Eulerian Formulation of the Torque and Drag Problem

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1 Introduction

The petroleum industry relies on several kilometer long drillstrings to transmit the axial force and torque necessary to drill the rock formations and reach deep hydrocarbon reservoirs. The assessment of the energy loss along the drillstring, known as the *torque and drag problem*, plays an essential role in well planning and drilling as the friction appearing at the contacts between the drillstring and the borehole may dramatically increase the costs or, even, be a limiting factor in some configurations [1].

The identification of the number of contacts as well as their positions and extents constitutes the central concern of this question. The nonlinearities associated with the large deflection of the drillstring and the non-penetration condition as well as the *a priori* unknown number of contacts, however, make the use of conventional numerical tools rather inefficient. Additionally, the commonly adopted division of the problem in segments bounded by two contacts [2, 3] requires to solve the governing equations on domains which are initially unknown and, therefore, leads to the establishment of integral constraints on the unknown length of a rod forced to go through fixed points in space. Finally, the assessment of the unilateral contact condition, which requires in principle the comparison of two curves parameterized by distinct curvilinear coordinates (e.g. the drillstring and borehole axes), prove to be a rather intensive computational task.

For inextensible drillstrings constrained in the vertical plane, the abovementioned difficulties have been circumvented by reformulating the problem within the Eulerian framework associated with the borehole and by describing the drillstring deformed configuration by means of its transverse position relative to the borehole axis [4, 5, 6]. Generalizing these concepts to three-dimensional configurations, the drillstring is modeled by means of the special Cosserat rod

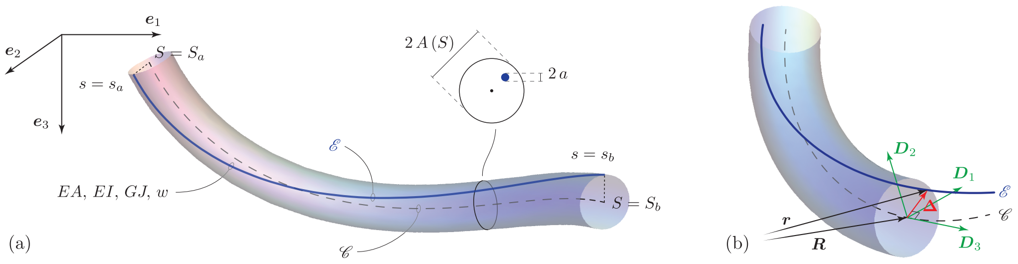

Figure 1: (a) Description of the canonical problem, and (b) Decomposition of the position vector $\boldsymbol{r}\,(s) = \boldsymbol{R}\,(S) + \boldsymbol{\Delta}\,(S)$.

theory, which is reformulated within the Eulerian framework of a, perfectly stiff, generic conduit. The axis $\mathscr{C}$ of this conduit is further assumed to be known and regular. The proposed Eulerian reformulation hinges on describing the drillstring deformed configuration by means of its deflection relative to the borehole axis and expressing the kinematical as well as mechanical quantities pertaining to the drillstring in terms of the curvilinear coordinate associated with the borehole.

Problem definition. The canonical problem considered in this paper consists of a segment of drillstring either *(i)* bounded by two contacts with the borehole and subjected to a known external loading (e.g. weight, applied torque and tension at one extremity), or *(ii)* along a continuous contact with the borehole, see Figure 1. These two distinct but complementary configurations essentially differ by the nature of the body force acting on the drillstring. While solely subjected to its weight in the absence of contact, the drillstring is additionally compelled to lay on the borehole surface along a continuous contact. Assuming frictionless interactions, the resulting contact pressure acts as an additional distributed body force operating normally to the wall of the borehole, its magnitude being however unknown. The length of the drillstring spanning these elementary boundary value problems and satisfying the associated boundary conditions is *a priori* unknown and, therefore, constitutes an inherent part of the solution.

2 Lagrangian Formulation

To solve this two-point boundary value problem, let us define the right-handed orthonormal basis $\{\boldsymbol{e}_k\}$ for the Euclidean space $\mathbb{E}^3$ and denote by $\boldsymbol{r}\,(s) = x_k\,\boldsymbol{e}_k$ the position vector of the drillstring axis with curvilinear coordinate s. The parameter s, which is referred to as the (natural) Lagrangian coordinate, identifies the drillstring cross section in its unstressed configuration $\boldsymbol{r}^0\,(s)$. To fully characterize the spatial configuration of the drillstring, one has to additionally supply the space curve $\mathscr{E}$, defined by $\boldsymbol{r}(s)$, with a vector characterizing the orientation of the cross section. Defining a pair of orthonormal vectors $\boldsymbol{d}_1(s)$, $\boldsymbol{d}_2(s)$ along the principal axes of inertia in the normal cross-section, the drillstring is entirely defined by the two vector-valued functions

$$[s_a, s_b] \ni s \mapsto \boldsymbol{r}(s), \boldsymbol{d}_1(s) \in \mathbb{E}^3, \tag{1}$$

with $s_a < s_b$ corresponding to the extremities of the canonical problem and such that the length of the drillstring, in its unstressed configuration, is $\ell = s_b - s_a$.

2.1 Governing Equations

The cross section orientation is then described by its normal $\boldsymbol{d}_3(s) = \boldsymbol{d}_1 \times \boldsymbol{d}_2$ such that the resulting triplet of *directors* $\{\boldsymbol{d}_k(s)\}$ constitutes an orthonormal basis for each cross section s. The twist vector $\boldsymbol{u} = u_k\,\boldsymbol{d}_k$, whose components measure flexure and twist, and the stretch vector $\boldsymbol{v} = \alpha\,\boldsymbol{d}_3$ are defined such that

$$\frac{\mathrm{d}\boldsymbol{d}_k}{\mathrm{d}s} = \boldsymbol{u} \times \boldsymbol{d}_k, \qquad \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \boldsymbol{v} \tag{2}$$

where the stretch $\alpha(s) > 0$ measures the rod extension. These strain variables are naturally related to the components of the internal force $\boldsymbol{F}(s) = F_k\,\boldsymbol{d}_k$ and moment $\boldsymbol{M}(s) = M_k\,\boldsymbol{d}_k$ through the following constitutive equations

$$\boldsymbol{M}(s) = EI\,(u_1\,\boldsymbol{d}_1 + u_2\,\boldsymbol{d}_2) + GJ\,u_3\,\boldsymbol{d}_3 \tag{3}$$

$$\boldsymbol{F}(s) = F_1\,\boldsymbol{d}_1 + F_2\,\boldsymbol{d}_2 + EA\,(\alpha - 1)\,\boldsymbol{d}_3 \tag{4}$$

with the bending stiffness EI, the torsional stiffness GJ and the axial stiffness EA of the drillstring. Finally, the conservation of linear and angular momenta yields

$$\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}s} + \boldsymbol{f} = 0 \tag{5}$$

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}s} + \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} \times \boldsymbol{F} = 0 \tag{6}$$

where $\boldsymbol{f}(s)$ is the body force per unit reference length, which embodies the drillstring weight and the eventual contact pressure along continuous contacts.

2.2 Shortcomings

The system (2-6) constitutes the set of equations that governs the drillstring deflection under prescribed boundary conditions. The two-point boundary value problem under consideration necessitates the imposition of the drillstring location at both extremities s_a and s_b of the domain. However, as the parametric coordinates of its axis $x_k(s)$ in the absolute reference frame $\{\boldsymbol{e}_k\}$ read

$$x_k(s) = x_k(s_a) + \int_{s_a}^{s} \boldsymbol{v} \cdot \boldsymbol{e}_k \, \mathrm{d}s, \tag{7}$$

these boundary conditions reduce to a set of integral constraints on the unknown length of the drillstring ℓ in its unstressed configuration. These stiff isoperimetric constraints result in ill-conditioned equations contributing to the numerical burden associated with the conventional Lagrangian formulation [5].

As emphasized in the introduction, the assessment of the unilateral contact condition constitutes a further source of difficulties. To ensure that the drillstring remains either free of contact or in continuous contact along the boundary value problem under consideration, this constraint necessitates the evaluation of the distance between the drillstring $\mathscr{E}$ and the borehole axis $\mathscr{C}$. However, these two curves being naturally parameterized by distinct curvilinear coordinates, the evaluation of this distance reveals to be computationally intensive.

3 Eulerian Formulation

In the global basis, the position vector of a point lying on the borehole axis $\mathscr{C}$ at S is denoted by $\boldsymbol{R}(S) = X_k \, \boldsymbol{e}_k$. The arc-length parameter S, referred to as the Eulerian coordinate in contrast to the Lagrangian coordinate s, identifies a section along $\mathscr{C}$ which consists of all points whose reference positions are on the plane perpendicular to the borehole axis at S. By analogy with the director basis, one may arbitrarily define a triplet $\{\boldsymbol{D}_j(S)\}$ constituting an orthonormal basis for each cross section S along the borehole axis and such that $\boldsymbol{D}_3 = \mathrm{d}\boldsymbol{R}/\mathrm{d}S$ is the unit vector tangent to $\mathscr{C}$. The kinematics of this frame can, therefore, be described by means of the *angular velocity* $\boldsymbol{U}(S) = U_k \, \boldsymbol{D}_k$ as $\boldsymbol{D}_j' = \boldsymbol{U} \times \boldsymbol{D}_j$, where the prime denotes differentiation with respect to the Eulerian coordinate.

As presented in Figure 1(b), the position vector for the rod cross section centroids $\boldsymbol{r}(s) = x_k \, \boldsymbol{e}_k$ can, naturally, be expressed as

$$\boldsymbol{r}(s(S)) = \boldsymbol{R}(S) + \boldsymbol{\Delta}(S), \tag{8}$$

where the *eccentricity vector* $\boldsymbol{\Delta}(S) = \Delta_1 \, \boldsymbol{D}_1 + \Delta_2 \, \boldsymbol{D}_2$ is a measure of the drillstring relative deflection in the cross section S [7]. Besides describing the space curve $\mathscr{E}$ with respect to the borehole axis $\mathscr{C}$, this decomposition of the position vector $\boldsymbol{r}(s)$ connects the Eulerian and Lagrangian formulations through the mapping $S \mapsto s(S)$.

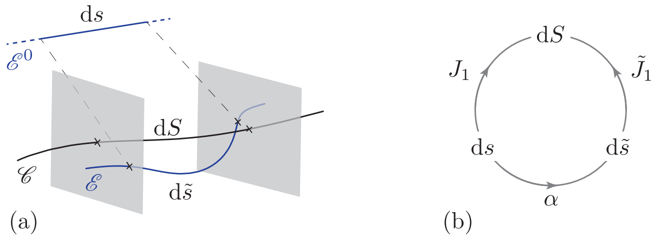

Figure 2: (a) An elementary segment of the reference curve (dS) compared to the corresponding elementary segment of rod in its undeformed (ds) and stretched configurations (d$\tilde{s}$). (b) Correspondence between these three elementary segments and relations between the Jacobians of the respective mappings.

3.1 Mappings and Jacobians

The reformulation of the local equilibrium (5-6) within the Eulerian framework requires to express the natural derivatives $\mathrm{d} \cdot /\mathrm{d}s$ in terms of Eulerian derivatives $\mathrm{d} \cdot /\mathrm{d}S$. While the Jacobian of the mapping $s \mapsto \tilde{s}\,(s)$ from Lagrangian to stretched (parametrizing the rod in its deformed configuration) coordinates can be identified as the stretch $\alpha = \mathrm{d}\tilde{s}/\mathrm{d}s$, the Jacobian of the mapping $S \mapsto s\,(S)$, from Eulerian to Lagrangian coordinates, is obtained by plugging the decomposition (8) in the definition (2) of the stretch vector and applying the chain rule differentiation

$$s'\,(S) = \pm \left\| \boldsymbol{D}_3 + \boldsymbol{\Delta}' \right\| / \left\| \boldsymbol{v} \right\| . \tag{9}$$

This expression emphasizes that the origin of the drift existing between the two curvilinear coordinates is two-fold: *(i)* the eccentricity between the rod and the reference curve, and *(ii)* the stretch $\|\boldsymbol{v}\| = \alpha$ of the drillstring itself, see Figure 2(a). Defining the functions $J_1\,(S) = 1/s'\,(S)$ as the Jacobian of the inverse mapping $s \mapsto S\,(s)$, from Lagrangian to Eulerian coordinates, and $\tilde{J}_1\,(S)$ as the Jacobian of the mapping $\tilde{s} \mapsto S\,(\tilde{s})$, from stretched to Eulerian coordinates, provides the circular relation $J_1 = \alpha\,\tilde{J}_1$ depicted in Figure 2(b).

3.2 Directors and Strain Variables

Through the introduction of the eccentricity vector, the space curve $\mathscr{E}$ characterizing the drillstring in its deformed configuration is expressed by reference to the borehole axis $\mathscr{C}$. Differentiating now (8) with respect to the Eulerian coordinate, the drillstring local inclination reads

$$\boldsymbol{d}_3\,(S) = \tilde{J}_1 \left(\boldsymbol{D}_3 + \boldsymbol{\Delta}' \right) , \tag{10}$$

such that the knowledge of the angle $\varphi(S)$ between either director $\boldsymbol{d}_1$ or $\boldsymbol{d}_2$ and a specified direction is sufficient to fully characterize the drillstring spatial configuration. This rotation of the cross section about $\boldsymbol{d}_3$ can, for instance, be described with respect to the pair $\{\boldsymbol{k}_1, \boldsymbol{k}_2\}$, defined as the image of $\{\boldsymbol{D}_1, \boldsymbol{D}_2\}$ through the rotation mapping $\boldsymbol{D}_3$ on $\boldsymbol{d}_3$.

The strain variables, which are derived from the kinematic relation (2), can in turn be reformulated within the Eulerian framework so that the drillstring configuration, entirely defined by $\boldsymbol{r}(s)$ and $\boldsymbol{d}_1(s)$ in the Lagrangian formulation (1), reduces to the knowledge of the three Eulerian functions

$$[S_a, S_b] \ni S \mapsto \Delta_1(S), \Delta_2(S), \varphi(S) \in \mathbb{R}, \tag{11}$$

with the boundaries $S_a < S_b$ of the canonical problem corresponding to the drillstring extremities s_a and s_b, respectively. While the components Δ_1, Δ_2 of the eccentricity vector describe the drillstring relative deflection with respect to the borehole axis, the angle φ characterizes the rotation of its cross section about the director $\boldsymbol{d}_3$.

3.3 Boundary Value Problem

Projecting the conservation of linear and angular momenta (5-6) in the directors basis, the following mixed order system is obtained

$$J_1\, F_j' + \mathcal{G}_j\left[\alpha, \varphi, \boldsymbol{\Delta}, \boldsymbol{F}, \boldsymbol{U}\right] + f_j = 0, \quad J_1^3\, \Delta_j''' + \mathcal{H}_j\left[\alpha, \varphi, \boldsymbol{\Delta}, \boldsymbol{F}, \boldsymbol{U}\right] = 0, \tag{12}$$

$$J_1\, \alpha' + \mathcal{G}_\alpha\left[\alpha, \varphi, \boldsymbol{\Delta}, \boldsymbol{F}, \boldsymbol{U}\right] + f_3/EA = 0, \qquad J_1^2\, \varphi'' + \mathcal{H}_\varphi\left[\alpha, \varphi, \boldsymbol{\Delta}, \boldsymbol{U}\right] = 0, \tag{13}$$

for $j = 1, 2$; the $\mathcal{H}$ and $\mathcal{G}$'s functionals being obtained by plugging the Eulerian reformulation of the strain variables in the conservation of linear and angular momenta (5-6). This set of six ODE's, involving exclusively Eulerian quantities, is subject to eleven nonlinear boundary conditions of the form

$$\mathcal{B}_i\left[S_i; \alpha, \varphi, \boldsymbol{\Delta}, \boldsymbol{F}, \boldsymbol{U}\right] = 0, \qquad S_i \in [S_a, S_b], \; i = 1, \ldots, 11. \tag{14}$$

Depending on the nature of the problem under consideration, the latter conditions prescribe the eccentricity vector $\boldsymbol{\Delta}$, the inclination of director $\boldsymbol{d}_3$, the internal force $\boldsymbol{F}$ or moment $\boldsymbol{M}$ at $S = S_i$. As an essential outcome of the proposed formulation, the isoperimetric constraints, a source of difficulty in the Lagrangian formulation, disappear and the system of equations (12)-(13) with the boundary conditions (14) constitute a *classical* boundary value problem.

Small Inclination Approximation. The previous developments lead to the rigorous reformulation of the special Cosserat rod theory within the Eulerian framework associated with the borehole axis. Although this reformulation hinges on the description of the drillstring deformed configuration by means of its relative deflection, no assumptions were made on the magnitude of the eccentricity vector. The drillstring is however expected to remain within close range of the borehole axis such that the norm $\|\boldsymbol{\Delta}\|$ be, at most, of the order of the clearance $c(S) = A - a$, with $A(S)$ and a the radii of the borehole and drillstring, respectively. It can actually be shown [7] that, for small values of $\varepsilon = c(S)/L$, the space curve $\mathscr{E}$ may be seen as a perturbation of the borehole axis $\mathscr{C}$. Therefore, provided that the borehole curvature remains reasonably small compared to the drillstring relative deflection, the distinction between Eulerian and the stretched coordinates becomes negligible, i.e., $\tilde{J}_1(S) = 1 + \mathcal{O}\left(\varepsilon^2\right)$ and $J_1(S) = \alpha + \mathcal{O}\left(\varepsilon^2\right)$, which results in a substantial simplification of the $\mathcal{G}$'s and $\mathcal{H}$'s functionals.

Numerical Implementation. To solve numerically the resulting nonlinear boundary value problem, a collocation method has been implemented. Defining a partition π of $[S_a, S_b]$ constituted of N subintervals, we seek an approximate solution $(\boldsymbol{\Delta}^*, \varphi^*, \boldsymbol{F}^*)$ such that $\boldsymbol{\Delta}^* \in \mathcal{P}_{k+3,\pi} \cap C^2[S_a, S_b]$, $\varphi^* \in \mathcal{P}_{k+2,\pi} \cap C^1[S_a, S_b]$ and $\boldsymbol{F}^* \in \mathcal{P}_{k+1,\pi} \cap C^0[S_a, S_b]$ where $k \geq 3$ is the number of collocation points per subinterval and $\mathcal{P}_{n,\pi}$ is the set of all piecewise polynomial functions of order n on the partition π. For reasons of efficiency, stability, and flexibility in order and continuity, B-splines are chosen as basis functions while collocation is applied at Gaussian points [8]. The unknown B-spline coefficients are then determined by solving the nonlinear system of $6\,k\,N$ collocation equations and imposing the 11 boundary conditions. For the continuous contact problem, the reaction pressure is eliminated from equations (12-a), $j = 1, 2$, and the system (12-13) is supplemented by the following equation $\Delta_1^2 + \Delta_2^2 = c^2$.

3.4 Applications

The Eulerian reformulation and the small inclination approximation are compared to the classic Lagrangian formulation in Figure 4. For the purpose of the present example, a fictitious weightless drillstring (OD = 5 in, ID = 4.276 in) is considered centered at both extremities of a helical borehole, which is characterized by its constant axis curvature $\kappa = 1/400\,\mathrm{m}^{-1}$, torsion $\tau = 1/200\,\mathrm{m}^{-1}$ and with a length $L = \pi/\sqrt{\kappa^2 + \tau^2}$. As expected, both Lagrangian and Eulerian formulations provide the same results, the small inclination approximation leading to slightly different fields.

Although the same helical borehole is considered in Figure 4, the drill-

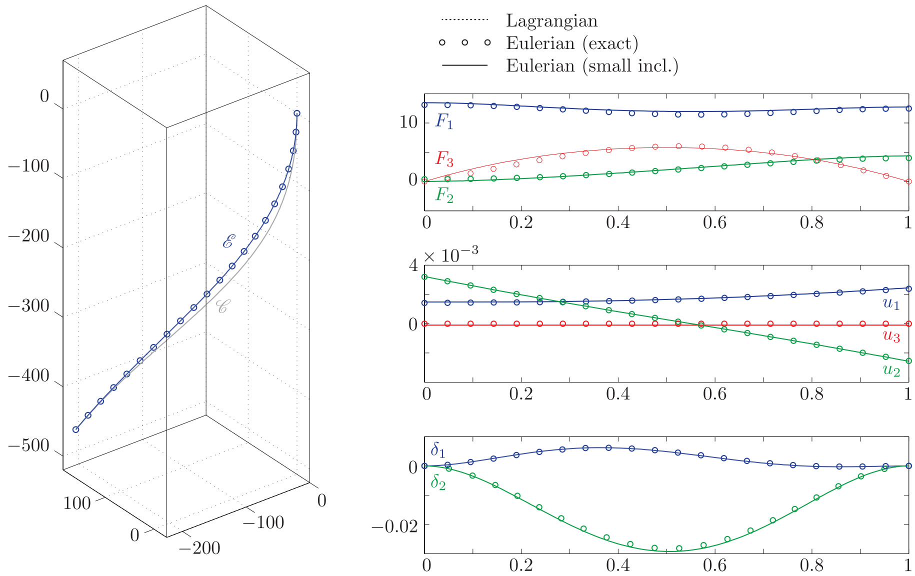

Figure 3: Comparison of the Eulerian reformulation and the small inclination approximation with the classic Lagrangian formulation ($[F_j] = \mathrm{N}$, $[u_j] = \mathrm{m}^{-1}$, $\delta_i = \Delta_i/L$, $k = 5$ and $N = 10$).

string weight ($w = 28.8\,\mathrm{kg/m}$) is now applied and the deformed configuration of the drillstring is computed for three distinct measured depths: $L = 100$, 200 and 300 m. Both extremities of the drillstring are assumed centered on the borehole axis and the axial force applied at the bottom is nul, i.e., $F_3\,(L) = 0$.

4 Conclusion

The Eulerian reformulation of the equations governing the three-dimensional deflection of an extensible drillstring has been achieved by *(i)* introducing the eccentricity vector $\mathbf{\Delta}\,(S)$ and *(ii)* expressing of the drillstring local equilibrium in terms of derivatives with respect to the curvilinear coordinate associated to the borehole axis. The originality of the proposed formulation, which resolves in one stroke a series of issues that afflict the classical Lagrangian approach (isoperimetric constraints and potential contact detection with the wall of a conduit), lays in the *self-feeding* character of the resulting system as the length of the drillstring is subject to self-adjustement in response to the external loading.

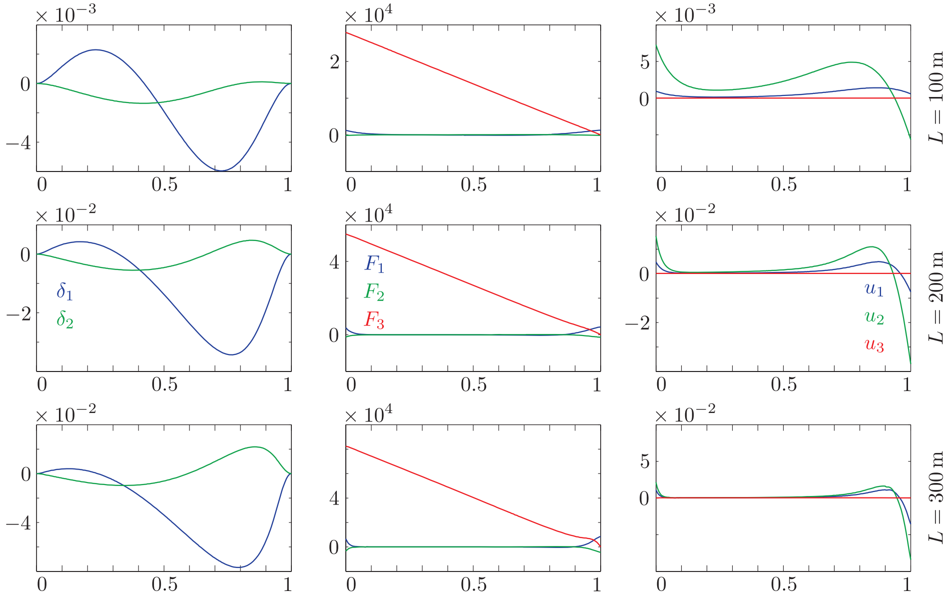

Figure 4: Components of the eccentricity vector and internal efforts for three distinct measured depths: $L = 100$, 200 and 300 m ($k = 5$ and $N = 10$).

References

[1] C. Mason and D. Chen, *Step changes needed to modernise T&D software*, in SPE/IADC Drilling Conference, 2007.

[2] J.-S. Chen and C.-W. Li, *Planar elastica inside a curved tube with clearance*, International Journal of Solids and Structures, 44 (2007), pp. 6173–6186.

[3] V. Denoël, *Advantages of a semi-analytical approach for the analysis of an evolving structure with contacts*, Communications in Numerical Methods in Engineering, 24 (2008), pp. 1667–1683.

[4] V. Denoël and E. Detournay, *The problem of a drillstring inside a curved borehole: a perturbed Eulerian approach*, in First International Colloquium on Non-linear Dynamics of Deep Drilling Systems, March 2009.

[5] V. Denoël and E. Detournay, *Eulerian formulation of constrained elastica*, International Journal of Solids and Structures, 48 (2011), pp. 625–636.

[6] V. Denoël and E. Detournay, *Recent advances in the modeling of a drillstring inside a curved borehole*, in Second International Colloquium on Nonlinear Dynamics and Control of Deep Drilling Systems, May 2012.

[7] A. Huynen, E. Detournay, and V. Denoël, *Eulerian Formulation of the Special Cosserat Theory of Rods*, International Journal of Solids and Structures, Submitted.

[8] U. Ascher, J. Christiansen, and R. D. Russell, *A collocation solver for mixed order systems of boundary value problems*, Mathematics of Computation, 33 (1979), pp. 659–679.