Computational & Multiscale
Mechanics of Materials

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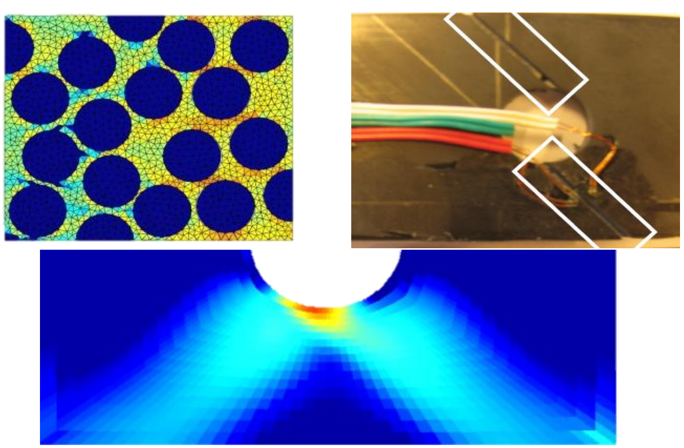
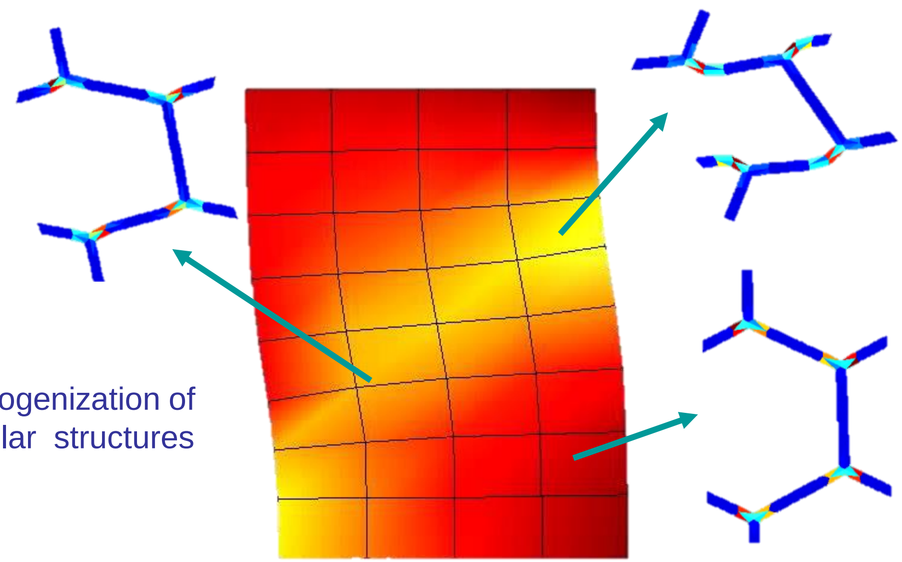
Muti-scale methods with strain-softening: damage-enhanced MFH for composite materials and computational homogenization for cellular materials with micro-buckling

L. Wu, V.-D. Nguyen, L. Adam (e-Xstream)
I. Doghri (UCL), L. Noels



Non-local damage mean-field-homogenization



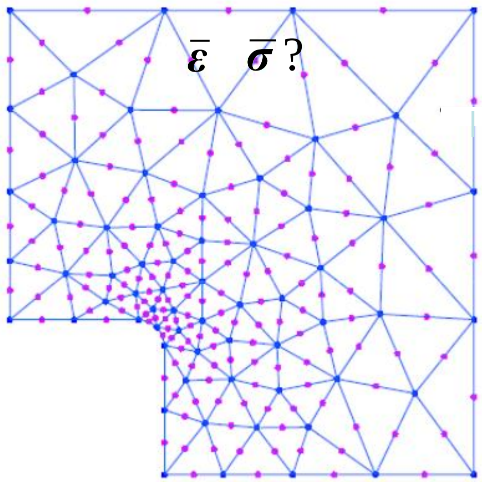
FE2 homogenization of cellular structures

Content

- Introduction
  - Finite element multi-scale modelling
  - Strain softening issues
- Non-local damage-enhanced mean-field-homogenization
- Computational homogenization for cellular materials
- Conclusions

Multi-scale modelling: How?

- Computational technique: FE^2
  - Macro-scale
    - FE model
    - At one integration point $\bar{\varepsilon}$ is know, $\bar{\sigma}$ is sought



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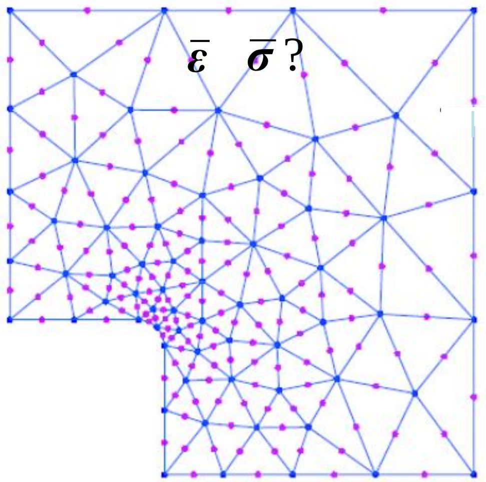
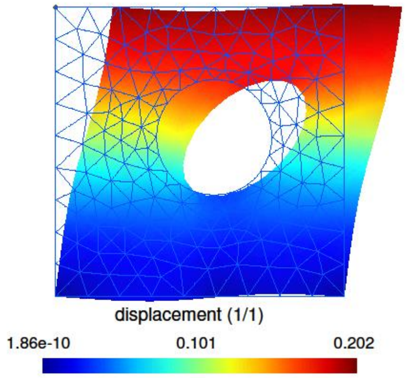


  - Micro-scale
    - Usual 3D finite elements
    - Periodic boundary conditions



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  - Micro-scale
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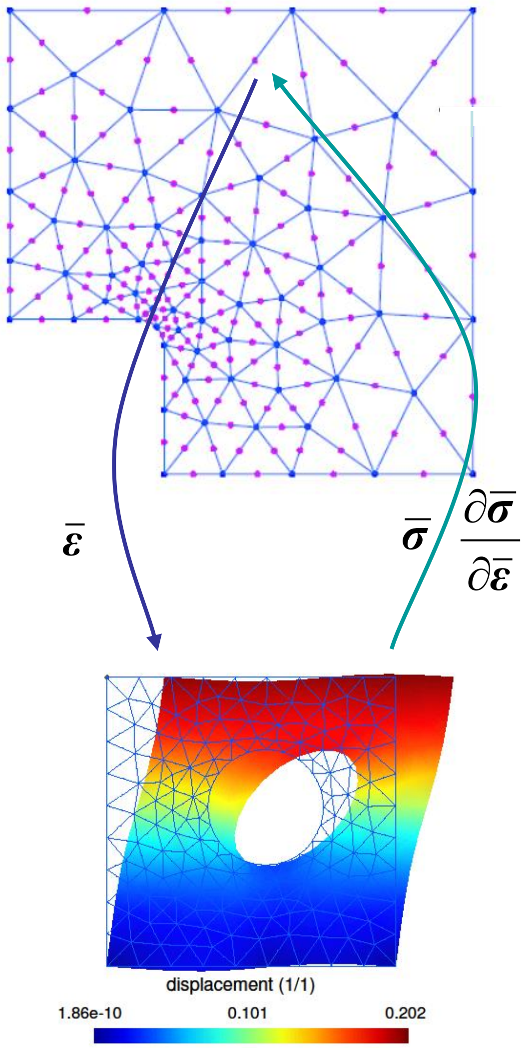

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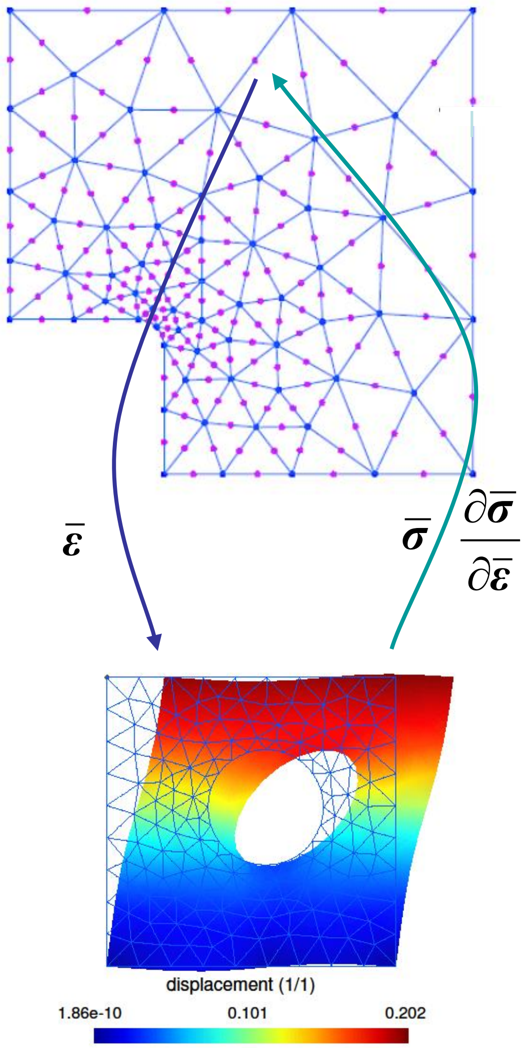
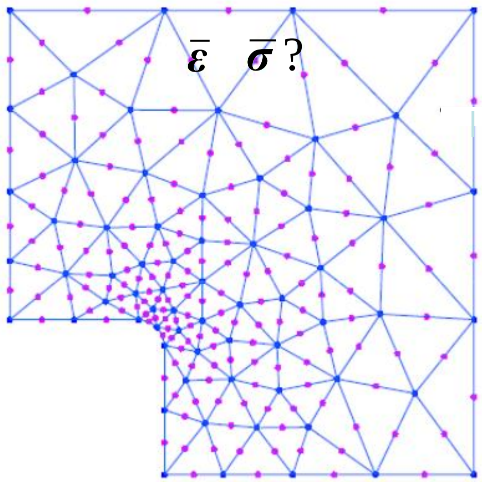
  - Advantages
    - Accuracy
    - Generality
  - Drawback
    - Computational time



Ghosh S et al. 95, Kouznetsova et al. 2002, Geers et al. 2010, …

- Mean-Field Homogenization
  - Macro-scale
    - FE model
    - At one integration point $\bar{\boldsymbol{\varepsilon}}$ is know, $\bar{\boldsymbol{\sigma}}$ is sought



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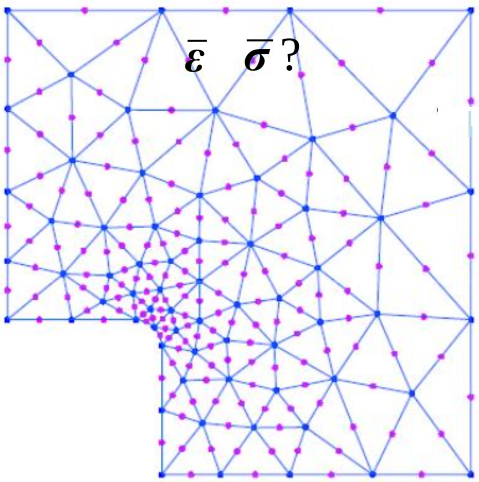
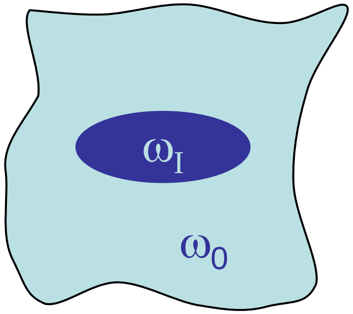


  - Micro-scale
    - Semi-analytical model
    - Predict composite meso-scale response
    - From components material models



Université de Liège

Multi-scale modelling: How?

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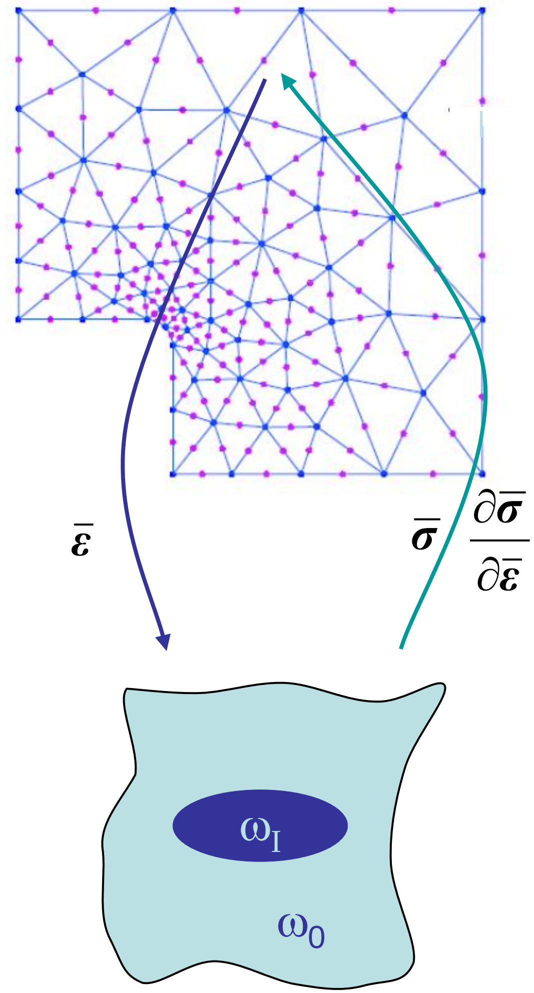

Mori and Tanaka 73, Hill 65, Ponte Castañeda 91, Suquet 95, Doghri et al 03, Lahellec et al. 11, Brassart et al. 12, …

Multi-scale modelling: How?

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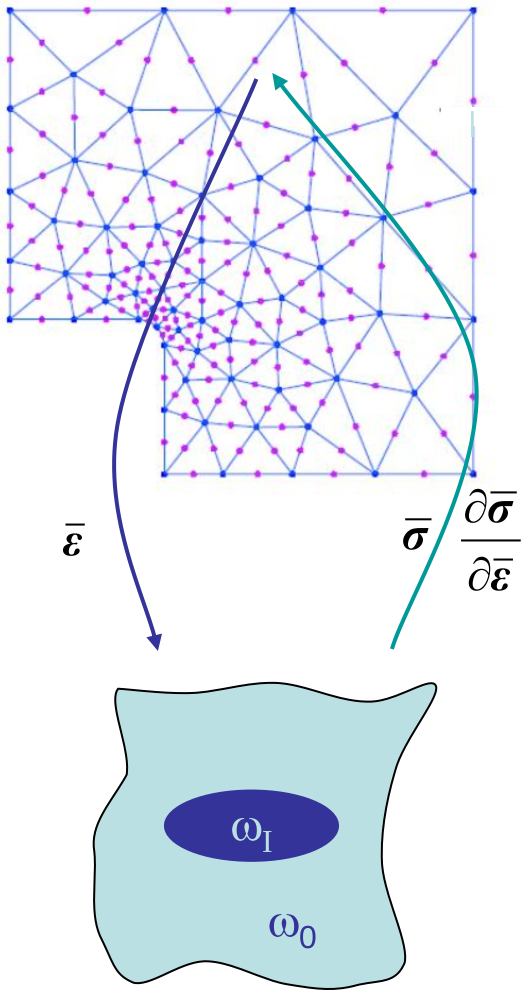
  - Micro-scale
    - Semi-analytical model
    - Predict composite meso-scale response
    - From components material models
  - Advantages
    - Computationally efficient
    - Easy to integrate in a FE code (material model)
  - Drawbacks
    - Difficult to formulate in an accurate way
      - Geometry complexity
      - Material behaviours complexity



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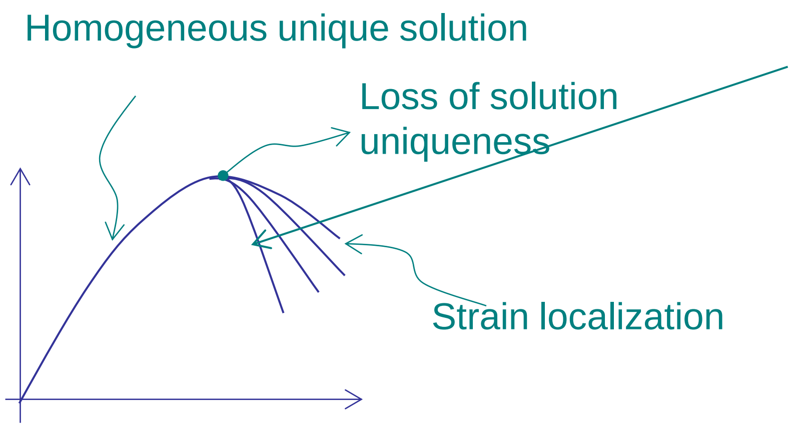
Strain softening of the microscopic response

- Finite element solutions for strain softening problems suffer from:
  - Loss of solution uniqueness and strain localization
  - Mesh dependence



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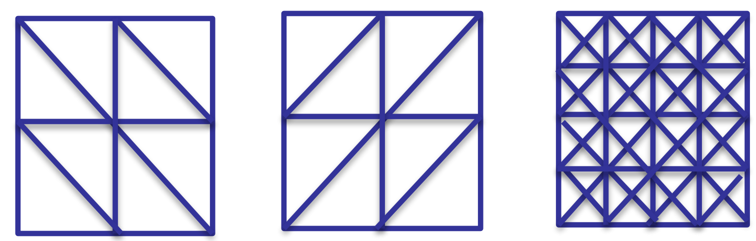
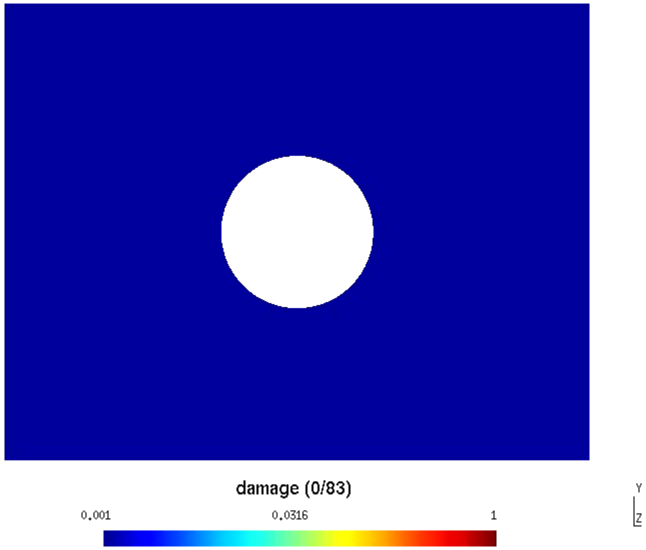


The numerical results change without convergence

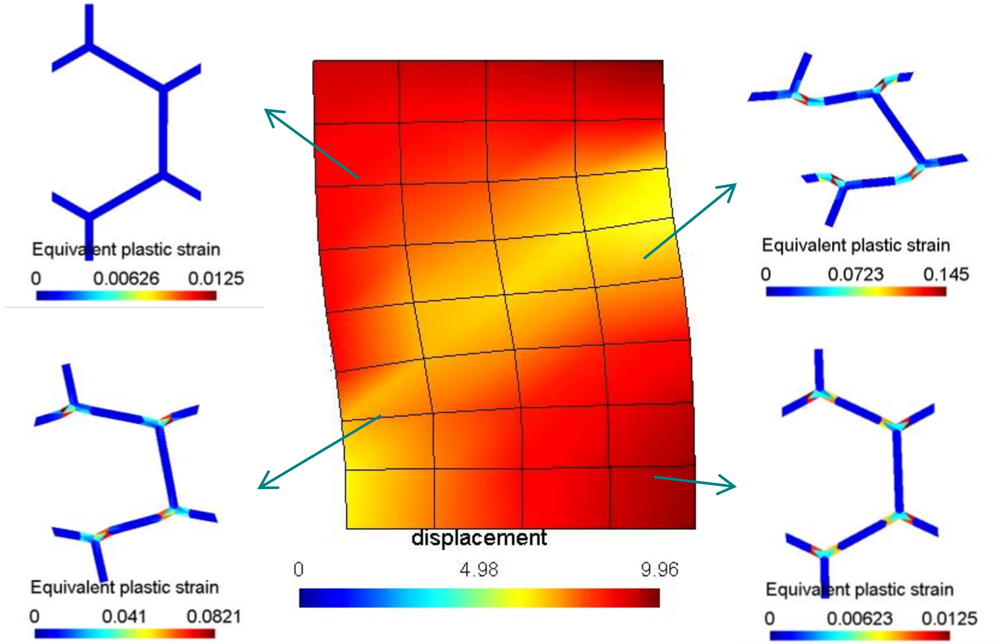
- Requires a non-local formulation of the macro-scale problem

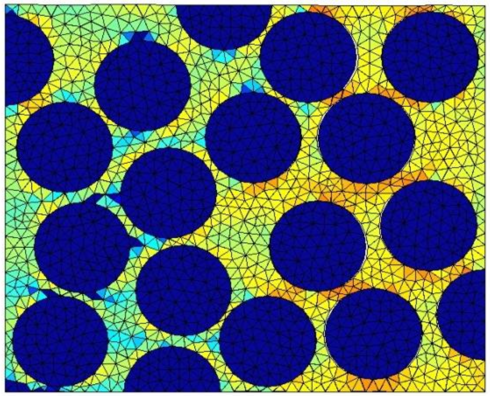
Multi-scale simulations with strain softening

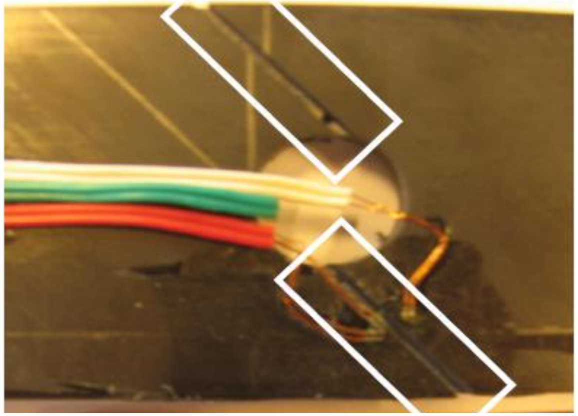
- Two cases considered
  - Composite materials
    - Mean-field homogenization
    - Non-local damage formulation

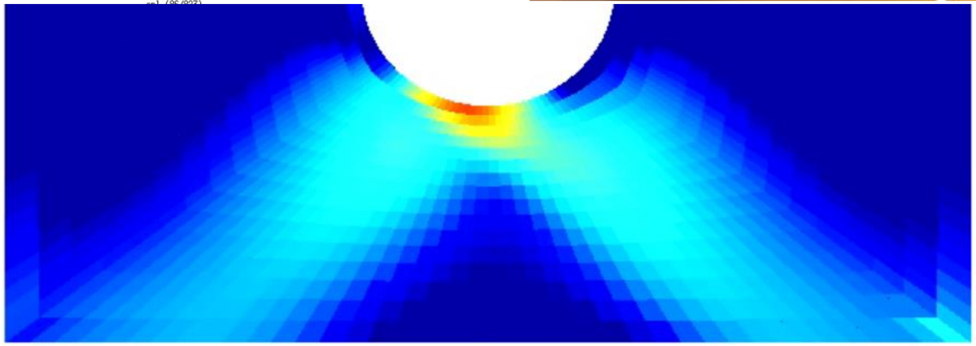


  - Honeycomb structures
    - Computational homogenization
    - Second-order FE2
    - Micro-buckling









Non-local damage-enhanced mean-field-homogenization

L. Wu (ULg), L. Noels (ULg), L. Adam (e-Xstream), I. Doghri (UCL)

SIMUCOMP The research has been funded by the Walloon Region under the agreement no 1017232 (CT-EUC 2010-10-12) in the context of the ERA-NET +, Matera + framework.

- Semi analytical Mean-Field Homogenization
  - Based on the averaging of the fields

$$\langle a \rangle = \frac{1}{V}\int_V a(\boldsymbol{X})\,\mathrm{d}V$$

  - Meso-response
    - From the volume ratios ($v_0 + v_\mathrm{I} = 1$)

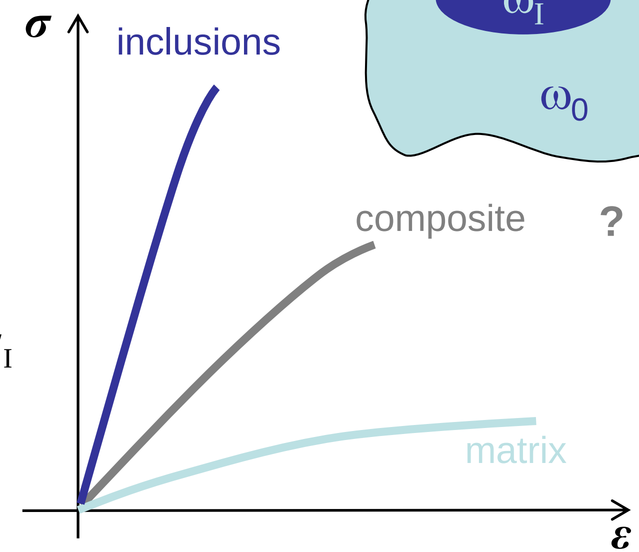
$$\begin{cases} \bar{\boldsymbol{\sigma}} = \langle \boldsymbol{\sigma} \rangle = v_0 \langle \boldsymbol{\sigma} \rangle_{\omega_0} + v_\mathrm{I} \langle \boldsymbol{\sigma} \rangle_{\omega_\mathrm{I}} = v_0 \boldsymbol{\sigma}_0 + v_\mathrm{I} \boldsymbol{\sigma}_\mathrm{I} \\ \bar{\boldsymbol{\varepsilon}} = \langle \boldsymbol{\varepsilon} \rangle = v_0 \langle \boldsymbol{\varepsilon} \rangle_{\omega_0} + v_\mathrm{I} \langle \boldsymbol{\varepsilon} \rangle_{\omega_\mathrm{I}} = v_0 \boldsymbol{\varepsilon}_0 + v_\mathrm{I} \boldsymbol{\varepsilon}_\mathrm{I} \end{cases}$$

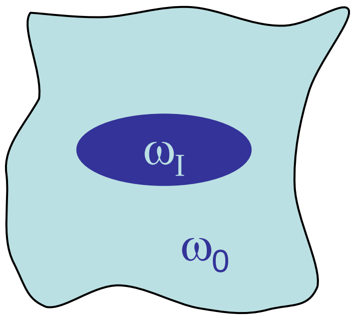
    - One more equation required

$$\boldsymbol{\varepsilon}_\mathrm{I} = \boldsymbol{B}^{\varepsilon} : \boldsymbol{\varepsilon}_0$$

  - Difficulty: find the adequate relations

$$\begin{cases} \boldsymbol{\sigma}_\mathrm{I} = f(\boldsymbol{\varepsilon}_\mathrm{I}) \\ \boldsymbol{\sigma}_0 = f(\boldsymbol{\varepsilon}_0) \\ \boldsymbol{\varepsilon}_\mathrm{I} = \boldsymbol{B}^{\varepsilon} : \boldsymbol{\varepsilon}_0 \end{cases} \qquad \boldsymbol{B}^{\varepsilon}\ ?$$





Non-local damage-enhanced mean-field-homogenization

- Mean-Field Homogenization for different materials
  - Linear materials
    - Materials behaviours

$$\begin{cases} \boldsymbol{\sigma}_\mathrm{I} = \overline{\boldsymbol{C}}_\mathrm{I} : \boldsymbol{\varepsilon}_\mathrm{I} \\ \boldsymbol{\sigma}_0 = \overline{\boldsymbol{C}}_0 : \boldsymbol{\varepsilon}_0 \end{cases}$$

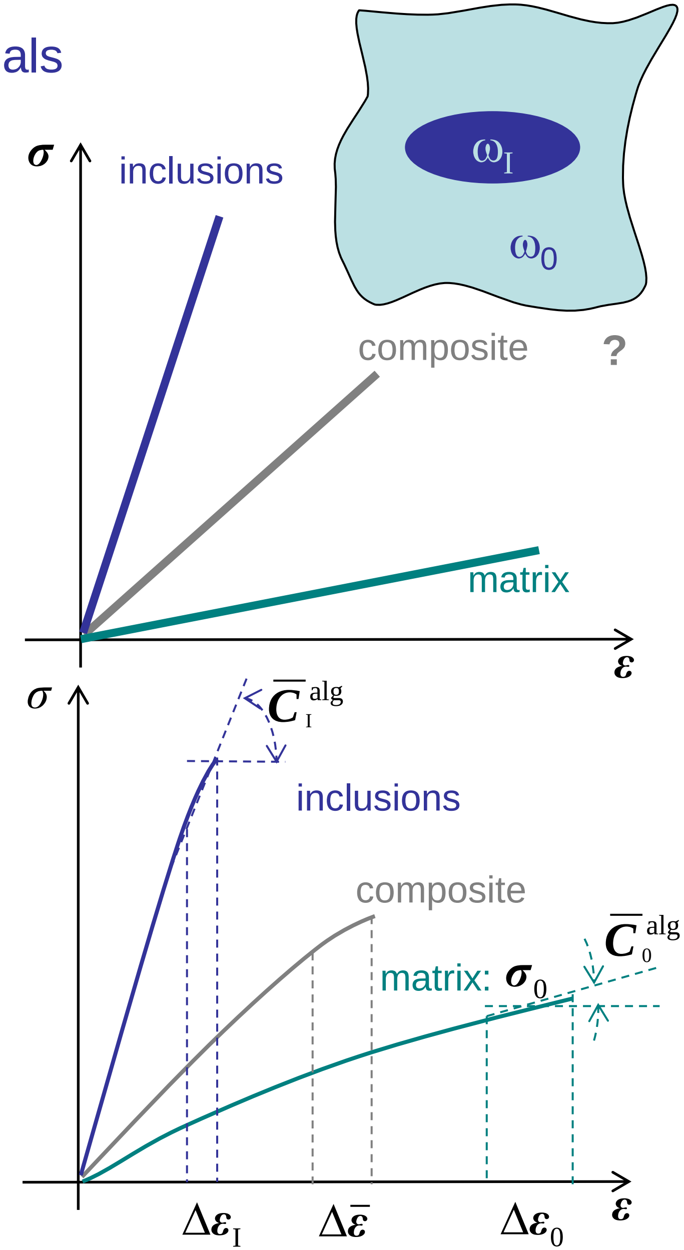
    - Mori-Tanaka assumption $\boldsymbol{\varepsilon}^\infty = \boldsymbol{\varepsilon}_0$
    - Use Eshelby tensor

$$\boldsymbol{\varepsilon}_\mathrm{I} = \boldsymbol{B}^\varepsilon\left(\mathrm{I}, \overline{\boldsymbol{C}}_0, \overline{\boldsymbol{C}}_\mathrm{I}\right) : \boldsymbol{\varepsilon}_0$$

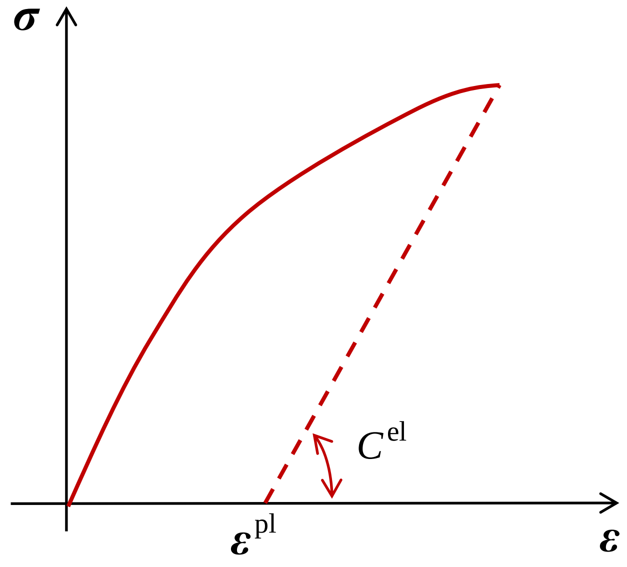
with $\boldsymbol{B}^\varepsilon = [\boldsymbol{I} + \boldsymbol{S} : \overline{\boldsymbol{C}}_0^{-1} : (\overline{\boldsymbol{C}}_1 - \overline{\boldsymbol{C}}_0)]^{-1}$

  - Non-linear materials
    - Define a Linear Comparison Composite
    - Common approach: incremental tangent

$$\Delta\boldsymbol{\varepsilon}_\mathrm{I} = \boldsymbol{B}^\varepsilon\left(\mathrm{I}, \overline{\boldsymbol{C}}_0^\mathrm{alg}, \overline{\boldsymbol{C}}_\mathrm{I}^\mathrm{alg}\right) : \Delta\boldsymbol{\varepsilon}_0$$



- Material models
  - Elasto-plastic material
    - Stress tensor $\boldsymbol{\sigma} = \boldsymbol{C}^{\mathrm{el}} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{pl}})$
    - Yield surface $f(\boldsymbol{\sigma}, p) = \boldsymbol{\sigma}^{\mathrm{eq}} - \sigma^{Y} - R(p) \leq 0$
    - Plastic flow $\Delta\boldsymbol{\varepsilon}^{\mathrm{pl}} = \Delta p \boldsymbol{N}$ & $\boldsymbol{N} = \frac{\partial f}{\partial \boldsymbol{\sigma}}$
    - Linearization $\delta\boldsymbol{\sigma} = \boldsymbol{C}^{\mathrm{alg}} : \delta\boldsymbol{\varepsilon}$



Non-local damage-enhanced mean-field-homogenization

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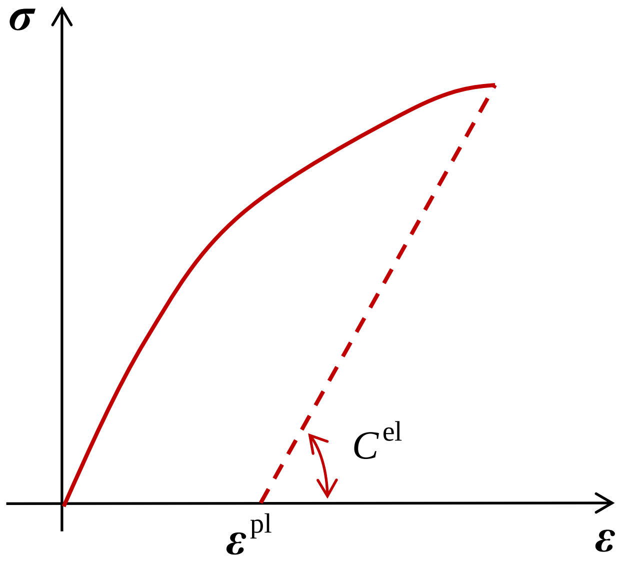
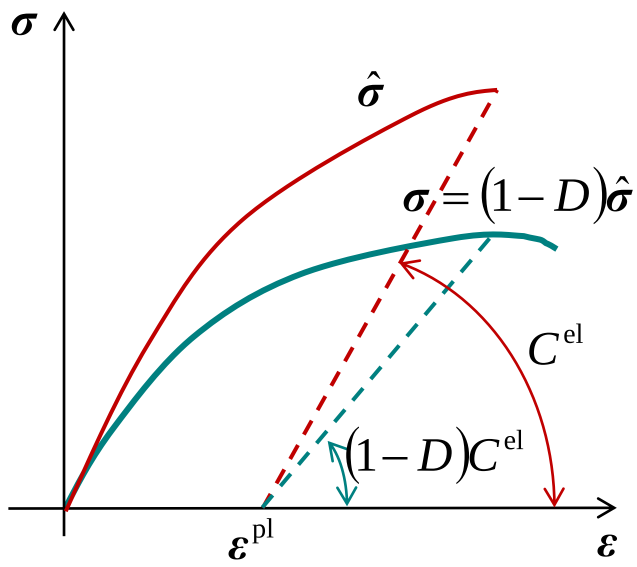
    - Linearization $\delta\boldsymbol{\sigma} = \boldsymbol{C}^{\mathrm{alg}} : \delta\boldsymbol{\varepsilon}$
  - Local damage model
    - Apparent-effective stress tensors $\boldsymbol{\sigma} = (1 - D)\hat{\boldsymbol{\sigma}}$
    - Plastic flow in the effective stress space
    - Damage evolution $\Delta D = F_D(\boldsymbol{\varepsilon}, \Delta p)$





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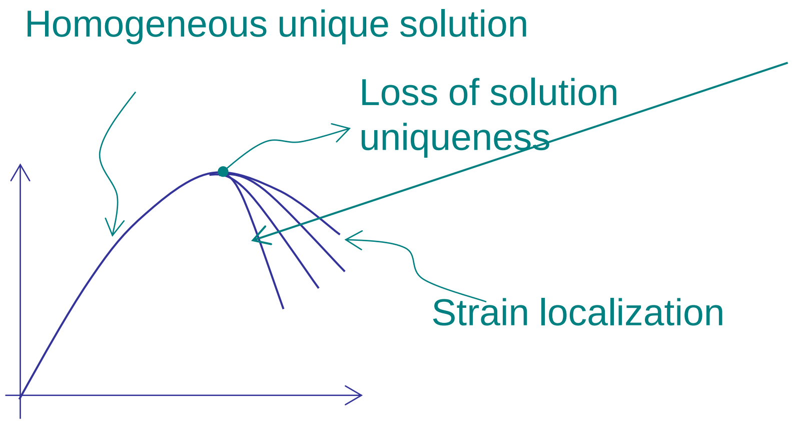

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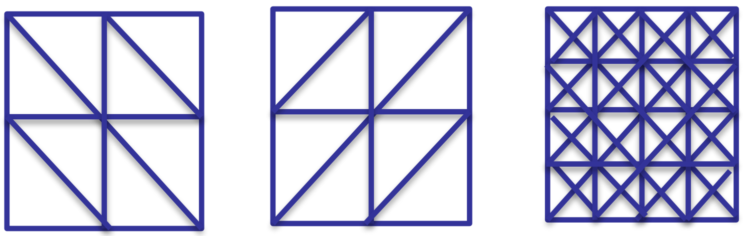

The numerical results change without convergence

- Implicit non-local approach [Peerlings et al 96, Geers et al 97, …]
  - A state variable is replaced by a non-local value reflecting the interaction between neighboring material points

$$\tilde{a}(\boldsymbol{x}) = \frac{1}{V_\mathrm{C}} \int_{V_\mathrm{C}} a(\boldsymbol{y}) w(\boldsymbol{y}; \boldsymbol{x}) \mathrm{d}V$$

  - Use Green functions as weight $w(y; x)$

⇨ $\tilde{a} - c\nabla^2 \tilde{a} = a$ ⇨ New degrees of freedom

Non-local damage-enhanced mean-field-homogenization

- Material models
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    - Stress tensor $\boldsymbol{\sigma} = \boldsymbol{C}^{\text{el}} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\text{pl}})$
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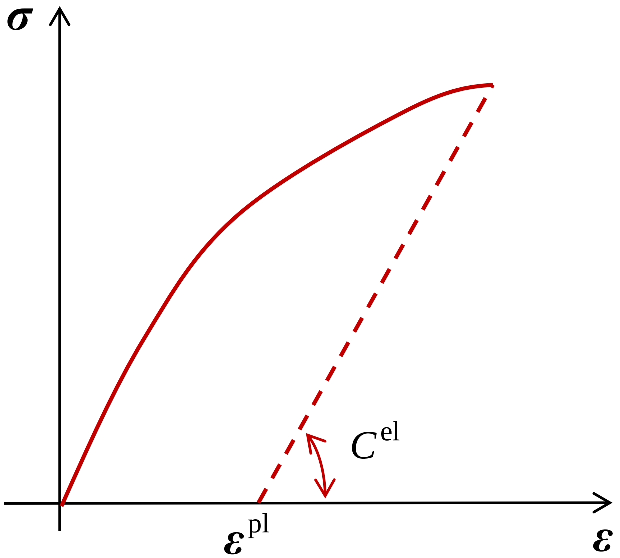
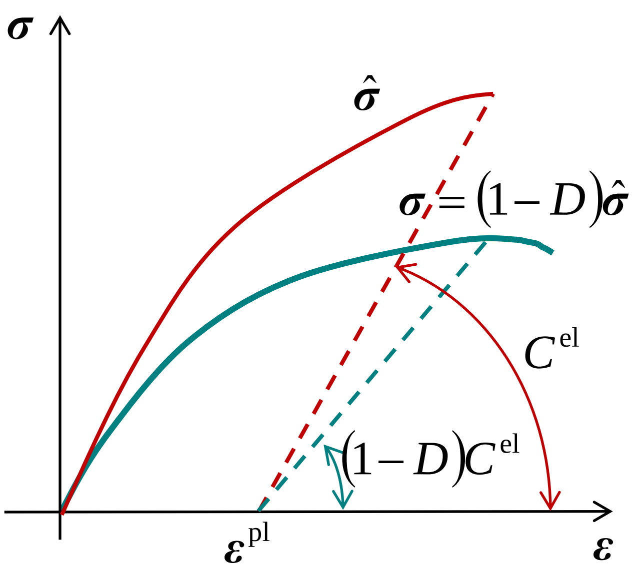
  - Non-Local damage model
    - Damage evolution $\Delta D = F_D(\boldsymbol{\varepsilon}, \Delta\tilde{p})$
    - Anisotropic governing equation $\tilde{p} - \nabla \cdot (\boldsymbol{c}_{\text{g}} \cdot \nabla\tilde{p}) = p$
    - Linearization

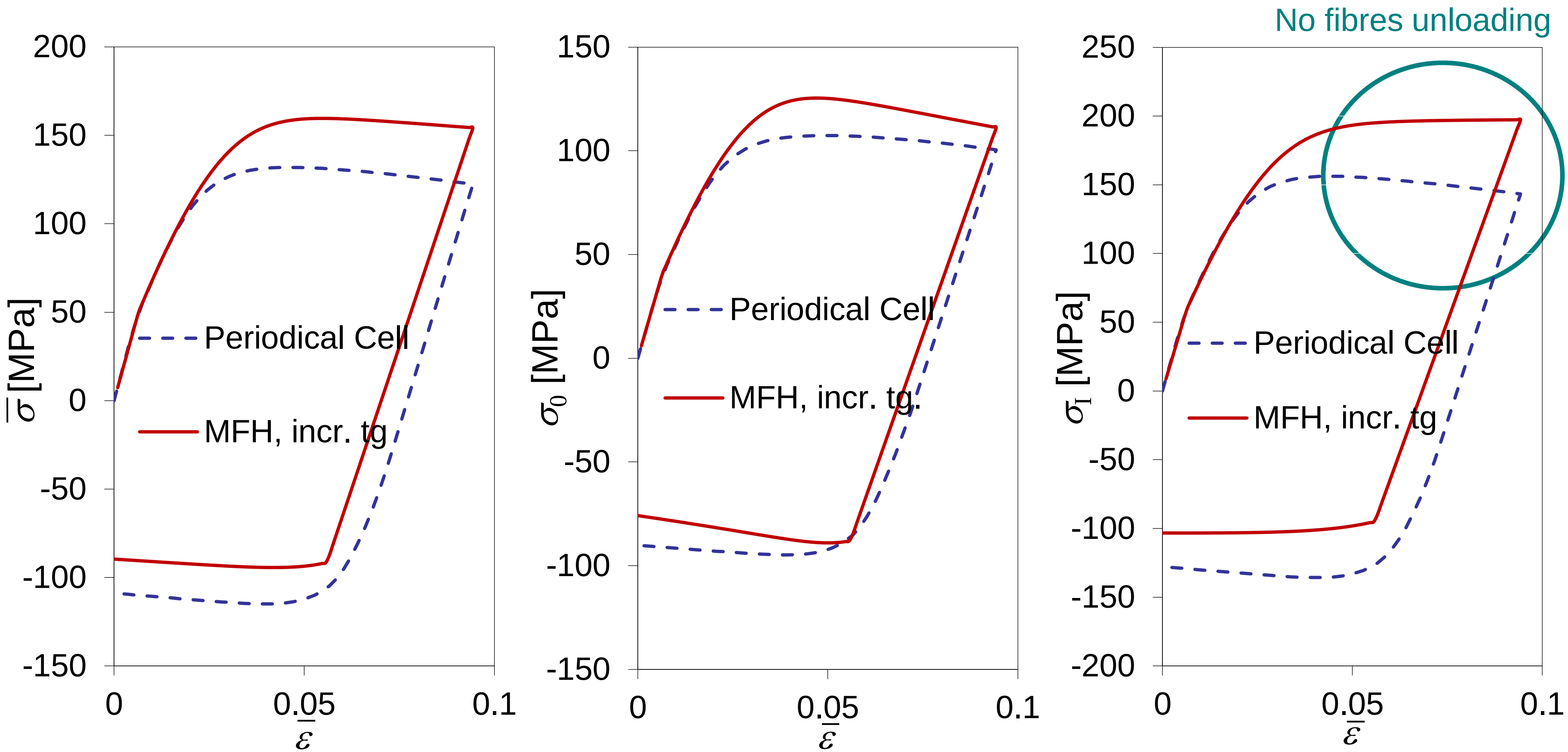
$$\delta\boldsymbol{\sigma} = \left[(1 - D)\boldsymbol{C}^{\text{alg}} - \hat{\boldsymbol{\sigma}} \otimes \frac{\partial F_D}{\partial \boldsymbol{\varepsilon}}\right] : \delta\boldsymbol{\varepsilon} - \hat{\boldsymbol{\sigma}} \frac{\partial F_D}{\partial \tilde{p}} \delta\tilde{p}$$





Non-local damage-enhanced mean-field-homogenization

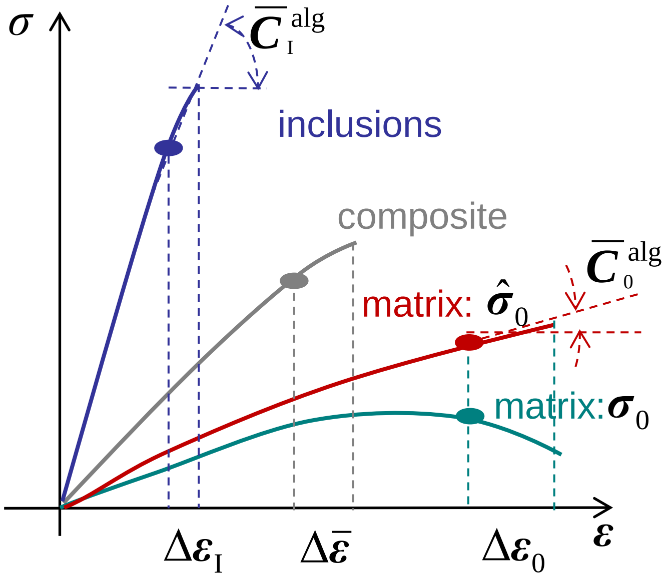
- Limitation of the incremental tangent method
  - Fictitious composite material
    - 50%-UD fibres
    - Elasto-plastic matrix with damage
  - Due to the incremental formalism, stress in fibres cannot decrease during loading



- Problem
  - We want the fibres to get unloaded during the matrix damaging process
    - For the incremental-tangent approach

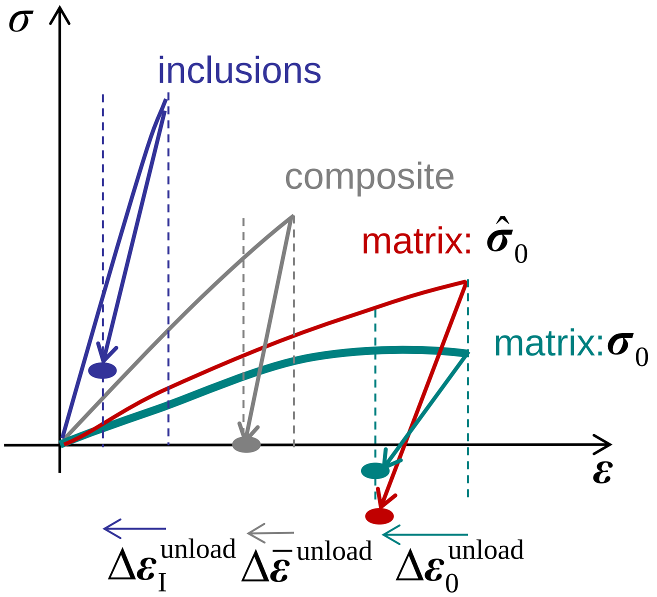
$$\Delta\boldsymbol{\varepsilon}_{\mathrm{I}} = \boldsymbol{B}^{\varepsilon}\left(\mathrm{I}, (1-D)\overline{\boldsymbol{C}}_{0}^{\mathrm{alg}}, \overline{\boldsymbol{C}}_{\mathrm{I}}^{\mathrm{alg}}\right) : \Delta\boldsymbol{\varepsilon}_{0}$$

    - To unload the fibres ($\boldsymbol{\varepsilon}_{\mathrm{I}} < 0$) with such approach would require $\overline{\boldsymbol{C}}_{\mathrm{I}}^{\mathrm{alg}} < 0$
    - We cannot use the incremental tangent MFH
  - We need to define the LCC from another stress state



Non-local damage-enhanced mean-field-homogenization

- Idea
  - New incremental-secant approach
    - Perform a virtual elastic unloading from previous solution
      - Composite material unloaded to reach the stress-free state
      - Residual stress in components



Non-local damage-enhanced mean-field-homogenization

- Idea
  - New incremental-secant approach
    - Perform a virtual elastic unloading from previous solution
      - Composite material unloaded to reach the stress-free state
      - Residual stress in components
    - Apply MFH from unloaded state
      - New strain increments (>0)

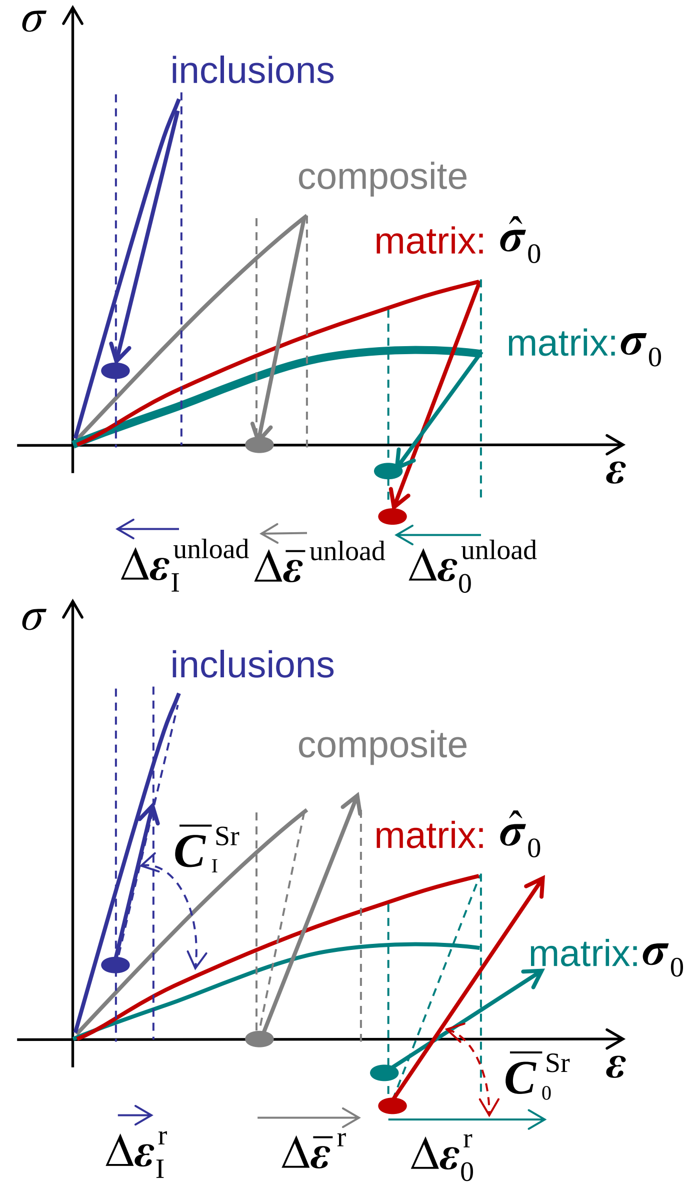
$$\Delta\boldsymbol{\varepsilon}^{\mathrm{r}}_{\mathrm{I/0}} = \Delta\boldsymbol{\varepsilon}_{\mathrm{I/0}} + \Delta\boldsymbol{\varepsilon}^{\mathrm{unload}}_{\mathrm{I/0}}$$

      - Use of secant operators

$$\Delta\boldsymbol{\varepsilon}^{\mathrm{r}}_{\mathrm{I}} = \boldsymbol{B}^{\varepsilon}\left(\mathrm{I},(1-D)\bar{\boldsymbol{C}}^{\mathrm{Sr}}_{0},\bar{\boldsymbol{C}}^{\mathrm{Sr}}_{\mathrm{I}}\right):\Delta\boldsymbol{\varepsilon}^{\mathrm{r}}_{0}$$

      - Possibility of have unloading

$$\begin{cases} \Delta\boldsymbol{\varepsilon}^{\mathrm{r}}_{\mathrm{I}} > 0 \\ \Delta\boldsymbol{\varepsilon}_{\mathrm{I}} < 0 \end{cases}$$

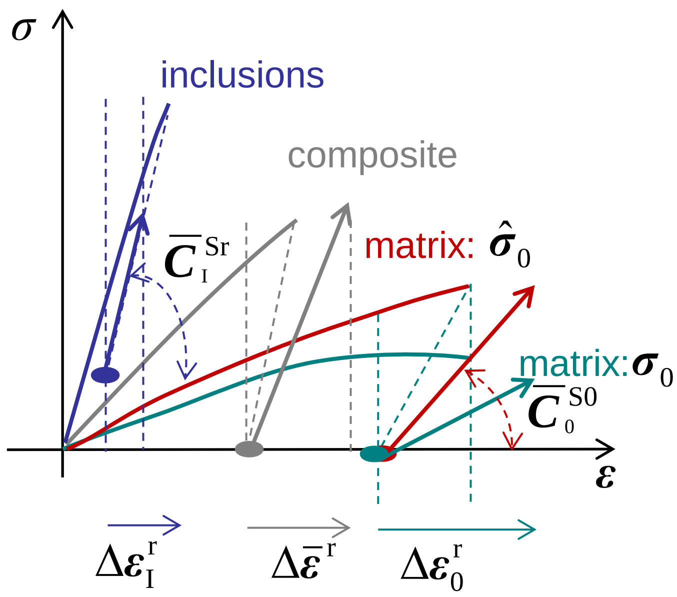


- Zero incremental-secant approach
  - For soft matrix response
    - Remove residual stress in matrix
    - Avoid adding spurious internal energy
    - Solve iteratively the system

$$\begin{cases} \Delta\bar{\boldsymbol{\varepsilon}}^{(\mathrm{r})} = v_0 \Delta\boldsymbol{\varepsilon}_0^{(\mathrm{r})} + v_\mathrm{I} \Delta\boldsymbol{\varepsilon}_\mathrm{I}^{(\mathrm{r})} \\ \Delta\boldsymbol{\varepsilon}_\mathrm{I}^\mathrm{r} = \Delta\boldsymbol{\varepsilon}_\mathrm{I} + \Delta\boldsymbol{\varepsilon}_\mathrm{I}^\mathrm{unload} \\ \Delta\boldsymbol{\varepsilon}_0^\mathrm{r} = \Delta\boldsymbol{\varepsilon}_0 + \Delta\boldsymbol{\varepsilon}_0^\mathrm{unload} \\ \Delta\boldsymbol{\varepsilon}_\mathrm{I}^\mathrm{r} = \boldsymbol{B}^\varepsilon\left(\mathrm{I}, (1-D)\bar{\boldsymbol{C}}_0^\mathrm{S0}, \bar{\boldsymbol{C}}_\mathrm{I}^\mathrm{Sr}\right) : \Delta\boldsymbol{\varepsilon}_0^\mathrm{r} \end{cases}$$

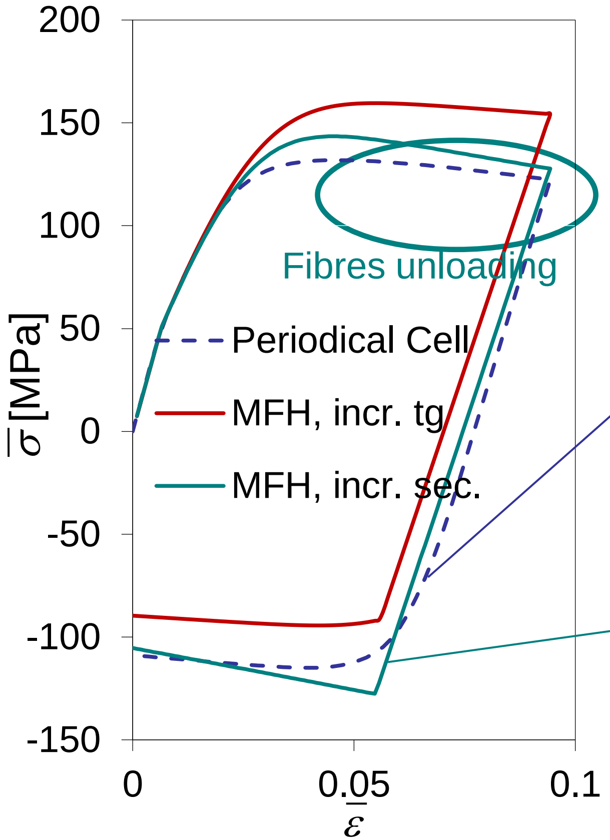
    - With the stress tensors

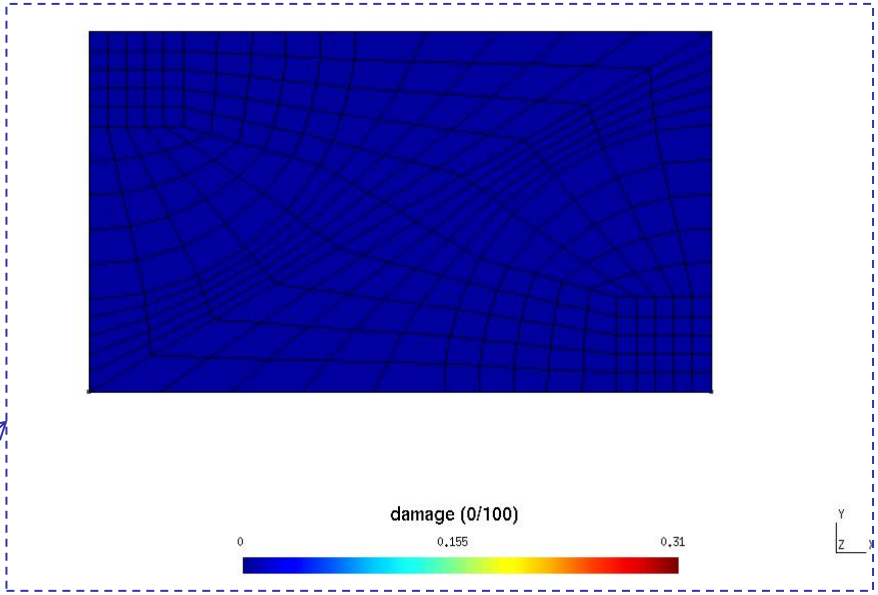
$$\begin{cases} \bar{\boldsymbol{\sigma}} = v_0 \boldsymbol{\sigma}_0 + v_\mathrm{I} \boldsymbol{\sigma}_\mathrm{I} \\ \boldsymbol{\sigma}_\mathrm{I} = \boldsymbol{\sigma}_\mathrm{I}^\mathrm{res} + \bar{\boldsymbol{C}}_\mathrm{I}^\mathrm{Sr} : \Delta\boldsymbol{\varepsilon}_\mathrm{I}^\mathrm{r} \\ \boldsymbol{\sigma}_0 = (1-D)\bar{\boldsymbol{C}}_0^\mathrm{S0} : \Delta\boldsymbol{\varepsilon}_0^\mathrm{r} \end{cases}$$

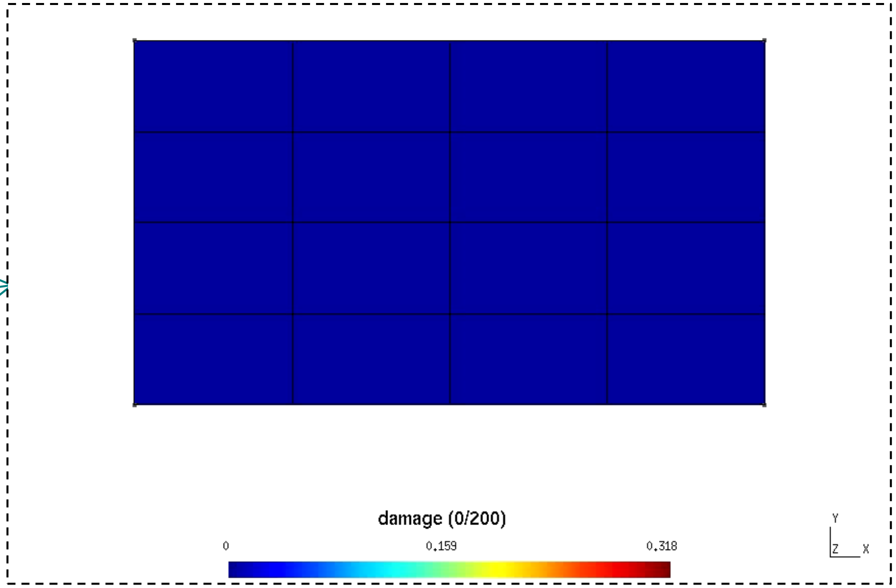


Non-local damage-enhanced mean-field-homogenization

- New results for damage
  - Fictitious composite material
    - 50%-UD fibres
  - **Elasto-plastic matrix with damage**

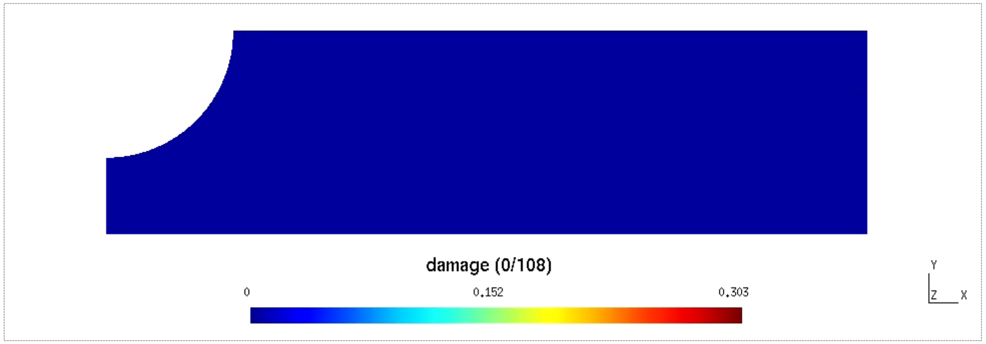


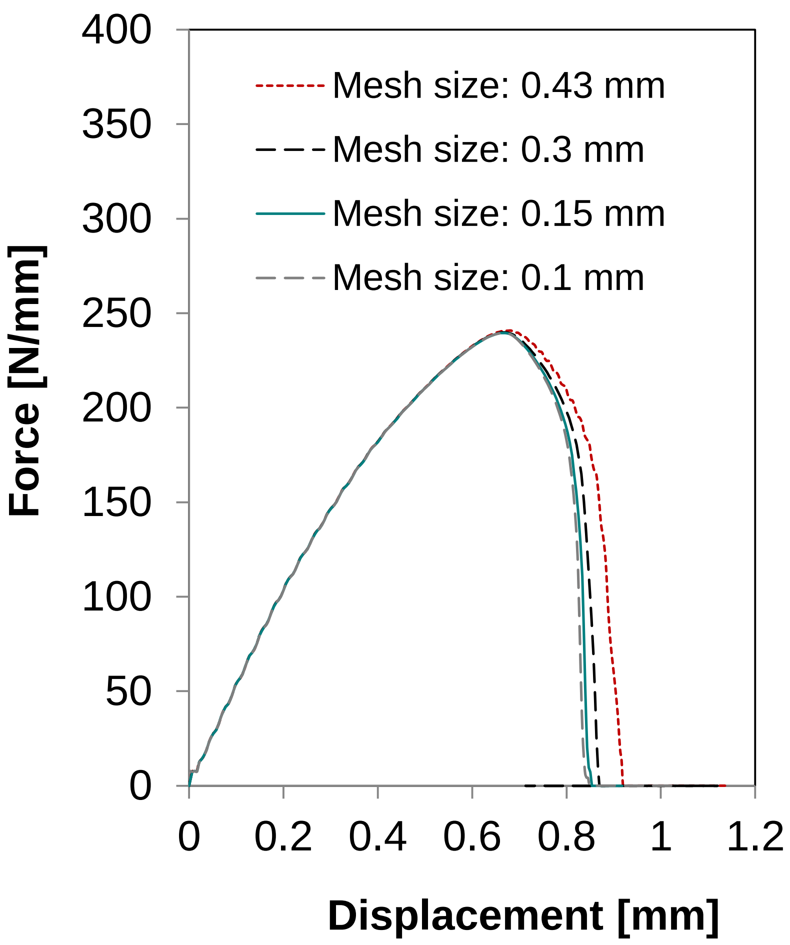




Non-local damage-enhanced mean-field-homogenization

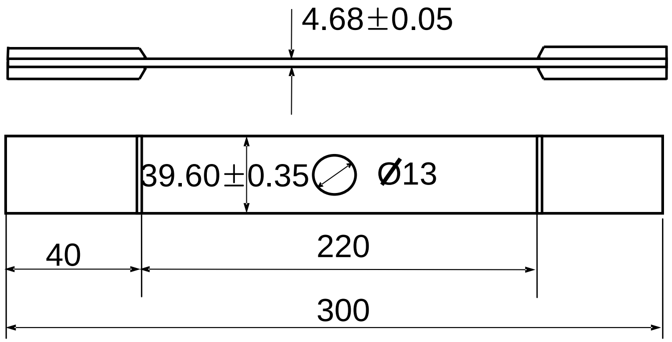
- Mesh-size effect
  - Fictitious composite material
    - 30%-UD fibres
    - Elasto-plastic matrix with damage
  - Notched ply

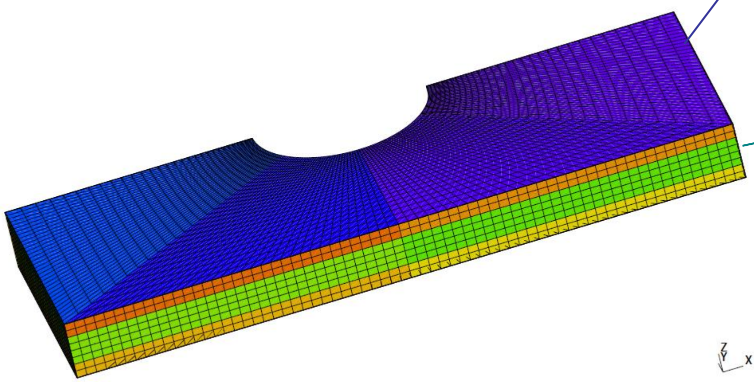


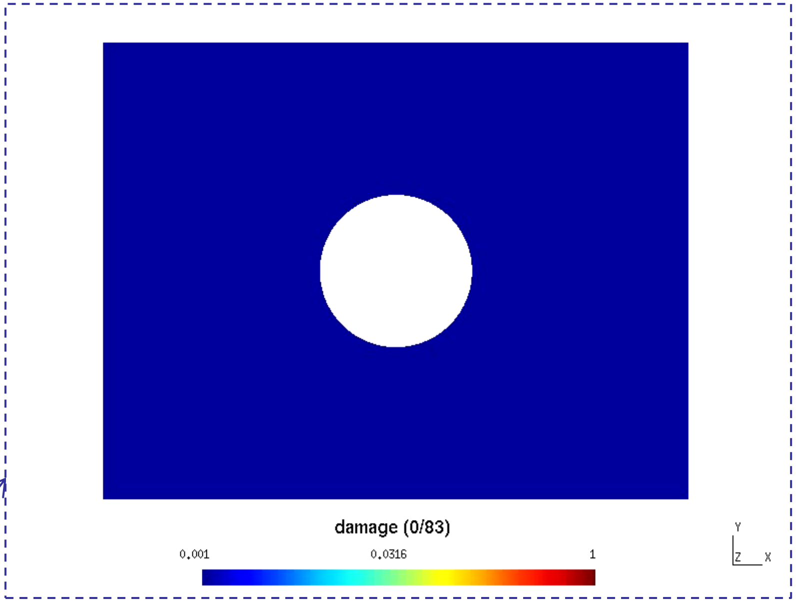


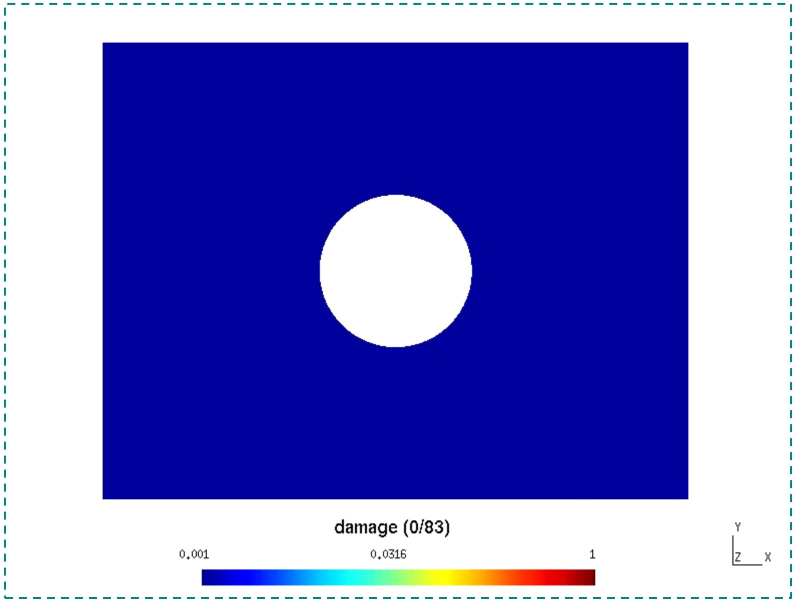
Non-local damage-enhanced mean-field-homogenization

- Laminate plate with hole
  - Carbon-fibres reinforced epoxy
    - 60%-UD fibres
    - Elasto-plastic matrix with damage
  - $[-45_2/45_2]_S$ staking sequence



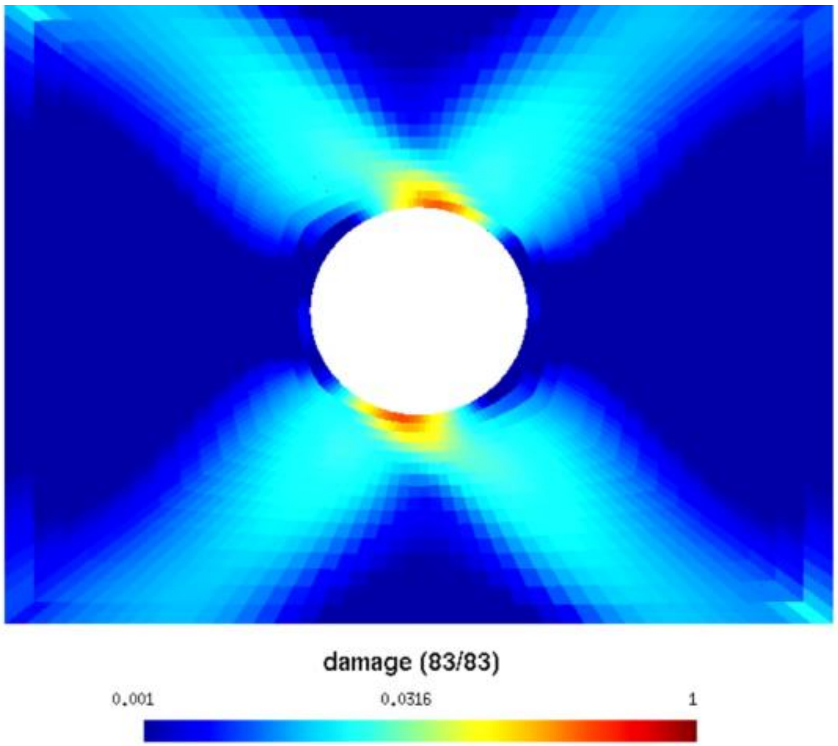


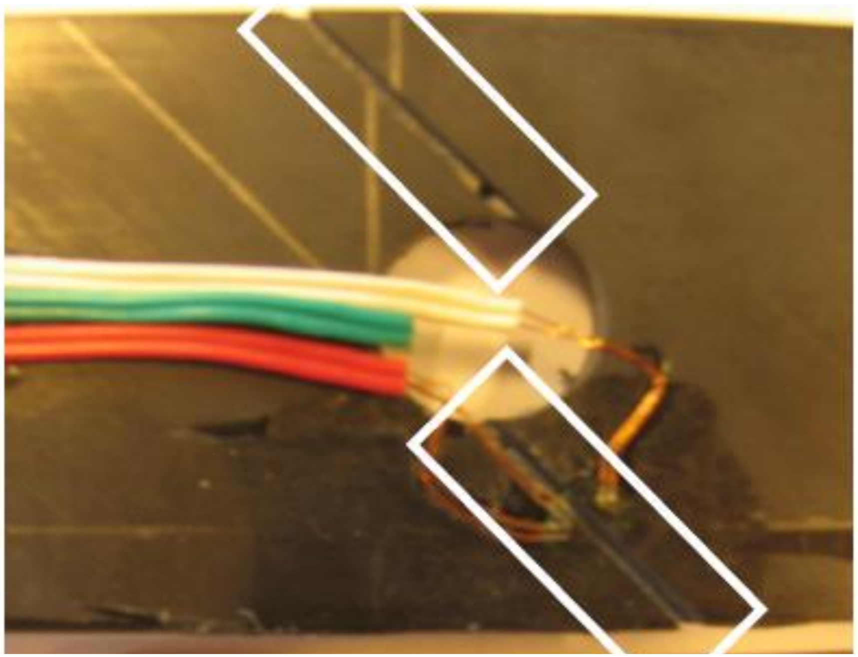


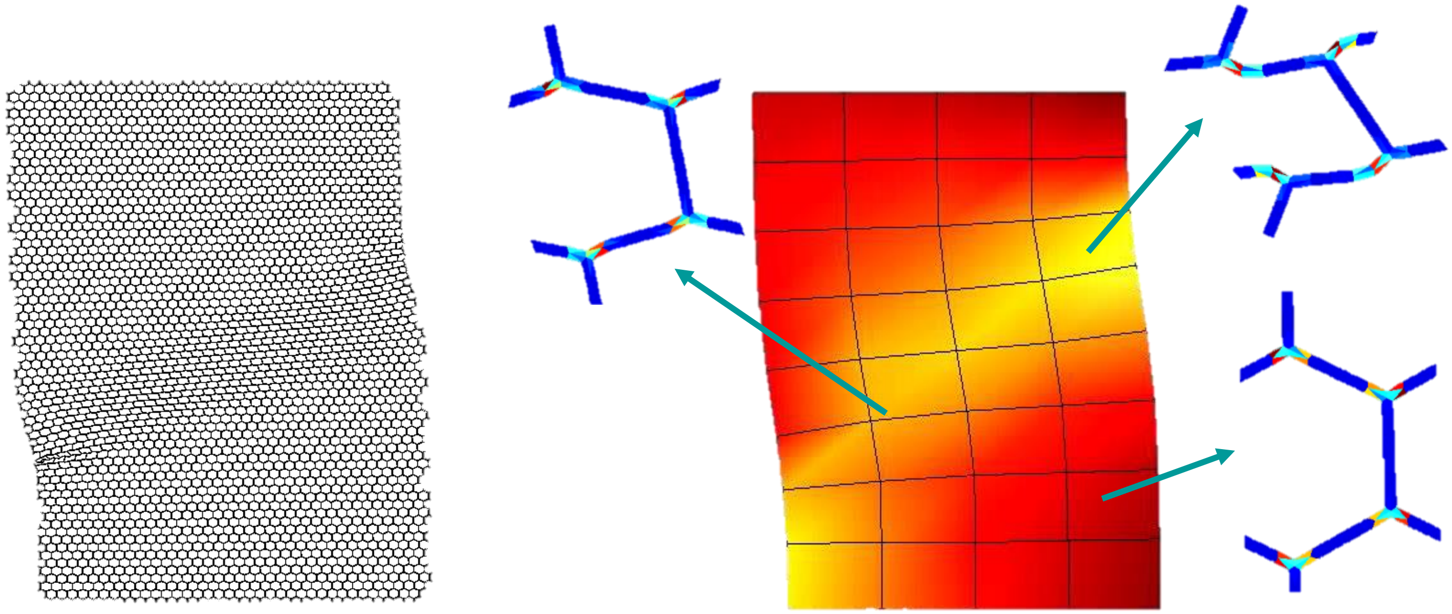


Non-local damage-enhanced mean-field-homogenization

- Laminate plate with hole (2)
  - Carbon-fibres reinforced epoxy
    - 60%-UD fibres
    - Elasto-plastic matrix with damage
  - $[-45_2/45_2]_S$ staking sequence







Computational homogenization for cellular materials

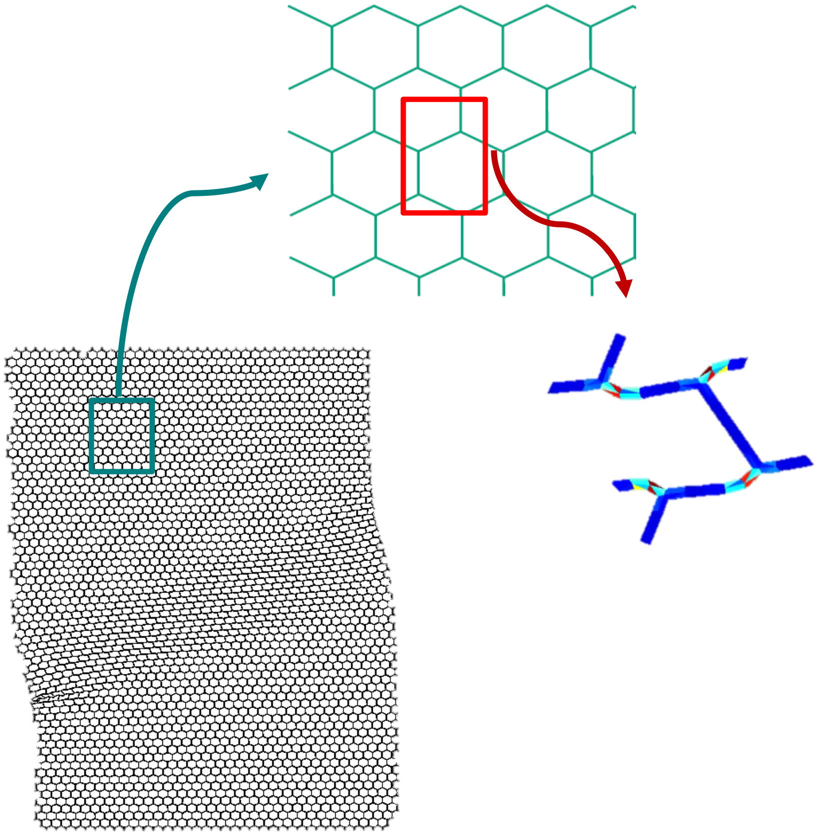
V.-D. Nguyen (ULg), L. Noels (ULg)

Les recherches ont été financées grâce à la subvention "Actions de recherche concertées ARC 09/14-02 BRIDGING- From imaging to geometrical modelling of complex micro structured materials: Bridging computational engineering and material science

Computational homogenization for foamed materials

- Challenges
  - Micro-structure
    - Not perfect with non periodic mesh

⇨ How to constrain the periodic boundary conditions?



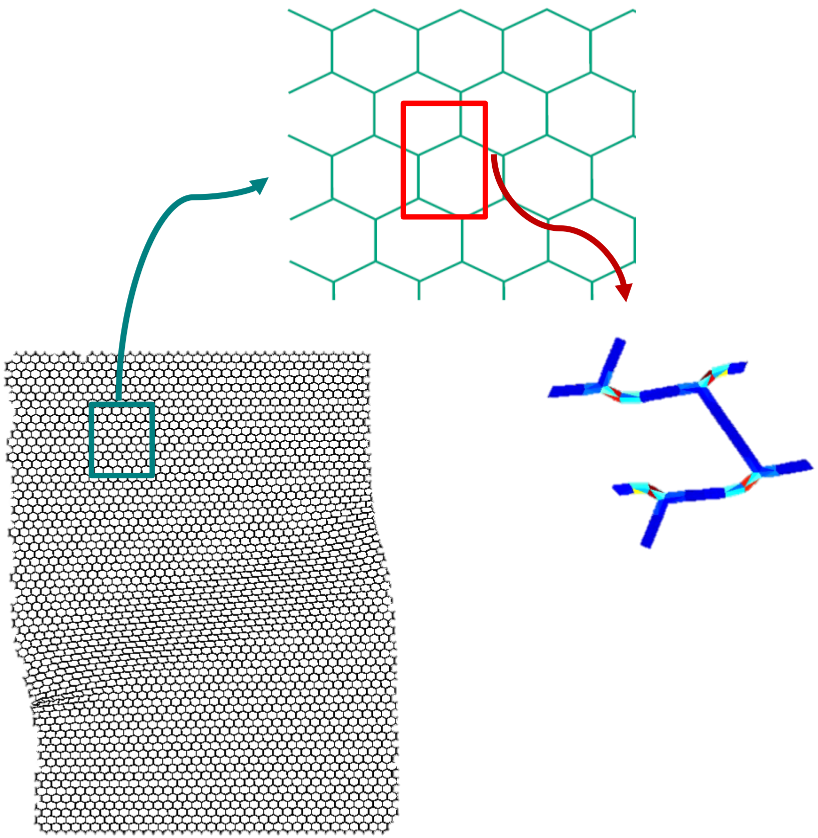
Computational homogenization for foamed materials

- Challenges
  - Micro-structure
    - Not perfect with non periodic mesh

      ⇨ How to constrain the periodic boundary conditions?

    - Thin components
    - Experiences micro-buckling

      ⇨ How to capture the instability?



Computational homogenization for foamed materials

- Challenges
  - Micro-structure
    - Not perfect with non periodic mesh

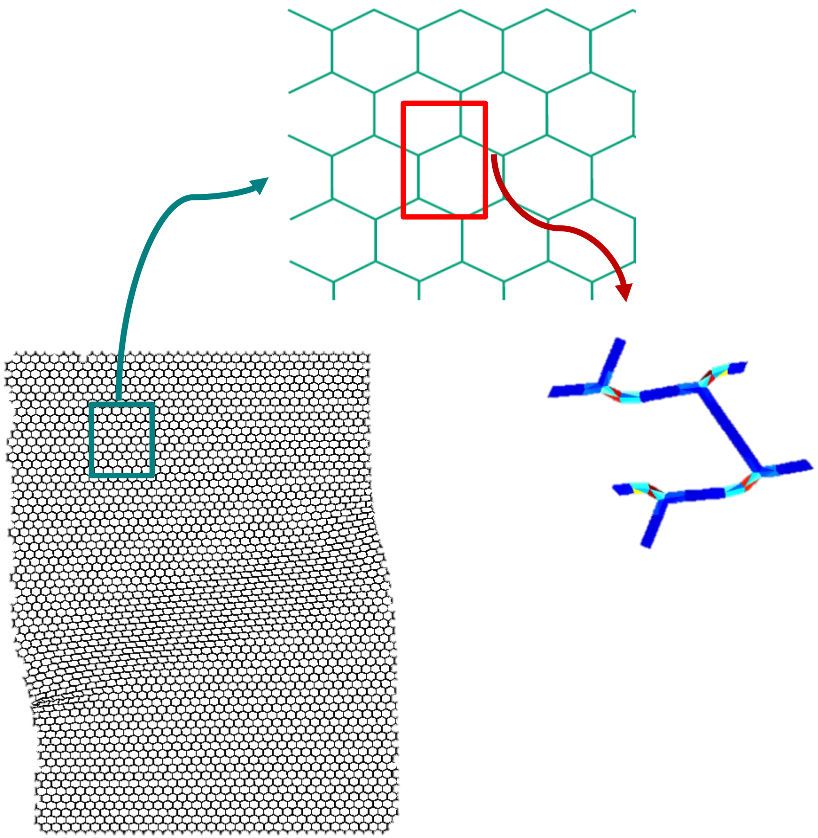
      ⇨ How to constrain the periodic boundary conditions?

    - Thin components
    - Experiences micro-buckling

      ⇨ How to capture the instability?

  - Transition
    - Homogenized tangent not always elliptic
    - Localization bands

      ⇨ How can we recover the solution unicity at the macro-scale?



Computational homogenization for foamed materials

- Challenges
  - Micro-structure
    - Not perfect with non periodic mesh

    ⇨ How to constrain the periodic boundary conditions?

    - Thin components
    - Experiences micro-buckling

    ⇨ How to capture the instability?

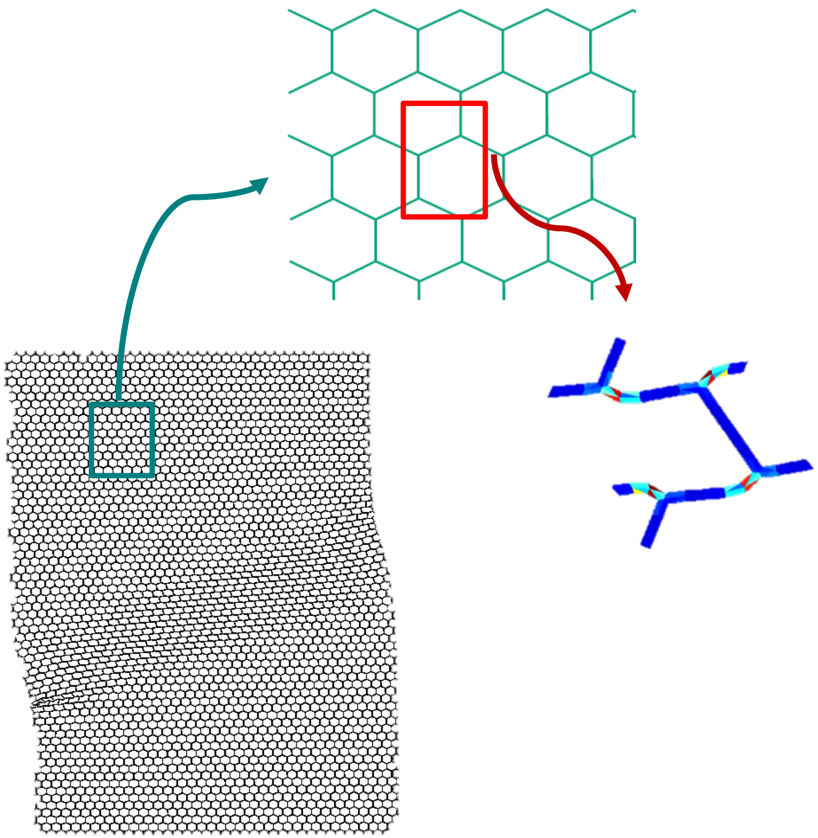
  - Transition
    - Homogenized tangent not always elliptic
    - Localization bands

    ⇨ How can we recover the solution unicity at the macro-scale?

  - Macro-scale
    - Localization bands

    ⇨ How to remain computationally efficient

    ⇨ How to capture the instability?



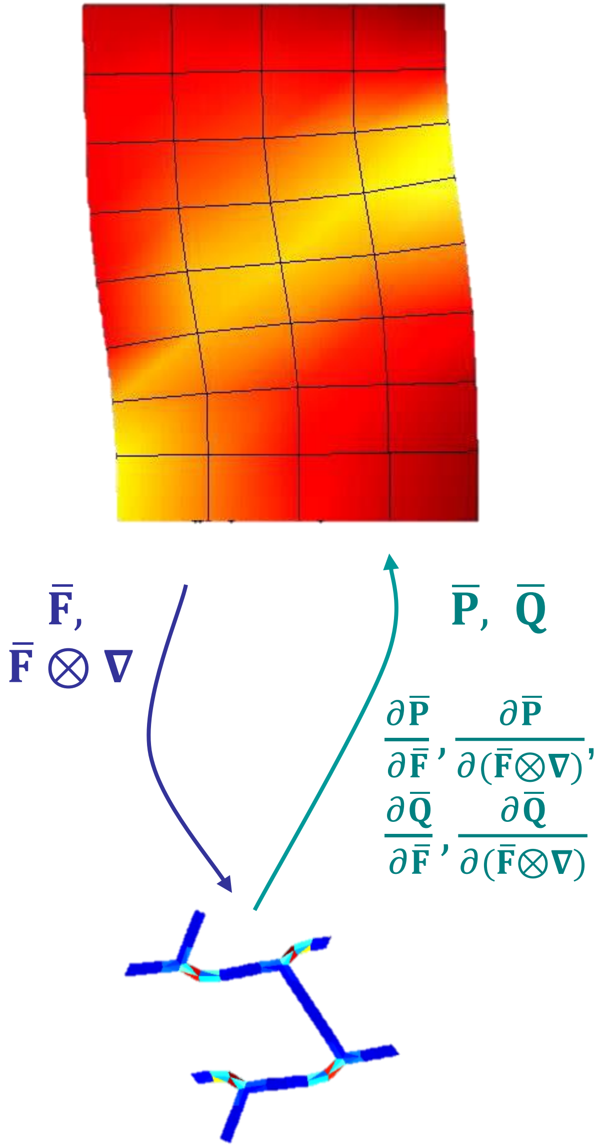
Computational homogenization for foamed materials

- Recover solution unicity: second-order FE²
  - Macro-scale
    - High-order Strain-Gradient formulation

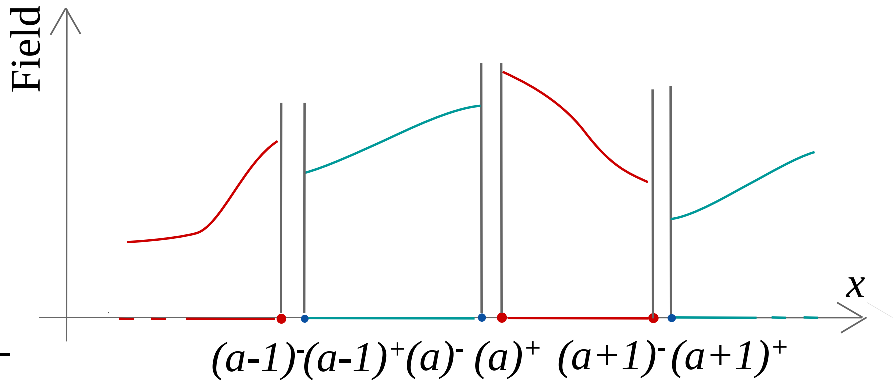
$$\bar{\mathbf{P}}(\bar{\boldsymbol{X}}) \cdot \boldsymbol{\nabla}_0 - \bar{\mathbf{Q}}(\bar{\boldsymbol{X}}) : (\boldsymbol{\nabla}_0 \otimes \boldsymbol{\nabla}_0) = 0$$

    - Partitioned mesh (//)
  - Transition
    - Gauss points on different processors
    - Each Gauss point is associated to one mesh and one solver
  - Micro-scale
    - Usual continuum

$$\mathbf{P}(\boldsymbol{X}) \cdot \boldsymbol{\nabla}_0 = 0$$



- Discontinuous Galerkin (DG) implementation of the second order continuum
  - Finite-element discretization
  - Same **discontinuous** polynomial approximations for the
    - **Test** functions φ_h and
    - **Trial** functions $\delta\varphi$
  - Definition of operators on the interface trace:
    - **Jump** operator: $[\![\cdot]\!] = \cdot^{+} - \cdot^{-}$
    - **Mean** operator: $\langle\cdot\rangle = \frac{\cdot^{+} + \cdot^{-}}{2}$



  - Continuity is weakly enforced, such that the method
    - Is consistent
    - Is stable
    - Has the optimal convergence rate
  - Can be used to weakly enforce higher discontinuities

- Second-order FE2 method
  - Macro-scale second order continuum

$$\bar{\mathbf{P}}(\bar{\boldsymbol{X}}) \cdot \boldsymbol{\nabla}_0 - \bar{\mathbf{Q}}(\bar{\boldsymbol{X}}) : (\boldsymbol{\nabla}_0 \otimes \boldsymbol{\nabla}_0) = 0$$

  - Requires $\mathbb{C}^1$ shape functions on the mesh
  - The $\mathbb{C}^1$ can be weakly enforced using the DG method

$$a(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) = a^{\mathrm{bulk}}(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) + a^{\mathrm{PI}}(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) + a^{\mathrm{QI}}(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) = b(\delta\bar{\boldsymbol{u}})$$

- Second-order FE2 method
  - Macro-scale second order continuum

$$\bar{\mathbf{P}}(\bar{\boldsymbol{X}}) \cdot \boldsymbol{\nabla}_0 - \bar{\mathbf{Q}}(\bar{\boldsymbol{X}}) : (\boldsymbol{\nabla}_0 \otimes \boldsymbol{\nabla}_0) = 0$$

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$$a(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) = a^{\text{bulk}}(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) + a^{\text{PI}}(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) + a^{\text{QI}}(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) = b(\delta\bar{\boldsymbol{u}})$$

- Usual volume terms

$$a^{\text{bulk}}(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) = \int_{\bar{V}} [\bar{\mathbf{P}}(\bar{\boldsymbol{u}}) : (\delta\bar{\boldsymbol{u}} \otimes \boldsymbol{\nabla}_0) + \bar{\mathbf{Q}}(\bar{\boldsymbol{X}}) \vdots (\delta\bar{\boldsymbol{u}} \otimes \boldsymbol{\nabla}_0 \otimes \boldsymbol{\nabla}_0)] dV$$

- Second-order FE2 method
  - Macro-scale second order continuum

$$\bar{\mathbf{P}}(\bar{\boldsymbol{X}}) \cdot \boldsymbol{\nabla}_0 - \bar{\mathbf{Q}}(\bar{\boldsymbol{X}}) : (\boldsymbol{\nabla}_0 \otimes \boldsymbol{\nabla}_0) = 0$$

  - Requires $\mathbb{C}^1$ shape functions on the mesh
  - The $\mathbb{C}^1$ can be weakly enforced using the DG method

$$a(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) = a^{\mathrm{bulk}}(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) + a^{\mathrm{PI}}(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) + a^{\mathrm{QI}}(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) = b(\delta\bar{\boldsymbol{u}})$$

    - Weak enforcement of the $\mathbb{C}^0$
      - Continuity
      - Consistency
      - Stability

    between the finite elements

$$a^{\mathrm{PI}}(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) = \int_{\partial_I \bar{V}} \begin{bmatrix} [\![\delta\bar{\boldsymbol{u}}]\!] \cdot \langle \bar{\mathbf{P}} - \bar{\mathbf{Q}} \cdot \boldsymbol{\nabla}_0 \rangle \cdot \bar{\boldsymbol{N}} + [\![\bar{\boldsymbol{u}}]\!] \cdot \langle \bar{\mathbf{P}}(\delta\bar{\boldsymbol{u}}) - \bar{\mathbf{Q}}(\delta\bar{\boldsymbol{u}}) \cdot \boldsymbol{\nabla}_0 \rangle \cdot \bar{\boldsymbol{N}} + \\ [\![\bar{\boldsymbol{u}}]\!] \otimes \bar{\boldsymbol{N}} : \langle \frac{\beta_P}{h_s} \mathbf{C}^{\mathbf{0}} \rangle : [\![\delta\bar{\boldsymbol{u}}]\!] \otimes \bar{\boldsymbol{N}} \end{bmatrix} dV$$

  - Allows efficient parallelization as elements are disjoint

- Second-order FE2 method
  - Macro-scale second order continuum

$$\bar{\mathbf{P}}(\bar{\boldsymbol{X}}) \cdot \boldsymbol{\nabla}_0 - \bar{\mathbf{Q}}(\bar{\boldsymbol{X}}) : (\boldsymbol{\nabla}_0 \otimes \boldsymbol{\nabla}_0) = 0$$

  - Requires $\mathbb{C}^1$ shape functions on the mesh
  - The $\mathbb{C}^1$ can be weakly enforced using the DG method

$$a(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) = a^{\text{bulk}}(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) + a^{\text{PI}}(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) + a^{\text{QI}}(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) = b(\delta\bar{\boldsymbol{u}})$$

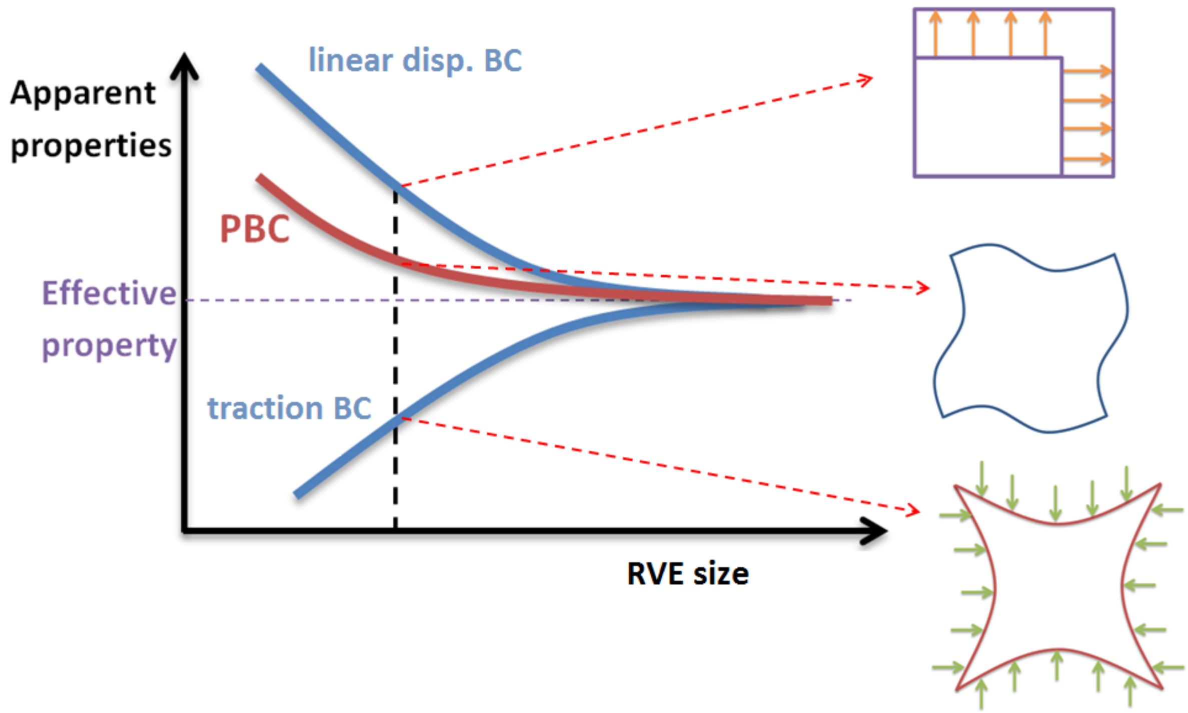
- Weak enforcement of the $\mathbb{C}^1$
  - Continuity
  - Consistency
  - Stability

between the finite elements

$$a^{\text{QI}}(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) = \int_{\partial_I \bar{V}} \begin{bmatrix} [\![\delta\bar{\boldsymbol{u}} \otimes \boldsymbol{\nabla}_{\mathbf{0}}]\!] \cdot \langle \bar{\mathbf{Q}} \rangle \cdot \bar{\boldsymbol{N}} + [\![\bar{\boldsymbol{u}} \otimes \boldsymbol{\nabla}_{\mathbf{0}}]\!] \cdot \langle \bar{\mathbf{Q}}(\delta\bar{\boldsymbol{u}}) \rangle \cdot \bar{\boldsymbol{N}} + \\ [\![\bar{\boldsymbol{u}} \otimes \boldsymbol{\nabla}_{\mathbf{0}}]\!] \otimes \bar{\boldsymbol{N}} : \langle \frac{\beta_P}{h_s} \mathbf{J}^{\mathbf{0}} \rangle : [\![\delta\bar{\boldsymbol{u}} \otimes \boldsymbol{\nabla}_{\mathbf{0}}]\!] \otimes \bar{\boldsymbol{N}} \end{bmatrix} dV$$

  - Allows efficient parallelization as elements are disjoint

- Micro-scale periodic boundary conditions
  - Convergence in terms of RVE size



  - Periodic boundary condition is the optimum choice for periodic structures
  - Periodic boundary condition remains the optimum choice for non-periodic structures

Computational homogenization for foamed materials

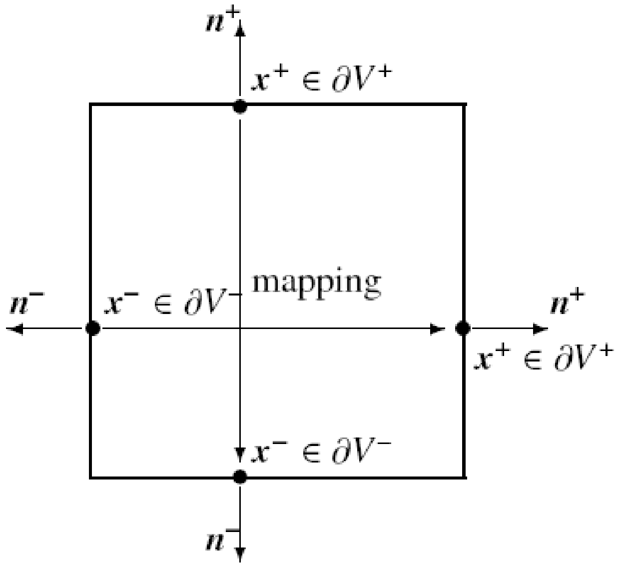
- Micro-scale periodic boundary conditions (2)
  - Defined from the fluctuation field

$$\boldsymbol{w} = \boldsymbol{u} - (\bar{\mathbf{F}} - \mathbf{I}) \cdot \boldsymbol{X} + \frac{1}{2}(\bar{\mathbf{F}} \otimes \boldsymbol{\nabla}_0) : (\boldsymbol{X} \otimes \boldsymbol{X})$$

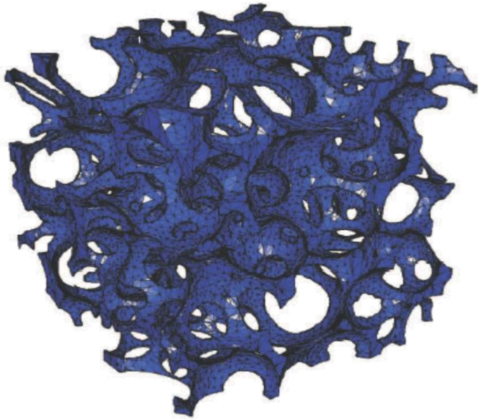
  - Stated on opposite RVE sizes

$$\begin{cases} \boldsymbol{w}(\boldsymbol{X}^+) = \boldsymbol{w}(\boldsymbol{X}^-) \\ \int_{\partial V^-} \boldsymbol{w}(\boldsymbol{X}) d\partial V = \boldsymbol{0} \end{cases}$$

  - Can be achieved by constraining opposite nodes

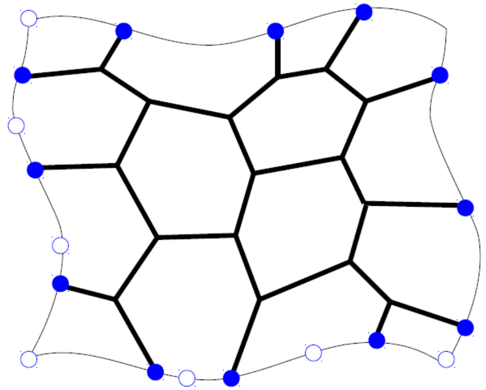


- Foamed materials
  - Usually random meshes
  - Important voids on the boundaries
- Honeycomb structures
  - Not periodic due to the imperfections



- Micro-scale periodic boundary conditions (2)
  - New interpolant method

$$\begin{cases} \boldsymbol{w}(\boldsymbol{X}^-) = \sum_k \mathrm{N}(\boldsymbol{X})\boldsymbol{w}^k \\ \boldsymbol{w}(\boldsymbol{X}^+) = \sum_k \mathrm{N}(\boldsymbol{X})\boldsymbol{w}^k \\ \int_{\partial V^-} \left(\sum_k \mathrm{N}(\boldsymbol{X})\boldsymbol{w}^k \right) \boldsymbol{d\partial V} = \boldsymbol{0} \end{cases}$$

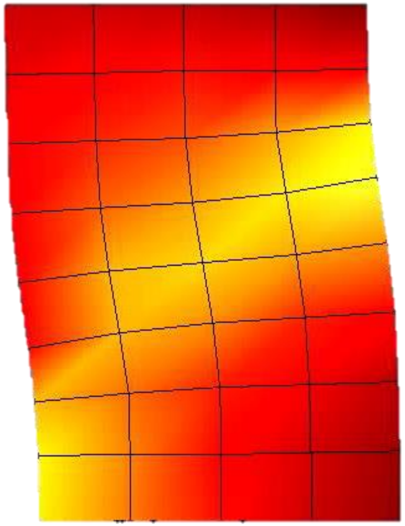


  - Use of Lagrange, cubic spline .. interpolations
  - Fits for
    - Arbitrary meshes
    - Important voids on the RVE sides
  - Results in new constraints in terms of the boundary and control nodes displacements

$$\tilde{\boldsymbol{C}}\, \tilde{\boldsymbol{u}}_b - \boldsymbol{g}(\bar{\mathbf{F}}, \bar{\mathbf{F}} \otimes \boldsymbol{\nabla}_{\mathbf{0}}) = 0$$

Computational homogenization for foamed materials

- Capturing instabilities
  - Macro-scale: localization bands



- Path following method on the applied loading

$$a(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) = \bar{\mu}\, b(\delta\bar{\boldsymbol{u}})$$

- Arc-length constraint on the load increment

$$\bar{h}(\Delta\bar{\boldsymbol{u}}, \Delta\bar{\mu}) = \frac{\Delta\bar{\boldsymbol{u}} \cdot \Delta\bar{\boldsymbol{u}}}{\bar{X}_0^2} + \Delta\bar{\mu}^2 - \Delta L^2 = 0$$

- Capturing instabilities
  - Macro-scale: localization bands
    - Path following method on the applied loading

$$a(\bar{\boldsymbol{u}}, \delta\bar{\boldsymbol{u}}) = \bar{\mu}\, b(\delta\bar{\boldsymbol{u}})$$

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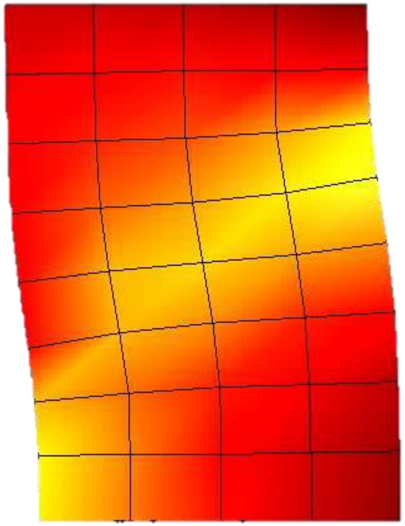
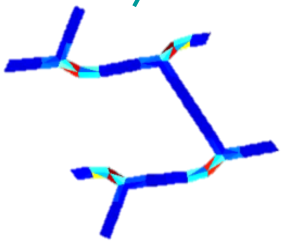


  - Micro-scale
    - Path following method on the applied boundary conditions

$$\tilde{\boldsymbol{C}}\,\tilde{\boldsymbol{u}}_b - \boldsymbol{g}(\bar{\mathbf{F}}, \bar{\mathbf{F}} \otimes \boldsymbol{\nabla}_0) = 0$$

$$\begin{cases} \bar{\mathbf{F}} = \bar{\mathbf{F}}_0 + \mu\, \Delta\bar{\mathbf{F}} \\ \bar{\mathbf{F}} \otimes \boldsymbol{\nabla}_0 = (\bar{\mathbf{F}} \otimes \boldsymbol{\nabla}_0)_0 + \mu\, \Delta(\bar{\mathbf{F}} \otimes \boldsymbol{\nabla}_0) \end{cases}$$

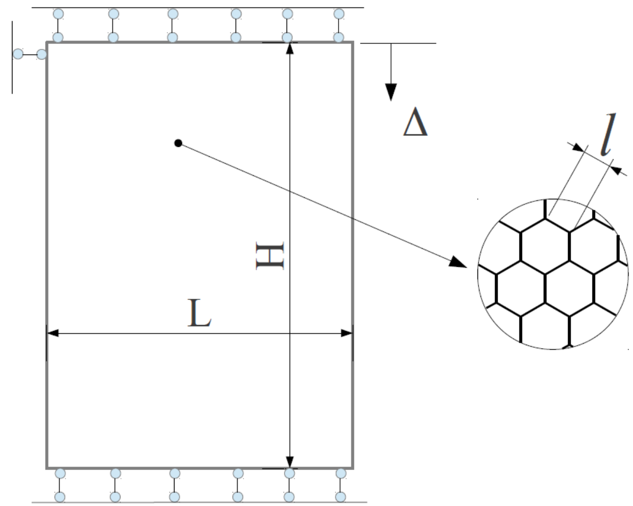


    - Arc-length constraint on the load increment

$$h(\Delta\boldsymbol{u}, \Delta\mu) = \frac{\Delta\boldsymbol{u} \cdot \Delta\boldsymbol{u}}{X_0^2} + \Delta\mu^2 - \Delta l^2 = 0$$

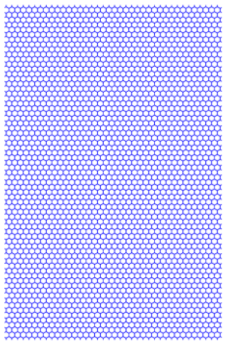
Computational homogenization for foamed materials

- Compression of an hexagonal honeycomb
  - Elasto-plastic material

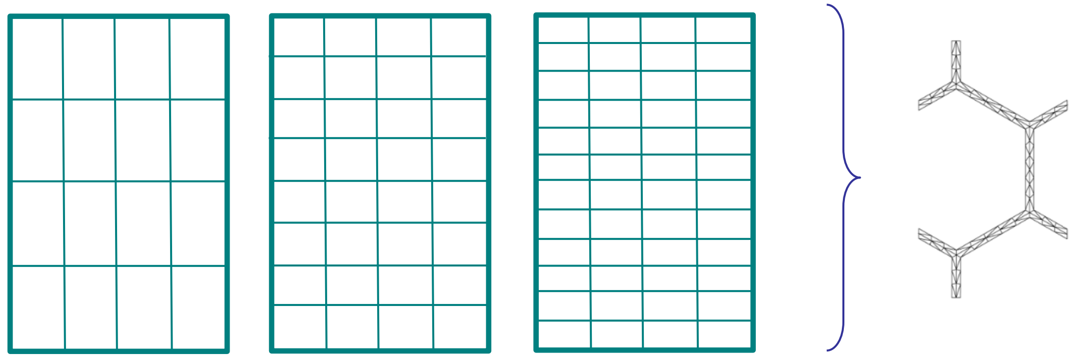


- Comparison of different solutions

Full direct simulation

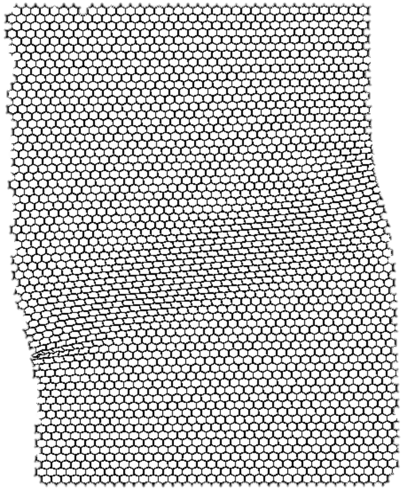


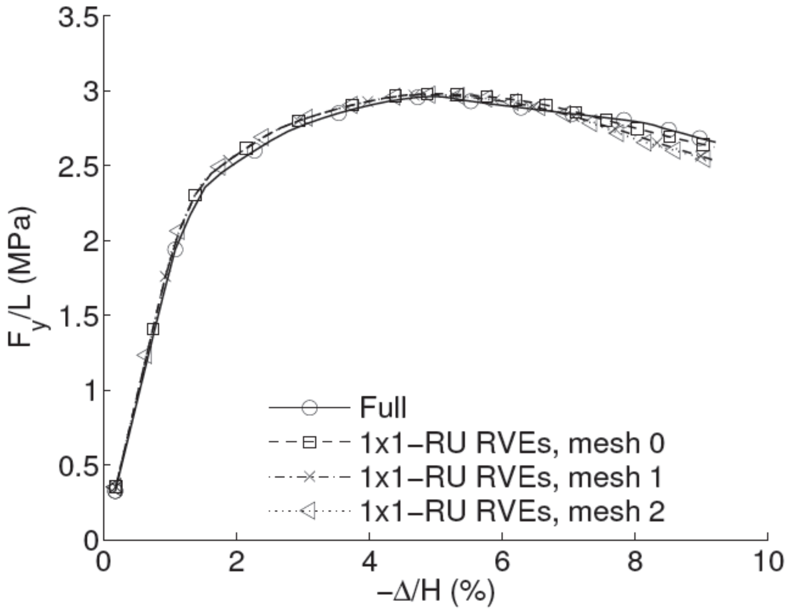
Multiscale with different macro-meshes

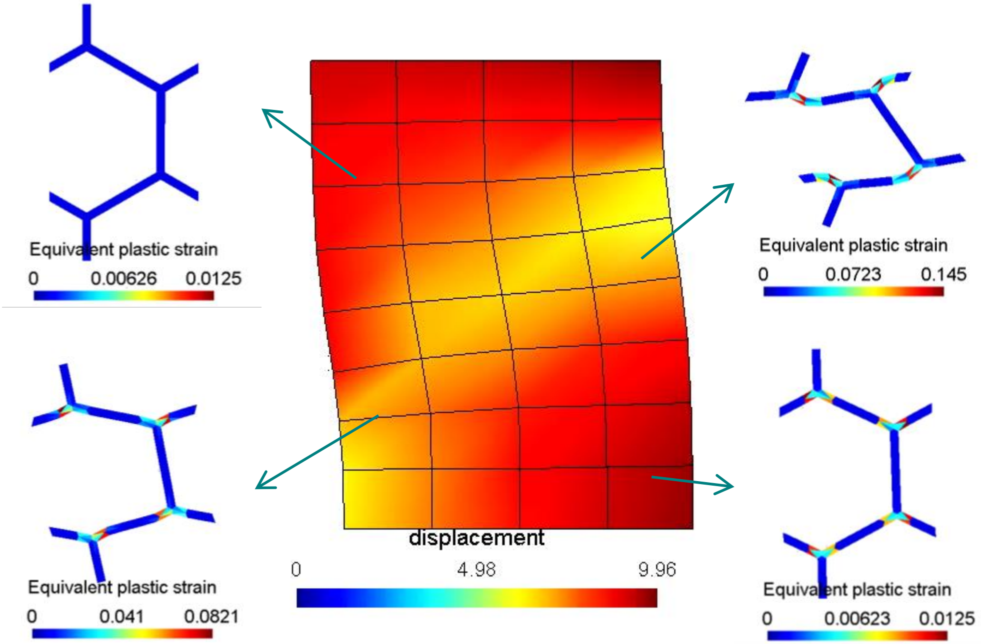


Computational homogenization for foamed materials

- Compression of an hexagonal honeycomb (2)
  - Captures of the softening onset
  - Captures the softening response
  - No macro-mesh size effect

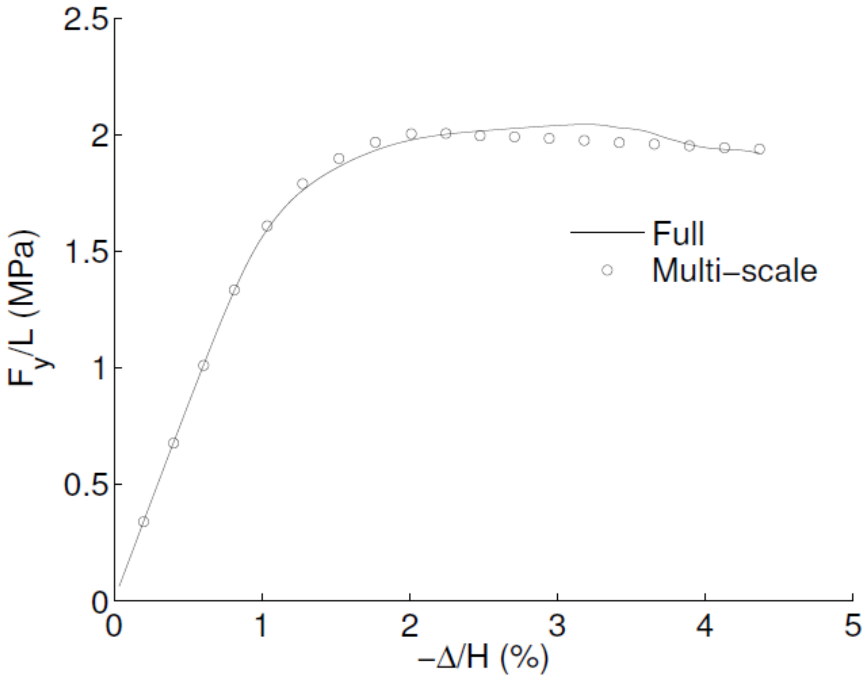


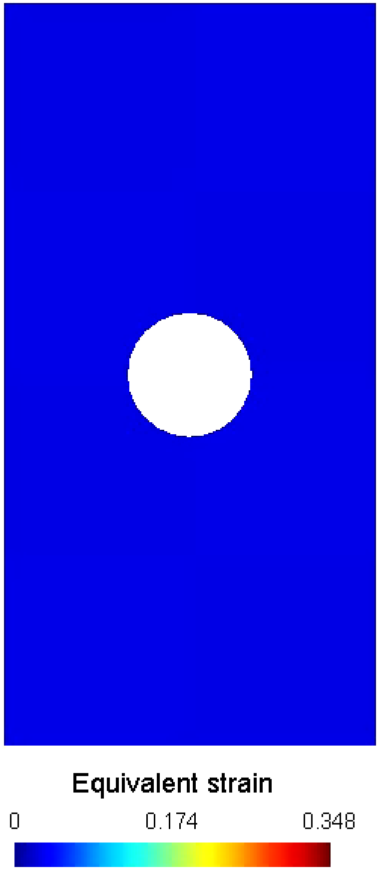


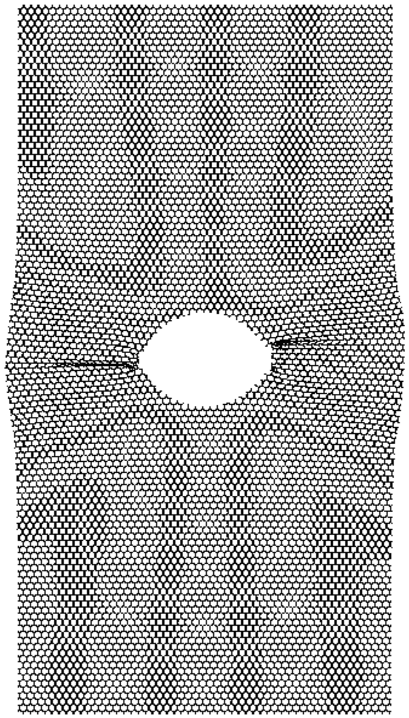


Computational homogenization for foamed materials

- Compression of an hexagonal honeycomb plate with a centered hole
  - Results given by full and multi-scale models are comparable







Conclusions

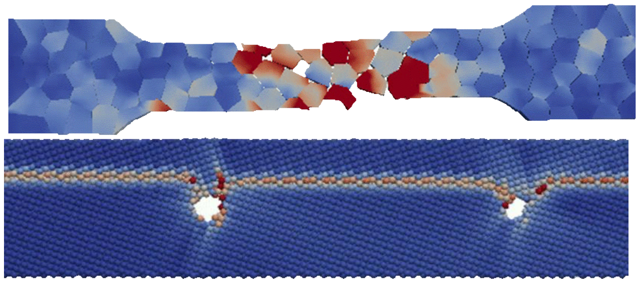
- Non-local damage-enhanced mean-field-homogenization
  - MFH with damage model for the matrix material
  - Non-local implicit formulation
  - Can capture the strain softening
  - More in
    - 10.1016/j.ijsolstr.2013.07.022
    - 10.1016/j.ijplas.2013.06.006
    - 10.1016/j.cma.2012.04.011
    - 10.1007/978-1-4614-4553-1_13
- Computational homogenization for foamed materials
  - Second-order FE^2 method
  - Micro-buckling propagation
  - General way of enforcing PBC
  - More in
    - 10.1016/j.cma.2013.03.024
    - 10.1016/j.commatsci.2011.10.017
    - 10.1016/j.ijsolstr.2014.02.029
- Open-source software
  - Implemented in GMSH
    - http://geuz.org/gmsh/

Computational & Multiscale Mechanics of Materials

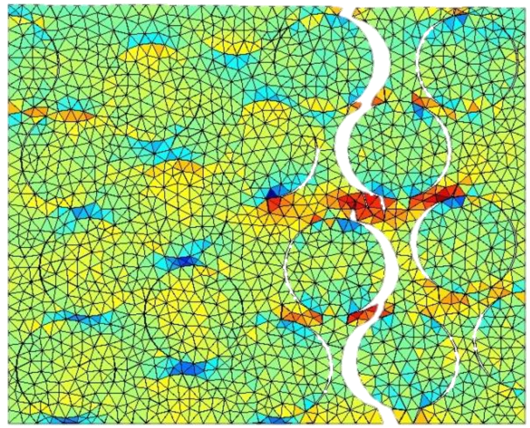
CM3

www.ltas-cm3.ulg.ac.be

Ludovic Noels, G. Becker,
L. Homsi, V. Lucas, S. Mulay,
V.-D. Nguyen, V. Péron-Lührs,
V.-H. Truong, F. Wan, L. Wu

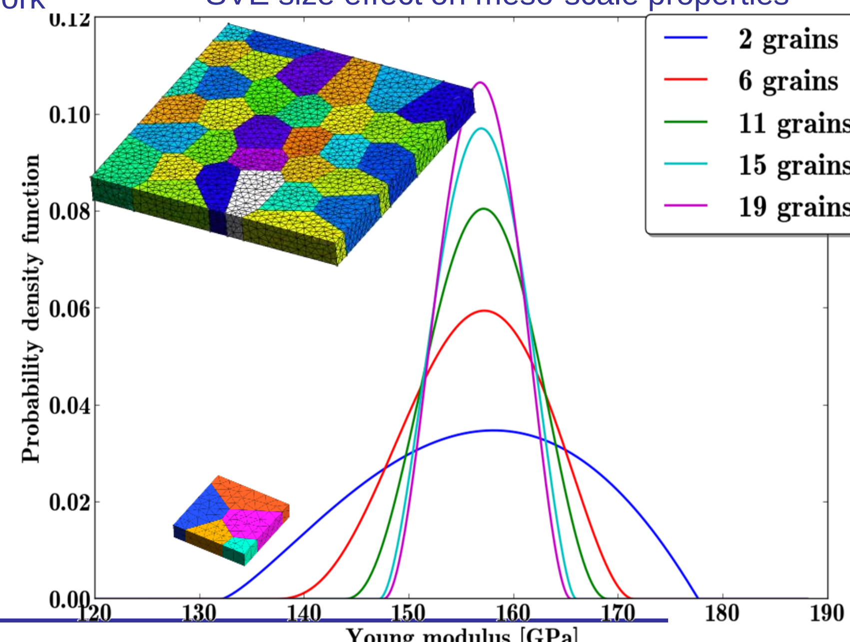


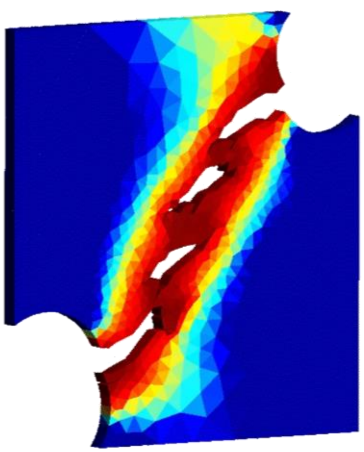
QC method for grain-boundary sliding



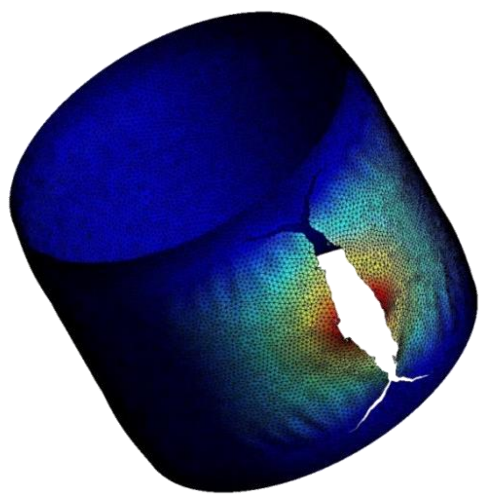
DG-based fracture framework

SVE size effect on meso-scale properties





Damage to crack transition



DG-based fracture framework

Université de Liège