A formulation on the special Euclidean group for dynamic analysis of multibody systems

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Abstract

This paper presents a finite element approach of multibody systems using the special Euclidean group $SE(3)$ framework. The development leads to a compact and unified mixed coordinate formulation of the rigid bodies and the kinematic joints. Flexibility in the kinematic joints is also introduced easily. The method relies on local description of motions, so that it provides a singularity-free formulation and exhibits important advantages regarding numerical implementation. A practical case is presented to illustrate the method.

1 Introduction

The finite element approach of multibody systems has been significantly studied, for instance in [1, 2, 3]. Nodal values, measuring positions and orientations with respect to an inertial frame, are introduced to describe the configuration of rigid and flexible multibody systems. Then, inertia properties are assigned to some nodes and kinematic constraints are used to model the physical links between the nodes of the system. Based on the finite element concept, this approach allows one to write systematically the equations of motion by collecting generic expressions which are slightly tuned according to a system part of interest. It results in second-order differential-algebraic equations for which suitable time integration solvers have been developed.

The finite element approach has been brought to a new level by exploiting the underlying Lie group structure of the configuration space, see e.g. [4]. The equations of motion are then formulated as a set of differential-algebraic equations on a Lie group, without the need to refer to a particular set of coordinates. These equations can then be solved using recently developed Lie group time integration schemes [5, 6]. A key aspect of this more abstract approach is that it systematically avoids the parameterization of the total motion but rather leads to local parameterizations. In particular, this framework reduces the non-linearity of the equations of motion and results in a naturally singularity-free formulation.

This paper presents the finite element approach of multibody systems using the special Euclidean group $SE(3)$ framework. This framework is widely used in robotics and is often referred to as the screw formulation [7, 8]. In the present work, the $SE(3)$ framework leads to a unified mixed coordinate formulation for multibody systems which complies with the Lie group setting. In contrast with similar approaches also based on the special Euclidean group such as [9, 10], here the kinematic constraint equations are systematically obtained from a general linear mapping which yields six constraint equations per kinematic constraint, involving the nodal values as well as kinematic joint transformations[11]. Thanks to the physical interpretation of

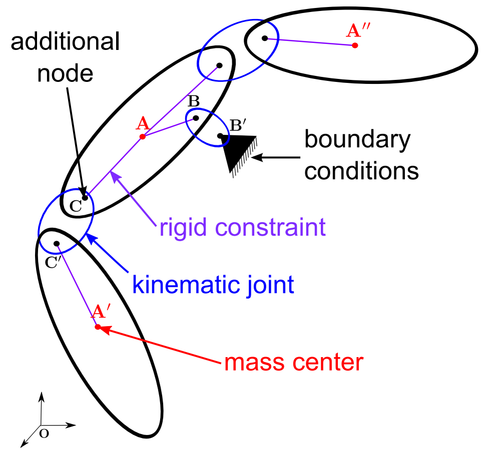


Figure 1: Representation of the nodes and the relationships involved in the finite element approach of multibody systems.

the relative coordinates, flexibility in the kinematic joints as well as actuators and control laws are easily introduced.

2 Description of Multibody Systems

In the present formulation, multibody systems are modelled using nodal absolute variables and kinematic joint relative transformations. The situation is represented in Fig. 1. First, a node is assigned at the mass center of each rigid body of the system, as nodes A, A' and A'' in Fig. 1. Using the rigid body assumption, the inertia forces of each body are expressed as a function of the nodal variables located at its mass center. Then, additional nodes are introduced to allow the description of the multibody system by specifying further constraints. For instance, in Fig. 1, nodes B and B' are used for boundary conditions and nodes C and C' are used to specify a kinematic joint between the two related bodies. The additional nodes are also linked to the mass center of the body by rigid constraints. In order to describe the restricted relative motion allowed by the kinematic joints, kinematic joint transformations are introduced and come in addition to the nodal variables.

In the following, a formalism is presented, based on the special Euclidean group $SE(3)$. The equations of motion and all terms involved are developed explicitly, and a suitable time integration method is recalled. Eventually, a practical case is presented to illustrate the method.

3 The special Euclidean group $SE(3)$ formalism

The variables used in this formalism, i.e. the nodal variables and the kinematic joint transformations, are material frames. Such material frames belong to the special Euclidean group $SE(3)$ and they can be represented as 4×4 homogeneous transformation matrices $\mathbf{H}$ denoted as

$$\mathbf{H} = \mathcal{H}(\mathbf{R}, \mathbf{x}) = \begin{bmatrix} \mathbf{R} & \mathbf{x} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \tag{1}$$

where $\mathbf{R}$ is a 3×3 rotation matrix and $\mathbf{x}$ is a 3×1 position vector which respectively account for an orientation and a position. For the nodal values, the rotation matrix and the position vector describe the orientation and the position of the node with respect to the inertial frame while for kinematic joint transformations, they represent the relative rotation and relative displacement inside the joint.

The group denomination comes from the fact that the composition, namely the matrix product, of two such 4×4 matrices yields another 4×4 matrix which has the same structure. Formally, one has $\mathbf{H}_3 = \mathbf{H}_2\mathbf{H}_1 \Leftrightarrow \mathcal{H}(\mathbf{R}_3, \mathbf{x}_3) = \mathcal{H}(\mathbf{R}_2\mathbf{R}_1, \mathbf{R}_2\mathbf{x}_1 + \mathbf{x}_2)$. In the rest of this section, the kinematics of multibody systems is introduced progressively as well as the necessary mathematical concepts, but the Lie group vocabulary is avoided on purpose for the sake of conciseness and simplicity. More abstract and rigorous treatments of the underlying mathematical framework can be found in [7, 6, 12, 8].

3.1 Kinematics of the Rigid Body

Derivatives of $SE(3)$ elements involve six-dimensional vectors which can be introduced, for node I, as follows

$$d(\mathbf{H}_I) = \mathbf{H}_I\widetilde{\mathbf{h}}_I = \begin{bmatrix} \mathbf{R}_I & \mathbf{x}_I \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{h}}_{I,\Omega} & \mathbf{h}_{I,U} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} \tag{2}$$

where $\mathbf{h}_I = [\mathbf{h}_{I,U}^T \ \mathbf{h}_{I,\Omega}^T]^T$ is a six-dimensional vector and the $\widetilde{\bullet}$ operator is a linear mapping. One may say that the 3×1 vector $\mathbf{h}_{I,U}$ and 3×1 vector $\mathbf{h}_{I,\Omega}$ are respectively the position and the rotation part of $\mathbf{h}_I$. $\widetilde{\mathbf{h}}_{I,\Omega}$ is the skew-symmetric matrix formed from the 3 components of $\mathbf{h}_{I,\Omega}$ as

$$\widetilde{\mathbf{h}}_{I,\Omega} = \begin{bmatrix} 0 & -\mathbf{h}_{I,\Omega 3} & \mathbf{h}_{I,\Omega 2} \\ \mathbf{h}_{I,\Omega 3} & 0 & -\mathbf{h}_{I,\Omega 1} \\ -\mathbf{h}_{I,\Omega 2} & \mathbf{h}_{I,\Omega 1} & 0 \end{bmatrix} \tag{3}$$

It is clear from the context whether the tilde operator refers to a six- or a three-dimensional vector. As a consequence of Eq. (2), the derivatives involve so-called material-frame elements. Indeed, considering $d(\mathbf{x}_I) = \mathbf{R}_I\mathbf{h}_{I,U}$ and $d(\mathbf{R}_I) = \mathbf{R}_I\widetilde{\mathbf{h}}_{I,\Omega}$, it is observed that the current rotation matrix multiplies the related parts of $\mathbf{h}_I$ which is therefore interpreted as being expressed in the material frame. As an alternative to Eq. (2), the derivatives could have been represented as $d(\mathbf{H}_I) = \widetilde{\mathbf{h}}_I\mathbf{H}_I$, and the variable $\mathbf{h}_I$ would then represent derivatives in the inertial frame. In this paper, the representation in the material frame is selected because it leads to a number of practical advantages that will appear later. In the same fashion, an arbitrary variation and the velocity associated to node I are introduced as

$$\delta(\mathbf{H}_I) = \mathbf{H}_I\widetilde{\delta\mathbf{h}}_I \qquad\qquad (\mathbf{H}_I)^{\cdot} = \mathbf{H}_I\widetilde{\mathbf{v}}_I \tag{4}$$

where $\delta\mathbf{h}_I = [\delta\mathbf{h}_{I,U}^T \ \delta\mathbf{h}_{I,\Omega}^T]^T$ and $\mathbf{v}_I = [\mathbf{v}_{I,U}^T \ \mathbf{v}_{I,\Omega}^T]^T$ are six-dimensional vectors which involve a 3×1 position and a 3×1 rotation part.

Considering Eq. (4), one obtains the second derivative relationship between the variation of the time derivative and the time derivative of the variation

$$\delta((\mathbf{H}_I)^{\cdot}) = (\delta(\mathbf{H}_I))^{\cdot} \Leftrightarrow \delta(\mathbf{v}_I) = (\delta\mathbf{h}_I)^{\cdot} + \widehat{\mathbf{v}}_I\delta\mathbf{h}_I \tag{5}$$

where $\widehat{\bullet}$ is a linear operator which maps a 6×1 vector into a 6×6 matrix as

$$\widehat{\mathbf{v}}_I = \begin{bmatrix} \widetilde{\mathbf{v}}_{I,\Omega} & \widetilde{\mathbf{v}}_{I,U} \\ \mathbf{0}_{3\times 3} & \widetilde{\mathbf{v}}_{I,\Omega} \end{bmatrix} \tag{6}$$

Because $\widehat{\mathbf{v}}_I$ does not vanish, one observes that derivatives (or variations) do not commute.

3.2 Kinematics of Joints

In the $SE(3)$ formalism, the relationship between two nodes A and B which are connected by a rigid constraint or by a kinematic joint I can conveniently be expressed as

$$\mathbf{H}_B = \mathbf{H}_A \mathbf{H}_{J,I} \tag{7}$$

where $\mathbf{H}_A$ and $\mathbf{H}_B$ are the nodal matrices of respectively node A and B, and $\mathbf{H}_{J,I}$ is a transformation matrix that describes the relative motion between the two nodes due to the joint I. The restricted relative rotations and displacements leave $k_I < 6$ degrees of freedom to the joint and accordingly, for usual joints, $\mathbf{H}_{J,I}$ belongs to a subgroup of $SE(3)$ [8]. Therefore, the derivative of $\mathbf{H}_{J,I}$ can be expressed as

$$d(\mathbf{H}_{J,I}) = \mathbf{H}_{J,I} \widetilde{\mathbf{h}}_{J,I} \tag{8}$$

where

$$\mathbf{h}_{J,I} = \mathbf{A}_I \mathbf{h}_{j,I} \tag{9}$$

In these equations, $\mathbf{h}_{J,I}$ is a 6×1 vector related to the derivative of the transformation matrix $\mathbf{H}_{J,I}$ while $\mathbf{h}_{j,I}$ is a $k_I \times 1$ vector representing the relative degrees of freedom. These vectors are expressed in the material frame since in Eq. (8) $\widetilde{\mathbf{h}}_{J,I}$ is on the right-hand side of $\mathbf{H}_{J,I}$. The relationship in Eq. (9) involves $\mathbf{A}_I$, a $6 \times k_I$ full-rank matrix whose columns are the 6×1 $\mathbf{e}_{Ii}$, with $i = 1...k_I$, vectors spanning the subalgebra of allowed infinitesimal motions. Each $\mathbf{e}_{Ii}$ can be seen as $\mathbf{e}_{Ii} = [\mathbf{e}_{Ii,U}^T \ \mathbf{e}_{Ii,\Omega}^T]^T$, where the 3×1 unit vectors $\mathbf{e}_{Ii,U}$ and $\mathbf{e}_{Ii,\Omega}$ are respectively related to the displacement and rotation parts of the relative motions. Since they are brought by material frame elements, their components are constant so that matrix $\mathbf{A}_I$ is also constant. In Table 1, some examples of usual joints are given, but others, such as the planar joint or the screw joint, can be easily formulated in this framework as well.

Accordingly, the variation and the time derivative of $\mathbf{H}_{J,I}$ are written

$$\delta(\mathbf{H}_{J,I}) = \mathbf{H}_{J,I} \big(\mathbf{A}_I \delta \mathbf{h}_{j,I}\big)^{\tilde{}} \quad ; \quad (\mathbf{H}_{J,I})^{\dot{}} = \mathbf{H}_{J,I} \big(\mathbf{A}_I \mathbf{v}_{j,I}\big)^{\tilde{}} \tag{10}$$

were $\mathbf{A}_I \delta \mathbf{h}_{j,I} = \delta \mathbf{h}_{J,I}$ and $\mathbf{A}_I \mathbf{v}_{j,I} = \mathbf{v}_{J,I}$. The non-commutativity of the derivative introduced in Eq. (5) leads to

$$\mathbf{A}_I \delta(\mathbf{v}_{j,I}) \quad = \mathbf{A}_I (\delta \mathbf{h}_{j,I})^{\dot{}} + (\widehat{\mathbf{A}_I \mathbf{v}_{j,I}}) \ \mathbf{A}_I \delta \mathbf{h}_{j,I} \tag{11}$$

For example, for the hinge joint, the prismatic joint and the cylindrical joint, one observes that $(\widehat{\mathbf{A}_I \mathbf{v}_{j,I}}) \mathbf{A}_I = \mathbf{0}$, so that derivatives commute, that is $\delta(\mathbf{v}_{J,I}) = (\delta \mathbf{h}_{J,I})^{\dot{}}$, and, since $\mathbf{A}_I$ has full rank, $\delta(\mathbf{v}_{j,I}) = (\delta \mathbf{h}_{j,I})^{\dot{}}$. This is consistent with the fact that, for each of them, the subgroup of relative motions is abelian.

3.3 Kinematics of Multibody Systems

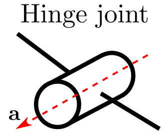
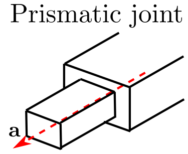
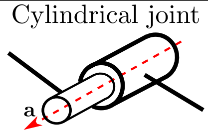
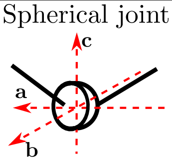
It is convenient to introduce a unified matrix notation to treat both the nodal variables and the kinematic joints at the same time. Let us consider M nodes and m kinematic joints, each of them characterized by an element of $SE(3)$. The state of the system can be described using the invertible block-diagonal square matrix

$$\mathbf{H} = \mathrm{diag}(\mathbf{H}_1, \cdots, \mathbf{H}_M, \mathbf{H}_{J,1}, \cdots, \mathbf{H}_{J,m}) \tag{12}$$

Similarly to Eq. (2) and Eq. (8), derivatives of $\mathbf{H}$ can be expressed as $d(\mathbf{H}) = \mathbf{H}\widetilde{\mathbf{h}}^*$. For instance, the velocities introduced by the time derivative take the form $(\mathbf{H})^{\dot{}} = \mathbf{H}\widetilde{\mathbf{v}}^*$ where

$$\widetilde{\mathbf{v}}^* = \mathrm{diag}\big(\widetilde{\mathbf{v}}_1, \cdots, \widetilde{\mathbf{v}}_M, \widetilde{\mathbf{v}}_{J,1}, \cdots, \widetilde{\mathbf{v}}_{J,m}\ \big) \tag{13}$$

Table 1: $\mathbf{A}_I$ matrix of rigid constraint and some kinematic joints

Joints	k-th column of $\mathbf{A}_J$	$\mathbf{e}_U$	$\mathbf{e}_\Omega$
Rigid constraint	0	-	-
Hinge joint 	1	$\mathbf{0}_{3\times 1}$	$\mathbf{a}$
Prismatic joint 	1	$\mathbf{a}$	$\mathbf{0}_{3\times 1}$
Cylindrical joint 	1 2	$\mathbf{a}$ $\mathbf{0}_{3\times 1}$	$\mathbf{0}_{3\times 1}$ $\mathbf{a}$
Spherical joint 	1 2 3	$\mathbf{0}_{3\times 1}$ $\mathbf{0}_{3\times 1}$ $\mathbf{0}_{3\times 1}$	$\mathbf{a}$ $\mathbf{b}$ $\mathbf{c}$

whose information can be collected in a $(6M+6m)$-dimensional vector as $\mathbf{v}^* = [\mathbf{v}_1^T \cdots \mathbf{v}_M^T\ \mathbf{v}_{J,1}^T\ \cdots \mathbf{v}_{J,m}^T]^T$. In the same fashion, the variation of $\mathbf{H}$ is given by $\delta(\mathbf{H}) = \mathbf{H}\widetilde{\delta\mathbf{h}}^*$ where $\delta\mathbf{h}^* = [\,\delta\mathbf{h}_1^T \cdots \delta\mathbf{h}_M^T\ \delta\mathbf{h}_{J,1}^T\ \cdots \delta\mathbf{h}_{J,m}^T]^T$.

From Eq. (9), one can introduce a constant linear mapping that relates $\mathbf{h}^*$ to a $(6M + k_1 + \cdots + k_m)$-dimensional velocity vector $\mathbf{h}$ as

$$\mathbf{h}^* = \mathbf{A}\mathbf{h} \tag{14}$$

where $\mathbf{A} = \mathrm{diag}(\mathbf{I}_{6\times 6}, \cdots, \mathbf{I}_{6\times 6}, \mathbf{A}_1, \cdots, \mathbf{A}_m)$ and $\mathbf{h} = [\mathbf{h}_1^T \cdots \mathbf{h}_M^T\ \mathbf{h}_{j,1}^T\ \cdots \mathbf{h}_{j,m}^T]^T$. This mapping also holds for the variation, that is $\delta\mathbf{h}^* = \mathbf{A}\delta\mathbf{h}$ where $\delta\mathbf{h} = [\,\delta\mathbf{h}_1^T \cdots \delta\mathbf{h}_M^T\ \delta\mathbf{h}_{j,1}^T\ \cdots \delta\mathbf{h}_{j,m}^T]^T$, and the velocity, that is $\mathbf{v}^* = \mathbf{A}\mathbf{v}$, where $\mathbf{v} = [\mathbf{v}_1^T \cdots \mathbf{v}_M^T\ \mathbf{v}_{j,1}^T\ \cdots \mathbf{v}_{j,m}^T]^T$.

Regarding the non-commutativity of the derivatives at system level, one has

$$\delta((\mathbf{H})^{\cdot}) = (\delta(\mathbf{H}))^{\cdot} \Leftrightarrow \delta(\mathbf{v}^*) = (\delta\mathbf{h}^*)^{\cdot} + \hat{\mathbf{v}}^*\delta\mathbf{h}^* \tag{15}$$

where the $\hat{\bullet}$ operator now reads

$$\hat{\mathbf{v}}^* = \mathrm{diag}\big(\hat{\mathbf{v}}_1, \cdots, \hat{\mathbf{v}}_M, \hat{\mathbf{v}}_{J,1}, \cdots, \hat{\mathbf{v}}_{J,m}\big) \tag{16}$$

Notice that in practical implementation, matrices $\mathbf{H}$ as in Eq. (12) would not be built as such for computation because only the block diagonal terms need to be evaluated. The matrix notation $(\mathbf{H})^{\cdot} = \mathbf{H}\tilde{\mathbf{v}}^*$ is used only to refer to the fact that there is a relationship $(\mathbf{H}_I)^{\cdot} = \mathbf{H}_I\tilde{\mathbf{v}}_I$ for each node and $(\mathbf{H}_{J,I})^{\cdot} = \mathbf{H}_{J,I}\widetilde{\big(\mathbf{A}_I\mathbf{v}_{j,I}\big)}$ for each kinematic joint.

4 Exponential Map and Tangent Map

4.1 $SE(3)$

Starting from a given six-dimensional vector $\mathbf{n} = [\mathbf{n}_U^T \ \mathbf{n}_\Omega^T]^T$, an element of $SE(3)$ can be built using the so-called exponential map which reads

$$\exp_{SE(3)}(\mathbf{n}) = \begin{bmatrix} \exp_{SO(3)}(\mathbf{n}_\Omega) & \mathbf{T}_{SO(3)}^T(\mathbf{n}_\Omega)\mathbf{n}_U \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \tag{17}$$

where, using $a = \sin(||\mathbf{n}_\Omega||)/||\mathbf{n}_\Omega||$ and $b = 2(1-\cos(||\mathbf{n}_\Omega||))/||\mathbf{n}_\Omega||^2$,

$$\exp_{SO(3)}(\mathbf{n}_\Omega) = \mathbf{I}_{3\times 3} + a\tilde{\mathbf{n}}_\Omega + \frac{b}{2}\tilde{\mathbf{n}}_\Omega^2 \tag{18}$$

$$\mathbf{T}_{SO(3)}(\mathbf{n}_\Omega) = \mathbf{I}_{3\times 3} - \frac{b}{2}\tilde{\mathbf{n}}_\Omega + \frac{1-a}{||\mathbf{n}_\Omega||^2}\tilde{\mathbf{n}}_\Omega^2 \tag{19}$$

The exponential map may be seen as a local parameterization in the sense that the argument of the exponential map belongs to a linear space while $SE(3)$ is a non-linear space. In practice, it means that standard vector calculus applies to the argument of the exponential map, such as multiplication by a scalar or addition of a another six-dimensional vector.

The derivative of the exponential map introduces the so-called tangent operator $\mathbf{T}_{SE(3)}$. Consider the transformation from $\mathbf{H}_0$ to $\mathbf{H}$ as $\mathbf{H} = \mathbf{H}_0 \exp_{SE(3)}(\mathbf{n})$. The derivative of $\mathbf{H}$ leads to $d(\mathbf{H}) = \mathbf{H}_0 \mathrm{Dexp}(\mathbf{n}) d(\tilde{\mathbf{n}}) = \mathbf{H} \exp_{SE(3)}^{-1}(\mathbf{n}) \mathrm{Dexp}(\mathbf{n}) d(\tilde{\mathbf{n}})$ where Dexp is the derivative of the exponential map, and this can be written as

$$d(\mathbf{H}) = \mathbf{H}\widetilde{\big(\mathbf{T}_{SE(3)}(\mathbf{n})d(\mathbf{n})\big)} \tag{20}$$

where $\mathbf{T}_{SE(3)}$ has the explicit expression

$$\mathbf{T}_{SE(3)}(\mathbf{n}) = \begin{bmatrix} \mathbf{T}_{SO(3)}(\mathbf{n}_\Omega) & \mathbf{T}_{U\Omega+}(\mathbf{n}_U, \mathbf{n}_\Omega) \\ \mathbf{0}_{3\times 3} & \mathbf{T}_{SO(3)}(\mathbf{n}_\Omega) \end{bmatrix} \tag{21}$$

in which $\mathbf{T}_{SO(3)}$ was given in Eq. (19) and $\mathbf{T}_{U\Omega+}(\mathbf{n}_U, \mathbf{n}_\Omega)$ reads

$$\begin{aligned} &\frac{-b}{2}\tilde{\mathbf{n}}_U + \frac{1-a}{||\mathbf{n}_\Omega||^2}(\tilde{\mathbf{n}}_U\tilde{\mathbf{n}}_\Omega + \tilde{\mathbf{n}}_\Omega\tilde{\mathbf{n}}_U) \\ &+\frac{\mathbf{n}_\Omega^T\mathbf{n}_U}{||\mathbf{n}_\Omega||^2}\Big((b-a)\tilde{\mathbf{n}}_\Omega + \big(\frac{b}{2} - \frac{3(1-a)}{||\mathbf{n}_\Omega||^2}\big)\tilde{\mathbf{n}}_\Omega^2\Big) \end{aligned} \tag{22}$$

with a and b as defined for Eq. (19). Notice that $\mathbf{T}_{U\Omega+}(\mathbf{n}_U, \mathbf{0}) = -\tilde{\mathbf{n}}_U/2$. Using the notation $d(\mathbf{H}) = \mathbf{H}\tilde{\mathbf{h}}$, one has

$$\mathbf{h} = \mathbf{T}_{SE(3)}(\mathbf{n})d(\mathbf{n}) \tag{23}$$

which expresses the derivatives of $\mathbf{H}$, namely $\mathbf{h}$, in terms of the derivative of $\mathbf{n}$, according to the representation of $\mathbf{H}$ as $\mathbf{H}_0 \exp_{SE(3)}(\mathbf{n})$.

4.2 Multibody System

At multibody system level, an exponential map can also be defined based on the exponential map defined in Eq. (18). Following the notations in section 3.3, the exponential map for a multibody system is given by

$$\begin{aligned} \exp(\mathbf{n}^*) = \mathrm{diag}\big(&\exp_{SE(3)}(\mathbf{n}_1), \cdots, \exp_{SE(3)}(\mathbf{n}_M), \\ &\exp_{SE(3)}(\mathbf{n}_{J,1}), \cdots, \exp_{SE(3)}(\mathbf{n}_{J,m})\big) \end{aligned} \tag{24}$$

where we introduce a $(6M+6m)-$dimensional vector $\mathbf{n}^* = [\mathbf{n}_1^T \cdots \mathbf{n}_M^T \; \mathbf{n}_{J,1}^T \cdots \mathbf{n}_{J,m}^T]^T$. $\mathbf{n}^*$ is related to a $(6M+k_1+\cdots+k_m)$-dimensional vector $\mathbf{n} = [\mathbf{n}_1^T \cdots \mathbf{n}_M^T \; \mathbf{n}_{j,1}^T \cdots \mathbf{n}_{j,m}^T]^T$ as $\mathbf{n}^* = \mathbf{A}\mathbf{n}$. Accordingly, $\exp_{SE(3)}(\mathbf{A}_I \mathbf{n}_{j,I})$ is a licit expression since $\mathbf{A}_I$ spans a subalgebra (see Section 3.2). Let us now consider a transformation from $\mathbf{H}_0$ to $\mathbf{H}$ as $\mathbf{H} = \mathbf{H}_0 \exp(\mathbf{n}^*)$. The linearization of the exponential map gives a relationship between the derivative of $\mathbf{n}^*$ and the derivative of $\mathbf{H}$, namely $d(\mathbf{H}) = \mathbf{H}\widetilde{\mathbf{h}}^*$, as $\mathbf{h}^* = \mathbf{T}^*(\mathbf{n}^*)d(\mathbf{n}^*)$ where

$$\begin{aligned}\mathbf{T}^*(\mathbf{n}^*) \quad = \quad & \mathrm{diag}\big(\mathbf{T}_{SE(3)}(\mathbf{n}_1), \cdots, \mathbf{T}_{SE(3)}(\mathbf{n}_M), \\ & \mathbf{T}_{SE(3)}(\mathbf{n}_{J,1}), \cdots, \mathbf{T}_{SE(3)}(\mathbf{n}_{J,m})\big)\end{aligned} \tag{25}$$

$\mathbf{h}^*$ can also be expressed as a function of the derivative of $\mathbf{n}$ as

$$\mathbf{h}^* = \mathbf{T}(\mathbf{n}^*)d(\mathbf{n}) \tag{26}$$

where $\mathbf{T}(\mathbf{n}^*) = \mathbf{T}^*(\mathbf{n}^*)\mathbf{A}$, that is

$$\begin{aligned}\mathbf{T}(\mathbf{n}^*) \quad = \quad & \mathrm{diag}\big(\mathbf{T}_{SE(3)}(\mathbf{n}_1), \cdots, \mathbf{T}_{SE(3)}(\mathbf{n}_M), \\ & \mathbf{T}_{SE(3)}(\mathbf{n}_{J,1})\mathbf{A}_1, \cdots, \mathbf{T}_{SE(3)}(\mathbf{n}_{J,m})\mathbf{A}_m\big)\end{aligned} \tag{27}$$

Notice thus that $\mathbf{T}(\mathbf{n}^*)$ is a $(6M+6m) * (6M+k_1+\cdots+k_m)$ rectangular matrix. In most frequent cases, such as the hinge joint, the prismatic joint and the cylindrical joint, It turns out that $\mathbf{T}_{SE(3)}(\mathbf{A}_I \mathbf{n}_{j,I})\mathbf{A}_I$ reduces to $\mathbf{A}_I$, such that $\mathbf{h}_{J,I} = \mathbf{A}_I d(\mathbf{n}_{j,I})$ and, since $\mathbf{A}_I$ has full rank, $\mathbf{h}_{j,I} = d(\mathbf{n}_{j,I})$.

5 Dynamics of a Multibody System

5.1 Equations of Motion

According to Hamilton's principle and following a Lagrange multiplier method, the actual trajectory of the system between two time instants t_i and t_f is such that the variation of the augmented action integral is stationary provided that the initial and final configurations are fixed, i.e.

$$\delta \int_{t_i}^{t_f} \left(\mathcal{K}(\mathbf{v}^*) - \mathcal{V}(\mathbf{H}) - \boldsymbol{\lambda}^T \boldsymbol{\Phi}(\mathbf{H})\right) dt = 0 \tag{28}$$

where $\mathcal{K}(\mathbf{v}^*) = \mathbf{v}^{*T}\mathbf{M}^*\mathbf{v}^*/2$ is the kinetic energy, $\mathcal{V}(\mathbf{H})$ is the potential energy due to external and internal forces, and $\boldsymbol{\lambda}$ are the Lagrange multipliers associated with the kinematic constraints $\boldsymbol{\Phi}(\mathbf{H}) = \mathbf{0}$. In the following, the relationship introduced in Eq. (14) is used to replace the $\bullet^*$ terms which appear when considering derivatives of the system. Taking the variations of each terms yields

$$\begin{aligned}\int_{t_i}^{t_f} \Big(& \delta(\mathbf{v})^{*T}\mathbf{M}^*\mathbf{v}^* - \delta\mathbf{h}^T\big(\mathbf{g}_{ext}(\mathbf{H}) + \mathbf{g}_{int}(\mathbf{H}) + \boldsymbol{\Phi}_q^T(\mathbf{H})\boldsymbol{\lambda}\big) \\ & -\delta\boldsymbol{\lambda}^T\boldsymbol{\Phi}(\mathbf{H})\Big)\, dt = 0\end{aligned} \tag{29}$$

where, using the directional derivative, we define

$$D\mathcal{V}(\mathbf{H})\cdot\delta\mathbf{h}^* \quad = \quad \delta\mathbf{h}^{*T}\big(\mathbf{g}^*_{ext}(\mathbf{H}) + \mathbf{g}^*_{int}(\mathbf{H})\big) \tag{30}$$

$$\quad = \quad \delta\mathbf{h}^{T}\big(\mathbf{g}_{ext}(\mathbf{H}) + \mathbf{g}_{int}(\mathbf{H})\big) \tag{31}$$

$$D\boldsymbol{\Phi}(\mathbf{H})\cdot\delta\mathbf{h}^* \quad = \quad \boldsymbol{\Phi}_q^*(\mathbf{H})\delta\mathbf{h}^* = \boldsymbol{\Phi}_q(\mathbf{H})\delta\mathbf{h} \tag{32}$$

where $\mathbf{g}_{ext}(\mathbf{H}) = \mathbf{A}^T\mathbf{g}^*_{ext}(\mathbf{H})$, $\mathbf{g}_{int}(\mathbf{H}) = \mathbf{A}^T\mathbf{g}^*_{int}(\mathbf{H})$ and $\boldsymbol{\Phi}_q(\mathbf{H}) = \boldsymbol{\Phi}_q^*(\mathbf{H})\mathbf{A}$. In order to compare with the other terms in Eq. (29), the velocity variation shall be expressed in terms of system variable variations as in Eq. (15). Integrating the contribution of the kinetic energy by parts yields

$$\int_{t_i}^{t_f} \delta(\mathbf{v})^{*T}\mathbf{M}^*\mathbf{v}^*\ dt = [\delta\mathbf{h}^T\mathbf{M}\mathbf{v}]_{t_f}^{t_i} - \int_{t_i}^{t_f} \delta\mathbf{h}^T(\mathbf{M}\dot{\mathbf{v}} - \hat{\mathbf{v}}^T\mathbf{M}\mathbf{v})\ dt \tag{33}$$

where $\mathbf{M} = \mathbf{A}^T\mathbf{M}^*\mathbf{A}$. As discussed in Section 6, Eq. (33) holds since no inertia is associated with the kinematic joints. The application of Hamilton's principle implies that the first term of the right hand side of Eq. (33) vanishes. Hence, inserting Eq. (33) into Eq. (29) yields

$$\begin{aligned}\int_{t_i}^{t_f} \Big(&\delta\mathbf{h}^T(\mathbf{M}\dot{\mathbf{v}} - \hat{\mathbf{v}}^T\mathbf{M}\mathbf{v} + \mathbf{g}_{ext}(\mathbf{H}) + \mathbf{g}_{int}(\mathbf{H}) + \mathbf{\Phi}_q^T(\mathbf{H})\boldsymbol{\lambda}) \\ &+\delta\boldsymbol{\lambda}^T\mathbf{\Phi}(\mathbf{H})\Big)\ dt = 0\end{aligned} \tag{34}$$

and since the variations $\delta\mathbf{h}$ and $\delta\boldsymbol{\lambda}$ are arbitrary, the equations of motion are obtained as a $(6M + k_1 + \cdots + k_m + 6m)$-dimensional second-order differential-algebraic equation

$$(\mathbf{H})^{\cdot} = \mathbf{H}\tilde{\mathbf{v}}^* \tag{35}$$

$$\mathbf{M}\dot{\mathbf{v}} - \hat{\mathbf{v}}^T\mathbf{M}\mathbf{v} + \mathbf{g}_{ext}(\mathbf{H}) + \mathbf{g}_{int}(\mathbf{H}) + \mathbf{\Phi}_q^T(\mathbf{H})\boldsymbol{\lambda} = \mathbf{0} \tag{36}$$

$$\mathbf{\Phi}(\mathbf{H}) = \mathbf{0} \tag{37}$$

where the first equations are kinematic compatibility equations and we recall that $\mathbf{v}^* = \mathbf{A}\mathbf{v}$ (see Eq. (14)). Explicit expressions of each of these terms are developed in the following.

5.2 Time Integration Method

The equations of motion can be solved using a suitable version of the generalized-α scheme, as proposed in Ref. [5]. It has a proven second-order convergence and some numerical damping can be used to lessen high frequency content. The integration method relies on the discretized equations of motion in Eq. (35–37)

$$\mathbf{H}_{n+1} = \mathbf{H}_n \exp(\mathbf{n}_{n+1}^*) \tag{38}$$

$$\begin{aligned}\mathbf{M}\dot{\mathbf{v}}_{n+1} - \hat{\mathbf{v}}_{n+1}^T\mathbf{M}\mathbf{v}_{n+1} + \mathbf{g}_{ext}(\mathbf{H}_{n+1})& \\ +\mathbf{g}_{int}(\mathbf{H}_{n+1}) + \mathbf{\Phi}_q^T(\mathbf{H}_{n+1})\boldsymbol{\lambda}_{n+1} &= \mathbf{0}\end{aligned} \tag{39}$$

$$\mathbf{\Phi}(\mathbf{H}_{n+1}) = \mathbf{0} \tag{40}$$

and the time integration formulae

$$\mathbf{n}_{n+1} = h\mathbf{v}_n + (0.5-\beta)h^2\mathbf{a}_n + \beta h^2\mathbf{a}_{n+1} \tag{41}$$

$$\mathbf{v}_{n+1} = \mathbf{v}_n + (1-\gamma)h\mathbf{a}_n + \gamma h\mathbf{a}_{n+1} \tag{42}$$

$$\mathbf{a}_{n+1} = \frac{1}{(1-\alpha_m)}\big((1-\alpha_f)\dot{\mathbf{v}}_{n+1} + \alpha_f\dot{\mathbf{v}}_n - \alpha_m\mathbf{a}_n\big) \tag{43}$$

where $\mathbf{n}_{n+1}^* = \mathbf{A}\mathbf{n}_{n+1}$. n refers to the time step and h is the time step size. The exponential map was defined in Eq. (24). The numerical parameters of the method can be selected to achieve a desired spectral radius $\rho \in [0, 1)$ at high frequency as

$$\alpha_m = \frac{2\rho - 1}{\rho + 1} \quad ; \quad \alpha_f = \frac{\rho}{\rho+1} \quad ; \quad \gamma = \frac{3-\rho}{2(1+\rho)} \quad ; \quad \beta = \frac{1}{(1+\rho)^2} \tag{44}$$

The time integration method is such that only the incremental motion between n and $n+1$ is parameterized, that is applying the exponential map to $\mathbf{n}^*$. Again, a practical implementation would be designed to obtain the result of Eq. (38) without explicitly building the matrices. Indeed, the operation can be carried out for each node and each kinematic joint by exploiting the block diagonal structure of the matrices.

Eq. (38–40) are non-linear and are solved at each time step for the unknown $\mathbf{n}_{n+1}$, $\mathbf{v}_{n+1}$, $\dot{\mathbf{v}}_{n+1}$ and $\mathbf{a}_{n+1}$ by a Newton iterative procedure. Denoting a finite variation due to the Newton procedure by $\Delta(\bullet)$, the linearization of Eq. (38–43) gives $\Delta\mathbf{h}^*_{n+1} = \mathbf{T}\Delta\mathbf{n}_{n+1}$, $\Delta\dot{\mathbf{v}}_{n+1} = \beta'\Delta\mathbf{n}_{n+1}$,$\Delta\mathbf{v}_{n+1} = \gamma'\Delta\mathbf{n}_{n+1}$ where $\mathbf{T} = \mathbf{T}(\mathbf{n}_{n+1})$ was defined in Eq. (26), $\beta' = ((1-\alpha_m)/(\beta h^2(1-\alpha_f))$ and $\gamma' = \gamma/(\beta h)$. Eventually, we denote Eq. (39) as $\mathbf{r} = \mathbf{0}$, and the linearized form of Eq. (39–40) is

$$\begin{bmatrix} \Delta\mathbf{r} \\ \Delta\mathbf{\Phi} \end{bmatrix} = \mathbf{S}_T \begin{bmatrix} \Delta\mathbf{n}_{n+1} \\ \Delta\boldsymbol{\lambda}_{n+1} \end{bmatrix} \tag{45}$$

where

$$\mathbf{S}_T = \begin{bmatrix} \beta'\mathbf{M} + \gamma'\mathbf{C}_T + \mathbf{K}^*_T\mathbf{T} & \mathbf{\Phi}_q^T \\ \mathbf{\Phi}^*_q\mathbf{T} & \mathbf{0} \end{bmatrix} \tag{46}$$

where $\mathbf{M}$, $\mathbf{C}_T$ and $\mathbf{K}^*_T$ are the tangent matrices obtained by linearizing Eq. (39) with respect to $\dot{\mathbf{v}}$, $\mathbf{v}$ and $\mathbf{n}^*$. The explicit form of the tangent matrices will be given in the following.

Regarding the initialization of the matrices, it is proposed to initialize all rotation matrices of the nodes at the identity matrix. This facilitates connections between elements and the description of relative configurations in the initial configuration. Indeed, the initialisation of the transformation matrices related to the kinematic joints is simply obtained from Eq. (7): $\mathbf{H}_{J,I} = \mathbf{H}_A^{-1}\mathbf{H}_B = \mathcal{H}(\mathbf{I}, \mathbf{x}_B - \mathbf{x}_A)$. Also, notice the axes used to build the 6×1 $\mathbf{e}_{Ii}$ vectors and $\mathbf{A}_I$ need thus to be specified with respect to the inertial frame in the initial configuration.

6 Inertia Forces

Let us consider that there are N rigid bodies. There is a node at the mass center of each rigid body, and let us choice that they appear first in $\mathbf{H}$. The $M - N$ additional nodes come after them and the kinematic joint transformations follow. The kinetic energy $\mathcal{K}$ of a multibody system is given by

$$\mathcal{K}(\mathbf{v}^*) = \sum_{I=1}^{N} \frac{1}{2} m_I \mathbf{v}_{I,U}^T \mathbf{v}_{I,U} + \frac{1}{2}\mathbf{v}_{I,\Omega}^T \mathbf{J}_I \mathbf{v}_{I,\Omega} = \frac{1}{2}\mathbf{v}^{*T}\mathbf{M}^*\mathbf{v}^* \tag{47}$$

where m_I is the mass and $\mathbf{J}_I$ is the constant 3×3 inertia tensor of node I expressed in the material frame. Since the rotation matrices are initially identity matrices, the inertia tensors must actually be provided with respect to the inertial frame in the initial configuration. If we denote as $\mathbf{R}_{0I}$ the rotation matrix built using the principal axes of the body I in the initial configuration as columns, then the inertia tensor $\mathbf{J}'_I$ computed in the principal axes leads to $\mathbf{J}_I = \mathbf{R}_{0I}\mathbf{J}'_I\mathbf{R}_{0I}^T$. The mass matrix of the system $\mathbf{M} = \mathbf{A}^T\mathbf{M}^*\mathbf{A}$ is straightforwardly obtained since no inertia is associated with the kinematic joint. It reads explicitly

$$\begin{aligned} \mathbf{M} \quad = \quad & \mathrm{diag}\big(\mathbf{M}_1, \cdots, \mathbf{M}_N, \\ & \mathbf{0}_{N+1}, \cdots, \mathbf{0}_M, \mathbf{0}_{j,1}, \cdots, \mathbf{0}_{j,m}\big) \end{aligned} \tag{48}$$

where the zeros stand for the additional nodes and the kinematic joints, which do not have any inertia, and $\mathbf{M}_I$ is the constant mass matrix

$$\mathbf{M}_I = \begin{bmatrix} m_I\mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{J}_I \end{bmatrix} \tag{49}$$

Notice then that Eq. (33) is valid since no inertia is associated with the kinematic joints. Indeed, using Eq. (15), the following relationship for the non-commutativity contribution in the right-hand side term in Eq. (33) holds

$$\int_{t_i}^{t_f} \delta\mathbf{h}^T \mathbf{A}^T (\hat{\mathbf{v}}^{*T}\mathbf{M}^*\mathbf{v}^*)\ dt = \int_{t_i}^{t_f} \delta\mathbf{h}^T (\hat{\mathbf{v}}^T\mathbf{M}\mathbf{v})\ dt \tag{50}$$

since all $\mathbf{A}_I^T \widehat{\mathbf{v}_{J,I}}^T$ of the kinematic joints are multiplied by a zero inertia.

The expression of the inertia forces in the equations of motion in Eq. (36) is obtained from the kinetic energy contribution in Eq. (34). For each node at the center of mass of a rigid body, it explicitly reads

$$\mathbf{M}_I\dot{\mathbf{v}}_I - \hat{\mathbf{v}}_I^T\mathbf{M}_I\mathbf{v}_I = \begin{bmatrix} m_I\dot{\mathbf{v}}_{I,U} + m_I\tilde{\mathbf{v}}_{I,\Omega}\mathbf{v}_{I,U} \\ \mathbf{J}_I\dot{\mathbf{v}}_{I,\Omega} + \tilde{\mathbf{v}}_{I,\Omega}\mathbf{J}_I\mathbf{v}_{I,\Omega} \end{bmatrix} \tag{51}$$

Quadratic gyroscopic terms are observed in both the translation and rotation parts.

In order to use the time integration method in section 5.2, the linearization of the inertia forces must be computed. The linearization with respect to $\dot{\mathbf{v}}$ is straightforward: it simply involves $\mathbf{M}$. The linearization with respect to $\mathbf{v}$ leads to the tangent damping matrix. Only the inertia nodes are involved

$$\begin{aligned} \mathbf{C}_T(\mathbf{v}) \quad = \quad & \mathrm{diag}\big(\mathbf{C}_{T,1}, \cdots, \mathbf{C}_{T,N}, \\ & \mathbf{0}_{N+1}, \cdots, \mathbf{0}_M, \mathbf{0}_{j,1}, \cdots, \mathbf{0}_{j,m}\big) \end{aligned} \tag{52}$$

where the nodal contribution is given by

$$\mathbf{C}_{T,I}(\mathbf{v}_I) = \begin{bmatrix} m_I\tilde{\mathbf{v}}_{I,\Omega} & -m_I\tilde{\mathbf{v}}_{I,U} \\ \mathbf{0}_{3\times3} & \tilde{\mathbf{v}}_{I,\Omega}\mathbf{J}_I - \widetilde{\mathbf{J}_I\mathbf{v}_{I,\Omega}} \end{bmatrix} \tag{53}$$

7 Kinematic Constraints

7.1 Constraint Equations

In order to include the kinematic joints of multibody systems in Hamilton's principle (Eq. (28)), a constraint equation vector $\mathbf{\Phi}(\mathbf{H})$ should be formulated. However, the kinematic joint formulation on $SE(3)$ in Eq. (7) brings matrix equations while the Lagrange multiplier method deals with vector constraints in the equations of motion. In order to obtain vector equations from Eq. (7), a vectorial map is introduced [11]. The constraint equations in Eq. (37) take the form

$$\mathbf{\Phi}(\mathbf{H}) = \begin{bmatrix} \mathbf{\Phi}_1 & \cdots & \mathbf{\Phi}_m \end{bmatrix}^T \tag{54}$$

where, for each joint I, one has 6 constraints given by

$$\mathbf{\Phi}_I(\mathbf{H}_A, \mathbf{H}_B, \mathbf{H}_{J,I}) = \mathrm{vect}_{SE(3)}(\mathbf{H}_B^{-1}\mathbf{H}_A\mathbf{H}_{J,I}) = \mathbf{0}_{6\times1} \tag{55}$$

where the vectorial map is defined as

$$\mathrm{vect}_{SE(3)}(\mathcal{H}(\mathbf{R},\mathbf{x})) = \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\phi}(\mathbf{R}) \end{bmatrix} \tag{56}$$

such that $\tilde{\boldsymbol{\phi}}(\mathbf{R}) = (\mathbf{R} - \mathbf{R}^T)/2$. This vectorial map introduces systematically six constraints for a kinematic joint or a rigid constraint. Notice that the argument of the vectorial map is a product of transformation matrices, which is then a transformation matrix itself. The vectorial map basically imposes that the relative displacements and relative rotations contained in $\mathbf{H}_{J,I}$

are exactly the relative configuration between node A and B by imposing 3 constraints at the position level and 3 constraints at the rotation level. The resulting constraint equations are invariant under rigid body motions, making them particularly appropriate to catch and enforce restricted relative motions in multibody systems.

This formulation leads systematically to six constraint equations, which is related to the mixed coordinates formulation. Indeed, for a joint I that has k_I degrees of freedom, the vectorial map provides $6-k_I$ equations to prevent non-allowed relative motions and k_I equations for the evolution of the allowed relative motions.

7.2 Constraint Gradient

The constraint contributions to the equations of motion in Eq. (36) involves the constraint gradient as in Eq. (32). When the constraint equations in Eq. (55) are satisfied, their directional derivative is given by $(D\mathbf{\Phi}_I \cdot \delta\mathbf{h}^*_{ABI})\big|_{\mathbf{\Phi}_I=\mathbf{0}} = \text{vect}_{SE(3)}\big(\mathbf{H}^{-1}_{J,I}\widetilde{\delta\mathbf{h}}_A\mathbf{H}_{J,I} - \widetilde{\delta\mathbf{h}}_B + \widetilde{\delta\mathbf{h}}_{J,I}\ \big)$ where $\delta\mathbf{h}^*_{ABI} = [\delta\mathbf{h}^T_A\ \delta\mathbf{h}^T_B\ \delta\mathbf{h}^T_{J,I}]^T$ and $\delta\mathbf{H}^{-1}_B = -\widetilde{\delta\mathbf{h}}_B\mathbf{H}^{-1}_B$. Since the vectorial map is linear, each contribution can be considered separately. Hence, the constraint gradient can be obtained from $(D\mathbf{\Phi}_I \cdot \delta\mathbf{h}^*_{ABI})\big|_{\mathbf{\Phi}_I=\mathbf{0}} = (\mathbf{\Phi}^*_{Iq}\delta\mathbf{h}^*_{ABI})\big|_{\mathbf{\Phi}_I=\mathbf{0}}$, namely $(\mathbf{\Phi}^*_{Iq})\big|_{\mathbf{\Phi}_I=\mathbf{0}} = \begin{bmatrix} Ad_{\mathbf{H}^{-1}_{J,I}} & -\mathbf{I}_{6\times 6} & \mathbf{I}_{6\times 6}\end{bmatrix}$ where $\mathbf{H}_{J,I} = \mathcal{H}(\mathbf{R}_{J,I}, \mathbf{x}_{J,I})$ and

$$Ad_{\mathbf{H}^{-1}_{J,I}} = \begin{bmatrix} \mathbf{R}^T_{J,I} & -\mathbf{R}^T_{J,I}\widetilde{\mathbf{x}}_{J,I} \\ \mathbf{0}_{3\times 3} & \mathbf{R}^T_{J,I} \end{bmatrix} \tag{57}$$

It is remarkable that the constraint gradient at equilibrium only depends on the relative configuration, and not on the global motion of nodes A and B. This shows that the non-linearity of the formulation is only related to the local motions. Introducing the variation of Eq. (14) in order to obtain $\mathbf{\Phi}_q$ used in Eq. (32), we have

$$(\mathbf{\Phi}_{Iq})\big|_{\mathbf{\Phi}_I=\mathbf{0}} = \begin{bmatrix} Ad_{\mathbf{H}^{-1}_{J,I}} & -\mathbf{I}_{6\times 6} & \mathbf{A}_I \end{bmatrix} \tag{58}$$

and the contribution of $\mathbf{\Phi}_I$ to the iterations matrix $(\mathbf{\Phi}^*_q\mathbf{T}(\mathbf{n}^*))\big|_{\mathbf{\Phi}_I=\mathbf{0}}$ in Eq. (45) is given by

$$\begin{bmatrix} Ad_{\mathbf{H}^{-1}_{J,I}}\mathbf{T}_{SE(3)}(\mathbf{n}_A) & -\mathbf{T}_{SE(3)}(\mathbf{n}_B) & \mathbf{T}_{SE(3)}(\mathbf{A}_I\mathbf{n}_{j,I})\mathbf{A}_I \end{bmatrix} \tag{59}$$

This contribution only depends on the relative configuration through $Ad_{\mathbf{H}^{-1}_{J,I}}$ and on the increments of motion through $\mathbf{n}_A$, $\mathbf{n}_B$ and $\mathbf{n}_{j,I}$. As discussed earlier, in some important cases such as the hinge joint, the cylindrical joint and the prismatic joint, it turns out that $\mathbf{T}_{SE(3)}(\mathbf{A}_I\mathbf{n}_{j,I})\mathbf{A}_I = \mathbf{A}_I$.

The expression of the gradient can be straightforwardly extended to multiple constraints by considering a finite element-like assembly procedure. Every 6 lines of the resulting $\mathbf{\Phi}_q$ matrix refers to corresponding 6 lines of constraint in $\mathbf{\Phi}$ and the columns corresponding to the nodes and the degrees of freedom of the kinematic joints are placed at the proper spot. The linearization of the term $\mathbf{\Phi}^T_q(\mathbf{H})\boldsymbol{\lambda}$ in Eq. (36) involves a contribution to the tangent stiffness matrix. However, it is not computed here for conciseness and, in our experience, it is actually not necessary for the time integration method to converge during the Newton iterative process. Furthermore, the possible non-satisfaction of the constraint equations during the iterative process is neglected and the constraint gradient is always computed as if the constraint equations were satisfied as in Eq. (58) and Eq. (59).

8 Relative Coordinates

Relative motions between two bodies can in general be described by relative coordinates which are directly available thanks to the kinematic joint formulation provided that the subgroup of relative motion is abelian. This holds for instance for the hinge joint, the prismatic joint and the cylindrical joint. In such a case, the increments of motion can be consistently summed up to obtain a relative coordinate description of the global motion. For such a joint I, the relationship in Eq. (26) reduces to $\mathbf{h}_{J,I} = \mathbf{A}_I d(\mathbf{n}_{j,I})$, which implies that $\mathbf{h}_{j,I} = d(\mathbf{n}_{j,I})$ since $\mathbf{A}_I$ has full rank. Thus, the following equation and its time discretization for the computation of the relative coordinates in time are introduced

$$\alpha_I(t) = \int_{t_i}^{t_f} \mathbf{h}_{j,I}\ dt \Leftrightarrow \boldsymbol{\alpha}_{I,n+1} = \boldsymbol{\alpha}_{I,n} + \mathbf{n}_{j,I,n+1} \tag{60}$$

where $\boldsymbol{\alpha}_I$ is the $k_I \times 1$ vector of relative coordinates related to joint I. Thanks to their physical interpretation, relative coordinates can be used for the description of local spring-damper elements or the definition of an actuator torque or force, as described in the following two sections. Their contribution to the equations of motion is straightforwardly included since $\delta\mathbf{h}_{j,I} = \delta\mathbf{n}_{j,I} = \delta\alpha_I$. Regarding the time derivative of the relative coordinates, we have $\mathbf{v}_{J,I} = \mathbf{A}_I\mathbf{v}_{j,I}$ and $\mathbf{v}_{J,I} = \mathbf{A}_I\dot{\alpha}_I$, such that

$$\dot{\alpha}_I(t) = \mathbf{v}_{j,I}(t) \Leftrightarrow \dot{\alpha}_{I,n+1} = \mathbf{v}_{j,I,n+1} \tag{61}$$

9 Internal Forces of Kinematic Joints

9.1 Elastic Forces of a Spring Element

The potential energy of a spring element can be introduced by considering stiffnesses related to the relative coordinates involved in kinematic joints. For instance, the stiffness in a hinge joint or a prismatic joint allows the modelling of respectively a rotational and a translational spring element. In general, the variation of the potential energy in the kinematic joints reads $\delta\mathcal{V} = \sum_{I=1}^{m}\ \delta\alpha_I^T\mathbf{K}_I(\boldsymbol{\alpha}_I - \boldsymbol{\alpha}_I^0)$ where $\mathbf{K}_I = \text{diag}(K_{I,1}, \cdots, K_{I,k_I})$ is the diagonal stiffness matrix of the kinematic joint I, $\boldsymbol{\alpha}_I = [\alpha_{I,1} \cdots \alpha_{I,k_I}]$ and $\boldsymbol{\alpha}_I^0 = [\alpha_{I,1}^0 \cdots \alpha_{I,k_I}^0]$ defines the reference value of the relative coordinates. The internal forces involved in the equations of motion Eq. (36) are thus simply given by $\mathbf{g}_{int} = \begin{bmatrix}\mathbf{0}_1^T & \cdots & \mathbf{0}_M^T & \mathbf{g}_{int,j,1}^T & \cdots & \mathbf{g}_{int,j,m}^T\end{bmatrix}^T$ where the kinematic joint contribution reads

$$\mathbf{g}_{int,j,I}(\boldsymbol{\alpha}_I) = \mathbf{K}_I(\boldsymbol{\alpha}_I - \boldsymbol{\alpha}_I^0) \tag{62}$$

The linearization of the internal forces leads to a contribution to the tangent stiffness matrix $\mathbf{K}_T^*\mathbf{T}$. By comparison of $\Delta\mathbf{g}_{int,j,I} = \mathbf{K}_I\Delta\alpha_I$ with $\Delta\mathbf{g}_{int} = \mathbf{K}_T^*\mathbf{T}\Delta\mathbf{n}$, we obtain directly

$$\mathbf{K}_T^*\mathbf{T} = \text{diag}(\mathbf{0}_1, \cdots, \mathbf{0}_M, \mathbf{K}_1, \cdots, \mathbf{K}_m) \tag{63}$$

since $\Delta\alpha_I = \Delta\mathbf{n}_{j,I}$.

9.2 Dissipation Forces of a Damper Element

Dissipation forces can be used to model viscous friction inside kinematic joints. Based on the relative velocity in the kinematic joint, the virtual work associated with the dissipation forces is then $\delta\mathcal{W} = \sum_{I=1}^{m}\ \delta\mathbf{h}_{j,I}^T\mathbf{C}_I\mathbf{v}_{j,I}$ where $\mathbf{C}_I = \text{diag}(C_{I,1}, \cdots, C_{I,k_I})$ is the diagonal matrix of damping coefficients associated with $\mathbf{v}_{j,I}$. This definition holds for all joints that can be described in the framework presented in Section 3.2. Following the discussion in Section 8, the time derivative of relative coordinates can be considered when the subgroup to which the kinematic joint transformation belongs is abelian and the virtual work associated with the dissipation forces reads then $\delta\mathcal{W} = \sum_{I=1}^{m}\ \delta\boldsymbol{\alpha}_I^T\mathbf{C}_I\dot{\alpha}_I$. The dissipation forces involved in the equations of motion in Eq. (36) are thus simply given by $\mathbf{g}_{int} = \begin{bmatrix}\mathbf{0}_1^T & \cdots & \mathbf{0}_M^T & \mathbf{g}_{int,j,1}^T & \cdots & \mathbf{g}_{int,j,m}^T\end{bmatrix}^T$ where the kinematic joint contribution reads

$$\mathbf{g}_{int,j,I}(\mathbf{v}_{j,I}) = \mathbf{C}_I \mathbf{v}_{j,I} \tag{64}$$

The linearization of the internal forces leads straightforwardly to a contribution to the tangent damping matrix

$$\mathbf{C}_T = \text{diag}(\mathbf{0}_1, \cdots, \mathbf{0}_M, \mathbf{C}_1, \cdots, \mathbf{C}_m) \tag{65}$$

10 External Forces

Let us consider the virtual work done by an external force $\mathbf{f}_I$ expressed in the inertial frame and an external torque $\mathbf{m}_I$ expressed in the material frame of node I. Let us also consider the virtual work done by actuators $\boldsymbol{\tau}_I$ acting in kinematic joint I. Hence, we have $\delta\mathcal{V} = \sum_{I=1}^{M} \delta\mathbf{h}_{I,U}^T(\mathbf{R}_I^T\mathbf{f}_I) + \delta\mathbf{h}_{I,\Omega}^T\mathbf{m}_I + \sum_{I=1}^{m} \delta\mathbf{h}_{j,I}^T\boldsymbol{\tau}_I$ where $\delta\mathbf{x}_I = \mathbf{R}_I\delta\mathbf{h}_{I,U}$ has been used. Denoting $\delta\mathcal{V} = \delta\mathbf{h}^T(-\mathbf{g}_{ext})$, the expression of the external forces $\mathbf{g}_{ext}$ follows immediately $\mathbf{g}_{ext} = -\begin{bmatrix}\mathbf{g}_{ext,1}^T \cdots & \mathbf{g}_{ext,M}^T & \mathbf{g}_{ext,j,1}^T & \cdots & \mathbf{g}_{ext,j,m}^T\end{bmatrix}^T$ where

$$\mathbf{g}_{ext,I} = \begin{bmatrix}\mathbf{R}_I^T\mathbf{f}_I \\ \mathbf{m}_I\end{bmatrix} \quad \text{and} \quad \mathbf{g}_{ext,j,I} = \boldsymbol{\tau}_I \tag{66}$$

The linearization of the external forces leads to a contribution to the tangent stiffness matrix: $\mathbf{K}_T^*(\mathbf{H})\mathbf{T} = \text{diag}\big(\mathbf{K}_{T,1}\mathbf{T}_{SE(3)}(\mathbf{n}_1), \cdots, \mathbf{K}_{T,M}\mathbf{T}_{SE(3)}(\mathbf{n}_M), \mathbf{0}_{j,1}, \cdots, \mathbf{0}_{j,m}\big)$ where the inertia node and additional node contribution is given by

$$\mathbf{K}_{T,I} = -\begin{bmatrix}\mathbf{0}_{3\times3} & \widetilde{\mathbf{R}_I^T\mathbf{f}_I} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3}\end{bmatrix} \tag{67}$$

11 Case Study: Slider-Crank Mechanism

11.1 Description of the model

A spatial slider-crank mechanism is considered as a case study. The model of this mechanism in the proposed formalism is given in Fig. 2.

Since there are three bodies (the crank, the rod and the slider), three nodes are defined at their mass center, namely nodes 3, 6 and 8. In order to describe the spherical joint that connects the crank and the rod, nodes 4 and 5 are introduced and connected by rigid constraints (3) and (5) to nodes 3 and 6, respectively. Then, the spherical joint is the kinematic joint (4) between nodes 4 and 5. The support of the crank is represented by the clamped node 1 and node 2 is introduced and connected by a rigid constraint (2) to node 3. Then, the horizontal axis hinge joint (1) is defined to connect nodes 1 and 2. For the slider, nodes 7, 8 and 9 are introduced. Node 7 is connected to node 6 by the rigid constraint (6) and connected to node 8 by the hinge joint (7). Eventually, nodes 8 and 9 are connected by a horizontal cylindrical joint (8) and node 9 is clamped by boundary conditions.

The system contains 9 nodes which leads to $9*6 = 54$ nodal vectorial variables. Since two nodes are clamped, only $54 - 2*6 = 42$ remain as unknowns. 8 constraints are involved such that $8*6 = 48$ Lagrange multipliers are needed. Considering the two hinge joints, the spherical joint and the cylindrical joint, $2*1+1*3+1*2 = 7$ relative coordinates come into play. The size of the residuals $\mathbf{r}$ and the increments of motion $\mathbf{n}$ is thus $42+7 = 49$ and the size of the constraint vector $\boldsymbol{\Phi}$ is 48, so that the size of the iteration matrix $\mathbf{S}_t$ is $49+48 = 97$.

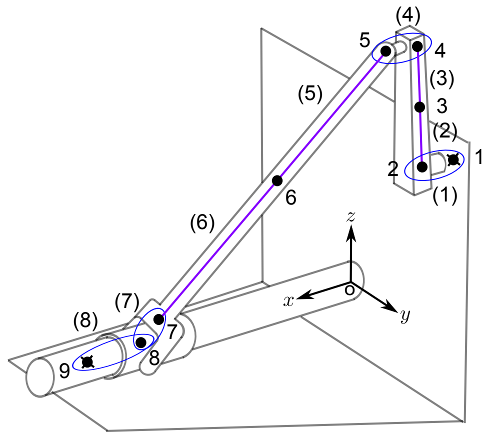


Figure 2: Model of the Spatial Slider Crank Mechanism

Table 2: Reference Configuration for the Spatial Slider Crank Mechanism

Node	x	y	z	Node	x	y	z
1,2	0	0.1	0.12	6	0.1	0.05	0.1
3	0	0.1	0.16	7,8,9	0.2	0	0
4,5	0	0.1	0.2				

11.2 Simulation case

In this simulation case, a 0.015 Nm constant torque is applied during the first 2 s in hinge joint (1) and the motion of the system is observed during 5 s. Moreover, a 1 Ns/m damping coefficient is introduced on the translation degree of freedom of the cylindrical joint (8). The geometric properties are given in Tables 2, 3 and 4 [13]. The system is subjected to gravity, acting downward z–axis with a 9.81 m/s^2 acceleration constant. The time integration is performed with $\rho = 0.9$ and $h = 1e - 2$ s.

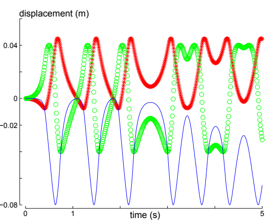
Fig. 3a shows the displacements of node 6. The motion is clearly three dimensional. In Fig. 3b, the relative coordinate of the hinge joint (1) is shown. This shows that the formulation is able to handle rotation angles which exceed 2π without problem. In Fig. 4a, the velocities at node 3 computed by the time integration method are plotted. Since they are computed in the material frame, only the local y-component does not vanish, while in Fig. 4b, the spatial velocities, that is $\mathbf{R}_3\mathbf{v}_{3,U}$, are plotted, and there the x– and y–components do not vanish.

Table 3: Inertia properties (mass in kg and rotation inertia in kgm) for the Spatial Slider Crank Mechanism

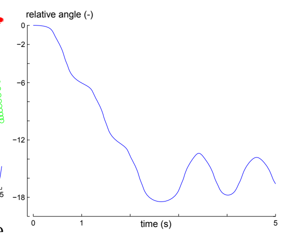
Node	m_I	$J_{I,11}$	$J_{I,22}$	$J_{I,33}$
3	0.12	1e-4	1e-4	1e-5
6	0.5	4e-3	4e-4	4e-3
8	2	1e-4	1e-4	1e-4

Table 4: Kinematic Joint Definition for the Spatial Slider Crank Mechanism

Joints	k-th column of $\mathbf{A}_J$	$\mathbf{e}_U$	$\mathbf{e}_\Omega$
(1)	1	$\mathbf{0}_{3\times 1}$	$[1\ 0\ 0]^T$
(4)	1	$\mathbf{0}_{3\times 1}$	$[1\ 0\ 0]^T$
	2	$\mathbf{0}_{3\times 1}$	$[0\ 1\ 0]^T$
	3	$\mathbf{0}_{3\times 1}$	$[0\ 0\ 1]^T$
(7)	1	$\mathbf{0}_{3\times 1}$	$[0\ \frac{0.2}{\sqrt{0.05}}\ \frac{-0.1}{\sqrt{0.05}}]^T$
(8)	1	$[1\ 0\ 0]^T$	$\mathbf{0}_{3\times 1}$
	2	$\mathbf{0}_{3\times 1}$	$[1\ 0\ 0]^T$

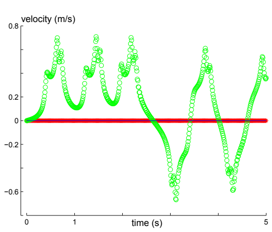


(a) Displacement of node 6. Stars (red): x, Circles (green): y, Solid line (blue): z

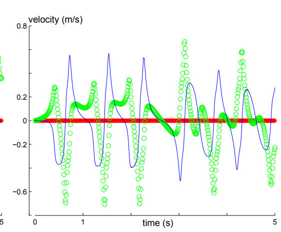


(b) Relative angle in hinge joint (1)

Figure 3: Simulation Results



(a) Material frame



(b) Inertial frame

Figure 4: Velocity of node 3. Stars (red): x, Circles (green): y, Solid line (blue): z

12 Conclusions

A finite element approach for the dynamic analysis of multibody systems has been presented in the framework of the special Euclidean group $SE(3)$. The modelling of rigid bodies and standard kinematic joints is addressed. The method allows a systematic derivation of the equations of motion of multibody systems by collecting generic expressions which are slightly tuned according to a system part of interest. The resulting differential-algebraic equations of motion involve mixed coordinates, which are nodal as well as kinematic joint transformations, and can be solved efficiently using appropriate time integration schemes. A unified treatment for these variables has been developed and yields a compact and easily implemented formulation.

The proposed formulation exhibits some important advantages. The velocity of a rigid body is expressed in the material frame so that the mass matrix is constant. Since no parameterization of the global motion is introduced, the gyroscopic forces are just a quadratic expression in the velocity. Regarding the kinematic joints, a vectorial map is proposed to impose systematically six constraint equations. The constraint gradient at equilibrium only depends on the relative motion, but it does not depend on the global motion of the constrained nodes, which reduces the non-linearity in the kinematic joints. In some frequent cases, the relative coordinates in the kinematic joints are straightforwardly available, and allow then an easy kinematic joint formulation of flexibility, actuators or control laws.

Since it relies on a finite element approach, the proposed framework can be extended to include flexible elements such as beam or shell elements, which benefit greatly of the reduction of geometric non-linearity provided by the $SE(3)$ framework [14].

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