Detection of non concave and non increasing multifractal spectra using wavelet leaders (Part II)

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FNRS Contact Group “Wavelets and applications”

Plan

1. **A new multifractal formalism based on wavelet leaders**
2. **Implementation of this new multifractal formalism**
3. **Numerical Simulations on Theoretical Examples**
4. **Conclusion**

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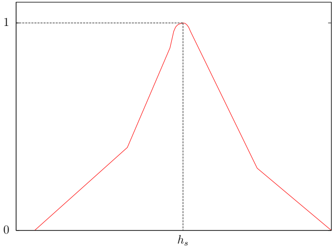
(Bastin, Esser, Jaffard, 2014 [4])
An approximation of spectrum of singularities of a function f is given by

$$\widetilde{\nu}_f(h) = \begin{cases} \lim\limits_{\epsilon \to 0^+} \limsup\limits_{j \to +\infty} \dfrac{\log \#\{k : d_{j,k} \geq C 2^{-(h+\epsilon)j}\}}{\log 2^j} & \text{if } h \leq h_s \\ \lim\limits_{\epsilon \to 0^+} \limsup\limits_{j \to +\infty} \dfrac{\log \#\{k : d_{j,k} < C 2^{-(h-\epsilon)j}\}}{\log 2^j} & \text{otherwise} \end{cases}$$

where $(d_{j,k})_{j \in \mathbb{N}, k \in \{0,\dots,2^j-1\}}$ are the **wavelet leaders** of f, h_s is the smallest positive real such that $\widetilde{\nu}_f(\beta) = 1$ and C is a positive constant.



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Proposition

The constant C in the definition of $\widetilde{\nu}_f$ is arbitrary.

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- Denote by $E_j^{\geq}(C,h)(f) = \{k : d_{j,k} \geq C2^{-hj}\}$. We must compute

$$\widetilde{\nu}_f(h) = \lim_{\epsilon \to 0^+} \limsup_{j \to +\infty} \frac{\log \# E_j^{\geq}(1, h+\epsilon)(f)}{\log 2^j}$$

for all $h \in [0; h_s]$ where h_s is the smallest $h \geq 0$ such that $\widetilde{\nu}_f(h) = 1$.

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This means that $\# E_j^{\geq}(C,h)(f) \sim 2^{-\widetilde{\nu}_f(h)j}$ for j "large enough".
So, we can approximate $\widetilde{\nu}_f(h)$ by the **slope** of

$$j \in \mathbb{N} \mapsto \frac{\log \# E_j^{\geq}(C,h)(f)}{\log 2}$$

for j "large enough". This slope is denote by $\widetilde{\nu}_f^C(h)$.

For a fixed h, the main problem is to determine a good constant C because we have only a **finite number** of wavelet leaders :

- If C is too small, the detected value of $\widetilde{\nu}_f^C(h)$ will be 1 ;
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We construct the function

$$C > 0 \mapsto \widetilde{\nu}_f^C(h).$$

In theory, the constant is arbitrary, so, in practice, this function must stabilize if $h \geq h_{min} = \inf\{h \geq 0 : \widetilde{\nu}_f(h) = 0\}$.

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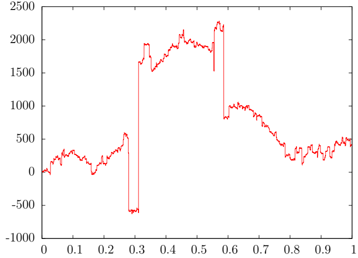
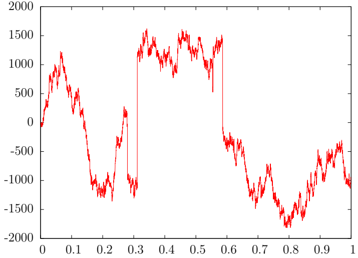
Lévy process

A Lévy process (associated to an index $\beta \in [0;2]$) can be decomposed as the sum of a (possibly vanishing) Brownian part and an independent pure jump process.



Lévy without part Brownian



Lévy with a part Brownian

Lévy process

A Lévy process (associated to an index $\beta \in [0;2]$) can be decomposed as the sum of a (possibly vanishing) Brownian part and an independent pure jump process.

Theorem (Jaffard, 1999 [7])

- If the Lévy process with index β has no Brownian part, then almost surely, the spectrum is given by

  $$d_f(h) = \beta h$$

  for all $h \in [0; 1/\beta]$.

- If the Lévy process with index β has a Brownian part, then almost surely, the spectrum is given by

  $$d_f(h) = \begin{cases} \beta h & \text{if } h \in [0; 1/2) \\ 1 & \text{if } h = 1/2 \end{cases} .$$

Lévy process without Brownian part ($\beta = 1.3$)

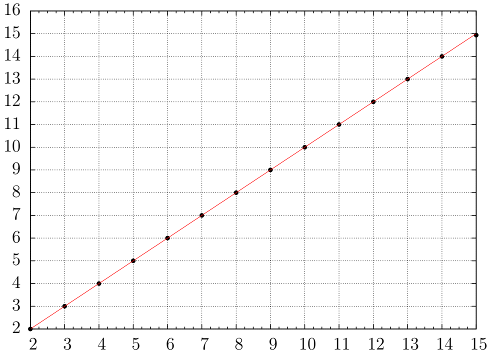
$$\text{Function } j \in \mathbb{N} \mapsto \frac{\log \#\{k : d_{j,k} \geq C\ 2^{-hj}\}}{\log 2}$$

Lévy process without Brownian part $(\beta = 1.3)$

$$\text{Function } j \in \mathbb{N} \mapsto \frac{\log \#\{k : d_{j,k} \geq C\ 2^{-0.55 j}\}}{\log 2}$$

Lévy process without Brownian part ($\beta = 1.3$)

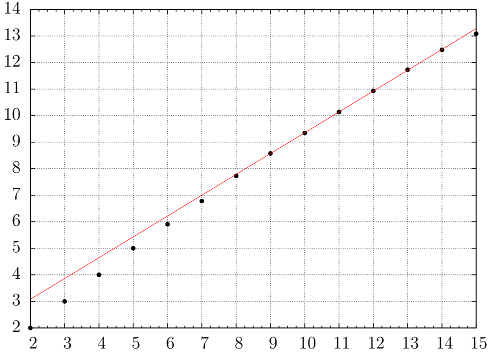
Function $j \in \mathbb{N} \mapsto \dfrac{\log \#\{k : d_{j,k} \geq 100\ 2^{-0.55j}\}}{\log 2}$



slope $= \widetilde{\nu}_f^{100}(0.55) = 1$

Lévy process without Brownian part ($\beta = 1.3$)

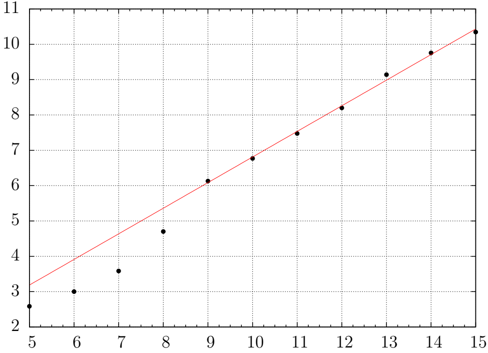
Function $j \in \mathbb{N} \mapsto \dfrac{\log \#\{k : d_{j,k} \geq 600 \ 2^{-0.55j}\}}{\log 2}$



$$\text{slope} = \widetilde{\nu}_f^{600}(0.55) = 0.784862$$

Lévy process without Brownian part ($\beta = 1.3$)

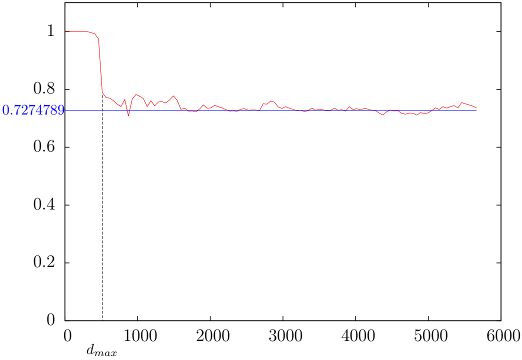
Function $j \in \mathbb{N} \mapsto \dfrac{\log \#\{k : d_{j,k} \geq 2500\ 2^{-0.55j}\}}{\log 2}$



slope $= \widetilde{\nu}_f^{2500}(0.55) = 0.724723$

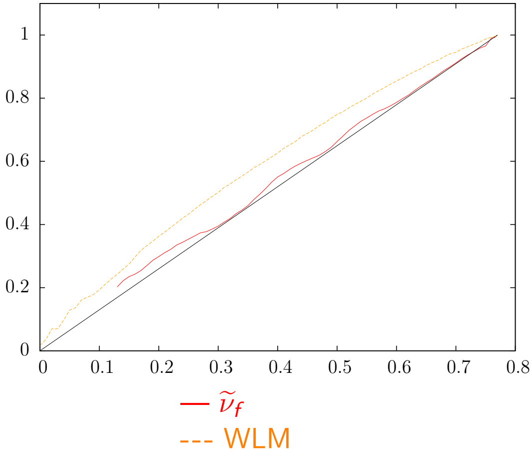
Lévy process without Brownian part ($\beta = 1.3$)

Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.55)$



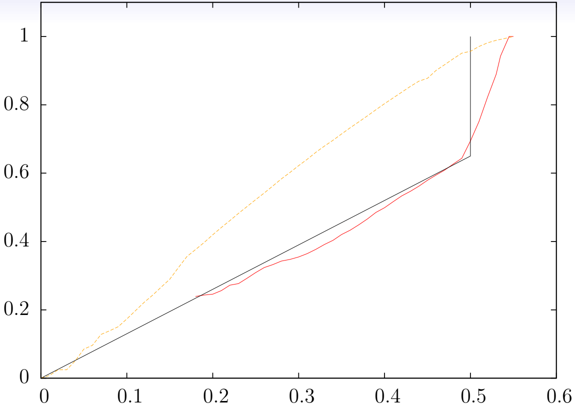
$$\widetilde{\nu}_f(0.55) = 0.715$$

Lévy process without Brownian part ($\beta = 1.3$)



The results are the averages over 100 realizations of length 2^{17}.

Lévy process with a Brownian part ($\beta = 1.3$)



— $\widetilde{\nu}_f$

--- WLM

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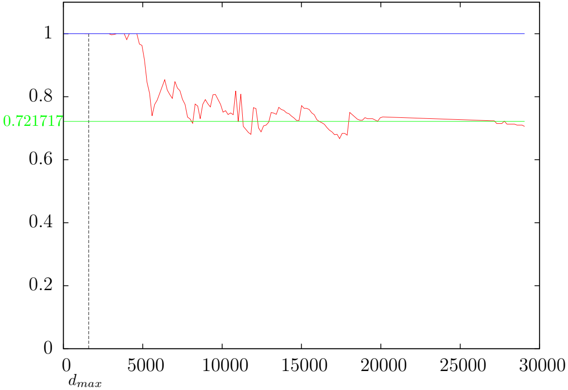

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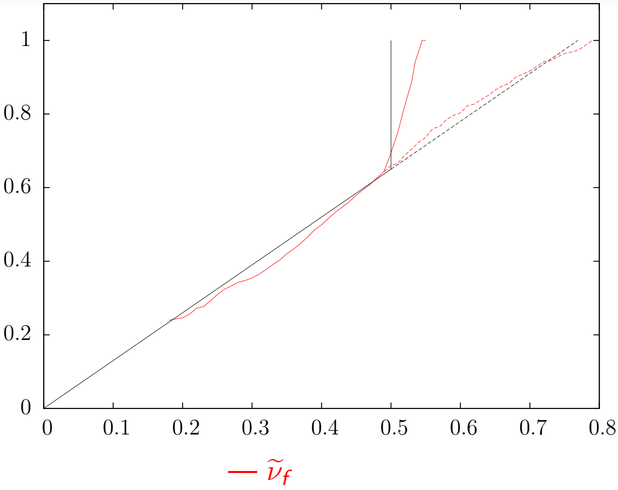
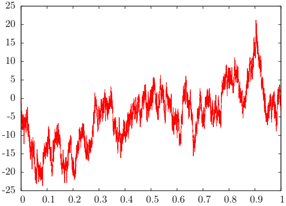


Fractional Brownian motion

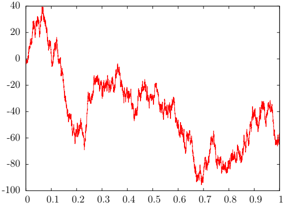
Theorem

Let a Fractional Brownian motion of the indice $H \in]0;1[$ define on a probability space $(\Omega, \mathcal{A}, \mu)$. For almost everywhere $\omega \in \Omega$, the Fractional Brownian Walk associated to ω is mono-Hölder with an exponent H.

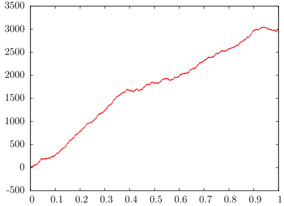
$H = 0.3$



$H = 0.5$



$H = 0.8$



Fractional Brownian motion

$H = 0.5$

Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.4)$

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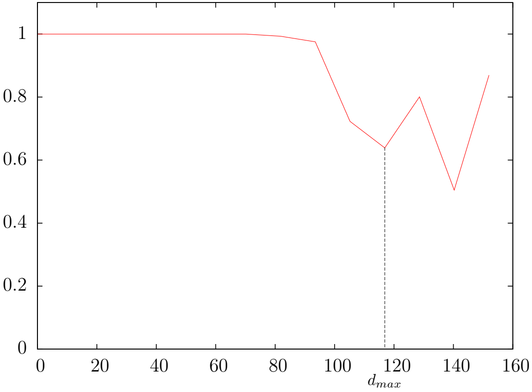
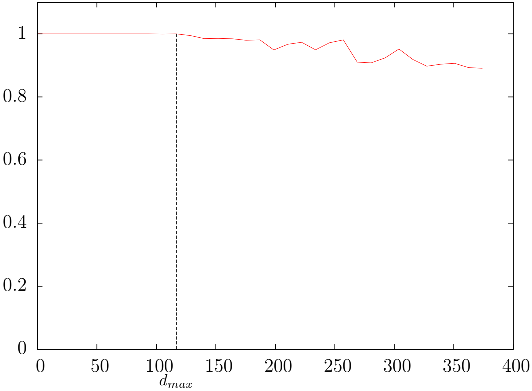


Fractional Brownian motion

$H = 0.5$

Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.48)$



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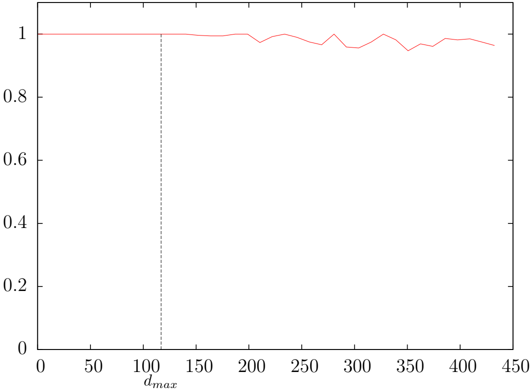

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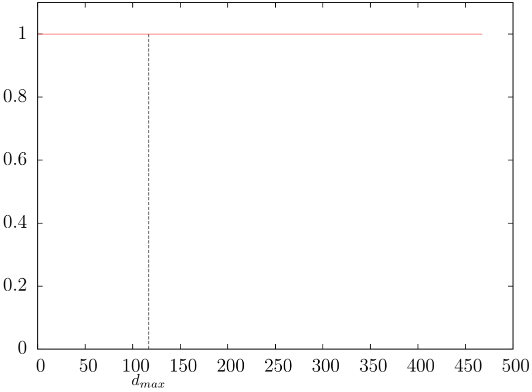

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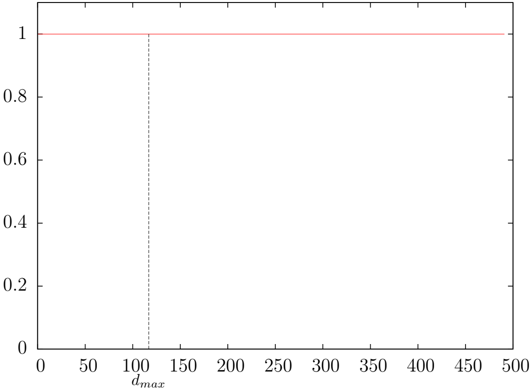

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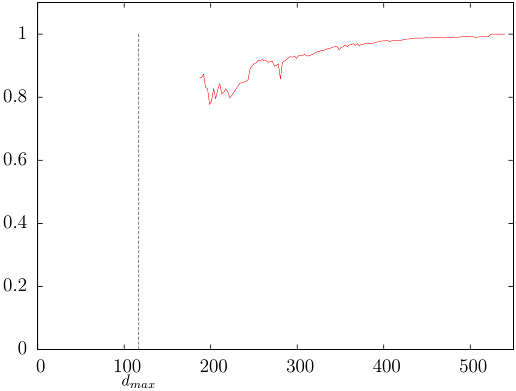

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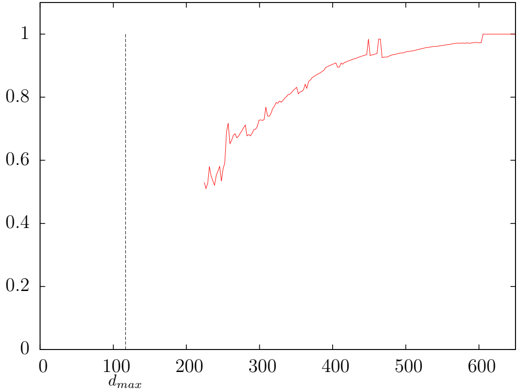
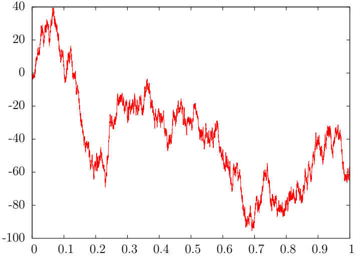


Fractional Brownian motion

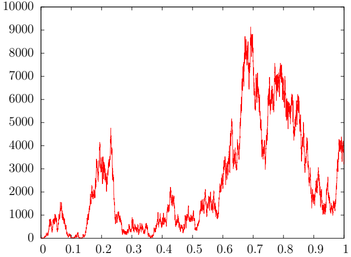
Test on 100 Fractional Brownian Walks of a size ≈ 1000000.

	L^ν Method	WLM
Error mean	0.011888889	0.016746522
Standard deviation of error	0.020542338	0.020469233

The square of a Brownian motion



Brownian Motion



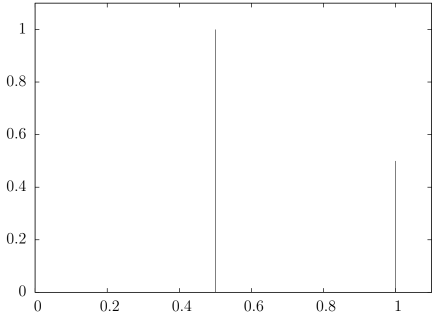
Square of a Brownian motion

The square of a Brownian motion

Theorem (Abry, Jaffard, Wendt, 2012 [1])

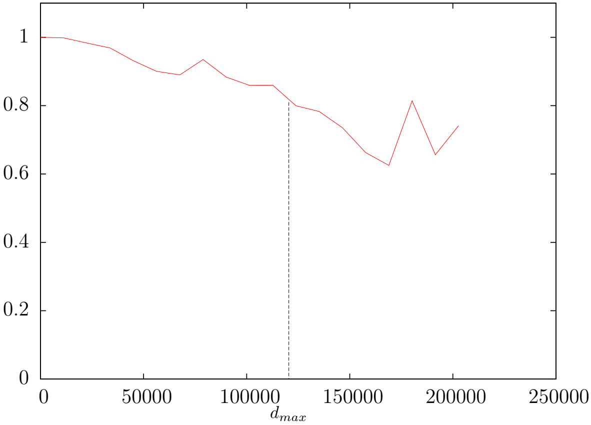
The spectrum of the square of a Brownian motion is given by

$$d(h) = \begin{cases} 1 & \text{if } h = 0.5 \\ 0.5 & \text{if } h = 1 \end{cases} .$$



The square of a Brownian motion

Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.4)$



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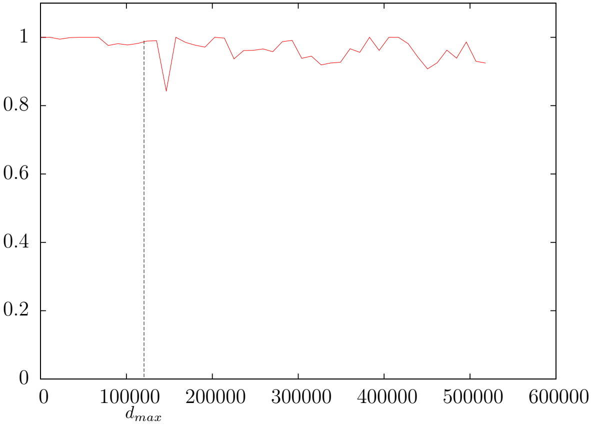
Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.48)$



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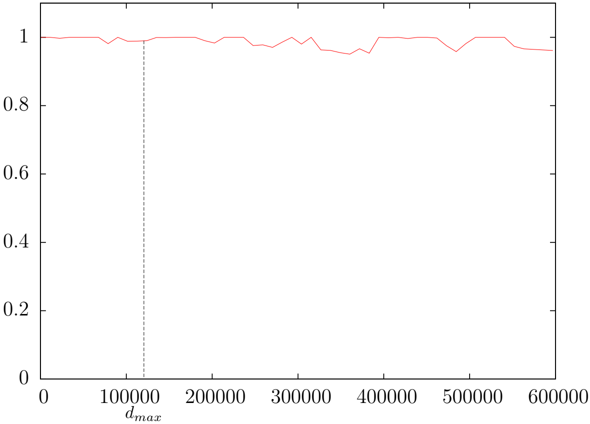
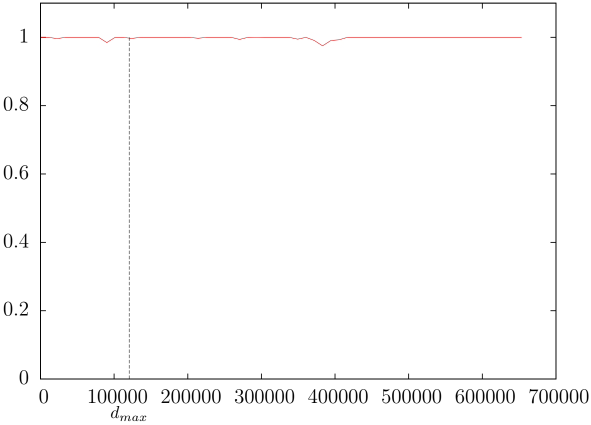


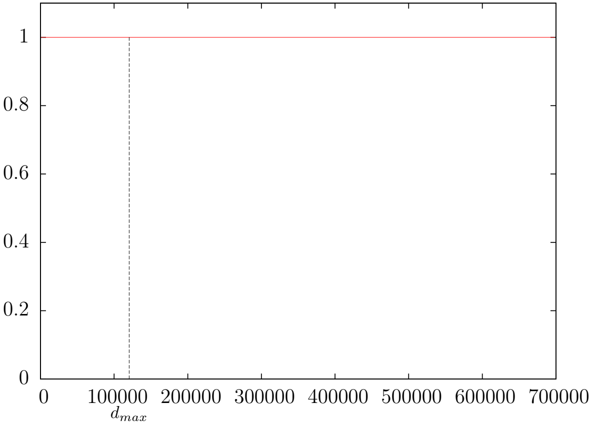
The square of a Brownian motion

Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.5)$



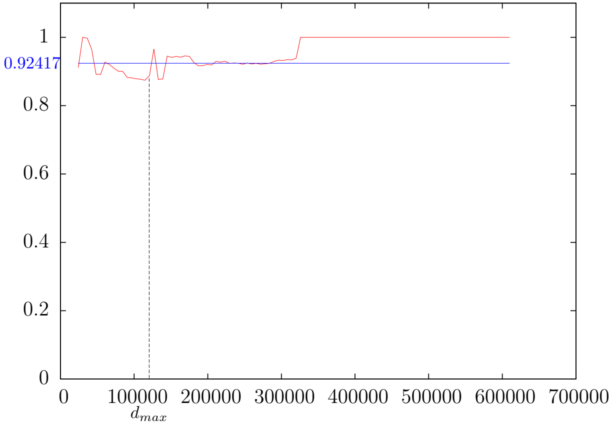
The square of a Brownian motion

Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.51)$



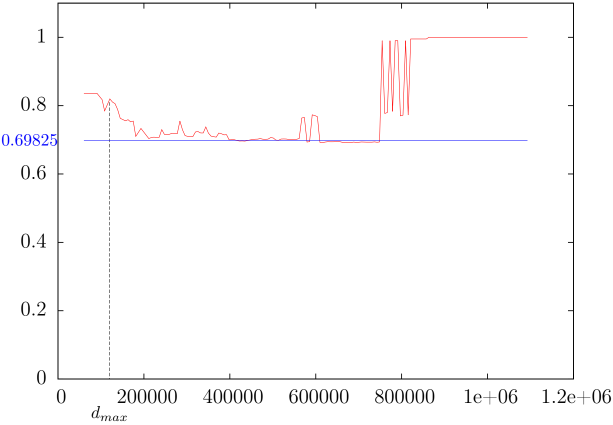
The square of a Brownian motion

Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.6)$



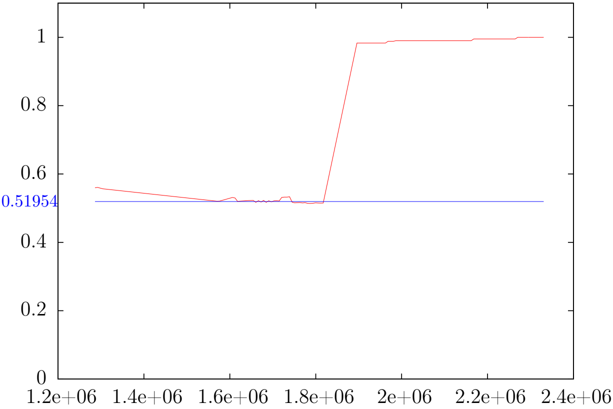
The square of a Brownian motion

Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.8)$

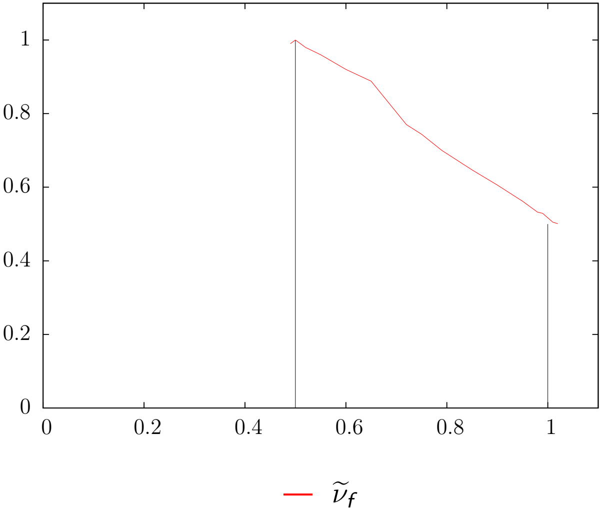


The square of a Brownian motion

Function $C > 0 \mapsto \widetilde{\nu}_f^C(1)$

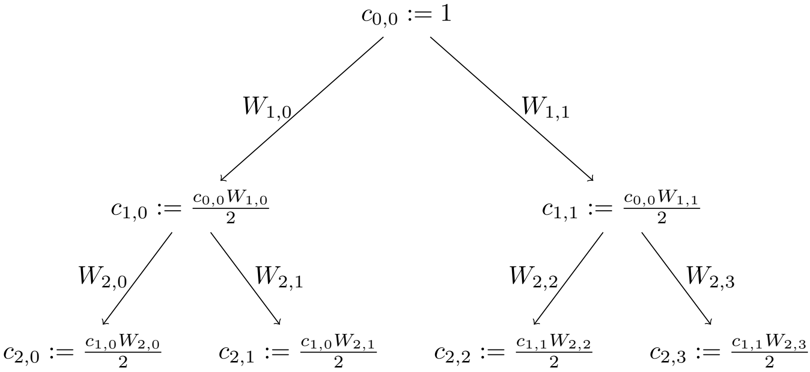


The square of a Brownian motion



Cascades of Mandelbrot

Take W a positive random variable such that $E[W] = 1$ and $W_{jk} \sim^{iid} W$ for all $j \in \mathbb{N}, k \in \{0, \ldots, 2^j - 1\}$. Almost surely, the following construction defines a Borel measure μ on $[0; 1]$:



$$c_{0,0} := 1$$

$W_{1,0}$ $W_{1,1}$

$$c_{1,0} := \frac{c_{0,0} W_{1,0}}{2} \qquad c_{1,1} := \frac{c_{0,0} W_{1,1}}{2}$$

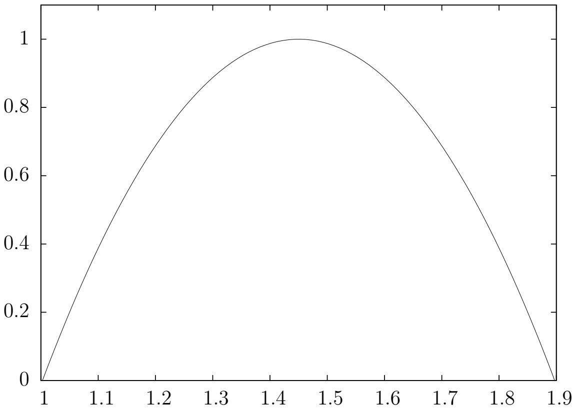
$W_{2,0}$ $W_{2,1}$ $W_{2,2}$ $W_{2,3}$

$$c_{2,0} := \frac{c_{1,0} W_{2,0}}{2} \qquad c_{2,1} := \frac{c_{1,0} W_{2,1}}{2} \qquad c_{2,2} := \frac{c_{1,1} W_{2,2}}{2} \qquad c_{2,3} := \frac{c_{1,1} W_{2,3}}{2}$$

with $\mu([k2^{-j}; (k+1)2^{-j}[) = c_{j,k}$. Moreover, $(c_{j,k})_{j,k}$ can be considered as wavelet coefficients of a function.

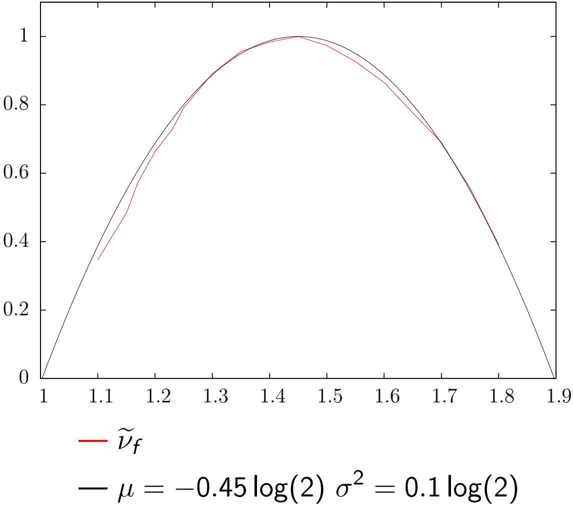
Cascades of Mandelbrot - log-normal

(Arneodo, Bacry, Muzy, 1998 [2] - Barral, Seuret, 2005 [3])



— $\mu = -0.45\log(2)\ \sigma^2 = 0.1\log(2)$

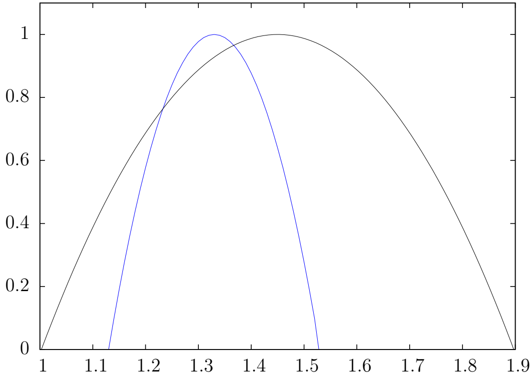
Cascades of Mandelbrot - log-normal



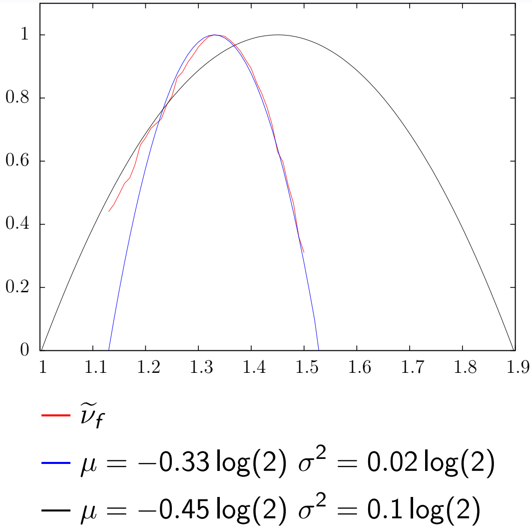
Cascades of Mandelbrot - sum of log-normal

Denote by μ_1 and μ_2 two borel measures on $[0;1]$. We can prove that

$$h_{\mu_1+\mu_2}(x) = \min\{h_{\mu_1}(x), h_{\mu_2}(x)\}.$$

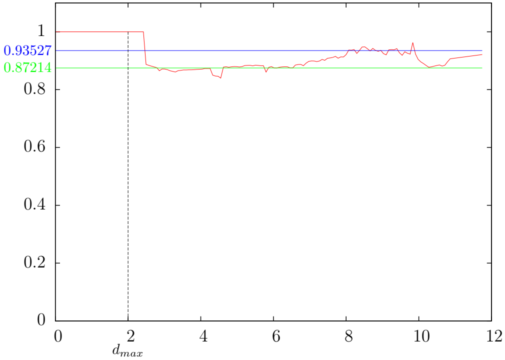


Cascades of Mandelbrot - sum of log-normal



Cascades of Mandelbrot - sum of log-normal

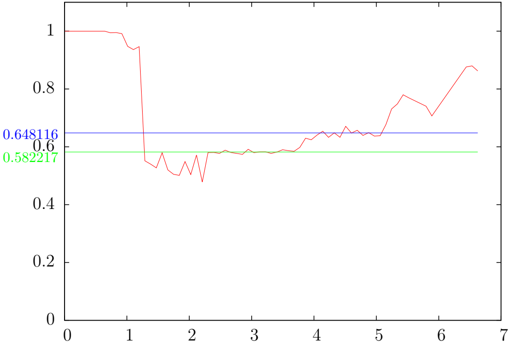
Function $C > 0 \mapsto \widetilde{\nu}_f^C(1.29)$



$$\mu = -0.33\log(2)\ \sigma^2 = 0.02\log(2) \rightarrow \widetilde{\nu}_f(1.29) = 0.96$$
$$\mu = -0.45\log(2)\ \sigma^2 = 0.1\log(2) \rightarrow \widetilde{\nu}_f(1.29) = 0.872$$

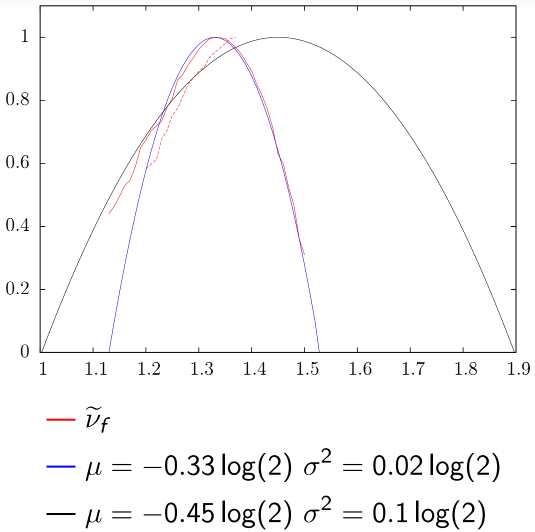
Cascades of Mandelbrot - sum of log-normal

Function $C > 0 \mapsto \widetilde{\nu}_f^C(1.2)$



$$\mu = -0.33\log(2)\ \sigma^2 = 0.02\log(2) \to \widetilde{\nu}_f(1.2) = 0.5775$$
$$\mu = -0.45\log(2)\ \sigma^2 = 0.1\log(2) \to \widetilde{\nu}_f(1.2) = 0.6875$$

Cascades of Mandelbrot - sum of log-normal



Plan

1. A new multifractal formalism based on wavelet leaders
2. Implementation of this new multifractal formalism
3. Numerical Simulations on Theoretical Examples
4. **Conclusion**

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In the future,

- it should also be tested on real-life signals (cardiac signals, image processing and so on) ;
- ...

Thank you for your attention !

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