Multifractal analysis of air temperature signals using the wavelet leaders method

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Hölder exponent

Definition

Let f be a signal and x_0 a real number. Then f belongs to the Hölder space $C^{\alpha}(x_0)$ if there exists a polynomial $P_{x_0,\alpha}$ of degree at most α, a positive constant C and a neighborhood V_{x_0} of x_0 satisfying

$$|f(x) - P_{x_0,\alpha}(x)| \leq C|x - x_0|^{\alpha}$$

for all $x \in V_{x_0}$.

Hölder exponent

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for all $x \in V_{x_0}$.

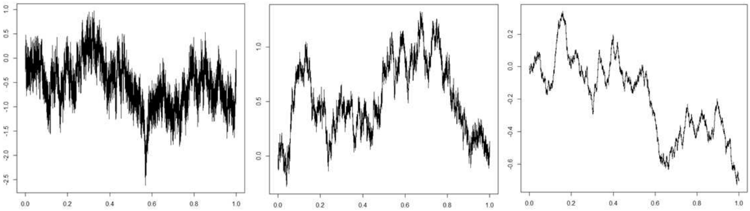
Definition

The Hölder exponent $h(x_0)$ of f at x_0 is defined as the supremum of the exponents α such that f belongs to $C^\alpha(x_0)$:

$$h(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}.$$

Monofractality

- Hölder exponent changes from point to point : f multifractal
- Constant Hölder exponent : f monofractal, i.e. f is regularly irregular
- Example of a monofractal function : fractional Brownian motion



Fractional Brownian motions with Hölder exponents 0.2, 0.4, 0.6 almost surely.

Spectrum of singularities

How to characterize the global regularity of a signal ?

Definition

The spectrum of singularities of f is the Hausdorff dimension of the set of points sharing the same Hölder exponent :

$$d_f : h \mapsto \dim_{\mathcal{H}}(\{x_0 \in \mathbb{R} : h(x_0) = h\}),$$

where $\dim_{\mathcal{H}}(X)$ denotes the Hausdorff dimension of the set X.

Corollary : f is monofractal if and only if its spectrum of singularities is reduced to a single point.

Wavelet leaders method (WLM)

1) Wavelet decomposition of the signal :

$$f(x) = \sum_{j,k \in \mathbb{Z}} c_{j,k} \psi(2^{-j}x - k) = \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda$$

where ψ is a wavelet and $c_{j,k}$ is the wavelet coefficient associated to the dyadic interval λ at scale j and position k :

$$\lambda = \lambda_{j,k} = [2^j k, 2^j(k+1)[$$

and

$$c_{j,k} = 2^{-j} \int_{\mathbb{R}} f(x) \psi(2^{-j}x - k) dx.$$

2) For each λ, compute the wavelet leaders

$$d_\lambda = \sup_{\lambda' \subset \lambda} |c_{\lambda'}|$$

Wavelet leaders method (WLM)

3) Remove the null wavelet leaders and compute

$$S(q,j) = 2^j \sum_{\lambda \in \Lambda_j} d_\lambda^q,$$

where Λ_j is the set of dyadic intervals at scale j.

4) Compute the function τ defined as

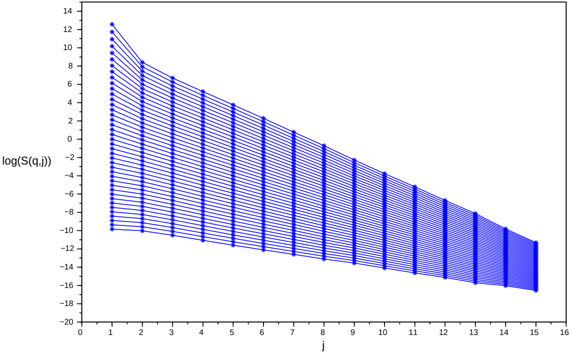
$$\tau(q) = \liminf_{j \to -\infty} \frac{\log(S(q,j))}{\log 2^j},$$

which is numerically obtained through the slopes of linear regressions at small scales of $\log(S(q,j))$ seen as a function of j.

5) One can hope to obtain the spectrum of singularities as

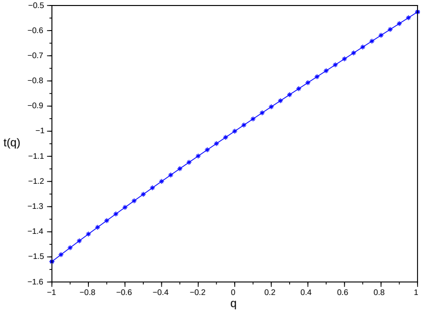
$$d(h) = \inf_q \{qh - \tau(q)\} + 1.$$

Wavelet leaders method (WLM)



$\log(S(q,j))$ for a fractional Brownian motion with Hölder exponent 0.5 with q ranging from -1 to 1.

Wavelet leaders method (WLM)



τ function associated to the previous signal. Linear regression gives a slope of 0.494021.

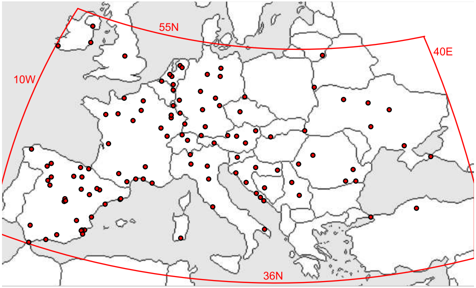
6) Remark : if τ is a straight line, then f is monofractal, in which case the Hölder exponent of f is the slope of τ.

Wavelet leaders method (WLM)

- Remark : f is monofractal if and only if τ is a straight line, in which case the Hölder exponent of f is the slope of τ.
- If f is a monofractal signal with Hölder exponent H, then f belongs to the uniform Hölder space C^H, and a norm in this space is defined by

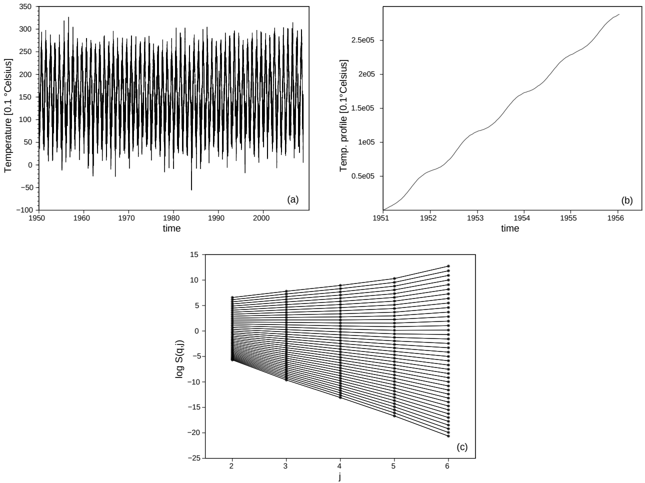
$$\|f\|_{C^H} = \sup_{j,k}\{|c_{j,k}|/2^{jH}\} := N$$

Analyzed data

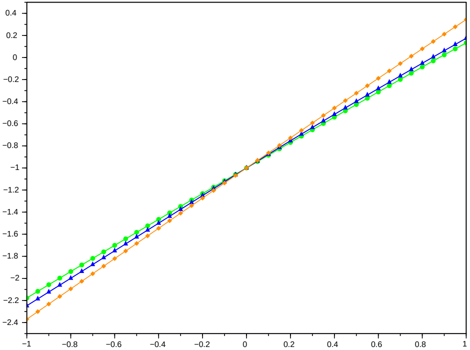


- Daily mean temperature data from 1951 to 2003, calculated as average of minimum and maximum daily temperatures
- Weather stations located below 1000 meters of altitude
- 115 stations selected
- Missing data up to 7%, less than 1% for 97 stations
- Temperature profile used for more stable numerical results (i.e. x_n replaced by $\sum_{j=1}^{n} x_j$.)

Analyzed data



Monofractal nature of the signals



τ functions associated to Aachen (green), Шепетивка (blue) and Rome (orange), with respective slopes 1.156, 1.218, 1.358.

Hölder exponents and norms

- τ linear $\Longrightarrow$ signals are monofractal
- Mean coefficient of determination : $R^2 = 0.9975 \pm 0.0028$
- Hölder exponents ranging from 1.093 to 1.43
- Norms ranging from 8.23 to 30.45

Comparison with the detrended fluctuation analysis ?

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Description of the method

- Another method to define a notion of regularity.
- Seasonal variation of the temperature signal f have to be removed first. If d is a calendar date (e.g. May 22nd), then $\langle f \rangle (d)$ is the average over the years of the values $f(t)$ such that t corresponds to the calendar date d (May 22nd 1951, May 22nd 1952,...).
  The corresponding trend is
  $$\Delta f(t) = f(t) - \langle f \rangle (d).$$
- "Trend profile" used to reduce the noise, i.e. $f(t)$ replaced by $\sum_{u=1}^{t} \Delta f(u)$.

Description of the method

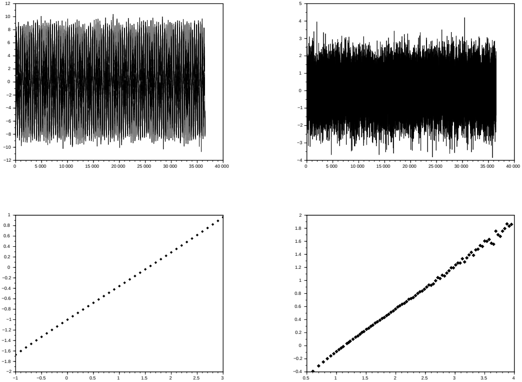
- For a given length l, divide the signal in n segments of length l.
- For each $i = 1, ...n$, determine the best linear fit of segment i and computes the standard deviation $F_i(l)$ of the profile from that straight line.
- Compute the standard deviation of the profile as

$$F(l) = \sqrt{\frac{\sum_{i=1}^{n} F_i(l)}{n}}.$$

- For monofractal signals of exponent γ, we have $F(l) \sim l^{\gamma}$.

Comparison bewteen WLM and DFA

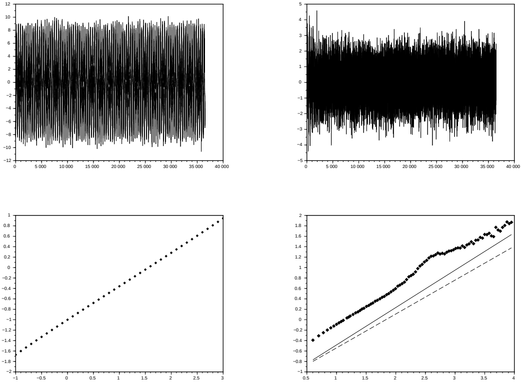
First simulation : Gaussian noise associated to LRC with index 0.65 + $7\sin(\frac{2\pi}{365}t - \frac{\pi}{2})$



Left : WLM. Slope = 0.651104 . Right : DFA. Slope = 0.657925

Comparison bewteen WLM and DFA

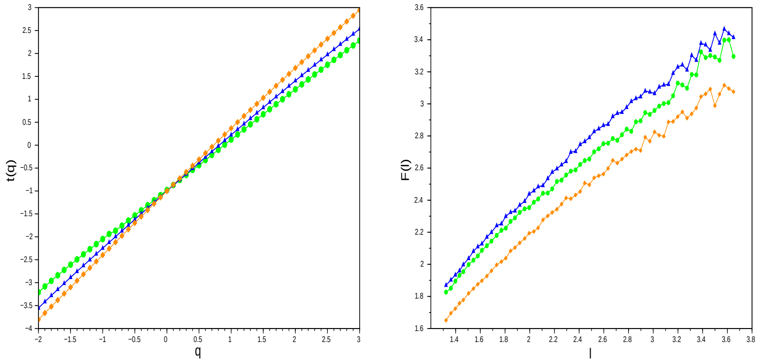
Second simulation : Gaussian noise associated to LRC with index 0.65 + $7\sin(\frac{2\pi}{365}t - \frac{\pi}{2} - \frac{1}{20}\log(t+1))$



Left : WLM. Slope = 0.647673 . Right : DFA. Slope (straight line) = 0.717090 .

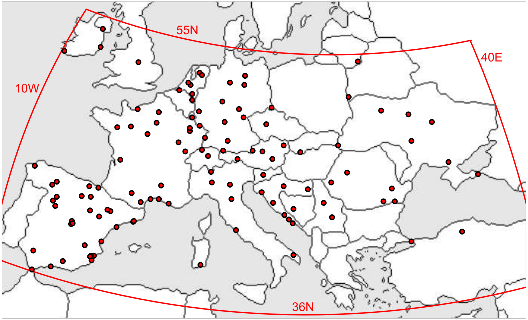
Comparison bewteen WLM and DFA

- On synthetic examples : DFA seems less accurate than WLM.
- On real temperature signals ?



- How to check the "interest" of the results ?
- Idea : temperature variability linked with standard deviation of pressure anomalies.

Relation with pressure anomalies



- Normalize the Hölder exponents from WLM between 0 and 1, as well as those from DFA, and standard deviation of pressure anomalies.
- Consider these values as matrices representing Europe (Mw, Md, Mp).

Relation with pressure anomalies

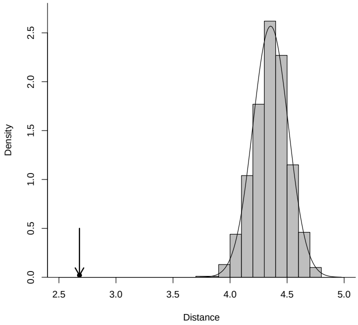
- Normalize the Hölder exponents from WLM between 0 and 1, as well as those from DFA, and standard deviation of pressure anomalies.
- Consider these values as matrices representing Europe (Mw, Md, Mp).
- Compute the Frobenius distance between these matrices :

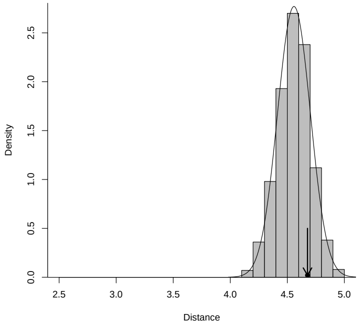
$$d(M,N) = \sqrt{\sum_{i,j}(M_{i,j} - N_{i,j})^2}.$$

  We get $d(Mw, Mp) = 2.68$ and $d(Md, Mp) = 4.67$.
- Are these distances significant ? Confirmation that Mw and Mp are correlated and Md and Mp are not ?

Relation with pressure anomalies

Mw and *Md* shuffled 1000 times and distance with *Mp* measured.





Correlation probable between *Mw* and *Mp* but not between *Md* and *Mp*. Hölder exponents seem to be linked to climate variability.

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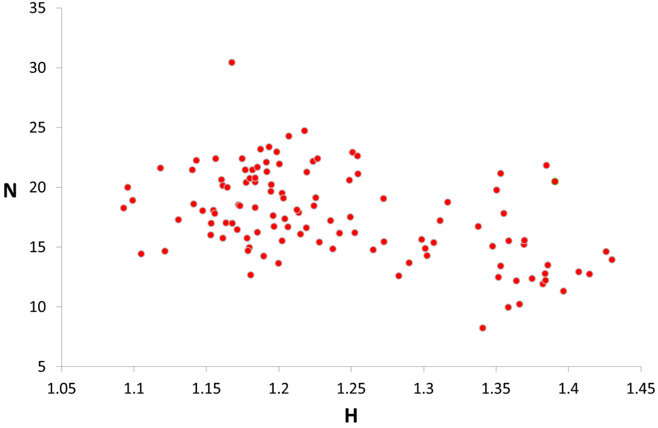
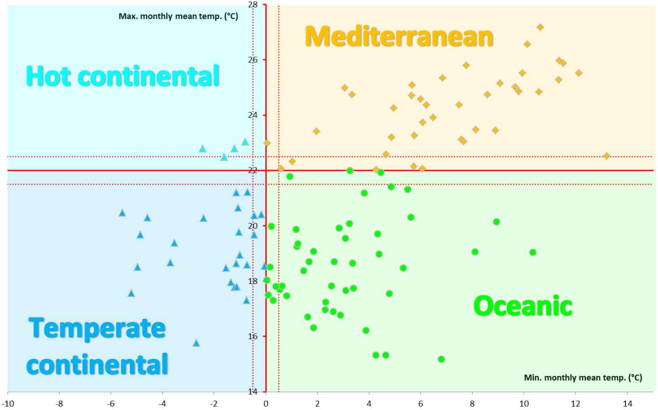
Distribution of the exponents and norms



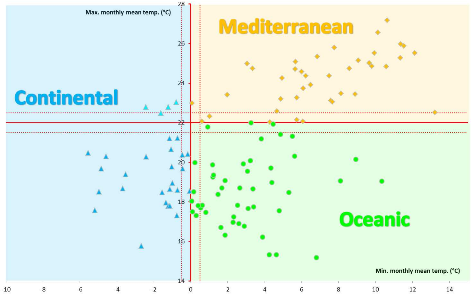
Link with climate types ?

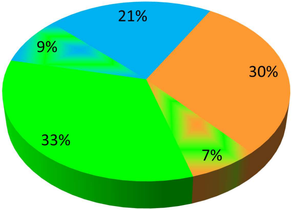
Köppen-Geiger climate classification

Classification based on maximum and minimum monthly mean temperatures (references fixed at 22°C and 0°C). Stations close to 0.5°C of another type of climate were also associated to this second category. Here, precipitations were not taken into account.



Climate distribution





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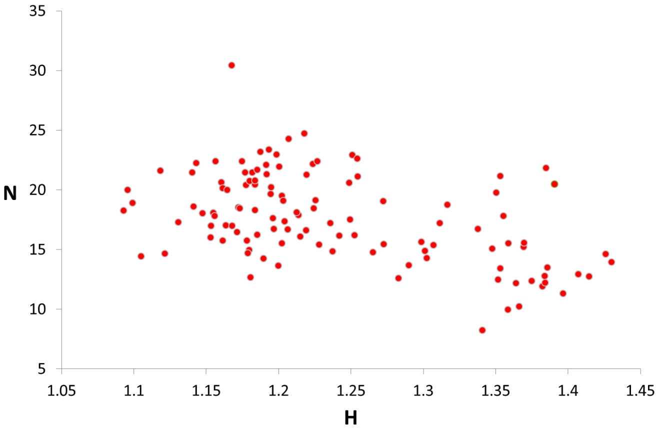

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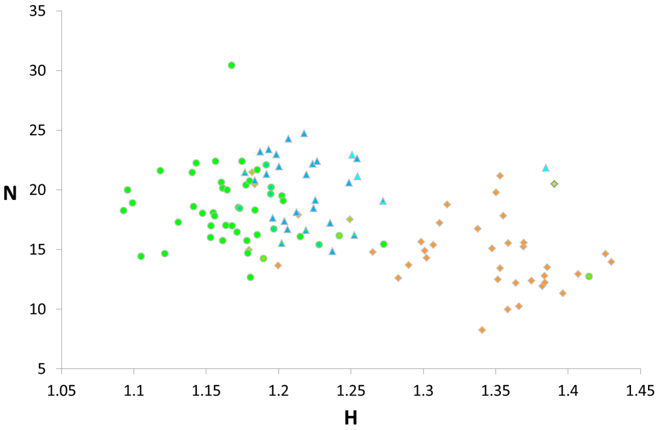
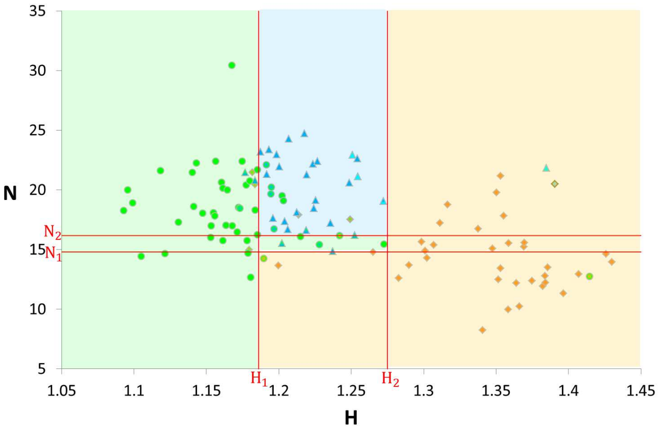


Distribution of the exponents and norms



Hölder spaces-based climate classification and results

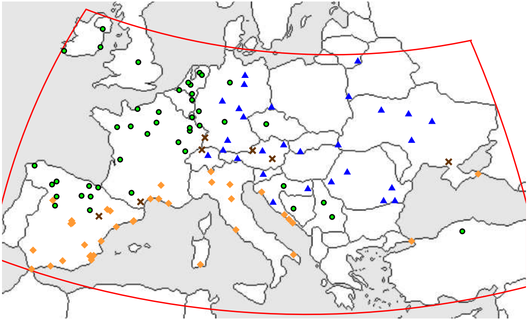
Maximum matching with Köppen-Geiger classification if

$$
\begin{aligned}
H_1 &= 1.186 \\
H_2 &= 1.275 \\
N_1 &= 14.81 \\
N_2 &= 16.18
\end{aligned}
$$

Result : 93.9% correctly associated

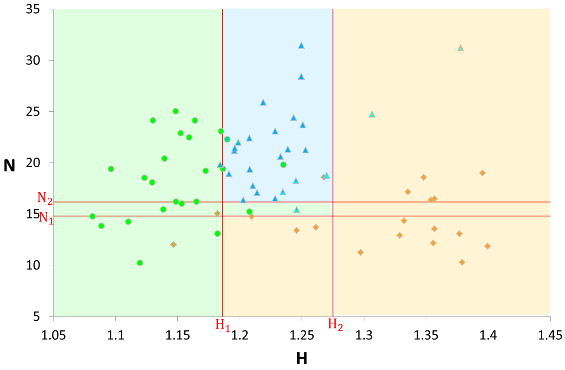
Remark : without the norm, 89.6% correctly associated.

Results on the map



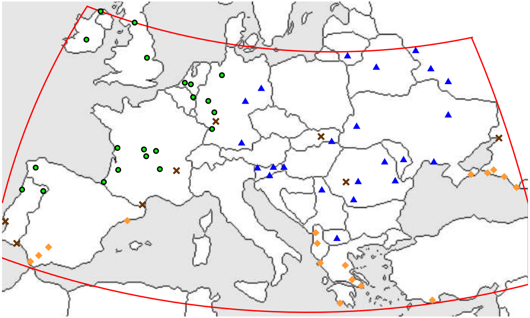
Stations marked with a brown cross are the ones whose type of climate was erroneously predicted. The others were correctly predicted ; green discs stand for Oceanic climate, blue triangles correspond to continental stations and orange orange diamonds are the Mediterranean ones.

Blind test



- 69 other stations
- 40 years of data between 1951 and 2003

Blind test



Result : 88.4% correctly associated

Remark : without the norm, 84.1% correctly associated.

Discussion of the results

Results

Oceanic stations	$\longleftrightarrow$	Lowest Hölder exponents
Continental stations	$\longleftrightarrow$	Intermediate Hölder exponents
Mediterranean stations	$\longleftrightarrow$	Largest Hölder exponents

Discussion

- On a daily basis, Oceanic climate is more irregular than the Continental weather, which is less regular than Mediterranean climate.
- Explanation could be the North Atlantic Oscillation (NAO), anticyclonic conditions in Southern Europe, ...

Conclusions and future work

Conclusions

- WLM shows surface air temperatures signals are monofractal signals
- WLM seems to be more accurate than DFA
- Their belonging to functional spaces reflects their temperature-based Köppen-Geiger climate type
- Algorithm and results confirmed through blind tests

Future work

- Checking of the validity of current climatic models
- Analysis of other climate indices (pressure, precipitation,...)
- Generalization to global temperatures

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