize that analytical expressions for $g_1(\xi)$ and $\rho(s)$ may not be available and even with analytical expressions, the double integrals may be awkward (or even impossible) to compute analytically. For simple cases, one may consider fitting those functions with simple polynomial functions which ensures integrability.

The third cumulant of the structural response is obtained as

$$\begin{aligned}\kappa_{3R} &= \iiint_0^1 I(\xi_i)I(\xi_j)I(\xi_k)\kappa_{3p}\left(s_{ij},s_{ik}\right)\mathrm{d}\xi_i\mathrm{d}\xi_j\mathrm{d}\xi_k \\ &\simeq 6\iiint_0^1 g_2(\xi_i)g_1(\xi_j)g_1(\xi_k)\rho\left(s_{ij}\right)\rho\left(s_{ik}\right)\mathrm{d}\xi_i\mathrm{d}\xi_j\mathrm{d}\xi_k\end{aligned} \tag{10}$$

where $g_2(\xi) = c(\xi)\sigma_u^2(\xi)I(\xi)$ and neglecting the fourth term in Eq. (4), which is marginal. Following the same strategy as discussed hereinbefore, this paper aims at extending the work of (Dyrbye and Hansen, 1988) to a third order analysis, i.e. simplifying triple integrals of Eq. (10) to double integrals as

$$\begin{aligned}\kappa_{3R} &= 2\int_0^1\int_0^{s_1}\left[(k_1(s_1,s_2)+k_2(s_1,s_2))\rho(s_1)\rho(s_2)\right]\mathrm{d}s_2\mathrm{d}s_1 \\ &\quad +2\int_0^1\int_0^{1-s_1} k_3(s_1,s_2)\rho(s_1)\rho(s_2)\mathrm{d}s_2\mathrm{d}s_1\end{aligned} \tag{11}$$

where we define the third order influence functions as

$$k_1(s_1, s_2) = \int_{s_1}^{1} g_2(\xi) g_1(\xi - s_1) g_1(\xi - s_2) \mathrm{d}\xi \tag{12}$$

$$k_2(s_1, s_2) = \int_{0}^{1-s_1} g_2(\xi) g_1(\xi + s_1) g_1(\xi + s_2) \mathrm{d}\xi \tag{13}$$

$$k_3(s_1, s_2) = \int_{s_2}^{1-s_1} g_2(\xi) g_1(\xi + s_1) g_1(\xi - s_2) \mathrm{d}\xi. \tag{14}$$

One could want to apply the same procedure for the fourth cumulant of the structural response, obtained as

$$\kappa_{4R} \quad = \quad \iiiint_0^1 I(\xi_i) I(\xi_j) I(\xi_k) I(\xi_l) \kappa_{4p_{ijkl}} \mathrm{d}\xi_i \mathrm{d}\xi_j \mathrm{d}\xi_k \mathrm{d}\xi_l \tag{15}$$

where $\kappa_{4p_{ijkl}}$ is the cross cumulants of order 4 between the aerodynamic pressures. However it comes out that this is quite challenging as interchange of the order of integration is tricky for a four-dimensional domain.

Notice that if analytical expressions are derived for the cumulants of order 2, thanks to Eq. (7), order 3, thanks to Eq. (11), and order 4, thanks to Eq. (15), one could obtain analytical expressions for the skewness coefficient defined as $\gamma_{3R} = \kappa_{3R}/\kappa_{2R}^{3/2}$ and for the excess coefficient defined as $\gamma_{eR} = \kappa_{4R}/\kappa_{2R}^2$. These two coefficients are of paramount importance to assess the non-gaussianity of the structural response and the impact on its extreme values through non-gaussian peak factors (Gurley et al., 1997).

3 Illustration

A beam with constant section and length l is considered. The mean velocity U, standard deviation σ_u and coefficient γ are assumed to be constant along the beam. Table 1 collects the influence functions for uniform and linear response influence functions.

	Uniform	Linear
$k(s)/\left(b^2\sigma^2\right)$	$2(1-s)$	$\frac{1}{3}\left(s^3 - 3s + 2\right)$
$k_1(s_1,s_2)/\left(b^2c\sigma^4\right)$	$(1-s_1)$	$\frac{1}{12}\left(s_1-1\right)^2\left(s_1^2 + 2s_1 - 2\left(s_1+2\right)s_2 + 3\right)$
$k_2(s_1,s_2)/\left(b^2c\sigma^4\right)$	$(1-s_1)$	$\frac{1}{12}\left(s_1-1\right)^2\left(4s_2 - s_1\left(s_1 - 2s_2 + 2\right) + 3\right)$
$k_3(s_1,s_2)/\left(b^2c\sigma^4\right)$	$(1-s_1-s_2)$	$-\frac{1}{12}\left(s_1+s_2-1\right)^2\left(s_1\left(s_1+2\right) - s_2\left(s_2+2\right) - 3\right)$

Table 1. Second order and third order influence functions for uniform and linear response influence functions.

In the calculation of the cumulants, the correlation function is considered as a decaying exponential function (Holmes, 2007), see Eq. (9). The parameter $\phi = l/L_u^x$ is the ratio between the length of the structure and L_u^x, the integral length scale for the longitudinal turbulence u in direction $x(=\xi l)$. The integral length scale L_u^x is a measurement of the averaged size of the vortices in the wind. In this case, the cross cumulants of order 4 between aerodynamic pressures is given by

$$\kappa_{4p_{ijkl}} \quad = \quad 4b^2c^2\sigma^6\left(12\rho_{ij}\rho_{ik}\rho_{jl}\right) + 16c^4\sigma^8\left(3\rho_{ij}\rho_{ik}\rho_{jl}\rho_{kl}\right) \tag{16}$$

and neglecting the second term in Eq. (16), which is marginal, leading to

$$\kappa_{4p_{ijkl}} \simeq 48b^2c^2\sigma^6 \iiiint_0^1 I(\xi_i)I(\xi_j)I(\xi_k)I(\xi_l)\rho\left(s_{ij}\right)\rho\left(s_{ik}\right)\rho\left(s_{jl}\right)\mathrm{d}\xi_i\mathrm{d}\xi_j\mathrm{d}\xi_k\mathrm{d}\xi_l. \tag{17}$$

Table 2 collects the analytical results for γ_{3R} and γ_{eR} for uniform and linear response influence functions.

	Uniform	Linear
$\frac{\gamma_{3R}}{3I_u}$	$\frac{e^{-2\phi}\left(2e^{\phi}(\phi+4)+e^{2\phi}(4\phi-7)-1\right)}{2\sqrt{2}\phi^3\left(\frac{\phi+e^{-\phi}-1}{\phi^2}\right)^{3/2}}$	$\frac{3\sqrt{3}e^{-2\phi}\left(-4e^{\phi}(\phi+5)+e^{2\phi}(2\phi(2\phi-7)+19)+1\right)}{4\phi^4\left(\frac{(2\phi-3)\phi^2+6(\phi+1)\sinh(\phi)-6(\phi+1)\cosh(\phi)+6}{\phi^4}\right)^{3/2}}$
$\frac{\gamma_{eR}}{12I_u^2}$	$\frac{e^{-3\phi}\left(e^{\phi}\left(e^{\phi}\left(2\phi(\phi+9)+2e^{\phi}(8\phi-19)+47\right)-2(2\phi+5)\right)+1\right)}{8\left(\phi+e^{-\phi}-1\right)^2}$	too long formula.

Table 2. Skewness and excess coefficients for uniform and linear response influence functions.

Figure 1 depicts the cumulants, γ_{3R} and γ_{eR} for ϕ ranging $[10^{-3}; 10^2]$. Notice for the limit case of quasi-full correlation, *i.e.* $\phi \ll 1; \rho(s) \simeq 1$, skewness and excess coefficients approach the values associated to an aerodynamic pressure (resp. $3I_u$ and $12I_u^2$) while in the limit case of no correlation, *i.e.* $\phi \gg 1; \rho(s) \simeq 0$, their values approach asymptotically the gaussian ones (*i.e.* zeros) explained by the central limit theorem (Papoulis, 1965) and the well-known scale effect.

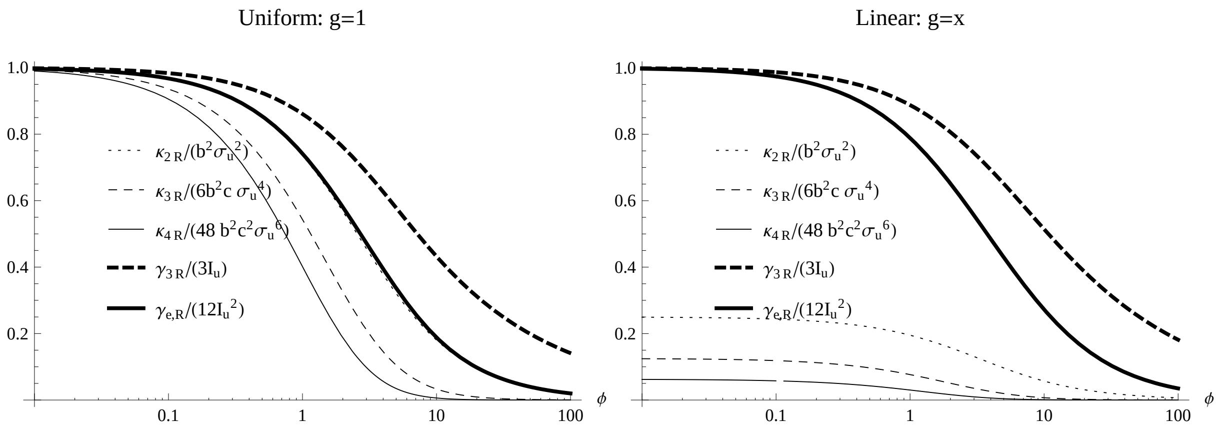

Figure 2. Cumulants and coefficients as function of ϕ for uniform and linear response influence functions.

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