International Congress « Engineering Mechanics Today 2004 »
Ho Chi Minh City, Viet Nam, August 16-20, 2004

THE EQUILIBRIUM FINITE ELEMENT MODEL AND ERROR ESTIMATION FOR PLATE BENDING

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Abstract: *This paper presents a procedure for computing approximate solution of bending Kirchhoff plate by equilibrium finite element model. The accuracy of the equilibrium approach is based on the concept of error estimator. Especially, a dual analysis by finite elements which leads to lower and upper bounds of the exact solution is presented in numerical examples.*

Key words: Plate Bending, Equilibrium Element, Dual Analysis, Error Estimation.

1. Introduction

Most of the today's engineering analysis deals with problems involving differential equations which are too difficult to be solved analytically. Currently the most widely used method of solving these problems, especially those with irregular geometries and complex boundary conditions, is the Finite Element Method (FEM) which over time has become an indispensable tool for today's engineer.

In linear elasticity, the FEM is divided into two basic models. First, the pure displacement models are based on conforming displacement elements where the compatibility equations of strains and displacements are verified exactly, relaxing on local equilibrium. Secondly, pure equilibrium models are based on stress element where the equilibrium equations are a priori satisfied and compatibility of strain is derived in weak form.

If displacement elements naturally suggest a stiffness matrix process, the same is not true for equilibrium elements, for which a force method, with structural self-stresses determination, seemed more natural. But the force method is very difficult to automate. Here, Fraeijs de Veubeke showed initially that it is possible to use equilibrium elements in a stiffness matrix formulation [1]so that the algorithm is still the displacement method.

In the present paper, we would like to establish an equilibrium element of plate bending and then show that this element may be identified with a (nonconforming) displacement element. Finally, an estimation of the error by *dual analysis* permits to obtain a bound for the global error of the solution for the equilibrium model.

2. The equilibrium Morley element

First, we introduce a triangular equilibrium plate element with the constant moment field. A complementary equilibrium approach to finite element modeling was proposed by Fraeijs de Veubeke [1]. The equilibrium approach is used inside the element, in order to find the stiffness matrix. The solution algorithm is then the displacement method. For this purpose the total complementary energy will be used in order to determine the element stiffness matrix [7]. For triangle e, one has

$$\Psi^e = \frac{1}{2}\int_e M^T H^{-1} M d\Omega - \left[\int_{\partial e} (K_n w - M_{nn} w,_n) ds + \sum_{i=1}^{3} Z_i w_i \right] \tag{1}$$

where $K_n, M_{nn}, Z_i, w,_n$ are respectively Krichhoff shear loads, normal moments, corner loads and normal derivatives.

Classically, the moment distribution is assumed to be constant within an element. The moments are thus written as

$$M = N\beta \tag{2}$$

where $N = [\delta_{ij}], i, j = 1,2,3$ is a constant matrix, $\beta = [\beta_1 \quad \beta_2 \quad \beta_3]^T$ are unknowns.

Next, consider the triangular element (Fig.3). These load corners are in the form [7]

$$\begin{aligned}
\overline{Z}_1 &= (C_3S_3 - C_1S_1)M_{xx} + (C_1S_1 - C_3S_3)M_{yy} + (C_1^2 - S_1^2 - C_3^2 + S_3^2)M_{xy} \\
\overline{Z}_2 &= (C_1S_1 - C_2S_2)M_{xx} + (C_2S_2 - C_1S_1)M_{yy} + (C_2^2 - S_2^2 - C_1^2 + S_1^2)M_{xy} \\
\overline{Z}_3 &= (C_2S_2 - C_3S_3)M_{xx} + (C_3S_3 - C_2S_2)M_{yy} + (C_3^2 - S_3^2 - C_2^2 + S_2^2)M_{xy}
\end{aligned} \tag{3}$$

Total normal moment on *ij* side is calculated by

$$\int_{side} M_n ds = \int_0^l M_n ds = (C^2 M_{xx} + S^2 M_{yy} + 2CSM_{xy})l \tag{4}$$

where $C_i = (y_j - y_i)/l_{ij}$, $S_i = (x_i - x_j)/l_{ij}$, ij =12, 23, 31 and l_{ij} is the length of side ij.

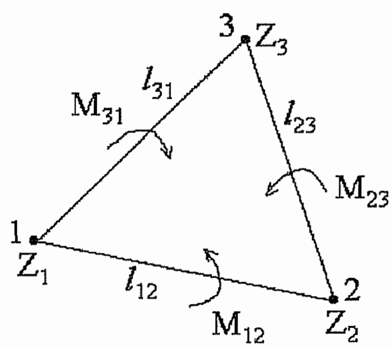

Fig.1: The generalized loads

However, with this moment fields, it is not possible to obtain exact equilibrium when applying a constant pressure. For this purpose, a special mode has to be complemented. It is constructed as follows. Let side 1-2 be the X axis and Y axis be perpendicular to it, passes through node 1 and orientate in such a manner that Y_3 is positive (Fig. 2). Let $c_1(X,Y)$, $c_2(X,Y)$ and $c_3(X,Y)$ be the *three areal coordinates* [3] and $c_i = 0$ be the equation of the side which i opposite to node i.

It is a second degree field as follows

$$M_{comp} = T\gamma \tag{5}$$

where M_{comp} is in equilibrium with a constant pressure p and amplitude γ refers to pressure p.

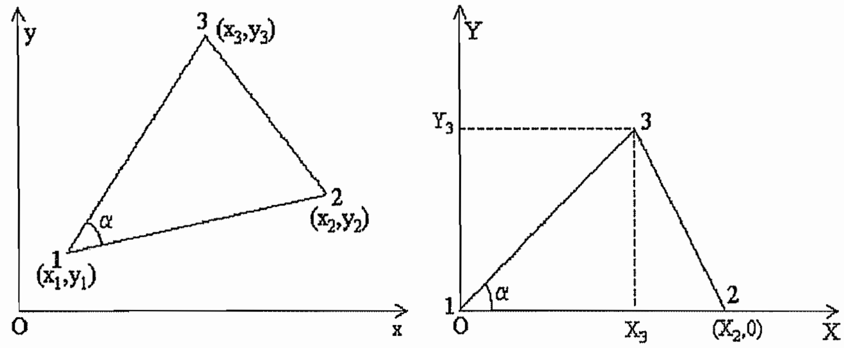

Fig.2: The between global system (Oxy)
and local system (OXY) generalized loads

For this equilibrium, The following special mode T has been developed by one of the authors,

$$
\begin{aligned}
T_x &= -\frac{1}{3}\left\{-\frac{X_3}{Y_3}c_1 + \frac{X_3 - X_2}{Y_3}c_2 - \frac{X_3}{Y_3}\frac{(X_3 - X_2)}{X_2}c_3 + \frac{1}{2S}(X^2 - X_2^2 c_2 - X_3^2 c_3)\right\} \\
T_y &= -\frac{1}{3}\left\{-\frac{Y_3}{X_2}c_3 + \frac{1}{2S}(Y^2 - Y_3^2 c_3)\right\} \\
T_{xy} &= -\frac{1}{3}\left\{-\tfrac{1}{2}c_1 + \tfrac{1}{2}c_2 - \frac{2X_3 - X_2}{2X_2}c_3 + \frac{1}{2S}(XY - X_3 Y_3 c_3)\right\}
\end{aligned} \tag{6}
$$

Let the three corner loads of triangular element be added by

$$Z_1 = Z_2 = Z_3 = -\frac{P}{3} = -\frac{1}{3}\gamma \tag{7}$$

and one implements a new load $g_e = \gamma$ where $P = pS$, S being the area of the triangle.
Here, these 7 loads may be assembled in a load vector $\bar{g}^T = [g^T \ \ g_e]$ is in the form

$$
\begin{bmatrix} g \\ g_e \end{bmatrix} =
\begin{bmatrix} \bar{Z}_1 \\ \bar{Z}_2 \\ \bar{Z}_3 \\ -\int_{12} M_n ds \\ -\int_{23} M_n ds \\ -\int_{31} M_n ds \\ P \end{bmatrix} =
\begin{bmatrix}
C_3S_3 - C_1S_1 & C_1S_1 - C_3S_3 & C_1^2 - S_1^2 - C_3^2 + S_3^2 & -1/3 \\
C_1S_1 - C_2S_2 & C_2S_2 - C_1S_1 & C_2^2 - S_2^2 - C_1^2 + S_1^2 & -1/3 \\
C_2S_2 - C_3S_3 & C_3S_3 - C_2S_2 & C_3^2 - S_3^2 - C_2^2 + S_2^2 & -1/3 \\
-C_1^2 l_{12} & -S_1^2 l_{12} & -2C_1 S_1 l_{12} & 0 \\
-C_2^2 l_{23} & -S_2^2 l_{23} & -2C_2 S_2 l_{23} & 0 \\
-C_3^2 l_{31} & -S_3^2 l_{31} & -2C_3 S_3 l_{31} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \gamma \end{bmatrix} \tag{8}
$$

or globally,

$$\bar{g} = Ca \tag{9}$$

where

$$C = \begin{bmatrix} C_\beta & C_\gamma \\ 0 & 1 \end{bmatrix},\ a^T = (\beta^T\ \gamma)$$

$$C_\gamma^T = [-1/3 \quad -1/3 \quad -1/3 \quad 0 \quad 0 \quad 0]$$

The work of reactions against the prescribed conjugate displacement and displacement conjugate to the distributed load will be

$$W = q^T g + q_e g_e = \bar{q}^T \bar{g} \tag{10}$$

where $\bar{q}^T = [q^T \quad q_e]$ and q, q_e are the conjugate boundary displacements and the displacement conjugate to the distributed load, respectively.

The complementary energy is then

$$V = \frac{1}{2}\int_S M^T H^{-1} M\, dS = \frac{1}{2}\beta^T F_{\beta\beta}\beta + \beta^T F_{\beta\gamma}\gamma + \frac{1}{2}\gamma F_{\gamma\gamma}\gamma \tag{11}$$

with the following flexibility matrices

$$F_{\beta\beta} = \int_S N^T H^{-1} N dS = SH^{-1},\ F_{\beta\gamma} = \int_S N^T H^{-1} T dS,\ F_{\gamma\gamma} = \int_S T^T H^{-1} T dS$$

where H^{-1} is the inverse plate Hooke matrix.

The total complementary energy

$$\Psi^e = \frac{1}{2}\begin{bmatrix}\beta^T & \gamma\end{bmatrix}\begin{bmatrix} F_{\beta\beta} & F_{\beta\gamma} \\ F_{\gamma\beta} & F_{\gamma\gamma}\end{bmatrix}\begin{bmatrix}\beta \\ \gamma\end{bmatrix} - \bar{q}^T g = \frac{1}{2}a^T F a - \bar{q}^T C a \quad \text{minimum} \tag{12}$$

Varying (12) by respect of β and γ leads to

$$\begin{aligned} F_{\beta\beta}\beta + F_{\beta\gamma}\gamma &= C_\beta^T q \\ F_{\beta\gamma}\beta + F_{\gamma\gamma}\gamma &= C_\gamma^T q + q_e \end{aligned} \tag{13}$$

or, in global form,

$$Fa = C^T \bar{q} \tag{14}$$

This implies

$$a = F^{-1} C^T \bar{q} \tag{15}$$

From (9) and (15), one obtains

$$\bar{g} = CF^{-1}C^T\bar{q} = \bar{K}^e \bar{q} \tag{16}$$

and $\bar{K}^e = CF^{-1}C^T$ is the stiffness matrix. The system (16) can be rewritten

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22}\end{bmatrix}\begin{bmatrix} q \\ q_e \end{bmatrix} = \begin{bmatrix} g \\ g_e \end{bmatrix}$$

By eliminating the last element of $\bar{q}$, that is q_e, we obtain

$$g^* = (g - K_{12}K_{22}^{-1}g_e) = (K_{11} - K_{12}K_{22}^{-1}K_{21})q = K^e q \tag{17}$$

Matrix $K^e = (K_{11} - K_{12}K_{22}^{-1}K_{21})$ is the (6x6) stiffness matrix of element and q may be interpreted as mean displacement. In fact, from virtual work,

$$\delta\Psi^e = \delta a^T J a = \delta a^T C^T q = q^T \delta(Ca) = q^T \delta g \tag{18}$$

As the work of reactions is

$$q^T \delta \bar{g} = q_1 \delta \bar{Z}_1 + q_2 \delta \bar{Z}_2 + q_3 \delta \bar{Z}_3 + q_4 .\delta \int_{l_{12}} M_n ds + q_5 .\delta \int_{l_{23}} M_n ds + q_6 .\delta \int_{l_{31}} M_n ds \qquad (19)$$

It implies that

q_1, q_2, q_3 are the corresponding displacements at node 1, 2, 3.

q_4, q_5, q_6 are the corresponding mean of $w_{,n}$ on side 12, 23, 31.

The connectors are thus the same as in Morley's element [3]. Now, when there is no pressure load, moments are constants. Any constant moment field is compatible, that is, there exist a displacement field of second degree, whose curvatures lead to this moment field. It is clear that the strain energy of the displacement field is equal to the above complementary energy. So that Morley's element is nothing other than a disguised form of the constant moment element. But a difference arises when a constant pressure is applied.

3. Numerical results

Problem 1

Consider a square plate clamped at all edges, length of edge L =10, thickness t = 0.1, Young module E=2.05x10^{11}, Poisson ratio ν = 0.3, uniform load $p = -1000$. Mesh M x M element over one quater were used with M =2, 4, 8,16,24 and 32.

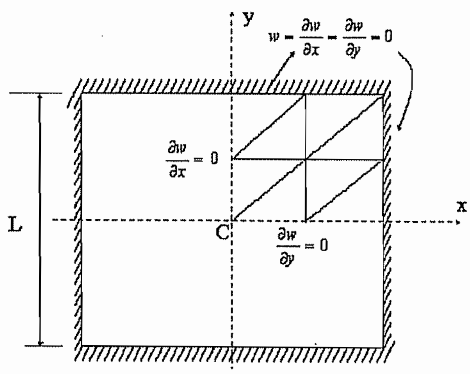

Fig. 3

The result of deflection at center and extenal work is determined by [2, 6]

$$w_C = \alpha \frac{pL^4}{D}, \; W = \int_\Omega pwd\Omega = \omega \frac{p^2 L^6}{D}$$

where Ω is area of plate, $D = Et^3 / 12(1-\nu^2)$ - the bending stiffness.

Exact solution (Hencky-Wojtaszak):

$$\alpha_{ex} = 1.26532 . 10^{-3}, \; \omega_{ex} = 3.89120 . 10^{-4}$$

The context of error estimate in energy norm

$$\|e_h\|_a^2 = \left| \|U_{ex}\|_a^2 - \|U_h\|_a^2 \right|$$

with the exact strain energy is $\frac{1}{2}\|U_{ex}\|_a^2$=10.36388, $\|U_h\|_a^2$ is the square energy norm of FEM solution.

The percentage of relative error in the deflection at center and strain energy is

$$\alpha_{rel} = 100x \frac{\alpha_{FEM} - \alpha_{ex}}{\alpha_{ex}}, \; \omega_{rel} = 100x \frac{\omega_{FEM} - \omega_{ex}}{\omega_{ex}}$$

Tables 1 to 2 assemble all the results of error estimation for a sequence of meshes uniformly refined from the initial ones (Fig. 3).

Table 1: The FEM solution and error energy for equilibrium model

Mesh	N.EL	D.O.F	$\frac{1}{2}\left\|U_h\right\|_a^2$	$\frac{1}{2}\left\|e_h\right\|_a^2$
2x2	8	12	25.534581643874	15.170702619483
4x4	32	56	14.997471297109	4.633592272718
8x8	128	240	11.472889637504	1.109010613113
16x16	512	992	10.558121861220	0.194242836829
24x24	1152	2256	10.408685734228	0.044806709837
32x32	2048	4032	10.365206787949	0.001327763558

Table 2: The percentage of relative error in the deflection at center and the energy

Mesh	N.EL	$\alpha_{FEM}\ 10^{-3}$	$\omega_{FEM}\ 10^{-4}$	$\alpha_{rel}\ (\%)$	$\omega_{rel}\ (\%)$
2x2	8	2.75306	10.06776	117.57809	158.73141
4x4	32	1.64496	5.66096	30.00375	45.48098
8x8	128	1.35301	4.30946	6.93038	10.74897
16x16	512	1.28114	3.96425	1.25048	1.87725
24x24	1152	1.26942	3.90805	0.32372	0.43294
32x32	2048	1.26595	3.89171	0.04956	0.01303

The following graph shows the energy convergence

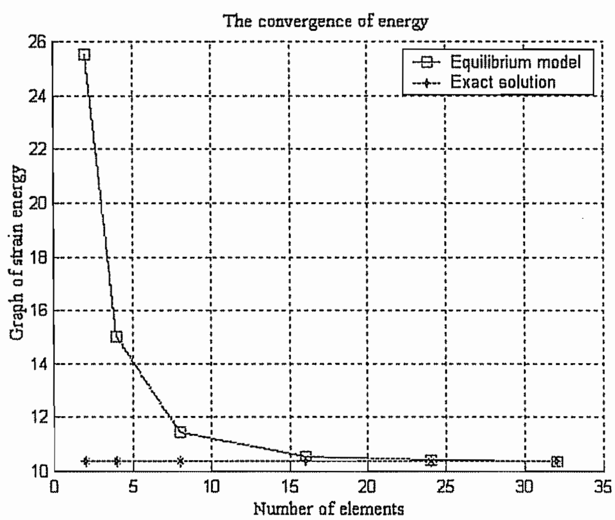

Fig.4

❖ Comments

• The equilibrium model leads to an *upper bound* of the exact solution. The FEM energy converges to the exact energy when mesh is refined.

• If the exact solution of a problem is known, then all the reliability indices defined above can be easily evaluated. However, most of the problems encountered in practice do not have analytical solutions. It is then more important to evaluate the reliability of the error estimators for problems whose exact solution is not available. In this case the uniformity index can not be evaluated, but the global effect index can still be precisely calculated if the strain energy of the structure can be precisely obtained. When the convergence of the energy norm of the error is monotonic and asymptotic, then there exists an asymptotic relation between the global energy of the error and the total number of degrees of freedom (*DOF*)

$$\|e_h\|_a^2 = C\left(\frac{1}{DOF}\right)^{2r_c}$$

where C is a constant independent of the mesh size, r_c is called the asymptotic convergence rate of the global energy norm of the error. The exact energy norm $\|U_{ex}\|_a$ of the structure can be estimated by a procedure called ***Richardson's extrapolation*** [5]. Three analyses are needed to determine the two constants C, r_c and the energy norm. Denote such an estimate by $\|U_R\|_a$. Then if the boundary displacement conditions of a structure are homogeneous and consistent loads are applied, the Pythagoras' theorem of the discretization error can be applied

$$\|e_h\|_a^2 = \|U_{ex}\|_a^2 - \|U_h\|_a^2$$

so that the exact error can be approximated by

$$\|e_h\|_a^2 \approx \|U_R\|_a^2 - \|U_h\|_a^2$$

In order that the approximate global effect indices be sufficiently precise, a sequence of fine meshes should be used in the procedure of Richardson's extrapolation.

The exact and estimated relative error (Fig. 4) is given by

$$\eta_{ex} = \sqrt{\frac{\|U_h\|^2 - \|U_{ex}\|^2}{\|U_{ex}\|^2}}, \eta_R = \sqrt{\frac{\|U_h\|^2 - \|U_R\|^2}{\|U_R\|^2}}$$

The convergence of the FEM and extrapolation strain energy

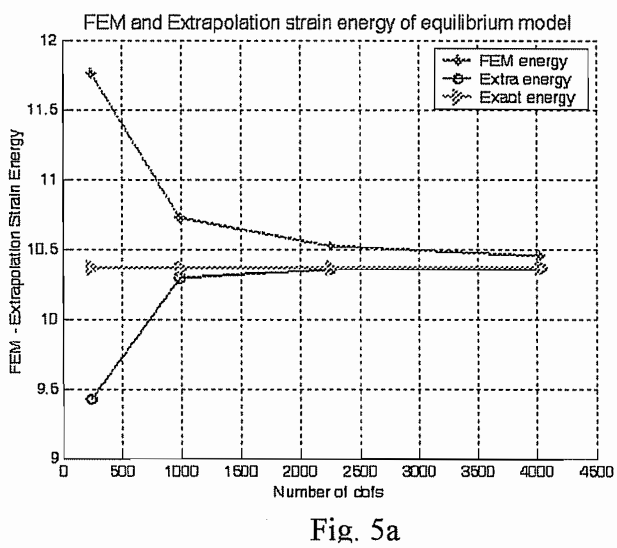

Fig. 5a

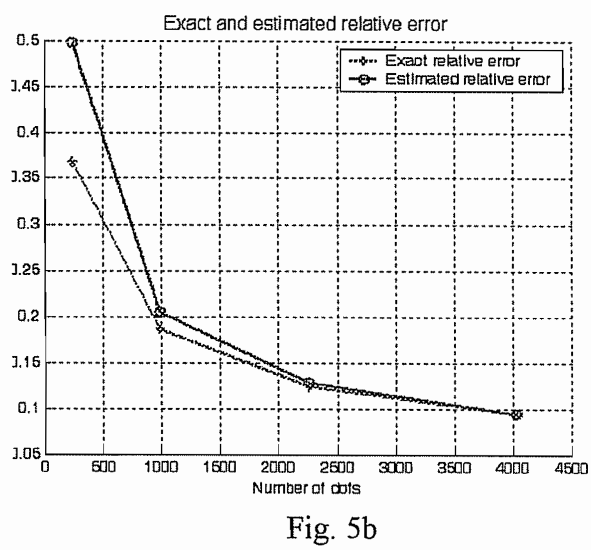

Fig. 5b

Fig. 5 shows that the exact energy norm can be replaced by the energy of Richardson's extrapolation.

Another method to obtain a precise estimation of the exact strain energy of plate is to perform *dual analyses* [3, 4] in which the same problem is solved by using both displacement models and equilibrium models. Then if the displacement boundary conditions are homogeneous and consistent loads are applied, a displacement model gives a lower bound to the exact strain energy while an equilibrium model gives an upper bound. Richardson's extrapolation may then be applied to both sequences of displacement and equilibrium models so as to obtain two estimates which bound the exact strain energy.

The steps of calculation in a brief of dual analysis as follow

- Compute the total strain energy $\Pi(w_h) = U(w_h) + P(w_h)$ which is obtained from the conforming HCT's element (displacement model) [7].
- Compute the total complementary energy $\Psi(M_h) = V(M_h) + P(M_h)$ which obtained with equilibrium Morley's element (equilibrium model).
- The relative error is calculated by

$$R.E = \left\{ \frac{[\Pi(w_h) + \Psi(M_h)]}{U(w_h) + V(M_h)} \right\}^{1/2}$$

In the case of a clamped plate, one type of boundary conditions is homogeneous (Fraeijs de Veubeke's particular case). Thus the relative error can be rewritten as

$$R.E = \left\{ \frac{[V(M_h) - U(w_h)]}{U(w_h) + V(M_h)} \right\}^{1/2}$$

where

$$U(w) = \frac{1}{2}\int_\Omega \chi^T H \chi d\Omega \text{ - strain energy of plate}$$

$$V(M) = \frac{1}{2}\int_\Omega M^T H^{-1} M d\Omega \text{ - strain complementary energy of plate}$$

The error is thus measured by the difference between the two obtained values of the elastic energy.

Table 3: The results on relative error of conventional dual analysis

Mesh N^0	Morley element	HCT element	$R.E$
	$V(M_h)$	$U(w_h)$	
2x2	25.53458	8.19668	0.716938622160
4x4	14.99747	9.77068	0.459378580234
8x8	11.47289	10.23024	0.239283760983
16x16	10.55812	10.33456	0.103442730131
24x24	10.40869	10.35166	0.052413797916
32x32	10.36521	10.35726	0.019588576616

Herein, both approaches converge when the mesh is refined. The distance between two curves is a measure of convergence.

The convergence behaviour of strain energy in dual analysis is illustrated in Figure. 6.

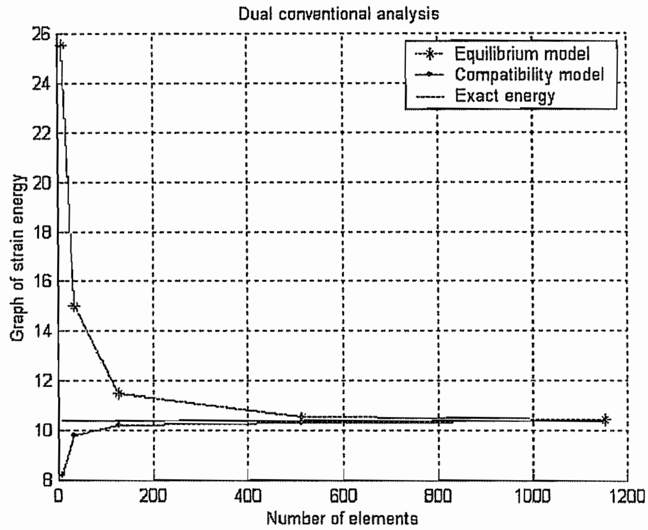

Fig. 6

Problem 2

Consider a L-shaped plate with a uniform pressure and clamped on a part of its boundary (Fig.6). Data of problem is the same of the first problem.

The meshes will generally be composed of 3-node or 6-node triangles with two different levels of refinement.

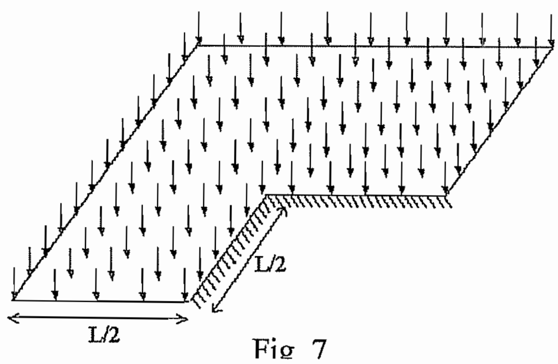

Fig. 7

Fig.7 shows that a uniform mesh is generated

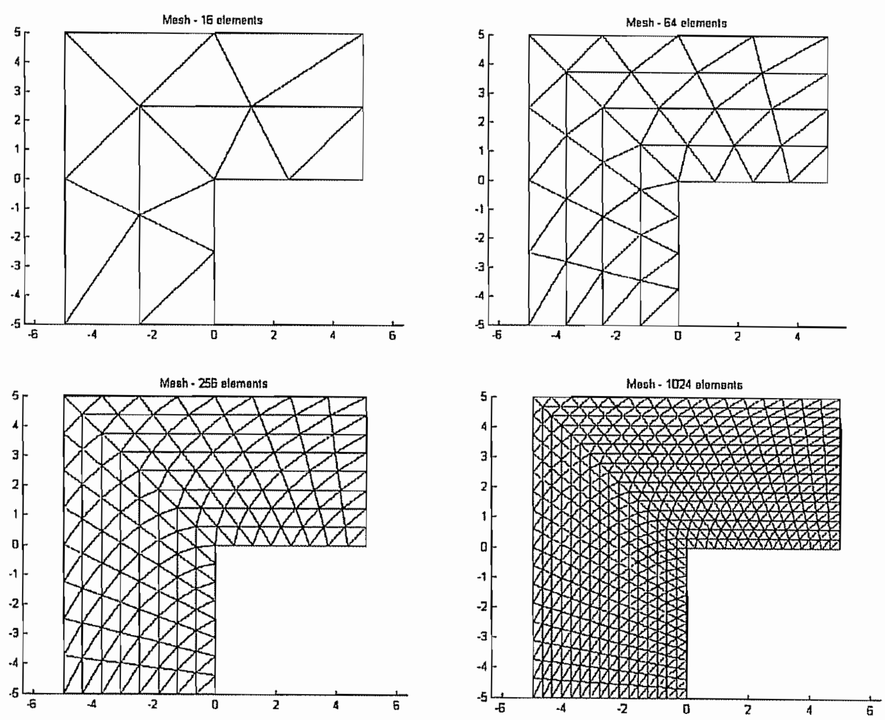

Fig. 8

Table 4: The results on relative error of conventional dual analysis

N.EL	Morley element	HCT element	$R.E$
	$V(M_h)$	$U(w_h)$	
16	113.853208442487	40.059883652542	0.692421839611
64	72.941612872744	48.533920209122	0.448248614913
256	60.007267746066	51.062060819190	0.283790675966
1024	55.707999760283	52.109129736621	0.182700273914

Fig. 8 plots the convergence of the energy of L-shaped plate in dual analysis.

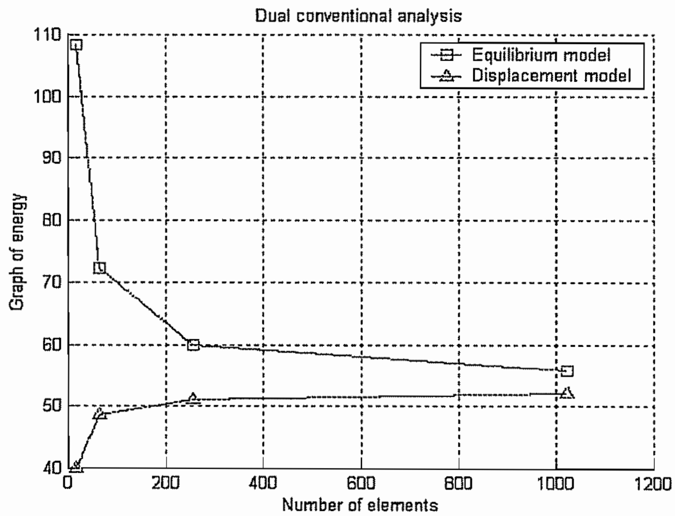

Fig. 9

The following mesh will be graded with smaller elements towards the inward corner.

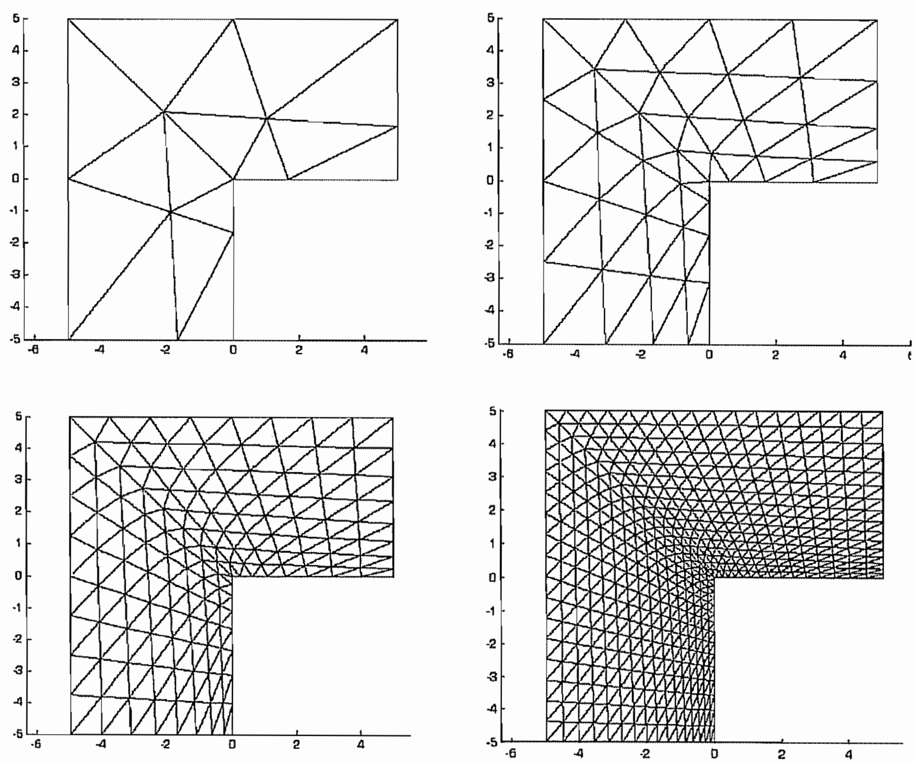

Fig. 10

Fig. 11 presents the results of relative error in two case the uniform mesh and other.

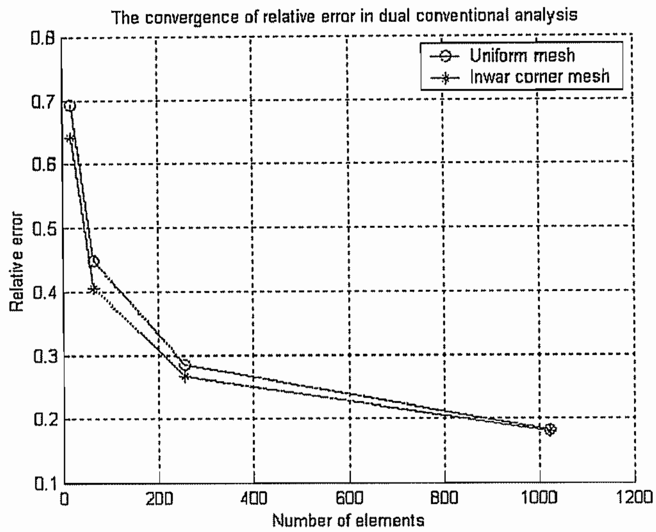

Fig.11

It shows that mesh is based on smaller elements towards the inward corner gives better values of convergence compared with a uniform mesh.

4. Conclusion

- This paper has showed how to obtain equilibrium element and error estimation for plate bending. Here, a new moment field was proposed, which permits to take pressures in account with corner loads. These fields may be complemented to equilibrium elements of constant moment field. This constitutes an appreciable extension of the application field of these elements. In the case of a constant pressure, the proposed field is only slightly more intricate than Sander's one [2] which in counterpart requires the first degree connectors at least. The obtained results not only apply to problems with the known exact solution but also solve most of the problems which do not have analytical solutions.

- Th errors may be estimated from the squared distance between the two approximations which refers to conventional dual analysis. The conventional dual analysis consists in a parallel analysis of an equilibrium finite element model and a compatible finite element model. A Rayleigh-Ritz process is performed in both cases, so the approaches do converge when the mesh is refined. The relative error is then obtained. It is therefore possible not only to detect too coarse meshes but also to know if a given solution is a good one. As most practical problems do not have analytical solutions, this method. is a very useful tool to estimate the error.

- However, the results only derive the evolution of the computed error of the plate using a uniform refinement of the mesh. We thus can combine with adaptive meshing that these error estimators prove to be effective parameters in mesh optimization for plate analysis.

- Further observations on the equilibrium element show that the errors are the lack of compatibility of the strain field and an estimation of the error that is called to be *compatibility error* [7]. This is the dual of *equilibrium error* present in the stress field obtained by using compatible finite element model. Thus, a compatible displacement field may be recovered from equilibrium element through a post-processing scheme. This procedure can be considered as a *dual* of *Ladevèze method* [8] for problem of plate bending. From this dual solution, an upper bound for the global error is obtained. Numerical examples on the dual-Ladevèze method will be shown in another paper.

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