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3D Shape Optimization with X-FEM and a Level Set Constructive Geometry Approach

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1. Abstract
This paper extends previous work on structural optimization with the eXtended Finite Element Method (X-FEM) and the Level Set description of the geometry. The proposed method takes advantage of fixed mesh approach by using an X-FEM structural analysis method and from the geometrical shape representation of the Level Set description. In order to allow the optimization of complex geometries represented with a Level Set description, we apply here a Constructive Solid Geometry (CSG) approach with the Level Set geometrical representation. Hence, this extension allows to optimize any boundary of the structure that is defined with a coumpound Level Set. Design variables are the parameters of basic geometric primitives which are described with a Level Set representation and/or the control points of the NURBS curves that act as the definition of an advanced Level Set primitive. The number of design variables of this formulation remains small whereas global (i.e. compliance or eigenfrequency) and local constraints (i.e. stresses) can be considered. Our results illustrate that fixed grid optimization with X-FEM coupled to a Level Set geometrical description is a promising technique to achieve structural shape optimization.

2. Keywords: Shape optimization, Level Set description, Extended Finite Element Method.

3. Introduction
To achieve the research of the optimal design, the optimization of structures relies generally on the modification of the boundaries, the topology or the material distribution of the model. In shape optimization, the internal and external boundaries of the model are modified while keeping a fixed intial topology. In the classical Finite Element Method (FEM), the modification of these boundaries implies successive adaptation and remeshing step of the mesh to follow the evolution of the model. While shape optimization has reached a certain degree of robustness and sophistication, a major difficulty appear in the mesh management coming from the large shape modifications. The main technical problems comes from the sensitivity analysis. Derivatives have to be regarded as material derivatives and the sensitivity analysis requires the calculation of the so-called velocity field [16]. Actually, 2-D problems are quite well mastered but 3-D problems are still difficult to handle in the most general way. It turns out that shape optimization remains generally quite fragile and delicate to use in industrial context. However, compared to topology optimization this method generally present few design variables and is therefore able to treat various criteria such as restricted displacements, stress criteria, eigenfrequencies, etc.
To circumvent the technical difficulties coming from successive mesh generation and adaptation, various authors have proposed a couple of methods based on fixed grid approach such as the fictitious domain [6], the fixed grid finite elements [7] and the projection methods[10]. In this paper, to avoid the main problems encountered with the classical shape optimization technique, we use another fixed grid approach: the extended finite element method (X-FEM [9]). Besides the advantage of working with of a fixed mesh, this technique allow to treat the void as real zero stiffness material whereas other methods generally simulates void as a weak material such as SIMP in topology optimization. Indeed, this method deals with a clearly defined interface between void and solid part whereas many other techniques do not. Finally, the successive mesh adaptation and generation is suppressed and the same mesh is kept during the whole optimization process.
The X-FEM method is naturally associated with the Level Set description of the geometry to provide an efficient treatment of problems involving moving boundaries or discontinuities. The description of the geometry is represented by the zero iso-contour of an implicit function called the Level Set function [12].

This method is very convenient for the topology optimization as it allows deep and complex changes of the global shape of the model while presenting the smooth curves representation of the Level Set description. In order to provide the structural optimization with the ability to modify any boundary of the structure, we extend here previous Level Set description with a CSG-like approach to build complex geometries. Hence, the whole structure can be implicitly modelled with a Level Set and be modified during the optimization. This work differs from other papers on the subject as we do not use the equation of motion of the Level Set method like [13], [15], [1] nor the nodal Level Set value as design variables like Belytschko *et al.* in [2]. In our work, the design variables are the parameters of basic level set primitives (circles, rectangles, etc.) or NURBS control points defining an advanced Level Set primitive, while various global (compliance, eigenfrequencies) and local responses (stress) can be considered in the formulation.
Numerical applications illustrate the great interest of using X-FEM and level set description on several 2D and 3D applications involving compliance, volume or structural stresses as objective or constraints functions. The outline of this paper is as follows. In the next section, we give a brief introduction of the Level Set Description and the Constructive Solid Geometry applied to Level Set. Then, follows an introduction to the X-FEM. Section 6 details the optimization approach and the applications are presented in section 7.

4. The Level Set Description

In the classical Finite Element Method, the geometry of the structure is generally explicitly defined and the topology of the structure is therefore fixed. Hence, only small geometrical modifications can be taken into account while preserving a correct topology. Moreover, any transformation of this topology usually asks for a wide modification of the model geometry definition. For example, any creation (removal) of holes needs the insertion (suppression) of a new geometrical entity in the model. These limitations are generally considered as the main reason of the low performance generally associated to the shape optimization compared to the performance obtained with a topology optimization method. In order to be able to treat large shape and topological modifications, the present work relies on the Level Set description. This method, which has been developed by Osher and Sethian [12], consists in representing the boundary of the structure with an implicit method thus allowing deep changes of boundaries.
The Level Set method is a numerical technique first developed for tracking moving interfaces. It is based upon the idea of representing implicitly the interfaces as a Level Set curve of a higher dimension function $\psi(\mathbf{x}, t)$. The boundaries of the structure is then conventionnally represented by the zero level ($\psi(\mathbf{x}, t)$=0) of this function ψ, whereas the filled region is then attached to the negative or the positive part of the ψ function. In practice, this function is approximated on a fixed mesh by a discrete function which is usually a signed distance function:

$$\psi(\mathbf{x}, t) = \pm \min_{\mathbf{x}_\Gamma \in \Gamma(t)} \|\mathbf{x} - \mathbf{x}_\Gamma\| \tag{1}$$

The sign is positive (negative) if $\mathbf{x}$ is inside (outside) the boundary defined by $\Gamma(t)$. The evolution of the interfaces is then embedded in the evolution equation for ψ, which is given in [12] by:

$$\frac{\partial \psi(\mathbf{x}, t)}{\partial t} + F \|\nabla \psi\| = 0 \tag{2}$$

$$\psi(\mathbf{x}, t) = 0 \quad given \tag{3}$$

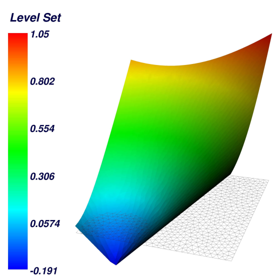
where F is the speed function defined on the interface $\Gamma(t)$ in the outward normal direction to the interface. Applied to the XFEM framework, the Level Set is defined on the structural mesh and each finite element node is associated a geometrical degree of freedom representing its Level Set function value. The Level Set is then interpolated on the whole design domain with the classical FEM shape function:

$$\psi(\mathbf{x}, t) = N_i(\mathbf{x})\psi_i \tag{4}$$

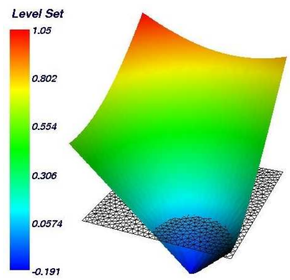
When more than one Level Set is defined on the structural mesh, it is possible to combine them with three different simple Boolean operators: union, intersection and difference. Hence, the union of two Level Set can be directly obtained from the nodal values by computing the minimum of the two Level Sets (see Fig. 1).

4.1. Constructive geometry based on the Level Set Description

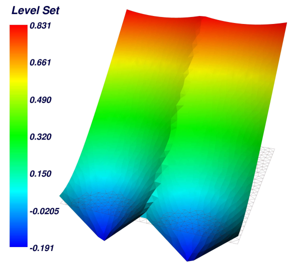
The constructive solid geometry (CSG) is a technique widely used in solid modelling and Computer Aided Desgin (CAD). The idea of this technique is to combine relatively simple objects called primitives



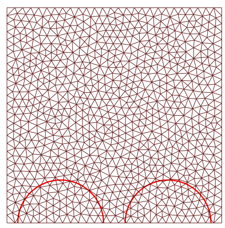
(a) Level Set representation of the first circle



(b) Level Set representation of the second circle



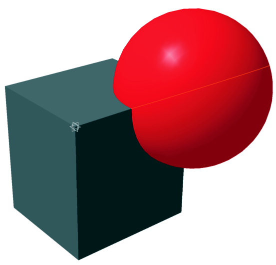
(c) Level Set representation of the union of two Level Set



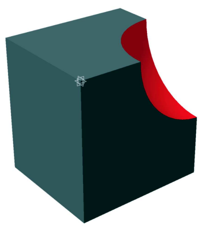
(d) Iso-zero of the Level Set interpolated on the mesh

Figure 1: Example of the union of two Level Set

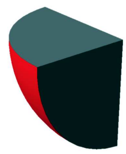
in order to define complex geometries. Typically, the primitives have a very simple shape such as cuboids, cylinders, prisms or spheres ... but this set of shapes can be extended to hold more complex geometries such as curved objects. Each combination of primitives is obtained by using Boolean operators: union, intersection, difference. These operators generally act upon two objects and produce a single compound object results. The union of two objects results in an object that encloses the space occupied by the two primitives. Intersection results in an object that encloses the common space of the two given objects. Then, the difference is an order dependent operator; it results in the first object minus the space where the second intersected the first (see Fig. 2).



(a) Union of a cube and a sphere



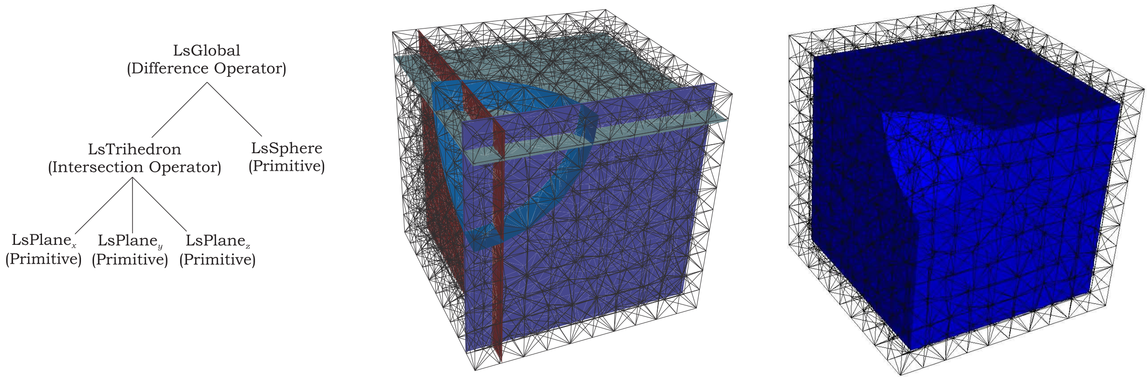
(b) Difference cube - sphere



(c) Intersection between the cube and the sphere

Figure 2: Boolean operations in constructive solid geometry

As one can see, in constructive solid geometry, the primitives always divide the space in 2 distinct parts and the related Boolean operators are used to combine or extract different space region surrounded by the primitives. Indeed, one can easily see that Level Set function also split a domain in 2 parts with respect to its sign value and that the Level Set operators act just as the Boolean CSG operators. Therefore, these operators can be used together with several Level Set primitives to represent complex geometries. Practically, following the methods used in CSG, the structure geometry is procedurally modelled as a compound of different Level Sets and organized as a binary tree where all leaves are Level Set primitive.



(a) Tree representation of the Level Set Model (b) Iso-zero value of the different Level Set primitive (c) Representation of the model after Boolean Operations

Figure 3: Example of a CSG Level Set Model

5. The extended finite element method

The extended finite element method developed by Moes, Belytschko and co-workers[9] is a recent method whose been firstly developed for the simulation and the analysis of structures presenting crack growth. The principal strength of this method is its capacity of including discontinuities inside the finite elements. Hence, this method enables the model to include geometric boundaries, material or phase changes that are not coincident with the mesh.

5.1. Basis of the method

In order to allow any types of discontinuities inside the elements and therefore to be able of representing discontinuities in the physics field of the problem's solution, it is necessary to add special properties to the shape functions. For example, in the case of cracked structures, the physical discontinuous field is the displacement field, hence, if one want to be able of modelling this discontinuity, we have to add discontinuous shape functions. The classical finite element approximation used is then extended to embed discontinuous shape function as in the following equation:

$$\mathbf{u}(\mathbf{x}) = \sum_i u_i N_i(\mathbf{x}) + \sum_j a_j N_j(\mathbf{x}) H(\mathbf{x}) \tag{5}$$

where N_i are the classical shape functions associated to degree of freedom u_i. The $N_j(\mathbf{x})H(\mathbf{x})$ are the discontinuous shape functions constructed by multiplying a classical $N_j(\mathbf{x})$ shape function with a Heaviside function presenting a switch value where the discontinuity lies.

In our case, we are only interested in modelling material-void interfaces with X-FEM (see [14]). To this end, the displacement field is then approximated by:

$$\mathbf{u}(\mathbf{x}) = \sum_i u_i N_i(\mathbf{x}) V(\mathbf{x}) \tag{6}$$

where

$$V(\mathbf{x}) \;=\; \begin{cases} 0 & \text{if} \quad \mathbf{x} \in \text{void} \\ 1 & \text{if} \quad \mathbf{x} \in \text{material} \end{cases}$$

The elements lying outside the material zone are removed from the system of equations, whereas the partially filled elements are integrated using the X-FEM integration procedure (see [8] and [14]). Modelling holes with the XFEM is a very appealing method for the shape optimization but also for the topology optimization as no remeshing is needed and no approximation is done on the nature of the voids in opposition to the SIMP method. The elements lying fully outside the material are removed from the system of equations, whereas the others are kept. One can observe that using a quadrature rule would lead to a totally unsatisfactory result even when increasing dramatically the number of Gauss points. At first, one has to notice that because of the zero displacement field in the void domain, the void part of the element does not contribute to the stiffness matrix. Thus the integration procedure is restricted to the solid sub-domain of the element (see Fig. 4).

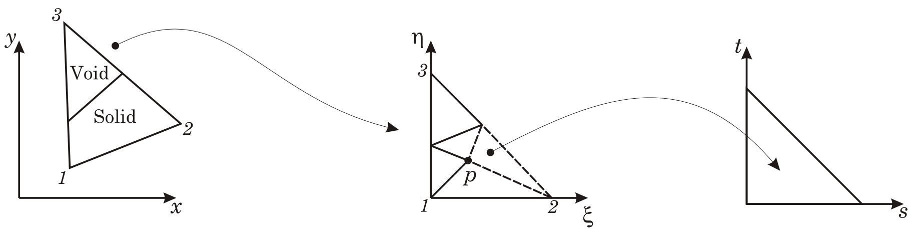


Figure 4: Triangle real space, reference space and second reference space

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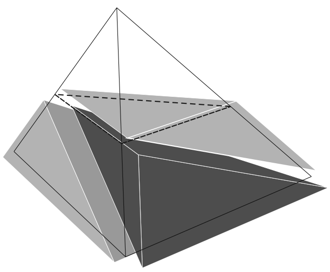
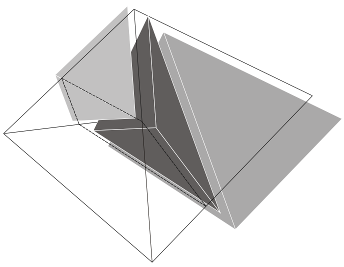
In the case of a triangular element, we now have to deal with an integration domain that is not anymore the reference triangle in space ξ, η. Hence, we divide the solid part of the element into several sub-triangles (with the barycenter P as one vertex for instance) conforming to the interface and the boundaries of the element itself. Then, a second reference space s,t is introduced in which we successively map the sub-triangles of space ξ,η. The classical FEM shape functions defined in this space are then integrated over each sub-triangles which define the integration domain. Therefore, the stiffness matrix is now given by:

$$K = \int_{\Omega_{Solid}} B^T H B dV = \sum_{\Delta} K_{\Delta} = \sum_{\Delta} \int_{\Delta} \hat{B}^T H \hat{B} \left|J_1\right| \left|J_2\right| dsdt \tag{7}$$

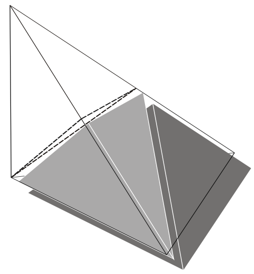
where Ω_{Solid} is the solid zone, Δ is a sub-triangle, $|J_1|$ and $|J_2|$ are the Jacobian of the transformation of x, y space to ξ, η and ξ, η to s, t respectively. Notice that the shape functions are still defined in the space ξ,η whereas the integration domain is defined by the second mapping in the space s,t. In case of 3D elements, the preceding method is extended to the third dimension and the sub-division of an element is then composed of sub-tetrahedrals domains. The following figures (Fig. 5) illustrate the sub-division of a 3D tetrahedron element for various intersecting plane positions (the positive nodes lie in the material whereas negative nodes are in the void part).



(a) One negative node



(b) Two negative nodes



(c) One negative node and one zero node

Figure 5: Intersection of a plane and a tetrahedron and sub-division

When using a complex model built with more than one Level set, each extended finite element is sub divided in order to conform the iso zero of the resulting Level Set defining the model. Hence, the sub division algorithm is called recursively on the element until all sub cells are conforming the model boundaries.

6. Formulation of optimization problem

The formulation of the optimization problem can be seen as a classical shape optimization problem formulation. Hence, the variables are geometrical parameters that modify the shape of structure. However, these parameters do not directly modify the geometry but change one of the Level Set used to define the

model. The research of an optimal solution is simplified thanks to the use of X-FEM and a Level Set description as no velocity field and/or mesh perturbation are needed. The geometry and the material repartition are specified using Level Sets representations. The positive part of the Level Set represents the region where lies the material and the negative part the void. To describe the structure, the user has a library of basic geometric primitives (in Level Sets) that can be combined to create almost any structural geometry. Beside these pre-defined Level Set shapes, the user can also build a Level Set from a Nurbs curve or a general set of points and combine them with Level Set operators union, intersection and difference. The optimization problem aims at finding the best shape for minimizing a given objective function while satisfying mechanical and geometrical design restrictions. The mechanical constraints can either be global responses (e.g. compliance, volume or eigenfrequency) or local ones such as displacements or stress constraints. The number of design variables is generally small as in shape optimization. However the number of constraints may be large if local stress restrictions (e.g. stress constraints) are considered. Nonetheless, large scale problems as in topology optimization are avoided.
The design problem is stated as a general constrained optimization problem:

$$\begin{array}{ll} \min\limits_{\mathbf{x}} & g_0(\mathbf{x}) \\ \text{s.t.:} & g_j(\mathbf{x}) \leq g_j^{max} \quad j = 1 \ldots m \\ & \underline{x}_i \leq x_i \leq \overline{x}_i \quad i = 1 \ldots n \end{array}$$

The solution to this problem is obtained using the so-called *sequential convex programming* [5]. At each iteration, the X-FEM analysis problem is solved and a sensitivity analysis is performed. The solution of the optimization problem is then found by using a CONvex LINearization approximation scheme of each constraint functions (CONLIN [4]). The solution becomes the new design and the procedure is repeated until convergence. Because of the X-FEM characteristics, the geometry has not to coincide with the mesh and the shape optimization problem is carried out on a *fixed mesh*. One works here in an Eulerian approach and not in a Lagrangian approach. This circumvents the mesh perturbation problems of classical shape optimization. Sensitivity analysis does not require the velocity field anymore. The present formulation is then, up to a certain point, simpler. However, some technical difficulties can be encountered if a finite difference or a semi-analytical scheme is used for sensitivity analysis. Basically, the problem is that the perturbation must not change the number of degrees of freedom of the X-FEM stiffness matrix (see [3] for more details). The Level Set approach is very convenient to modify the geometry because the Level Sets (and so the holes) can penetrate each other or disappear. Creation of new holes is more problematic since it leads to a non smooth problem. Topological derivatives (see [11]) have then to be used for a rigorous treatment of the problem. This capability has not been implemented in this study.

6.1. The sensitivity analysis method

As in classical shape optimization, the sensitivity analysis of mechanical responses (such as compliance, displacement, stress, ...) is carried out using a semi-analytic approach. In this approach the derivative of stiffness matrix ($\mathbf{K}$) is calculated by finite differences with respect to a small perturbation δx of Level Set parameters:

$$\frac{\partial \mathbf{K}}{\partial x} \simeq \frac{\mathbf{K}(x+\delta x) - \mathbf{K}(x)}{\delta x}$$

In the case of invariant loading forces, the derivative of static equilibrium equation gives the expression of the generalized displacements sensitivity as a function of the stiffness matrix derivative:

$$\frac{\partial \mathbf{u}}{\partial x} = \mathbf{K}^{-1}(-\frac{\partial \mathbf{K}}{\partial x}\mathbf{u}) \tag{8}$$

If the objective function or constraint involves the stresses of the problem, the sensitivity of this response is needed. In this work, we compute this sensitivities with a finite difference over the stresses while the perturbed displacements are computed using eq. (8):

$$\begin{aligned} \sigma(x) &= \mathbf{H}\mathbf{B_j}\mathbf{u}(x) \\ \sigma(x+\delta x) &\simeq \mathbf{H}\mathbf{B_j}\mathbf{u}(x+\delta x) \\ \frac{\partial \sigma}{\partial x} &\simeq \frac{\sigma(x+dx) - \sigma(x)}{\partial x} \end{aligned}$$

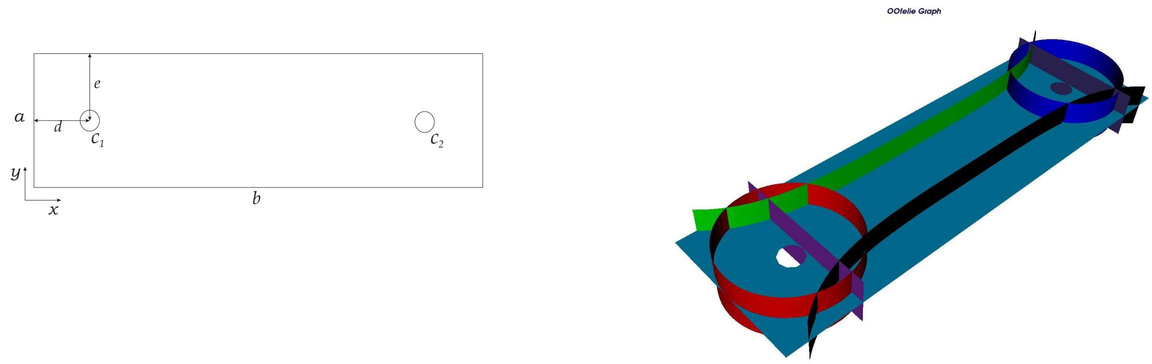
where $\mathbf{H}$ is the Hooke's matrix and $\mathbf{B_j}$ the matrix of the derivated shape functions of the element j.

7. Applications

The X-FEM method for the modelling of material-void discontinuity and its level set description have been implemented in an object-oriented (C++), multiphysics finite element code, OOFELIE, which is commercialized by *Open Engineering* (information available on the web site www.open-engineering.com). In this software, various mechanical responses can be chosen as objective functions and design restrictions, that is, compliance and potential energy, all stress components and Von Mises equivalent stress, displacements, eigenfrequencies and geometric results. The implementation of the X-FEM method is available in 2 and 3D, with a library of first degree quadrangle, triangle and tetrahedron elements. The level set description can be defined in different ways. They can be constructed classically from functions (circle, quadrangles, ellipses, ...) or from a set of points which are interpolated by a NURBS curve. The CONLIN optimizer by C. Fleury described in [4] has been coupled in the OOFELIE environment to realize the numerical applications.

7.1. Shape optimization of a 3D rod in traction and bending

The following application presents an example of shape optimization where the model is designed with a complex CSG Level Set built as a compound of simple and NURBS Level Set primitives. The geometry of the domain and the description of the Level Set primitive is presented on figure 6. The rod is clamped on the left circle C_1 while the loading is applied on the circle C_2 with values F_x equal to 220 kN and F_y 60kN. To obtain a symmetry with the xy-plane, we only study a half rod by applying symmetry condition in this plane and we prescribe a width along z axis of 20 mm. The circles C_1 and C_2 have a radius of 5 mm, d=25 mm, e=30 mm a=60 mm and b=200 mm.

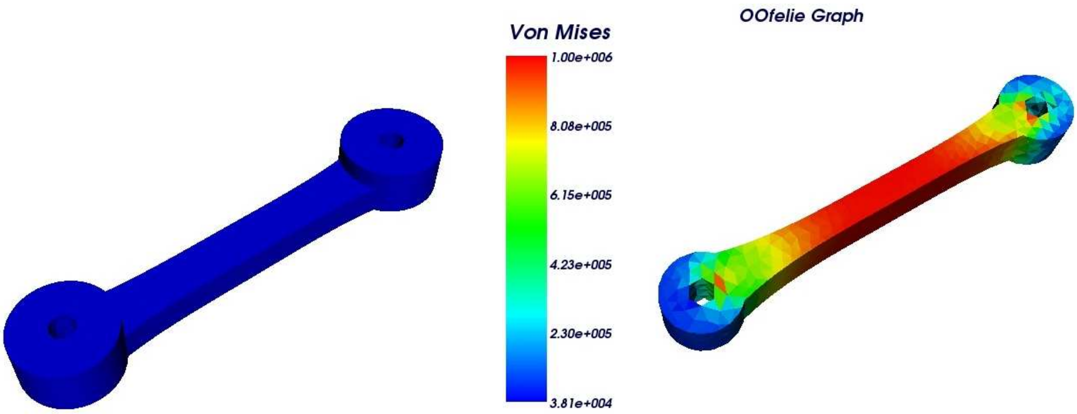


(a) Geometry of domain

(b) Parametrization of the rod with 7 Level Set primitives

Figure 6: Representation of the rod

The goal of the optimization is to minimize the mass (the volume) by modifying the exterior contour shape. The restriction the structure should verify is a limitation on the Von Mises stress (10^3 kN/mm^2) and prescribed side constraints on the design variables. The constraint on the stresses is carried out by selecting only a certain number of elements. In this case, we select at each iteration 50% of the total number of elements that present the highest value of the Von Mises stress. As mentioned before, the contour of the structure is described with a compound of simple and advanced Level Set primitives. Two cylinders aligned with circle C_1 and C_2 are placed at both extremity of the rod while the arm is modelled with two NURBS curve extruded in the z direction and a plane parallel to xy plane. Two planes are also placed to limit the arm length. The optimization variables are the external radius of the two cylinders, the control points of the Nurbs curves and the elevation z of the Level Set plane (xy) . The internal diameters of the two circles C_1 and C_2 are fixed. Hence, the mesh is conforming the geometry described in figure 6 (a). This enables us to apply the boundary conditions on these surfaces. We obtain the following shape (see Fig.7(b) after 22 iterations. The volume reduction is equal to 25% and the prescribed maximum Von Mises stress is respected. It is also interesting to remark that the connectivity of the geometry has been modified as the xy plane primitive has moved to obtain a thicker arm. We can see here an advantage of the Level Set representation as this kind of change could not be achieved with



(a) Initial geometry of rod

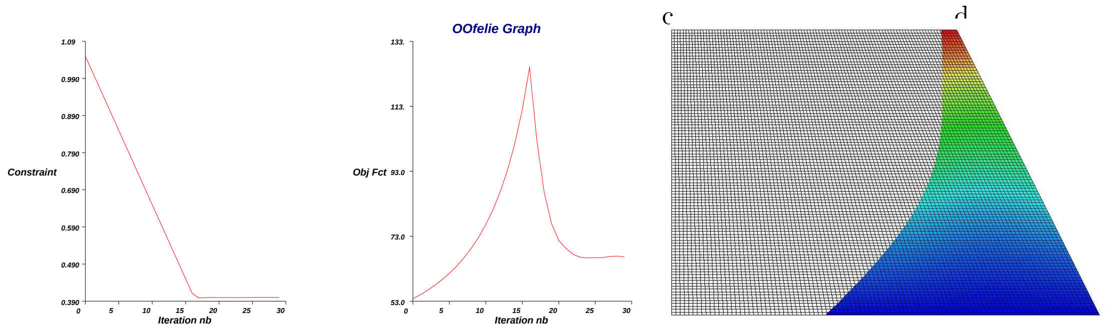
(b) Solution obtained

Figure 7: Initial model and solution obtained

a classical shape optimization method.

7.2. Shape optimization of a 2D dam

The second test case illustrates the application of a shape optimization with loads depending on the shape of the structure i.e. configurational loads. Here, the objective is to minimize the compliance of the dam which is subject to an hydrostatic pressure $\rho g h$. This pressure is applied on the iso-zero Level Set and acts perpendicularly to the structure (h is aligned with line ca). The geometry of the model is depicted in figure 8 (c) with the mesh corresponding to the initial shape and the displacement field representing the optimal boundary obtained. The line ab has a length of 1.5, ac is equal to 1 and cd=1. The interface is modelled with a NURBS curve primitive with 7 control points K_i placed at y=0,0.1,0.3,0.5,0.7,0.9,1. The optimization variables are these control points with a movement limited to the x axis. Moreover, the volume is constrained to be less or equal to 40% of the initial volume. The figure 8 (a) and (b) shows the evolution of the constraint and the objective function respectively.



(a) Evolution of the Volume

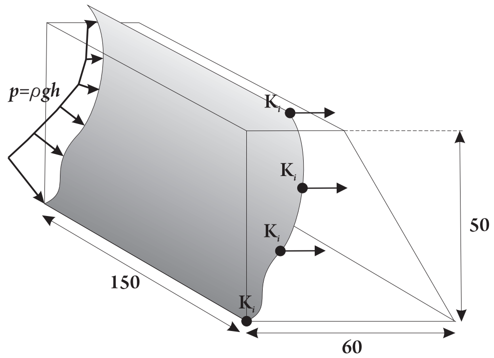
(b) Evolution of the Compliance

(c) Initial domain (mesh) and optimal geometry

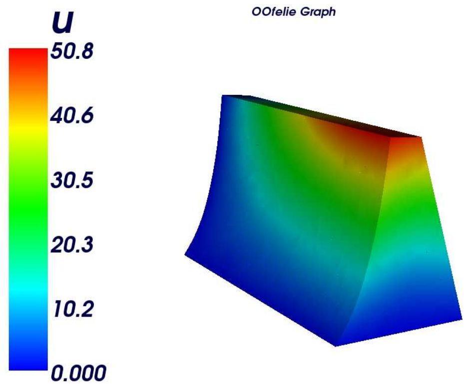
Figure 8: Optimization of a 2D dam

7.3. Shape optimization of a 3D dam

This last application is the extension of the previous one in 3D. The model geometry is described in figure 9 (a). The Level Set boundary is represented by a NURBS curve that is extruded in the z direction. As a consequence, the iso-zero surface of the Level Set can only be modified in the x, y plane which is a limitation related to the set of primitive currently available. The variables of the optimization problem are the control points K_i (see Fig. 9 (a)). The objective function is to minimize the compliance with a constrain on the volume $V \leq 0.3 V_i$, where V_i is the initial volume. The applied load is the hydrostatic pressure $\rho g z$.



(a) Geometry of the 3D dam



(b) Solution obtained

Figure 9: Geometry and solution of the 3D dam

8. Conclusions and perspectives

The approach presented here takes place between shape and topology optimization as it presents some characteristics of both methods. Hence, this method can be related to shape optimization as it relies on a geometric representation of the structure thus limiting the number of variables introduced in the optimization problem. Moreover, the geometric modeller is based on a Level Set description that allows more freedom than a classical geometry definition. Like the topology optimization, this method shares also the interesting advantage of working on a fixed grid thanks to the X-FEM method but do not approximate void as a smooth material. Finally, the Level Set is able to represent smooth curves description. This feature is seen to be very interesting when a load has to be applied on an evolving boundary. In the present paper, we have presented an application of the Constructive Solid Geometry approach applied to a Level Set Description to model complex geometries with Level Sets. The geometrical primitives from CSG are replaced by primitives described with Level Sets while the CSG Boolean operators are the Level Set combination operators. Some applications have been presented to illustrate the advantages of using both X-FEM and Level Set in shape optimization. The semi-analytic sensitivity analysis with X-FEM and level set has been developed for the majority of common objective function such as the compliance, the stresses or the displacements.

The fixed grid approach is a very interesting feature in shape optimization as it removes all mesh problems occurring when Finite Element method. However, when using an implicit representation of the boundaries, it can be necessary to modify the mesh in order to obtain accurate representation. Hence, our future work should consider the introduction of an automatic mesh refinement in order to avoid shape presenting a very poor level of refinement during the evolution of the structure. Moreover, to take fully advantage of the fixed grid approach, hexahedral elements should be developed to use octree meshing technique to improve the performance of the method in terms of CPU time. The optimization should also be improved to be able to take into account geometrical constraint between Level Set. Finally, the enrichment of Level Set primitives and the generation of these primitives should be studied.

9. Acknowledgments

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10. References

[1] Grégoire Allaire, François Jouve, and Anca-Maria Toader. Structural optimization using sensitivity analysis and a level-set method. *Journal of Computational Physics*, 194(1):363–393, 2004.

[2] Ted Belytschko, Chandu Parimi, Nicolas Moës, N. Sukumar, and Shuji Usui. Structured extended finite element methods for solids defined by implicit surfaces. *International Journal for Numerical Methods in Engineering*, 56:609–635, 2003.

[3] Pierre Duysinx, Laurent Van Miegroet, Thibault Jacobs, and Claude Fleury. Generalized shape optimization using x-fem and level set methods. In *IUTAM Symposium on Topological Design, Optimization of Structures, Machines and Materials.*, pages 23–32. Springer, Berlin Hiedelberg New York, 2006.

[4] Claude Fleury. Conlin: An efficient dual optimizer based on convex approximation concepts. *Structural and Multidisciplinary Optimization*, 1:81–89, Jun 1989.

[5] Claude Fleury. Sequential convex programming for structural optimization problems. *Optimization of large structural systems*, 1:531–553, 1993.

[6] J. T. Haslinger, K. Kunisch, and G. Peichl. Shape optimization and fictitious domain approach for solving free boundary problems of bernoulli type. *Computational Optimization and Applications*, 26(3):231–251, 2003.

[7] H. Kim, O. M. Querin, G. P. Steven, and Y. M. Xie. Improving efficiency of evolutionary structural optimization by implementing fixed grid mesh. *Structural and Multidisciplinary Optimization*, 24:441–448, Dec 2002.

[8] Laurent Van Miegroet and Pierre Duysinx. Stress concentration minimization of 2d filets using x-fem and level set description. *Structural and Multidisciplinary Optimization*, 33(4):425–438, 2007.

[9] Nicolas Moës, John Dolbow, and Ted Belytschko. A finite element method for crack growth without remeshing. *International Journal for Numerical Methods in Engineering*, 46:131–150, 1999.

[10] J. Norato, R. Haber, D. Tortorelli, and M. P. Bendsøe. A geometry projection method for shape optimization. *International Journal for Numerical Methods in Engineering*, 60:2289–2312, 2004.

[11] A. A. Novotny, R. A. Feijoo, E. Taroco, and C. Padra. Topological sensitivity analysis. *Computer Methods in Applied Mechanics and Engineering*, 192:803–829, Feb 2003.

[12] Stanley Osher and James A Sethian. Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulations. *Journal of Computational Physics*, 79:12–49, 1988.

[13] J. A. Sethian and A. Wiegmann. Structural boundary design via level set and immersed interface methods. *Journal of Computational Physics*, 163:489–528, 2000.

[14] N. Sukumar, D. L. Chopp, N. Moes, and T. Belytschko. Modeling holes and inclusions by level sets in the extended finite-element method. *Computer Methods in Applied Mechanics and Engineering*, 190:6183–6200, Sep 2001.

[15] Michael Yu Wang, Xiaoming Wang, and Dongming Guo. A level set method for structural topology optimization. *Computer Methods in Applied Mechanics and Engineering*, 192:227–246, Jan 2003.

[16] S. Zhang and A. D. Belegundu. A systematic approach for generating velocity fields in shape optimization. *Structural Optimization*, 5:84–94, 1992.