Higher symmetries of the conformal Laplacian

Joint work with J.-P. Michel and J. Silhan

Second order conformal symmetries of $\Delta_Y$
Conformal Killing tensors
Natural and conformally invariant quantization
Structure of the conformal symmetries

Examples
DiPirro system
Conformal Stäckel metrics in dimension 3

Application to the $R$-separation

Varna, June 2014
Introduction

On \((\mathbb{R}^2, g_0)\), we consider the Helmholtz equation

\[ \Delta \phi = E \phi, \]

where

\[ \Delta = \partial_x^2 + \partial_y^2, \quad E \in \mathbb{R}. \]
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Coordinates \((u, v)\) separate this equation \iff \exists solution of the form \(f(u)g(v)\)
Introduction

- On \((\mathbb{R}^2, g_0)\), we consider the Helmholtz equation
  \[ \Delta \phi = E \phi, \]
  where
  \[ \Delta = \partial_x^2 + \partial_y^2, \quad E \in \mathbb{R}. \]

- Coordinates \((u, v)\) separate this equation \(\iff\) \(\exists\) solution of the form \(f(u)g(v)\)

- Coordinates \((u, v)\) orthogonal \(\iff g_0(\partial_u, \partial_v) = 0\)
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Application to the \( R \)-separation

- There exist 4 families of orthogonal separating coordinates systems:
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1. Cartesian coordinates
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1. Cartesian coordinates
2. Polar coordinates \((r, \theta)\):

\[
\begin{align*}
x &= r \cos(\theta) \\
y &= r \sin(\theta)
\end{align*}
\]
There exist 4 families of orthogonal separating coordinates systems:

1. Cartesian coordinates
2. Polar coordinates \((r, \theta)\):
   \[
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   x &= r \cos(\theta) \\
   y &= r \sin(\theta)
   \end{align*}
   \]
3. Parabolic coordinates \((\xi, \eta)\):
   \[
   \begin{align*}
   x &= \xi \eta \\
   y &= \frac{1}{2}(\xi^2 - \eta^2)
   \end{align*}
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   x &= \xi \eta \\
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   \end{align*}
   \]

4. Elliptic coordinates \((\alpha, \beta)\):
   \[
   \begin{align*}
   x &= \sqrt{d} \cos(\alpha) \cosh(\beta) \\
   y &= \sqrt{d} \sin(\alpha) \sinh(\beta)
   \end{align*}
   \]
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Figure: Coordinates lines corresponding to the parabolic coordinates system
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**Figure:** Coordinates lines corresponding to the elliptic coordinates system
Separating coordinates systems allow to simplify the resolution of the Helmholtz equation:
Separating coordinates systems allow to simplify the resolution of the Helmholtz equation:

Example: in cartesian coordinates \((x, y)\), \(f(x)g(y)\) is a solution of \(\Delta \phi = E \phi\) iff

\[
\begin{align*}
\partial_x^2 f - E_1 f &= 0 \\
\partial_y^2 g - (E - E_1) g &= 0
\end{align*}
\]
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- Bijective correspondence
  \[
  \{\text{Separating coordinates systems}\} \leftrightarrow \{\text{Second order symmetries of } \Delta : \text{second order differential operators } D \text{ such that } [\Delta, D] = 0\} 
  \]
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**Bijective correspondence**

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\begin{align*}
\{&\text{Second order symmetries of } \Delta : \text{second order} \\
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\leftrightarrow &
\{\text{Separating coordinates systems}\}
\end{align*}
\]

<table>
<thead>
<tr>
<th>Coordinates system</th>
<th>Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x, y)$</td>
<td>$\partial_x^2$</td>
</tr>
<tr>
<td>$(r, \theta)$</td>
<td>$L_\theta^2$</td>
</tr>
<tr>
<td>$(\xi, \eta)$</td>
<td>$\frac{1}{2}(\partial_x L_\theta + L_\theta \partial_x)$</td>
</tr>
<tr>
<td>$(\alpha, \beta)$</td>
<td>$L_\theta^2 + d \partial_x^2$</td>
</tr>
</tbody>
</table>

with $L_\theta = x \partial_y - y \partial_x$
Link between the symmetry and the coordinates system: if the second-order part of $D$ reads as

$$
\left( \begin{array}{c} \partial_x \\ \partial_y \end{array} \right) A \left( \begin{array}{c} \partial_x \\ \partial_y \end{array} \right),
$$

the eigenvectors of $A$ are tangent to the coordinates lines.
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- Link between the symmetry and the coordinates system: if the second-order part of $D$ reads as

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\partial_x & \partial_y
\end{pmatrix}
A
\begin{pmatrix}
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\partial_y
\end{pmatrix},
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the eigenvectors of $A$ are tangent to the coordinates lines.

- Example: second-order part of $L^2_\theta$:

$$
\begin{pmatrix}
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\end{pmatrix}
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y^2 & -xy \\
-xy & x^2
\end{pmatrix}
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\partial_y
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$$
Link between the symmetry and the coordinates system: if the second-order part of $D$ reads as

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-xy & x^2
\end{pmatrix}
\begin{pmatrix}
\partial_x \\
\partial_y
\end{pmatrix},
\]

eigenvectors of $A$ in this case:

\[
\begin{pmatrix}
x \\
y
\end{pmatrix}, \begin{pmatrix}
-y \\
x
\end{pmatrix}
\]
On a $n$-dimensional pseudo-Riemannian manifold $(M, g)$,

$$\Delta_Y := \nabla_i g^{ij} \nabla_j - \frac{n - 2}{4(n - 1)} Sc,$$

where $Sc$ is the scalar curvature of $g$. 
On a $n$-dimensional pseudo-Riemannian manifold $(M, g)$,

\[
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Symmetry of $\Delta_Y$ : $D \in \mathcal{D}(M)$ such that $[\Delta_Y, D] = 0$
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On a \( n \)-dimensional pseudo-Riemannian manifold \( (M, g) \),

\[
\Delta_Y := \nabla_i g^{ij} \nabla_j - \frac{n-2}{4(n-1)} Sc,
\]

where \( Sc \) is the scalar curvature of \( g \).

Symmetry of \( \Delta_Y : D \in \mathcal{D}(M) \) such that \([\Delta_Y, D] = 0\)

Conformal symmetry of \( \Delta_Y : D_1 \in \mathcal{D}(M) \) such that \( \exists D_2 \in \mathcal{D}(M) \) such that \( \Delta_Y \circ D_1 = D_2 \circ \Delta_Y \)
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- $(M, g)$ conformally flat: for each $x \in M$, there exist a neighborhood $U$ of $x$ and a function $f$ on $U$ such that $e^{2f}g$ is flat on $U$

Conformal symmetries of $\Delta_Y$ known (M. Eastwood, J.-P. Michel)
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- \((M, g)\) conformally flat: for each \( x \in M \), there exist a neighborhood \( U \) of \( x \) and a function \( f \) on \( U \) such that \( e^{2f}g \) is flat on \( U \)

Conformal symmetries of \( \Delta_Y \) known (M. Eastwood, J.-P. Michel)

- \((M, g)\) Einstein: \( \text{Ric} = \frac{1}{n} \text{Sc} \ g \)

Existence of a second order symmetry (B. Carter)
1 Second order conformal symmetries of $\Delta \gamma$
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Application to the $R$-separation

If $D \in \mathcal{D}^k(M)$ reads

$$\sum_{|\alpha| \leq k} D^\alpha \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n},$$
If $D \in \mathcal{D}^k(M)$ reads

$$\sum_{|\alpha| \leq k} D^{\alpha} \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n},$$

then

$$\sigma(D) = \sum_{|\alpha| = k} D^{\alpha} p_1^{\alpha_1} \ldots p_n^{\alpha_n},$$

where $(x^i, p_i)$ are the canonical coordinates on $T^*M$. 

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If $D$ is a conformal symmetry of $\Delta_Y$, there exists an operator $D'$ such that $\Delta_Y \circ D = D' \circ \Delta_Y$. 
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Application to the $R$-separation

- If $D$ is a conformal symmetry of $\Delta_Y$, there exists an operator $D'$ such that $\Delta_Y \circ D = D' \circ \Delta_Y$

- $\sigma(\Delta_Y) = H = g^{ij} p_ip_j$, then $\{H, \sigma(D)\} \in (H)$, i.e. $\sigma(D)$ is a conformal Killing tensor
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If $D$ is a symmetry of $\Delta_Y$, $[\Delta_Y, D] = 0$, then $\{H, \sigma(D)\} = 0$, i.e. $\sigma(D)$ is a Killing tensor
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- If $D$ is a conformal symmetry of $\Delta_\gamma$, there exists an operator $D'$ such that $\Delta_\gamma \circ D = D' \circ \Delta_\gamma$
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- If $D$ is a symmetry of $\Delta_\gamma$, $[\Delta_\gamma, D] = 0$, then $\{H, \sigma(D)\} = 0$, i.e. $\sigma(D)$ is a Killing tensor
- The existence of a (conformal) Killing tensor is necessary to have the existence of a (conformal) symmetry of $\Delta_\gamma$
If $D$ is a conformal symmetry of $\Delta_Y$, there exists an operator $D'$ such that $\Delta_Y \circ D = D' \circ \Delta_Y$

- If $D$ is a symmetry of $\Delta_Y$, $[\Delta_Y, D] = 0$, then $\{H, \sigma(D)\} = 0$, i.e. $\sigma(D)$ is a Killing tensor

The existence of a (conformal) Killing tensor is necessary to have the existence of a (conformal) symmetry of $\Delta_Y$

Is this condition sufficient?
Definition

A quantization on $M$ is a linear bijection $Q^M$ from the space of symbols $\text{Pol}(T^*M)$ to the space of differential operators $\mathcal{D}(M)$ such that

$$\sigma(Q^M(S)) = S, \quad \forall S \in \text{Pol}(T^*M)$$
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Definition

A natural and conformally invariant quantization $Q^M(g)$:

- $Q^M(\Phi^* g)(\Phi^* S) = \Phi^* (Q^N(g)(S))$
- $Q^M(g) = Q^M(\tilde{g})$ whenever $\tilde{g} = e^{2\gamma} g$
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Application to the $R$-separation

- Proof of the existence of $Q^M$:
  1. Work by A. Cap, J. Silhan
  2. Work by P. Mathonet, R.
If $K$ is a (conformal) Killing tensor of degree 2, there exists a (conformal) symmetry of $\Delta_Y$ with $K$ as principal symbol iff $\text{Obs}(K)^b$ is an exact one-form, where

$$
\text{Obs} = \frac{2(n - 2)}{3(n + 1)} p_i \partial_{p_j} \partial_{p_l} \left( C^k_{jl} \nabla_k - 3A_{jl,i} \right)
$$
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  $$\text{Obs} = \frac{2(n-2)}{3(n+1)} p_i \partial_{p_j} \partial_{p_l} \left( C^{k \ j \ l \ i}_j \nabla_k - 3 A_{j \ l \ i} \right)$$

- $C$ : Weyl tensor :

  $$C_{abcd} = R_{abcd} - \frac{2}{n-2} (g_{a[c} R_{i c \ d]b} - g_{b[c} R_{i c \ d]a})$$

  $$+ \frac{2}{(n-1)(n-2)} S_c \ g_{a[c \ g \ d]b}$$
If $K$ is a (conformal) Killing tensor of degree 2, there exists a (conformal) symmetry of $\Delta_Y$ with $K$ as principal symbol iff $\text{Obs}(K)^b$ is an exact one-form, where

$$\text{Obs} = \frac{2(n-2)}{3(n+1)} p_i \partial_{p_j} \partial_{p_l} \left( C^k_{jli} \nabla_k - 3 A_{jl}^i \right)$$

- $C$ : Weyl tensor :

$$C_{abcd} = R_{abcd} - \frac{2}{n-2} (g_a[c \text{Ric}_d]b - g_b[c \text{Ric}_d]a) + \frac{2}{(n-1)(n-2)} S_c g_{a[cg_d]b}$$

- $A$ : Cotton-York tensor :

$$A_{ijk} = \nabla_k \text{Ric}_{ij} - \nabla_j \text{Ric}_{ik} + \frac{1}{2(n-1)} \left( \nabla_j S_c g_{ik} - \nabla_k S_c g_{ij} \right)$$
If $\text{Obs}(K)^b = 2df$, the (conformal) symmetries of $\Delta Y$ whose the principal symbol is given by $K$ are of the form

$$Q(K) - f + LX + c,$$

where $X$ is a (conformal) Killing vector field, where $c \in \mathbb{R}$ and where $Q$ denotes the natural and conformally invariant quantization.
On $\mathbb{R}^3$, diagonal metrics admitting diagonal Killing tensors are classified.
On $\mathbb{R}^3$, diagonal metrics admitting diagonal Killing tensors are classified:

Hamiltonian $H = g^{ij} p_i p_j :$

$$\frac{1}{2(\gamma(x_1, x_2) + c(x_3))} (a(x_1, x_2)p_1^2 + b(x_1, x_2)p_2^2 + p_3^2),$$
On $\mathbb{R}^3$, diagonal metrics admitting diagonal Killing tensors are classified:

Hamiltonian $H = g^{ij} p_i p_j$:

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Killing tensor $K$:

$$\frac{c(x_3)a(x_1, x_2)p_1^2 + c(x_3)b(x_1, x_2)p_2^2 - \gamma(x_1, x_2)p_3^2}{\gamma(x_1, x_2) + c(x_3)},$$

$a, b, \gamma \in C^\infty(\mathbb{R}^2), c \in C^\infty(\mathbb{R})$. 

On $\mathbb{R}^3$, diagonal metrics admitting diagonal Killing tensors are classified:

Hamiltonian $H = g^{ij} p_i p_j$:

$$\frac{1}{2(\gamma(x_1, x_2) + c(x_3))} \left( a(x_1, x_2) p_1^2 + b(x_1, x_2) p_2^2 + p_3^2 \right),$$

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$$\frac{c(x_3) a(x_1, x_2) p_1^2 + c(x_3) b(x_1, x_2) p_2^2 - \gamma(x_1, x_2) p_3^2}{\gamma(x_1, x_2) + c(x_3)},$$

$a, b, \gamma \in C^\infty(\mathbb{R}^2)$, $c \in C^\infty(\mathbb{R})$.

In this situation, $\text{Obs}(K)^\flat$ exact $\Rightarrow$ existence of symmetries.
Conformal Stäckel metric $g : g$ s.t. the Hamilton-Jacobi equation

$$g^{ij}(\partial_i W)(\partial_j W) = 0$$

admits additive separation in an orthogonal coordinate system.
- Conformal Stäckel metric $g : g$ s.t. the Hamilton-Jacobi equation
  
  $$g^{ij}(\partial_i W)(\partial_j W) = 0$$

  admits additive separation in an orthogonal coordinate system.

- Coordinate $x$ ignorable for $g : \partial_x$ is a conformal Killing vector field for $g$. 

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- Conformal Stäckel metric \( g : g \) s.t. the Hamilton-Jacobi equation
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g^{ij}(\partial_i W)(\partial_j W) = 0
  \]
  admits additive separation in an orthogonal coordinate system.

- Coordinate \( x \) ignorable for \( g : \partial_x \) is a conformal Killing vector field for \( g \).

- If \( g \) admits one ignorable coordinate \( x_1 \), then
  \[
g = Q \left( dx_1^2 + (u(x_2) + v(x_3))(dx_2^2 + dx_3^2) \right).
  \]
$\partial_{x_1}$ is a conformal Killing vector field and
\[ K = (u(x_2) + v(x_3))^{-1}(v(x_3)p_2^2 - u(x_2)p_3^2) \]
a conformal Killing 2-tensor.
\[ \partial_{x_1} \text{ is a conformal Killing vector field and} \]

\[ K = (u(x_2) + v(x_3))^{-1}(v(x_3)p_2^2 - u(x_2)p_3^2) \]

a conformal Killing 2-tensor.

\[ \text{In general, } \text{Obs}(K)^b \text{ not closed} \Rightarrow \text{no conformal symmetries with principal symbol } K. \]
Example of such a metric:

\[
\begin{align*}
\text{Minkowski metric } g &\text{ in Rindler helical coordinates } (t, r, \phi, z) : \\
g &= dr^2 + r^2 d\phi^2 + dz^2 + 2ar^2 d\phi dt + (a r - z)(a r + z) dt^2 \\
\text{Killing vector field: } &\partial_t \\
\text{Symplectic reduction: } &\left\{ f \in C^\infty(T^*\mathbb{R}^3) : \left\{ f, p_t \right\} = 0 \right\}/\langle p_t \rangle \sim C^\infty(T^*\mathbb{R}^3) \\
\text{Reduction of } g &\text{: } h = \ldots \\
\text{Reduction of the Killing tensor } K &\text{: } K' = \ldots 
\end{align*}
\]
- Example of such a metric:
- Minkowski metric $g$ in Rindler helical coordinates $(t, r, \phi, z)$:
  $$g = dr^2 + r^2 d\phi^2 + dz^2 + 2ar^2 d\phi dt + (ar - z)(ar + z)dt^2$$
Example of such a metric:

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Killing vector field: $\partial_t \in \mathfrak{p}$
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Symplectic reduction:

$$\{f \in C^\infty(T^*\mathbb{R}^{1,3}) : \{f, p_t\} = 0\} / \langle p_t \rangle \cong C^\infty(T^*\mathbb{R}^3)$$
Example of such a metric:

- Minkowski metric $g$ in Rindler helical coordinates $(t, r, \phi, z)$:
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- Killing vector field: $\partial_t \in \mathfrak{p}$

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- Reduction of $g$: $h = dr^2 + \frac{r^2z^2}{z^2 - a^2}r^2 d\phi^2 + dz^2$
Example of such a metric:

Minkowski metric $g$ in Rindler helical coordinates $(t, r, \phi, z)$:
$$g = dr^2 + r^2d\phi^2 + dz^2 + 2ar^2d\phi dt + (ar - z)(ar + z)dt^2$$

Killing vector field: $\partial_t \in p$

Symplectic reduction:
$$\{f \in C^\infty(T^*\mathbb{R}^{1,3}) : \{f, p_t\} = 0\}/\langle p_t \rangle \cong C^\infty(T^*\mathbb{R}^3)$$

Reduction of $g$:
$$h = dr^2 + \frac{r^2z^2}{z^2 - a^2r^2}d\phi^2 + dz^2$$

Reduction of the Killing tensor $K = p_r^2 + \frac{1}{r^2}p_\phi^2$:
$$K' = p_r^2 + \frac{1}{r^2}p_\phi^2$$
Schrödinger equation: \((\Delta_Y + V)\psi = E\psi, \ V \in C^\infty(M)\)

is a fixed potential and \(E \in \mathbb{R}\) a free parameter
Schrödinger equation: \((\Delta_Y + V)\psi = E\psi, \ V \in C^\infty(M)\)
is a fixed potential and \(E \in \mathbb{R}\) a free parameter

Schrödinger equation at zero energy: \((\Delta_Y + V)\psi = 0, \ V \in C^\infty(M)\) is a fixed potential
Schrödinger equation $R$-separable in an orthogonal coordinates system $(x^i)$ ($g_{ij} = 0$ if $i \neq j$)

$$\iff$$

$\forall \ E \in \mathbb{R}, \ \exists \ n + 1 \ functions \ R, h_i \in C^\infty(M) \ and \ n \ differential \ operators \ L_i := \partial_i^2 + l_i(x^i)\partial_i + m_i(x^i)$ such that

$$R^{-1}(\Delta_Y + V)R - E = \sum_{i=1}^{n} h_i L_i.$$
Schrödinger equation at zero energy $R$-separable in an orthogonal coordinates system $(x^i)$ ($g_{ij} = 0$ if $i \neq j$)

\[ \exists \ n + 1 \text{ functions } R, h_i \in C^\infty(M) \text{ and } n \text{ differential operators } L_i := \partial_i^2 + l_i(x^i)\partial_i + m_i(x^i) \text{ such that } \]

\[ R^{-1} (\Delta_Y + V) R = \sum_{i=1}^{n} h_i L_i. \]
Schrödinger equation at zero energy $R$-separable in an orthogonal coordinates system $(x^i)$ ($g_{ij} = 0$ if $i \neq j$)

\[ \iff \]

\[ \exists \ n + 1 \text{ functions } R, h_i \in C^\infty(M) \text{ and } n \text{ differential operators } L_i := \partial_i^2 + l_i(x^i)\partial_i + m_i(x^i) \text{ such that} \]

\[ R^{-1}(\Delta Y + V)R = \sum_{i=1}^{n} h_i L_i. \]

\[ R \prod_{i=1}^{n} \phi_i(x^i) \text{ solution of one of the two previous equations} \]

\[ \iff \]

\[ L_i \phi_i = 0 \ \forall i \]
Higher symmetries of the conformal Laplacian

Joint work with J.-P. Michel and J. Silhan

Second order conformal symmetries of \( \Delta \gamma \)

- Conformal Killing tensors
- Natural and conformally invariant quantization
- Structure of the conformal symmetries

Examples
- DiPirro system
- Conformal Stäckel metrics in dimension 3

Application to the \( R \)-separation

- Schrödinger equation (resp. at zero energy) \( R \)-separates in an orthogonal coordinate system if and only if:

\[ \exists a \text{-dimensional linear space of (resp. conformal) Killing 2-tensors } I \text{ such that } \{ K_1, K_2 \} = 0 (resp. } \in (H_2)) \text{ for all } K_1, K_2 \in I \text{, as endomorphisms of } TM \text{, the tensors in } I \text{ admit a basis of common eigenvectors.} \]

\[ (b) \text{ For all } K \in I, \exists \text{ second order (resp. conformal) symmetry } D, \text{ i.e. an operator such that } [\Delta \gamma + V, D] = 0 (resp. } \in (\Delta \gamma + V)), \text{ with principal symbol } \sigma_2(D) = K. \]
Higher symmetries of the conformal Laplacian
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Application to the $R$-separation

- Schrödinger equation (resp. at zero energy) $R$-separates in an orthogonal coordinate system if and only if:
  (a) $\exists$ a $n$-dimensional linear space of (resp. conformal) Killing 2-tensors $\mathcal{I}$ such that
  \[
  \{K_1, K_2\} = 0 \text{ (resp. } \in (H)) \text{ for all } K_1, K_2 \in \mathcal{I},
  \]
Schrödinger equation (resp. at zero energy) $R$-separates in an orthogonal coordinate system if and only if:

(a) $\exists$ a $n$-dimensional linear space of (resp. conformal) Killing 2-tensors $\mathcal{I}$ such that

- $\{K_1, K_2\} = 0$ (resp. $\in (H)$) for all $K_1, K_2 \in \mathcal{I}$,
- as endomorphisms of $TM$, the tensors in $\mathcal{I}$ admit a basis of common eigenvectors.
Schrödinger equation (resp. at zero energy) $R$-separates in an orthogonal coordinate system if and only if:

(a) $\exists$ a $n$-dimensional linear space of (resp. conformal) Killing 2-tensors $\mathcal{I}$ such that

- $\{K_1, K_2\} = 0$ (resp. $\in (H)$) for all $K_1, K_2 \in \mathcal{I}$,
- as endomorphisms of $TM$, the tensors in $\mathcal{I}$ admit a basis of common eigenvectors.

(b) For all $K \in \mathcal{I}$, $\exists$ second order (resp. conformal) symmetry $D$, i.e. an operator such that $[\Delta_Y + V, D] = 0$ (resp. $\in (\Delta_Y + V)$), with principal symbol $\sigma_2(D) = K$. 

- Link between the (conformal) symmetries and the R-separating coordinate systems:
Higher symmetries of the conformal Laplacian

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Application to the \( R \)-separation

- Link between the (conformal) symmetries and the \( R \)-separating coordinate systems:

- Hyperplans orthogonal to the eigenvectors of the tensors in \( \mathcal{I} \leftrightarrow \) integrable distributions
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Application to the $R$-separation

- Link between the (conformal) symmetries and the $R$-separating coordinate systems:
- Hyperplans orthogonal to the eigenvectors of the tensors in $\mathcal{I} \leftrightarrow$ integrable distributions
- Leaves of the corresponding foliations $\leftrightarrow$ Coordinate hyperplans of the $R$-separating coordinate systems