A Statistically adaptive sampling policy to the Hotelling's T^2 Control Chart: Markov Chain Approach

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Abstract

This paper proposes an alternative sampling scheme to the Hotelling's T^2 control chart with variable parameters (VP T^2). Indeed, the sampling interval h, the sample size n and the control limit k vary between minimum and maximum values while the warning line is kept fixed over time. The proposed method uses only one measurement scale and therefore overcomes the usual difficulties of using two scales. Finally, we show the merits of the proposed method as a good option for its ease of application and its quick responses to small and moderate shifts in a multivariate process.

Keywords: Multivariate Control Chart, Hotelling's T^2 Control Chart, Variable Parameters (VP), Genetic Algorithm (GA), Adjusted Average Time to Signal (AATS).

1. Introduction

Quality control problems in industry may involve more than a single quality characteristic, i.e., a vector of characteristics. The Hotelling's (1947) T^2 control chart is one of the most widely used tools in multivariate statistical process control. Consider a process in which p correlated characteristics are being measured simultaneously and controlled jointly. It is assumed that the joint probability distribution of the quality characteristics is a p-variate normal distribution with the vector of in-control means $\boldsymbol{\mu}_0' = (\mu_{01},..., \mu_{0p})$ and the variance-covariance matrix $\sum$. The procedure requires computing the sample means for each of the p quality characteristics from a sample of size n. The vector $\bar{\mathbf{x}}' = (\bar{x}_1,..., \bar{x}_p)$ gives the p sample

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means. Then the subgroup statistics $T^2 = \mathbf{n}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)' \sum^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$ are plotted on a control chart in sequential order. For the sake of simplicity, we assume here that $\boldsymbol{\mu}_0$ and $\sum$ are known or are estimated from large enough samples. In this case, T^2 is distributed as a chi square random variable with p degrees of freedom. Each of the T^2 values is compared with the upper α percentage of the chi square distribution ($k = \chi^2(\alpha, p)$) and if the sample values fall below the control limit k the process is considered in control, otherwise the process is said to be out of control and the corresponding subgroup(s) investigated. It is usually assumed that the variance-covariance structure of the quality characteristics being charted does not change and that assignable causes are manifested by a shift at least in one component of the mean vector of the process. The magnitude of this shift is often expressed by $d^2 = (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)' \sum^{-1} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)$, the Mahalanobis distance, and $\boldsymbol{\mu}_1$ is the out of control mean vector. In this case, the T^2 statistic is distributed as a non-central chi-square distribution with p degrees of freedom and non-centrality parameter $\eta = nd^2$ (Faraz and Parsian, 2006).

The traditional practice in applying the T^2 chart is to obtain samples of fixed size n_0 at constant intervals h_0 which is called a Fixed Ratio Sampling (FRS) scheme, but it is slow in detecting a small to moderate process shifts. Recently, the ideas of the variable sample sizes (VSS) (see, Faraz and Moghadam, 2009; Faraz et al. 2010), double sampling scheme (DS) (see, Faraz et al. 2012a; Tornga and Lee, 2009), variable sampling intervals (VSI) (see, Amin and Hemasinha, 1993; Chengular et al. 1989; Faraz et al., 2011); variable sample sizes and sampling intervals (VSSI) (see, Costa 1998; Faraz et al. 2012b), double warning lines scheme (DWL) (see, Faraz and Parsian, 2006; Faraz and Saniga, 2011), variable sample sizes and control limits (VSSCL) (see, Chen and Hsieh, 2007; Seif et al. 2011 a&b), variable sampling intervals and control limits (VSICL) (see, Torabian et al. 2010) and variable parameters VP (see, Costa 1999; Chen, 2007; Lin, 2009; Faraz et al. 2013 a&b) has been proposed in literature to provide users with a tool that detects small shifts more quickly than the classical FRS scheme. The question arises that which scheme is the most effective and powerful. Faraz and Parsian (2006) through a comparison between the FRS, VSS, VSI, VSSI and DWL schemes (the VP and

VSSCL are not included) have shown that the DWL scheme is more powerful than the other schemes in detecting small, moderate and even large shifts. Chen (2007) extended the VP scheme proposed by Costa (1999) to multivariate case and compared the two-scaled variable parameter (2VP) T^2 chart with the VSS, VSI, and VSSI T^2 charts (the DWL scheme is not included) and the results indicate that the 2VP T^2 chart outperforms the VSS, VSI, and VSSI T^2 charts, especially in detecting small shifts. However, the 2VP scheme is dominated by the VSI scheme when the process change in the mean vector is moderate or large. The same result is obtained by Chen and Hsieh (2007) and they showed that the two-scaled variable sample sizes and control limits (2VSSC) T^2 chart presents a similar performance to the 2VP T^2 chart. The proposed 2VP and 2VSSC schemes require the user to construct a T^2 chart with two different measuring scales. One on left hand side and the other on the right hand side. This approach is a tedious and of course unwilling for the practitioners. Hence, in this paper we propose the VP, VSSCL and VSICL T^2 control charts in a way that they gains from a single measuring scale, a problem heretofore not addressed. Furthermore, through a comparison between all existing T^2 charts using variable ratio sampling schemes, the adaptive sampling policy to the T^2 chart is proposed. The paper is organized as follows: in the next section, the Markov chain approach to construct VP, VSSCL and VSICL T^2 charts are discussed. The performance measure and GA approach to statistically optimal design of the schemes are then studied afterwards. The section 3 makes a comparison between the VP, VSSCL, VSICL, DWL, VSSI, VSS and VSI schemes and finally the optimal sampling policy is then proposed.

2. The VP, VSSCL and VSICL T^2 Schemes: Markov Chain Approach

The VP T^2 control chart is an extension of the VSSI T^2 control chart discussed by Faraz and Parsian (2006). Let h_1 and h_2 be maximum and minimum sampling intervals, k_1 and k_2 be maximum and minimum control limits, and n_1 and n_2 be maximum and minimum sample size respectively, such that $0 < h_2 < h_1$, $0 < k_2 < k_1$ and $n_1 < n_2$. Here we refer to the set (k_1, h_1, n_1) as minimum sampling plan and the set (k_2, h_2, n_2) as maximum sampling plan. The decision to switch between maximum and minimum sampling plans depends on position of the prior sample point on the control chart. If the prior sample point (i-1) falls in the safe region, we use the minimum sampling plan and if the prior sample point (i-1) falls in the warning region, we use the maximum sampling plan for the current sample. Finally, if a sample point falls in the action region, then the process is considered out of control. Here the safe, warning and action regions are given by the warning limit w and the control limit k_j. The safe region is given by [0,w), the warning region is given by [w, k_j), and the action region is given by $[k_j, \infty)$, where $j = 1$ if the prior sample point comes from the minimum plan and $j = 2$ if the prior sample point comes from the maximum plan (see figure 1). The following function summarizes the control scheme of the VP T^2 control chart:

$$(h_i, k_i) = \begin{cases} (k_1, h_1, n_1) & \text{if } 0 \le T_{i-1}^2 < w \\ (k_2, h_2, n_2) & \text{if } w \le T_{i-1}^2 < k_{i-1} \end{cases} \quad \textbf{(1)}$$

2.1 Performance Measure

The average time from the process mean shifts until the chart produces a signal is used to measure its statistical efficiency. This statistical measure is called AATS and determines the speed with which a control chart detects a process mean shift. The average time of the cycle (ATC) is the average time from the start of the production until the first signal after the process shift. If the assignable cause occurs according to an exponential distribution with parameter λ then the expected time interval that the process remains in control is $1/\lambda$. Therefore,

$$AATS = ATC - \frac{1}{\lambda} \quad \textbf{(2)}$$

The memoryless property of the exponential distribution allows the computation of the ATC using the Markov chain approach. The Markov chain approach we employ here is similar to that originally proposed by Faraz and Saniga (2009) in which they made a unification and some corrections to Markov chain approaches to develop control charts with variable ration sampling policies. The fundamental concepts of the Markov chain approach can be found in Cinlar (1975). Here, at each sampling stage, one of the following transient states is reached according to the status of the process (in or out of control), length of the sampling interval (short or long) and quantity of the control limit (k_1 or k_2):

State 1: $0 \le \mathrm{T}^2 < w$ and the process is in control;

State 2: $w \le \mathrm{T}^2 < k_j$ and the process is in control;

State 3: $0 \le \mathrm{T}^2 < w_j$ and the process is out of control;

State 4: $w_j \le \mathrm{T}^2 < k_j$ and the process is out of control;

State 5: (absorbing state): $\mathrm{T}^2 \ge k_j$ and the process is out of control.

The transition probability matrix is given as follows:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} \\ 0 & 0 & p_{33} & p_{34} & p_{35} \\ 0 & 0 & p_{43} & p_{44} & p_{45} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Where p_{ij} denotes the transition probability that i is the prior state and j is the current state. In what follows, $F(x, p, \eta)$ will denote the cumulative probability distribution function of a non-central chi-square distribution with p degrees of freedom and non-centrality parameter $\eta_i = n_i d^2$ and $q_i = \exp(-\lambda h_i)$; $i = 1,2$. Then, p_{ij}'s are

$$p_{11} = \Pr(T^2 < w \mid T^2 < k_1) \times q_1 = \frac{F(w, p, 0)}{F(k_1, p, 0)} \times q_1$$

$$p_{12} = \Pr(w \le T^2 < k_1 \mid T^2 < k_1) \times q_1 = q_1 - \frac{F(w, p, 0)}{F(k_1, p, 0)} \times q_1$$

$$p_{13} = \Pr(T^2 < w) \times (1 - q_1) = F(w, p, \eta_1) \times (1 - q_1)$$

$$p_{14} = \Pr(w \le T^2 < k_1) \times (1 - q_1) = \left[F(k_1, p, \eta_1) - F(w, p, \eta_1)\right] \times (1 - q_1)$$

$$p_{15} = \Pr(T^2 \ge k_1) \times (1 - q_1) = \left[1 - F(k_1, p, \eta_1)\right] \times (1 - q_1)$$

$$p_{21} = \Pr(T^2 < w \mid T^2 < k_2) \times q_2 = \frac{F(w, p, 0)}{F(k_2, p, 0)} \times q_2$$

$$p_{22} = \Pr(w \le T^2 < k_2 \mid T^2 < k_2) \times q_2 = q_2 - \frac{F(w, p, 0)}{F(k_2, p, 0)} \times q_2$$

$$p_{23} = \Pr(T^2 < w) \times (1 - q_2) = F(w, p, \eta_2) \times (1 - q_2)$$

$$p_{24} = \Pr(w \le T^2 < k_2) \times (1 - q_2) = \left[F(k_2, p, \eta_2) - F(w, p, \eta_2)\right] \times (1 - q_2)$$

$$p_{25} = \Pr(T^2 \ge k_2) \times (1 - q_2) = \left[1 - F(k_2, p, \eta_2)\right] \times (1 - q_2)$$

$$p_{33} = \Pr(T^2 < w) = F(w, p, \eta_1)$$

$$p_{34} = \Pr(w \le T^2 < k_1) = F(k_1, p, \eta_1) - F(w, p, \eta_1)$$

$$p_{35} = \Pr(T^2 \ge k_1) = 1 - F(k_1, p, \eta_1)$$

$$p_{43} = \Pr(T^2 < w) = F(w, p, \eta_2)$$

$$p_{44} = \Pr(w \le T^2 < k_2) = F(k_2, p, \eta_2) - F(w, p, \eta_2)$$

$$p_{45} = \Pr(T^2 \ge k_2) = 1 - F(k_2, p, \eta_2)$$

The expected number of trials needed in each state to reach the absorbing state can be obtained from $\mathbf{b}'(\mathbf{I} - \mathbf{Q})^{-1}$ where $\mathbf{Q}$ is the matrix obtained from **P** on deleting the elements corresponding to the absorbing state, **I** is the identity matrix of order 4 and $\mathbf{b}' = (p_1,\ p_2,\ 0,\ 0)$ is a vector of initial probabilities, with $\sum_{i=1}^{2} p_i = 1$. Hence,

$$ATC = \mathbf{b}'(\mathbf{I} - \mathbf{Q})^{-1}\mathbf{h} \qquad \textbf{(3)}$$

Where $\mathbf{h}' = (h_1,\ h_2,\ h_1,\ h_2)$ is the vector of sampling time intervals. In this paper the vector $\mathbf{b}'$ is set to $(0,\ 1,\ 0,\ 0)$, for providing an extra protection and preventing problems that are encountered during start-up.

The VP scheme should be such designed that satisfies two conditions. First, it should guaranty that the both FRS and VP schemes have the same ratio sampling (sampled items and sampling frequencies) as long as the process is in control. By using the elementary properties of Markov chains, the average number of samples (ANS) for the VP scheme during the in control period is calculated as follows:

$$ANS = \mathbf{b}'(\mathbf{I}-\mathbf{Q})^{-1}(1,1,0,0)' \quad (4)$$

Faraz and Saniga (2009) showed that the ANS and ATC for the FRS schemes are easily determined by

$$ANS = \frac{1}{1-q_0} \quad (5)$$

$$ATC = \left(\frac{q_0}{1-q_0} + \frac{1}{1-\beta_0}\right) \times h_0 \quad (6)$$

Where here $q_0 = \exp(-\lambda h_0)$ and $\beta_0 = F(k_0, p, \eta)$. Now, by equating expressions (4) and (5), the parameter w is obtained as follows:

$$w = F^{-1}\left(\frac{F(k_2,p,0)F(k_1,p,0)(q_0-q_2)}{F(k_2,p,0)q_1(q_0-q_2) - F(k_1,p,0)q_2(q_0-q_1)}, p, 0\right) \quad (7)$$

The second condition is that the VP schemes should have the average Type I error rate equals to α_0 during in-control period. Now, assume that the probability of having the minimum sampling plan while the process is in control is p_0. Therefore, the maximum sampling plan occurs with the probability ($1-p_0$) as long as the process is in control. Hence, we should have

$$\begin{cases} h_1 p_0 + h_2(1-p_0) = h_0 \\ n_1 p_0 + n_2(1-p_0) = n_0 \\ \alpha_1 p_0 + \alpha_2(1-p_0) = \alpha_0 \end{cases} \quad (8)$$

Where $\alpha_i = 1 - F(k_i, p, 0); \quad i = 0,1,2$. Hence, expressions for the calculation of n_2 and k_2 are obtained by

$$n_2 = \frac{n_0 - p_0 n_1}{1-p_0} \quad (9)$$

$$k_2 = F^{-1}\left[\frac{F(k_0) - p_0 F(k_1)}{1-p_0}, p, 0\right] \quad (10)$$

where $p_0 = \frac{h_0 - h_2}{h_1 - h_2}$.

The VSICL T^2 chart, as an especial case of the VP scheme, is obtained by letting $n_1 = n_2 = n_0$. In this case, the expression of w is the same as (7). When $h_1 = h_2 = h_0$, the VP T^2 chart is called the VSSCL T^2 chart. The expression of w for this particular case is obtained as fallows:

$$w = F^{-1}\left(\frac{F(k_2,p,0)F(k_1,p,0)(n_0-n_2)}{F(k_2,p,0)q_0(n_0-n_2)-F(k_1,p,0)q_0(n_0-n_1)},p,0\right) \quad (11)$$

Other special cases of the VP scheme are VSSI, VSS and VSI schemes which are obtained by letting $k_1 = k_2 = k_0$; $k_1 = k_2 = k_0$ & $h_1 = h_2 = h_0$ and $k_1 = k_2 = k_0$ & $n_1 = n_2 = n_0$, respectively (see Faraz and Parsian, 2006; Faraz and Moghadam, 2009 and Faraz et al, 2011).

2.2 Optimization Problem and Genetic Algorithm Approach

In this paper, we have limited the value of long sampling interval h_1 to maximum hours available in a work shift, i.e. $h \le 8$. The short sampling interval h_2 is limited to 0.1 because periods less than 0.1 hours can be problematic in the field. In fact, a minimum period should be set such that the process can generate the required sample size. Therefore, the general optimization problem is defined as follows:

$$\begin{aligned}
&\min AATS \\
&s.t: \\
&0.1 \le h_2 \le h_0 \le h_1 \le 8 \\
&0 \le w \le k_2 \le k_0 \le k_1 \\
&n_1 \le n_0 \le n_2 \in Z^+
\end{aligned} \quad (12)$$

The optimization problem has both continuous and discrete decision variables and a discontinuous and non-convex solution space. This problem can be solved with Meta heuristic search techniques which are the most widely used tools in this area; examples include taboo search, simulated annealing, artificial neural network, genetic algorithms, etc. The genetic algorithm approach (GA) is a method for solving both constrained and unconstrained optimization problems, which is based on natural selection the process that drives biological evolution. GA repeatedly modifies a population of individual solutions. At each step, GA selects individuals at random from the current population to be parents and uses them produce the children for the next generation. Over successive generations, the population evolves toward an optimal solution. GA can be applied to solve a variety of optimization problems that are not well suited for standard optimization algorithms, including problems in which the objective function is discontinuous, non differentiable, stochastic, or highly nonlinear. GA has received a great deal of attention in the recent literature and we apply it here as an appropriate technique for solving the optimization problem (Faraz et al., 2010). For the details of the solution method, the readers are referred to Faraz et al. (2010).

The procedure to solve the optimization problem (12) using genetic algorithm is as follows:

For a given FRS parameters (k_0, h_0, n_0) and VP parameters (k_1, h_1, h_2, n_1), first the value of k_2 is determined from equation (10) which allows the parameter w takes a value from equation (7). The value of parameter n_2 is determined using equation (9). Hence, the objective is to find the four chart parameters (k_1, h_1, h_2, n_1) that minimize the AATS measure. In the VSICL scheme, for given h_1 and h_2 values, the parameters w and k_2 take a value using equations (7) and (10), respectively. Finally, in the case of VSSCL, for given n_1 and n_2 values, the parameter k_2 takes a value from equation (10) in that $p_0 = \dfrac{n_0 - n_2}{n_1 - n_2}$. The parameter w is then determined using equation (11). This procedure ensures that the comparison of the different schemes is meaningful and unbiased. That is, the in-control performance of schemes are matched.

Tables 1, 2 and 3 provide the statistically optimal design for the VP, VSSCL and VSICL T^2 charts with a comparison to the FRS T^2 chart. The numerical solutions for p = 2, 3, 4, 5, 10 and 20; n_0 = 2, 3, 4, 5 and 10 and λ = 0.0001, 0.001, 0.01 and 0.1 are available upon request from the second author.

The results indicate that the proposed VP scheme with one warning limit performs well to the two-scaled VP (2SVP) scheme and furthermore it considers implementing considerations due to its fewer parameters and its one-scaled property. The differences between the two schemes are not much even for detecting small shifts. If varying the sampling intervals is not practical in the field, the VSSCL scheme is then an alternative to the VP T^2 chart. Table 2 presents the results. If varying sample sizes is not desired, the VSICL scheme is recommended. Table 3 gives the optimal parameters of the VSICL scheme. With a comparison to the VSI scheme, varying the control limit improves the power of that chart. Our findings indicate that the VP scheme is more powerful than the VSICL and consequently VSI scheme in detecting small shifts; however, the differences between two schemes to detect moderate to large shifts are not significant. The surprising finding is that the

VSSCL scheme is more powerful than the VP scheme in detecting small mean shifts. This is discussed in the next section.

3. A numerical comparison between the different variable ratio sampling schemes

The practitioners are faced to a group of seven variable ratio sampling policies: VP, DWL, VSSCL, VSICL, VSSI, VSS and VSI. Each sampling scheme overcomes the FRS schemes (except in detecting large shifts) and can be used upon application considerations and the amount of the specific shift, which is of concern. Table 4 summarizes the comparison results between these schemes and provides information to help practitioners in selecting the most powerful scheme in order to have the maximum protection over a pre-determined value of the shift. The results are somewhat opposed to what we expected. Comparison results indicate that for small shifts ($d \leq 0.5$) the VSSCL scheme is superior to the DWL, VP, VSICL, VSSI, VSS and VSI schemes. The VP and then the DWL schemes show better performance than the others in detecting moderate shifts ($0.5 < d \leq 1.5$). For detecting larger shifts ($d > 1.5$) the VSICL and VSI schemes are good options. However, the differences between schemes in detecting large shifts are not significant, but due to their simplified structures can be recommended in practice.

According to Tables 1 - 4, it is clear that different schemes have different structures for detecting different magnitude of shifts in the process. So, when estimating the magnitude of shift is costly or a process is subject to multiple shifts, then it is recommended to apply a scheme that shows good performance for all shifts. We have to take into account that a different curve AATS versus d is obtained in function of selected value of d employed to optimize the power of the chart. After analyzing the diverse available options, Faraz and Parsian (2006) showed that the optimum charts are obtained for $d = 0.75$ and these charts generally show a very good performance. Furthermore, as mentioned by Costa (1999), the parameter λ has minor influence on the AATS and hence on comparing results. Hence, in this section a comparison is made for the case $p = 2$, $n_0 = 2$, $h_0 = 1$ and $\lambda = 0.01$. Hence, the following charts are compared:

1. The VP T^2 chart: $k_1 = 25.87$; $k_2 = 6.51$; $w = 4.17$; $h_1 = 1.13$; $h_2 = 0.1$; $n_1 = 1$ and $n_2 = 9$.

2. The DWL T^2 chart: $k = 10.60$; $w_h = 1.27$; $w_n = 5.20$; $h_1 = 2.03$; $h_2 = 0.1$; $n_1 = 1$ and $n_2 = 14$.
3. The VSSCL T^2 chart: $k_1 = 25.14$; $k_2 = 6.27$; $w = 4.41$; $h = 1$; $n_1 = 1$ and $n_2 = 10$.
4. The VSICL T^2 chart: $k_1 = 13.51$; $k_2 = 9.69$; $w = 1.14$; $h_1 = 2.20$; $h_2 = 0.1$; and $n = 2$.
5. The VSSI T^2 chart: $k = 10.60$; $w = 4.71$; $h_1 = 1.10$; $h_2 = 0.1$; $n_1 = 1$ and $n_2 = 11$.
6. The VSS T^2 chart: $k = 10.60$; $w = 5.48$; $h = 1$; $n_1 = 1$ and $n_2 = 15$.
7. The VSI T^2 chart with parameters $k = 10.60$; $w = 0.63$; $h_1 = 3.47$; $h_2 = 0.1$; and $n = 2$.

Figure 2 illustrates the comparison results when d differs from small to large shifts. The comparison shows that the VSICL and VSI schemes are unable to detects small to moderate shifts. These two schemes have a good power in detecting the shifts $d \geq 1.5$. As Faraz and Parsian (2006) indicated, the DWL schemes is always a better option in detecting almost all process mean shifts while compared to the VSS, VSI and VSSI schemes, but when it comes to the VP and VSSCL T^2 charts, other schemes are dominated. In fact, adopting variable action limits provides a great improvement for the DWL and VSSI T^2 charts. Even more, the VSSCL and VP schemes have smaller value of the parameter n_2 with respect to the VSS, VSSI and DWL schemes. In fact, letting the control limits vary between maximum and minimum values, smaller sample sizes are then required to detect out of control states, which is more economical too. By a comparison between the both VP and VSSCL schemes, we find out that the power of the VSSCL scheme, with not large differences, is almost smaller than the VP scheme and on the average the VSSCL scheme alarms almost 42 minutes sooner than the VP T^2 chart in detecting small mean shifts ($d = 0.5$). On the other hand, the VP scheme shows better performance than the VSSCL in detecting moderate to large shifts ($0.75 \leq d \leq 2$) and on the average nearly 25 minutes sooner alarms. Of course, the VP scheme imposes some difficulties in application since the sampling intervals changes during the monitoring process. These adaptive changes in sampling intervals cause to increase the complexity of the chart. Hence, the VSSCL scheme, which always takes samples at fixed sampling intervals h_0, is more convenient than the VP scheme. As a result, it can be concluded that the VSSCL scheme is a statistically optimal sampling scheme for the T^2 control chart. It quickly detects small shifts as soon as possible and shows a very good performance when compared to the VP scheme in detecting for moderate to large shifts. Note that, the proposed VSSCL scheme does not require two charts with different measurement scales, since it only has one warning limit and that enables users to monitor the process in a single measurement scale.

4. Concluding remarks

The Hotelling's T^2 control chart, a direct along of univariate shewhart $\overline{X}$ control chart, is perhaps the most commonly used tool in industry for simultaneous monitoring of several quality characteristics. Recent studies have shown that applying variable ratio sampling (VRS) schemes yield faster detection of small to moderate shifts with respect to the FRS T^2 control charts. Among existing schemes, the variable parameters (VP) has been proved to have a very good performance on detecting small to moderate shifts, however applying the VP T^2 charts encounters some difficulties in the field. In this paper, we proposed an alternative to the VP T^2 chart in that the sampling interval h, the Sample size n and control limit k vary between minimum and maximum values while keeping the warning line fixed over time. The proposed method uses only one measurement scale to overcome the applied difficulties of using two scales in the field. This idea is also applied to the VSS and VSI schemes to form the VSSCL and VSICL T^2 charts. Through numerical comparisons between the seven existing VRS schemes in the literature, we show that the VSSCL scheme is statistically optimal and performs excellent for small shifts in process mean. Moreover, the VSSC T^2 chart performs well to the VP T^2 chart in detecting moderate to large shifts and that the VSSC T^2 chat is a more popular scheme in practice than the VP charts due to its fewer parameters and ease of application. In fact, one may give up the merits of the VP scheme in detecting moderate to large shifts for the ease of application of the VSSCL scheme in the field.

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Figures

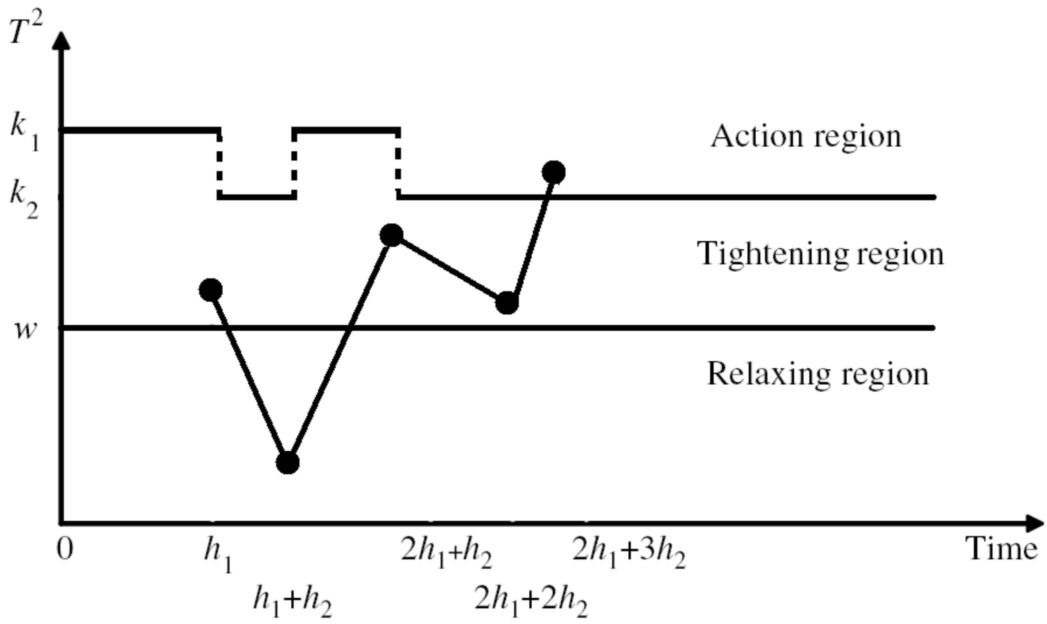
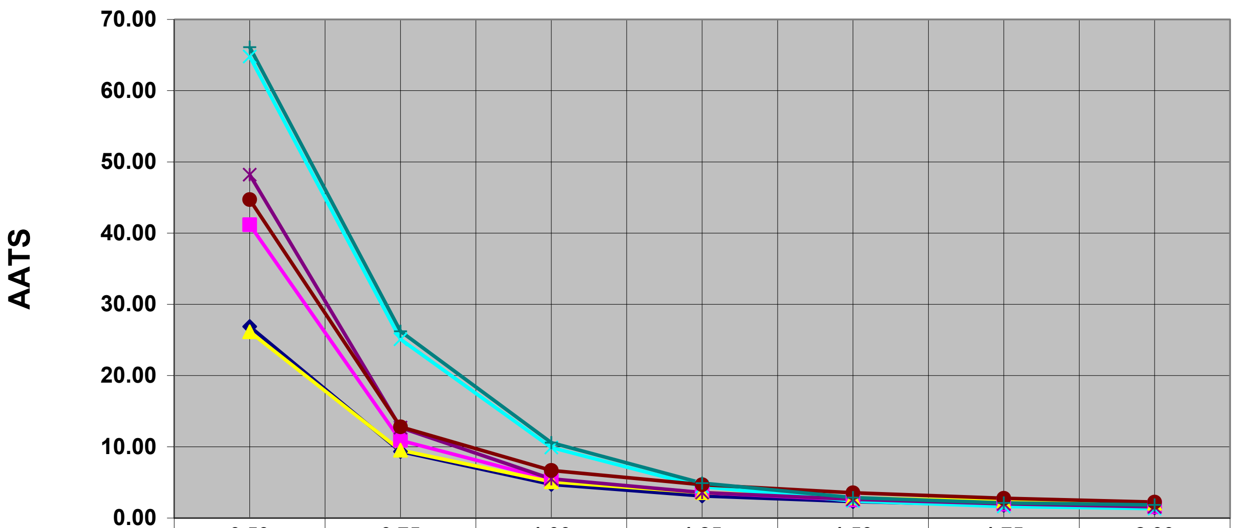


Figure 1. A schematic VP T^2 control chart



	0.50	0.75	1.00	1.25	1.50	1.75	2.00
VP	26.88	9.33	4.72	3.11	2.31	1.80	1.45
DWL	41.18	10.90	5.38	3.56	2.56	1.95	1.58
VSSCL	26.16	9.52	5.10	3.57	2.83	2.38	2.07
VSICL	64.79	25.10	9.91	4.41	2.40	1.63	1.32
VSSI	48.22	12.69	5.50	3.58	2.63	2.01	1.57
VSS	44.72	12.80	6.69	4.68	3.54	2.78	2.23
VSI	66.10	26.21	10.58	4.90	2.87	2.13	1.85

d

Figure 2. A comparison between different schemes the case p = 2, n_0 = 2, h_0 = 1 and λ = 0.01

Tables

Table 1. A Comparison between the VP , 2SVP and FRS schemes in accordance with AATS performance for p =2 and 4, n_0 =2, 3 and 5, h_0 =1, α = 0.005 and λ = 0.01.

p	n_0	d	k_1	k_2	w	h_1	h_2	n_1	n_2	$AATS_{VP}$	$AATS_{2SVP}$	$AATS_{FRS}$
2	2	0.5	32.64	5.99	4.70	1.10	0.10	1	11	26.29	24.43	76.36
		1.0	22.32	7.35	3.30	1.22	0.10	1	6	4.47	4.36	17.99
		1.5	15.31	8.47	2.29	1.43	0.10	1	4	1.66	1.66	5.26
		2.0	11.09	9.82	2.33	1.42	0.10	1	4	0.93	0.93	2.01
	3	0.5	42.45	6.36	4.31	1.12	0.10	1	18	14.32	13.45	54.82
		1.0	25.45	8.37	2.26	1.44	0.10	1	7	2.69	2.6	10.01
		1.5	13.01	9.16	1.86	1.60	0.10	1	6	1.08	1.08	2.68
		2.0	10.60	10.60	2.20	1.46	0.10	1	7	0.69	0.69	1.06
	5	0.5	80.11	7.36	3.28	1.22	0.10	1	21	7.64	6.41	32.44
		1.0	23.63	9.04	1.58	1.77	0.10	1	10	1.48	1.4	4.42
		1.5	10.60	10.60	1.73	1.67	0.10	1	10	0.72	0.72	1.17
		2.0	10.60	10.60	2.18	1.47	0.10	1	13	0.56	0.56	0.60
4	2	0.5	48.53	9.49	7.90	1.10	0.10	1	11	38.09	37.78	100.77
		1.0	32.15	10.62	6.65	1.17	0.10	1	7	6.45	6.39	28.20
		1.5	21.68	11.85	5.31	1.32	0.10	1	4	2.23	2.23	8.32
		2.0	16.42	12.55	5.76	1.26	0.10	2	3	1.15	1.15	3.01
	3	0.5	96.98	9.49	7.90	1.10	0.10	1	21	20.02	19.88	76.87
		1.0	31.86	11.65	5.47	1.30	0.10	1	8	3.70	3.70	15.96
		1.5	17.34	12.27	5.53	1.29	0.10	2	5	1.38	1.38	4.09
		2.0	15.30	13.34	6.73	1.17	0.10	3	4	0.79	0.79	1.46
	5	0.5	83.07	10.57	6.70	1.17	0.10	1	26	10.25	10.14	48.69
		1.0	20.16	11.93	5.33	1.32	0.10	3	10	1.93	1.96	6.95
		1.5	15.11	14.08	6.00	1.23	0.10	4	9	0.83	0.83	1.64
		2.0	14.86	14.86	7.80	1.10	0.10	4	14	0.59	0.59	0.70

Table 2. A Comparison between the VSSCL and FRS schemes in accordance with AATS performance for p =2 and 4, n_0 =2, 3 and 5, h_0 =1, α = 0.005 and λ = 0.01.

p	n_0	d	k_1	k_2	w	n_1	n_2	$AATS_{VSSCL}$	$AATS_{FRS}$
2	2	0.5	30.08	5.52	5.19	1	14	25.59	76.36
		1.0	21.03	6.92	3.74	1	7	4.90	17.99
		1.5	14.96	8.04	2.76	1	5	2.19	5.26
		2.0	12.14	9.31	1.99	1	4	1.39	2.01
	3	0.5	96.36	6.21	4.46	1	19	14.11	54.82
		1.0	24.52	7.86	2.77	1	9	3.12	10.01
		1.5	15.48	8.99	1.73	1	6	1.59	2.68
		2.0	12.52	10.21	0.60	1	4	1.01	1.06
	5	0.5	59.76	7.09	3.56	1	24	7.55	32.44
		1.0	26.87	8.80	1.81	1	11	2.05	4.42
		1.5	16.18	10.04	0.60	1	6	1.07	1.17
		2.0	10.60	10.60	0.00	1	5	0.60	0.60
4	2	0.5	94.37	8.53	8.92	1	16	35.69	100.77
		1.0	30.00	10.25	7.06	1	8	6.73	28.20
		1.5	20.69	11.28	6.04	1	6	2.74	8.32
		2.0	15.15	13.56	7.08	1	6	1.75	3.01
	3	0.5	84.91	9.31	8.08	1	23	19.46	76.87
		1.0	90.58	11.39	5.76	1	10	3.97	15.96
		1.5	21.15	12.48	4.55	1	6	1.88	4.09
		2.0	14.99	14.10	7.43	3	6	1.16	1.46
	5	0.5	95.58	10.35	6.94	1	29	10.05	48.69
		1.0	26.52	12.07	4.95	2	13	2.43	6.95
		1.5	15.23	13.73	6.18	4	9	1.23	1.64
		2.0	14.86	14.86	3.37	4	6	0.71	0.70

Table 3. A Comparison between the VSICL and FRS schemes in accordance with AATS performance for p =2 and 4, n_0 =2, 3 and 5, h_0 =1, α = 0.005 and λ = 0.01.

p	n_0	d	k_1	k_2	w	h_1	h_2	$AATS_{VSICL}$	$AATS_{FRS}$
2	2	0.5	87.41	9.77	0.84	2.75	0.10	65.99	76.36
		1.0	13.80	9.59	1.22	2.09	0.10	9.90	17.99
		1.5	12.96	9.22	1.80	1.63	0.10	2.28	5.26
		2.0	11.69	8.93	2.86	1.29	0.10	1.01	2.01
	3	0.5	13.15	9.73	1.14	2.19	0.10	42.53	54.82
		1.0	13.61	9.44	1.41	1.90	0.10	4.72	10.01
		1.5	12.05	8.99	2.47	1.38	0.10	1.24	2.68
		2.0	11.01	9.01	4.09	1.14	0.10	0.70	1.06
	5	0.5	13.59	9.68	1.14	2.19	0.10	21.49	32.44
		1.0	12.74	9.16	1.94	1.56	0.10	1.92	4.42
		1.5	11.10	8.97	3.86	1.16	0.10	0.73	1.17
		2.0	10.60	10.60	4.71	1.10	0.10	0.56	0.60
4	2	0.5	17.64	14.16	2.43	2.77	0.10	88.25	100.77
		1.0	18.85	14.09	2.36	2.87	0.10	16.63	28.20
		1.5	18.22	13.62	3.28	1.97	0.10	3.62	8.32
		2.0	16.70	13.11	4.68	1.44	0.10	1.35	3.01
	3	0.5	18.08	14.18	2.29	2.98	0.10	62.36	76.87
		1.0	18.78	13.91	2.69	2.44	0.10	7.93	15.96
		1.5	17.19	13.25	4.19	1.57	0.10	1.75	4.09
		2.0	15.68	12.97	6.18	1.21	0.10	0.83	1.46
	5	0.5	18.57	14.18	2.22	3.09	0.10	34.29	48.69
		1.0	17.99	13.52	3.48	1.85	0.10	2.97	6.95
		1.5	15.82	12.97	5.91	1.24	0.10	0.88	1.64
		2.0	14.97	14.07	7.91	1.10	0.10	0.59	0.70

Table 4. A Comparison between the VP, DWL, VSSCL, VSICL, VSSI, VSS, VSI and FRS schemes in accordance with AATS performance for p =2 and 4, n_0 =2, 3 and 5, h_0 =1, α = 0.005 and λ = 0.01.

p	n_0	d	VP	DWL	VSSCL	VSICL	VSSI	VSS	VSI	FRS
2	2	0.5	26.29	37.67	25.59*	65.99	48.22	39.97	66.10	76.36
		1.0	4.47*	4.72	4.90	9.90	5.18	5.88	10.54	17.99
		1.5	1.66*	1.73	2.19	2.28	1.83	2.38	2.43	5.26
		2.0	0.93*	0.98	1.39	1.01	1.17	1.43	1.04	2.01
	3	0.5	14.32	18.71	14.11*	42.53	20.98	19.79	43.87	54.82
		1.0	2.69*	2.69*	3.12	4.72	3.06	3.61	5.06	10.01
		1.5	1.08*	1.19	1.59	1.24	1.57	1.69	1.29	2.68
		2.0	0.69*	0.69*	1.01	0.69*	1.15	1.02	0.69*	1.06
	5	0.5	7.64	9.09	7.55*	21.49	9.45	9.72	22.53	32.44
		1.0	1.48	1.46*	2.05	1.92	2.21	2.27	2.03	4.42
		1.5	0.72*	0.72*	1.07	0.73	1.51	1.09	0.74	1.17
		2.0	0.56*	0.56*	0.60	0.56*	1.15	0.60	0.56*	0.60
4	2	0.5	38.09	54.97	35.69*	88.25	73.12	57.36	89.55	100.77
		1.0	6.45*	6.92	6.73	16.63	7.75	8.29	17.38	28.20
		1.5	2.23*	2.27	2.46	3.62	2.38	3.03	3.85	8.32
		2.0	1.15*	1.17	1.23	1.35	1.17	1.76	1.40	3.01
	3	0.5	20.02	26.94	19.48*	62.36	33.65	27.92	63.67	76.87
		1.0	3.71*	3.84	3.81	7.93	4.15	4.69	8.39	15.96
		1.5	1.38*	1.38*	1.42	1.75	1.42	2.00	1.84	4.09
		2.0	0.79*	0.79*	0.84	0.83	0.79*	1.16	0.84	1.46
	5	0.5	10.25	12.66	9.96*	34.29	13.12	13.13	35.38	48.69
		1.0	1.93	1.93	1.80*	2.97	2.07	2.70	3.16	6.95
		1.5	0.83*	0.83*	0.90	0.88	0.87	1.24	0.90	1.64
		2.0	0.59*	0.59*	0.70	0.59*	0.63	0.71	0.59*	0.70