New perspectives on probabilistic methods for nonlinear transient dynamics in civil engineering

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May 6th, 2014
Liège, Belgium

New perspectives on probabilistic methods

for nonlinear transient dynamics in civil engineering

Structural dynamics in civil engineering

Dynamics = study of body motion under forces

Equation of motion $$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}(t)$$

Structural dynamics in civil engineering

Dynamics = study of body motion under forces

Equation of motion

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}(t)$$

inertial forces   damping forces   restoring forces

Structure

Structural dynamics in civil engineering

Dynamics = study of body motion under forces

Equation of motion

$$\underbrace{\mathbf{M}\ddot{\mathbf{y}}}_{\text{inertial forces}} + \underbrace{\mathbf{C}\dot{\mathbf{y}}}_{\text{damping forces}} + \underbrace{\mathbf{K}\mathbf{y}}_{\text{restoring forces}} = \underbrace{\mathbf{f}(t)}_{\text{external forces}}$$

Structure

External forces wind pressure, ground acceleration, wave, crowd

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Nonlinear behavior in civil engineering

Equation of motion $\mathbf{M}\ddot{\mathbf{y}} + \underbrace{\mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y}}_{\text{linear internal forces}} = \mathbf{f}(t)$

Nonlinear behavior in civil engineering

Equation of motion $\mathbf{M}\ddot{\mathbf{y}} + \underbrace{\mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y}}_{\text{linear internal forces}} + \underbrace{\mathbf{g}(\mathbf{y}, \dot{\mathbf{y}})}_{\text{nonlinear internal forces}} = \mathbf{f}(t)$

Nonlinear behavior in structures

- large displacement $\mathbf{K}(\mathbf{y})$: $\mathbf{y}$ is large w.r.t. the static equilibrium
- material nonlinearities : platicity, cracking
- damping devices : nonlinear damping law $\mathbf{C}(\dot{\mathbf{y}})$, tuned liquid dampers

Nonlinear behavior in civil engineering

Equation of motion $$\mathbf{M}\ddot{\mathbf{y}} + \underbrace{\mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y}}_{\text{linear internal forces}} + \underbrace{\mathbf{g}(\mathbf{y}, \dot{\mathbf{y}})}_{\text{nonlinear internal forces}} = \mathbf{f}(t)$$

Nonlinear behavior in structures

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Sources of difficulties

- superposition principle is broken
- Fourier analysis and modal superposition cannot be applied

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Uncertainty in civil engineering

Uncertainty

↳ a lack of knowledge → it is impossible to exactly describe future outcomes of an experiment

Uncertainty in civil engineering

Uncertainty

$\hookrightarrow$ a lack of knowledge $\rightarrow$ it is impossible to exactly describe future outcomes of an experiment

Methods

- Non-probabilistic : fuzzy arithmetics, random sets
- Probabilistic : based on the probability theory, the uncertainty can be *measured*

Uncertainty in civil engineering

Uncertainty

↳ a lack of knowledge → it is impossible to exactly describe future outcomes of an experiment

Methods

- Non-probabilistic : fuzzy arithmetics, random sets
- Probabilistic : based on the probability theory, the uncertainty can be *measured*

Sources of uncertainty

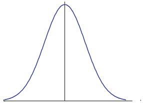
- Structure : $\mathbf{M}(\theta)$, $\mathbf{K}(\theta)$, $\mathbf{C}(\theta)$ uncertain
- Excitation : $\mathbf{f}(t, \theta)$ uncertain

Notation

$(\ , \theta)$ → function of the randomness in the universe (probability space)

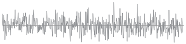
Uncertainty in civil engineering

Random excitations $\mathbf{f}(t, \theta)$

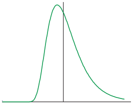


Gaussian
$\boldsymbol{\Sigma}_{\mathbf{f}}\ \boldsymbol{\mu}_{\mathbf{f}}$

stationary

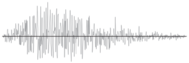


statistics time-independent



non-Gaussian
$\psi_{\mathbf{f}}$

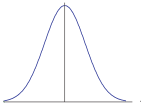
non-stationary



statistics time-dependent

Uncertainty in civil engineering

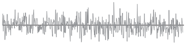
Random excitations $\mathbf{f}(t, \theta)$



Gaussian

$\boldsymbol{\Sigma}_{\mathbf{f}}$ $\boldsymbol{\mu}_{\mathbf{f}}$

stationary



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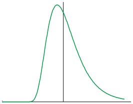

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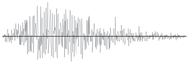

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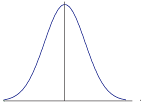
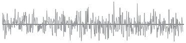
Uncertainty in civil engineering

Random excitations $\mathbf{f}(t,\theta)$

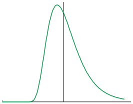


Gaussian $\leftrightarrow$ stationary

$\boldsymbol{\Sigma}_{\mathbf{f}}$ $\boldsymbol{\mu}_{\mathbf{f}}$

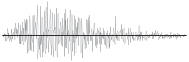


statistics time-independent



non-Gaussian $\leftrightarrow$ non-stationary

$\psi_{\mathbf{f}}$

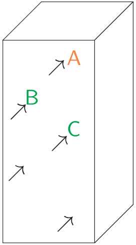


statistics time-dependent

Spatial Coherence

forces at different locations
are not statistically independent

Wind blowing on building
pressure A is correlated
to pressures B, C,...



Coherence $\longleftrightarrow$ Correlation

(frequency domain) (time domain)

Probabilistic Methods

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} + \mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}) = \mathbf{f}(t, \theta)$$

How to solve the equation of motion ?

$\mathbf{f}$ is Gaussian and $\mathbf{g} = \mathbf{0}$ $\longrightarrow$ $\mathbf{y}(t, \theta)$ is Gaussian

$\mathbf{f}$ is Gaussian and $\mathbf{g} \neq \mathbf{0}$ $\longrightarrow$ $\mathbf{y}(t, \theta)$ is non-Gaussian

		FPK equation	MC simulation	Equ. Linearization
Size	small large			
Transience				
Data	variance pdf			

Probabilistic Methods

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} + \mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}) = \mathbf{f}(t, \theta)$$

How to solve the equation of motion ?

Fokker-Planck Equation

$\psi(t)$ the probability density function of $\mathbf{y}$ and $\dot{\mathbf{y}}$

$$\frac{\partial \psi}{\partial t} + \sum_i \frac{\partial}{\partial x_i} (a_i \psi) = \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} (D_{ij} \psi)$$

		FPK equation		
Size	small	possible !		
	large	forget it !		
Transience		adapted		
Data	variance pdf	ideal		

Probabilistic Methods

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} + \mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}) = \mathbf{f}(t, \theta)$$

How to solve the equation of motion ?

Monte Carlo simulation

$$\underset{\{\mathbf{f}(t,\theta_1),\, \mathbf{f}(t,\theta_2), \ldots \mathbf{f}(t,\theta_n)\}}{\text{realizations of } \mathbf{f}} \xrightarrow{\text{solver}} \underset{\{\mathbf{y}(t,\theta_1),\, \mathbf{y}(t,\theta_2), \ldots \mathbf{y}(t,\theta_n)\}}{\text{realizations of } \mathbf{y}}$$

statistics

		FPK equation	MC simulation	
Size	small	possible !	possible	
	large	forget it !		
Transience		adapted	possible	
Data	variance	ideal	possible	
	pdf			

Probabilistic Methods

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} + \mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}) = \mathbf{f}(t, \theta)$$

How to solve the equation of motion ?

Equivalent Statistical Linearization

- Approximate method
- Nonlinear forces $\mathbf{g}(\mathbf{y}, \dot{\mathbf{y}})$ are replaced by Equivalent Linear forces
- Statistical Equivalence $\longrightarrow$ minimize the error on the variance

		FPK equation	MC simulation	Equ. Linearization
Size	small	possible !	possible	inappropriate !
	large	forget it !		pertinent !
Transience		adapted	possible	challenging
Data	variance	ideal	possible	targeted !
	pdf			too approximate !

New perspectives on probabilistic methods

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Motivations

		FPK equation	MC simulation	Equ. Linearization
Size	small	possible !	possible	inappropriate !
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Transience		adapted	possible	challenging
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“ *I force myself to contradict myself in order*

to avoid conforming to my own taste “

M. Duchamp

Motivations : Equivalent statistical linearization

		FPK equation	MC simulation	Equ. Linearization
Size	small	possible !	possible	inappropriate !
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Why ? Large structures + coherent random loads $\longrightarrow$ MC inefficient !

advantages of the linearity

$\swarrow$ $\downarrow$ $\searrow$

Gaussianity | Fourier analysis | Superposition principle

Fourier analysis $\downarrow$ Coherent random excitations modeled in the frequency domain

Superposition principle $\downarrow$ Projection in a modal basis

Goal (Part I)

- Develop a method for linear transient analysis
- Extension for equivalent statistical linearization

Motivations : Fokker-Planck Equation

		FPK equation	MC simulation	Equ. Linearization
Size	small	possible !	possible	inappropriate !
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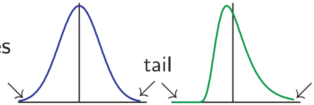
Why ?

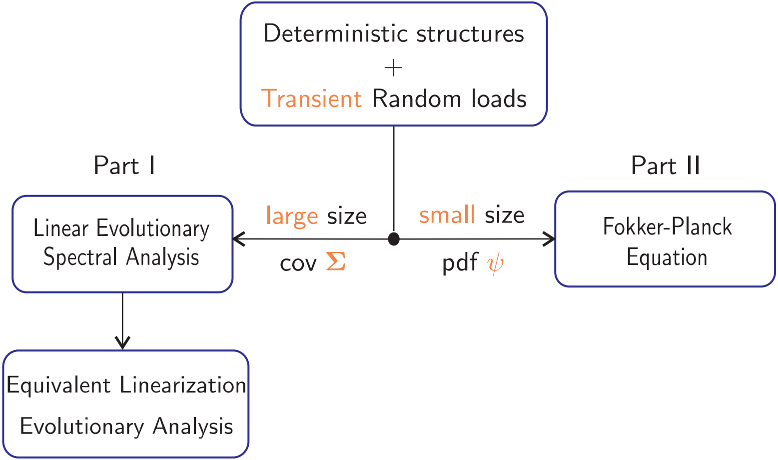
MC : slow convergence in low probability zones

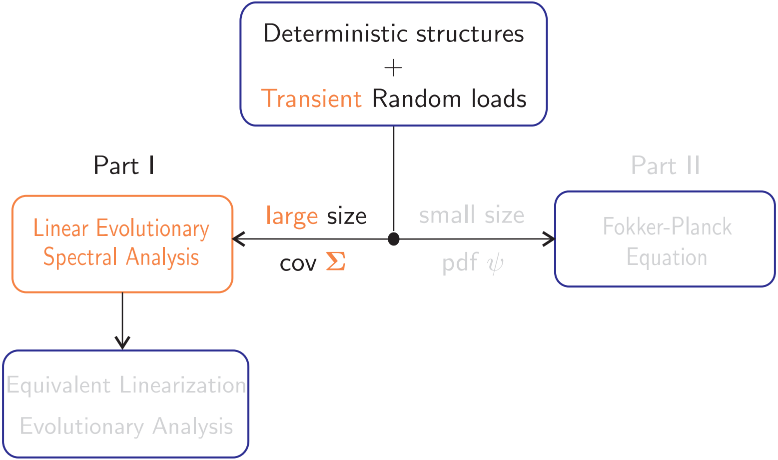
FPK : pdf $\longrightarrow$ tails of the distribution



Goal (Part II)

Develop and apply an accurate numerical solver


Deterministic structures
+
Transient Random loads
Part I
Part II
Linear Evolutionary Spectral Analysis
large size
small size
cov Σ
pdf ψ
Fokker-Planck Equation
Equivalent Linearization
Evolutionary Analysis



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Linear Evolutionary Spectral Analysis
Fokker-Planck Equation
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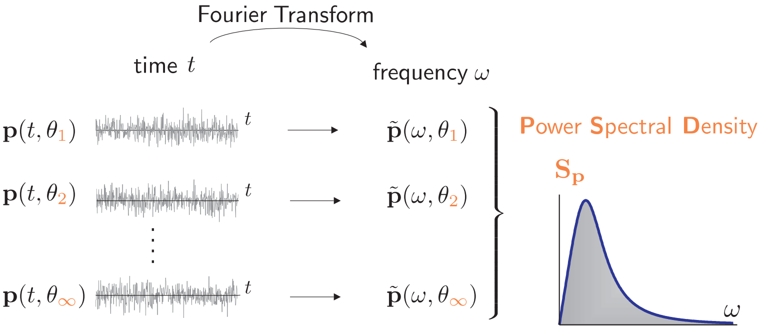
Evolutionary Analysis

Spectral representation of stationary random processes

Fourier analysis and random vectors



Properties

Covariance $\rightarrow \Sigma_{\mathbf{p}} = \int_{\mathbb{R}} \mathbf{S_p} d\omega$

$\mathbf{p}$ coherent $\longrightarrow \mathbf{S_p}$ is full

Stationary spectral analysis and modal projection

Equation of motion in basis $\mathbf{\Phi}$

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\,\dot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{f}(t,\theta) \longrightarrow \mathbf{K}\mathbf{\Phi} = \omega^2\mathbf{M}\mathbf{\Phi}$$

Eigenvalue problem

$$\ddot{\mathbf{q}} + \mathbf{D}\,\dot{\mathbf{q}} + \mathbf{\Omega}\,\mathbf{q} = \mathbf{p}(t,\theta) \longleftarrow \mathbf{x} = \mathbf{\Phi}\mathbf{q}$$

size $\mathbf{q} \ll$ size $\mathbf{x}$

nodal modal

Stationary spectral analysis and modal projection

Equation of motion in basis $\mathbf{\Phi}$

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\,\dot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{f}(t,\theta) \longrightarrow \mathbf{K\Phi} = \omega^2\mathbf{M\Phi}$$

Eigenvalue problem

$$\downarrow$$

$$\ddot{\mathbf{q}} + \mathbf{D}\,\dot{\mathbf{q}} + \mathbf{\Omega}\,\mathbf{q} = \mathbf{p}(t,\theta) \longleftarrow \mathbf{x} = \mathbf{\Phi q}$$

size $\mathbf{q} \ll$ size $\mathbf{x}$

nodal modal

Transfer Function in modal basis

$$\mathbf{H}(\omega) = \left(\mathbf{\Omega} - \omega^2\mathbf{I} + \imath\omega\mathbf{D}\right)^{-1}$$

Stationary spectral analysis and modal projection

Equation of motion in basis $\boldsymbol{\Phi}$

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\,\dot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{f}(t,\theta) \longrightarrow \mathbf{K}\boldsymbol{\Phi} = \omega^2\mathbf{M}\boldsymbol{\Phi}$$

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Transfer Function in modal basis

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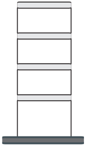
Structure $\mathbf{H}$

Spectral Analysis

$\mathbf{S}_{\mathbf{p}}(\omega)$ $\longrightarrow$  $\longrightarrow$ $\mathbf{S}_{\mathbf{q}}(\omega) = \mathbf{H}(\omega)\mathbf{S}_{\mathbf{p}}(\omega)\mathbf{H}(\omega)^*$

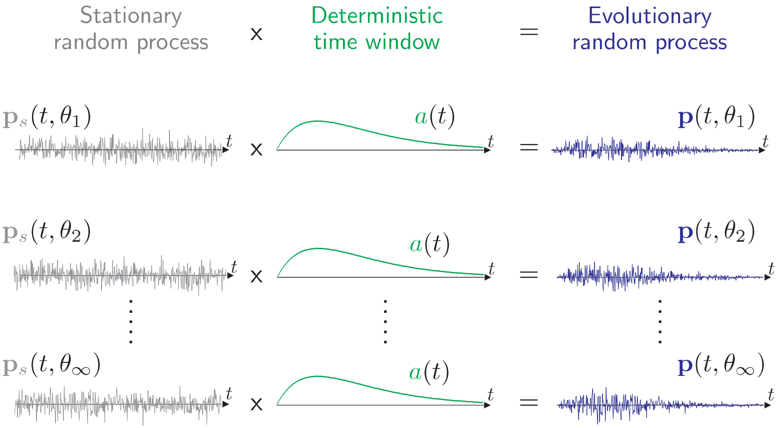
stationary process

$$\boldsymbol{\Sigma}_{\mathbf{q}} = \int_{\mathbb{R}} \mathbf{S}_{\mathbf{q}}(\omega)d\omega$$

covariance

Evolutionary random processes

How is built an evolutionary process ?



Evolutionary random processes

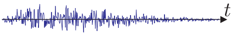
Evolutionary process

$$\mathbf{p}(t,\theta) = \underbrace{a(t)}_{\text{time window}} \; \underbrace{\mathbf{p}_s(t,\theta)}_{\text{stationary process}}$$

Extended Fourier Representation

$$\mathbf{p}(t) = \int_{\mathbb{R}} a(t,\omega)\, e^{\iota\omega t} d\tilde{\mathbf{p}}(\omega)$$

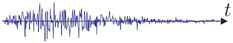
$\mathbf{p}(t,\theta_1)$



$\mathbf{p}(t,\theta_2)$

$\vdots$

$\mathbf{p}(t,\theta_\infty)$



Evolutionary Power Spectral Density

$$\widehat{\mathbf{S}}_{\mathbf{p}}(t,\omega) = |a(t,\omega)|^2 \underbrace{\mathbf{S}_{\mathbf{p}}(\omega)}_{\text{Embedded stationary process}}$$

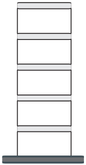
$$\boldsymbol{\Sigma}_{\mathbf{p}}(t) = \int_{\mathbb{R}} \widehat{\mathbf{S}}_{\mathbf{p}}(t,\omega) d\omega$$

Evolutionary spectral analysis

Impulse response matrix $(\mathbf{D}, \boldsymbol{\Omega})$

Structure
$\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \boldsymbol{\Omega}\mathbf{q}$

$\mathbf{p}(t,\theta) \longrightarrow$

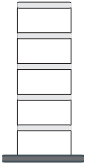


$$\longrightarrow \quad \mathbf{q}(t,\theta) = \int_{t_0}^{t} \mathbf{h}(t-u)\mathbf{p}(u)du$$

Evolutionary spectral analysis

Structure

$\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \boldsymbol{\Omega}\mathbf{q}$



Impulse response matrix $(\mathbf{D}, \boldsymbol{\Omega})$

$$\mathbf{p}(t,\theta) \longrightarrow \quad \longrightarrow \quad \mathbf{q}(t,\theta) = \int_{t_0}^{t} \mathbf{h}(t-u)\mathbf{p}(u)du$$

Embedded stationary process

$$a(t),\ \mathbf{S}_{\mathbf{p}}(\omega) \longrightarrow \quad \longrightarrow \quad \widehat{\mathbf{S}}_{\mathbf{q}}(t,\omega) = \mathbf{G}(t,\omega)\mathbf{S}_{\mathbf{p}}(\omega)\mathbf{G}(t,\omega)^{*}$$

$$\boldsymbol{\Sigma}_{\mathbf{q}}(t) = \int_{\mathbb{R}} \widehat{\mathbf{S}}_{\mathbf{q}}(t,\omega)d\omega$$

Evolutionary Transfer Function

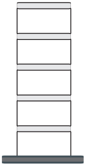
$$\mathbf{G}(t,\omega) = \int_{t_0}^{t} \mathbf{h}(t-u)a(u,\omega)\, e^{\iota\omega u}du$$

Evolutionary spectral analysis

Impulse response matrix $(\mathbf{D}, \mathbf{\Omega})$

Structure

$\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{\Omega}\mathbf{q}$



$\mathbf{p}(t,\theta) \longrightarrow$ $\longrightarrow$ $\mathbf{q}(t,\theta) = \int_{t_0}^{t} \mathbf{h}(t-u)\mathbf{p}(u)du$

Embedded stationary process

$a(t)$, $\mathbf{S_p}(\omega) \longrightarrow$ $\longrightarrow$ $\widehat{\mathbf{S}}_{\mathbf{q}}(t,\omega) = \mathbf{G}(t,\omega)\mathbf{S_p}(\omega)\mathbf{G}(t,\omega)^*$

$$\mathbf{\Sigma_q}(t) = \int_{\mathbb{R}} \widehat{\mathbf{S}}_{\mathbf{q}}(t,\omega)d\omega$$

Evolutionary Transfer Function $\mathbf{G}(t,\omega) = \int_{t_0}^{t} \mathbf{h}(t-u)a(u,\omega)\, e^{\iota\omega u}du$

How to *efficiently* compute $\mathbf{h}(t)$ and $\mathbf{G}(t,\omega)$?

How to compute $\mathbf{h}(t)$?

$$\mathbf{h}(t) = \int_{\mathbb{R}} \mathbf{H}(\omega)\, e^{\iota\omega t} d\omega$$

Impulse response matrix = Inverse Fourier Transfom of the transfer matrix

Decoupled modes : $\mathbf{\Omega}_d$, $\mathbf{D}_d$ diagonal

$$\mathbf{H}_d(\omega) = \left(\mathbf{\Omega}_d - \omega^2\mathbf{I} + \iota\omega\mathbf{D}_d\right)^{-1} \xrightarrow{\text{EASY}} \left(\mathbf{h}_d(t)\right)_i = \frac{e^{\Omega_i^+ t} - e^{\Omega_i^- t}}{2\iota\omega_i\sqrt{1-\xi_i^2}}$$

closed form expressions closed form expressions

How to compute $\mathbf{h}(t)$?

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Impulse response matrix = Inverse Fourier Transfom of the transfer matrix

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closed form expressions closed form expressions

Mechanically Coupled modes : $\mathbf{\Omega}$, $\mathbf{D}$ non-diagonal

$$\mathbf{H}(\omega) = \left(\mathbf{\Omega} - \omega^2\mathbf{I} + \iota\omega\mathbf{D}\right)^{-1} \xrightarrow{?}$$ NO closed form expression

How to compute $\mathbf{h}(t)$?

$$\mathbf{h}(t) = \int_{\mathbb{R}} \mathbf{H}(\omega)\, e^{\iota\omega t} d\omega$$

$$\downarrow$$

$$\mathbf{H} = \Big(\underbrace{\mathbf{\Omega} - \omega^2 \mathbf{I} + \iota\omega \mathbf{D}}_{\mathbf{J}} \Big)^{-1}$$

How to compute $\mathbf{h}(t)$?

$$\mathbf{h}(t) = \int_{\mathbb{R}} \mathbf{H}(\omega)\, e^{\iota\omega t} d\omega$$

$$\mathbf{H} = \Big(\underbrace{\mathbf{\Omega} - \omega^2\mathbf{I} + \iota\omega\mathbf{D}}_{\mathbf{J}} \Big)^{-1}$$

$$\underbrace{\begin{pmatrix} \blacksquare & \square & \square \\ \square & \blacksquare & \square \\ \square & \square & \blacksquare \end{pmatrix}}_{\mathbf{J}} = \underbrace{\begin{pmatrix} \blacksquare & & \\ & \blacksquare & \\ & & \blacksquare \end{pmatrix}}_{\mathbf{J}_d} + \underbrace{\begin{pmatrix} & \square & \square \\ \square & & \square \\ \square & \square & \end{pmatrix}}_{\mathbf{J}_o}$$

Decoupled

Coupling
small

$$\mathbf{H}_d = \mathbf{J}_d^{-1}$$

How to compute $\mathbf{h}(t)$?

$$\mathbf{h}(t) = \int_{\mathbb{R}} \mathbf{H}(\omega)\, e^{\iota\omega t} d\omega$$

$$\mathbf{H} = \Big(\underbrace{\boldsymbol{\Omega} - \omega^2 \mathbf{I} + \iota\omega\mathbf{D}}_{\mathbf{J}} \Big)^{-1}$$

$$\overset{\mathbf{J}}{\begin{pmatrix} \blacksquare & \square & \square \\ \square & \blacksquare & \square \\ \square & \square & \blacksquare \end{pmatrix}} = \overset{\mathbf{J}_d}{\begin{pmatrix} \blacksquare & & \\ & \blacksquare & \\ & & \blacksquare \end{pmatrix}} + \overset{\mathbf{J}_o}{\begin{pmatrix} & \square & \square \\ \square & & \square \\ \square & \square & \end{pmatrix}}$$

Decoupled

Coupling *small*

$$\mathbf{H}_d = \mathbf{J}_d^{-1}$$

$$\boxed{\mathbf{H} = (\mathbf{I} + \mathbf{H}_d \mathbf{J}_o)^{-1}\, \mathbf{H}_d}$$

Asymptotic Expansion-based Method

$$(\mathbf{I}+\mathbf{H}_d\mathbf{J}_o)^{-1}$$

$$\downarrow$$

?

$$\frac{1}{1+\varepsilon}$$

$$\downarrow \ \varepsilon < 1$$

$$\frac{1}{1+\varepsilon} = 1 - \varepsilon + \cdots$$

Asymptotic Expansion-based Method

$$(\mathbf{I} + \mathbf{H}_d \mathbf{J}_o)^{-1} \qquad\qquad \frac{1}{1+\varepsilon}$$

$$\Big\downarrow\ \rho_{\mathbf{J}} < 1 \qquad\qquad \Big\downarrow\ \varepsilon < 1$$

$$\mathbf{I} + \sum_{k=1}^{\infty} (-1)^k \underbrace{(\mathbf{H}_d \mathbf{J}_o)^k}_{\text{closed form}} \qquad\qquad \frac{1}{1+\varepsilon} = 1 - \varepsilon + \cdots$$

Convergence criterion

$\rho_{\mathbf{J}}$ = maximum eigenvalue of $\mathbf{H}_d \mathbf{J}_o$ over all ω (in norm)

Asymptotic Expansion-based Method

$$\mathbf{H} = \left(\mathbf{I} + \mathbf{H}_d \mathbf{J}_o\right)^{-1} \mathbf{H}_d \qquad \frac{1}{1+\varepsilon}$$

$$\downarrow \rho_{\mathbf{J}} < 1 \qquad \downarrow \varepsilon < 1$$

$$\mathbf{H} = \mathbf{H}_d + \sum_{k=1}^{\infty} (-1)^k \underbrace{\left(\mathbf{H}_d \mathbf{J}_o\right)^k \mathbf{H}_d}_{\text{closed form}} \qquad \frac{1}{1+\varepsilon} = 1 - \varepsilon + \cdots$$

Convergence criterion

$\rho_{\mathbf{J}}$ = maximum eigenvalue of $\mathbf{H}_d \mathbf{J}_o$ over all ω (in norm)

Asymptotic Expansion-based Method

$$\mathbf{H} = (\mathbf{I} + \mathbf{H}_d \mathbf{J}_o)^{-1} \mathbf{H}_d \qquad \frac{1}{1+\varepsilon}$$

$$\downarrow \rho_{\mathbf{J}} < 1 \qquad \downarrow \varepsilon < 1$$

$$\mathbf{H} = \mathbf{H}_d + \sum_{k=1}^{\infty} (-1)^k \underbrace{(\mathbf{H}_d \mathbf{J}_o)^k \mathbf{H}_d}_{\text{closed form}} \qquad \frac{1}{1+\varepsilon} = 1 - \varepsilon + \cdots$$

Convergence criterion

$\rho_{\mathbf{J}}$ = maximum eigenvalue of $\mathbf{H}_d \mathbf{J}_o$ over all ω (in norm)

Asymptotic expansion + Cauchy's theorem

$$\int_{\mathbb{R}} \mathbf{H} \quad e^{\iota \omega t} d\omega \longrightarrow \mathbf{h} = \mathbf{h}_d + \sum_{k=1}^{\infty} \Delta \mathbf{h}_k$$

decoupled system

closed form correction terms

Asymptotic Expansion-based Method

$$\mathbf{H} = \left(\mathbf{I} + \mathbf{H}_d \mathbf{J}_o\right)^{-1} \mathbf{H}_d \qquad \frac{1}{1+\varepsilon}$$

$$\downarrow \; \rho_{\mathbf{J}} < 1 \qquad \downarrow \; \varepsilon < 1$$

$$\mathbf{H} = \mathbf{H}_d + \sum_{k=1}^{\infty} (-1)^k \underbrace{\left(\mathbf{H}_d \mathbf{J}_o\right)^k \mathbf{H}_d}_{\text{closed form}} \qquad \frac{1}{1+\varepsilon} = 1 - \varepsilon + \cdots$$

Convergence criterion

$\rho_{\mathbf{J}}$ = maximum eigenvalue of $\mathbf{H}_d \mathbf{J}_o$ over all ω (in norm)

Asymptotic expansion + Cauchy's theorem + Truncation

$$\int_{\mathbb{R}} \mathbf{H}_N \, e^{\iota \omega t} d\omega \longrightarrow \underbrace{\mathbf{h}_N = \mathbf{h}_d}_{\text{decoupled system}} + \sum_{k=1}^{N} \underbrace{\Delta \mathbf{h}_k}_{\text{closed form correction terms}}$$

How to compute $\mathbf{G}(t, t_0, \omega)$?

$$\mathbf{G}(t, t_0, \omega) = \int_{t_0}^{t} \mathbf{h}(t-u)a(u)e^{\iota\omega u}du$$

How to compute $\mathbf{G}(t, t_0, \omega)$?

$$\mathbf{G}_N(t, t_0, \omega) = \int_{t_0}^{t} \mathbf{h}_N(t-u)a(u)e^{\iota\omega u}du$$

How to compute $\mathbf{G}(t, t_0, \omega)$?

$$\mathbf{G}_N(t, t_0, \omega) = \int_{t_0}^{t} \mathbf{h}_N(t-u) a(u) e^{\iota \omega u} du$$

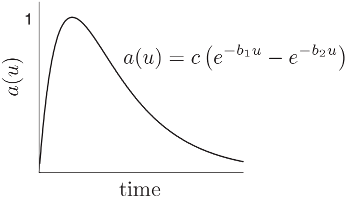
$a(u)$ is a tractable function

$\downarrow$

closed form expression $\mathbf{G}_N$

DONE !

Shinozuka window



How to compute $\mathbf{G}(t, t_0, \omega)$?

$$\mathbf{G}_N(t, t_0, \omega) = \int_{t_0}^{t} \mathbf{h}_N(t-u) a(u) e^{\iota \omega u} du$$

$a(u)$ is a tractable function

$\downarrow$

closed form expression $\mathbf{G}_N$

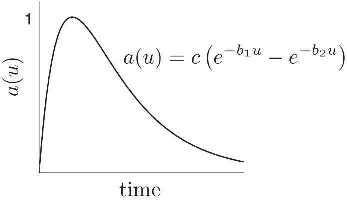
DONE !

$a(u)$ known numerically

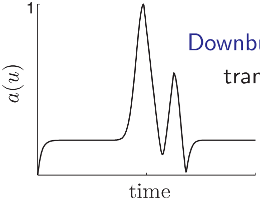
$\downarrow$

$\mathbf{G}_N$ cannot be computed from t_0 to t

Shinozuka window



Holmes window



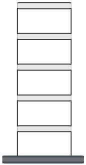
Downburst = transient wind

Efficient procedure : state space formalism

State space formalism $\quad \mathbf{z}(t,\theta) = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}$

state transition matrix

$$\mathbf{p}(t,\theta) \longrightarrow$$



$$\longrightarrow \mathbf{z}(t,\theta) = \int_{t_0}^{t} \boldsymbol{\Psi}(t,u) \begin{bmatrix} \mathbf{0} \\ \mathbf{p}(u) \end{bmatrix} du$$

Efficient procedure : state space formalism

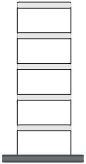
State space formalism $\mathbf{z}(t,\theta) = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}$

state transition matrix

$\mathbf{p}(t,\theta) \longrightarrow$

$a(t)$

$\mathbf{S_p}(\omega) \longrightarrow$



$$\longrightarrow \mathbf{z}(t,\theta) = \int_{t_0}^{t} \mathbf{\Psi}(t,u) \begin{bmatrix} \mathbf{0} \\ \mathbf{p}(u) \end{bmatrix} du$$

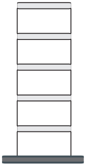
$$\longrightarrow \widehat{\mathbf{S}}_{\mathbf{z}}(t,\omega) = \mathbf{\Upsilon}(t,\omega) \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S_p} \end{bmatrix} \mathbf{\Upsilon}(t,\omega)^*$$

Evolutionary state transfer matrix $\mathbf{\Upsilon}(t,t_0,\omega) = \int_{t_0}^{t} a(u,\omega)\mathbf{\Psi}(t,u) \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} e^{\iota\omega u} du$

Efficient procedure : state space formalism

State space formalism $\mathbf{z}(t,\theta) = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}$

state transition matrix

$\mathbf{p}(t,\theta) \longrightarrow$  $\longrightarrow$ $\mathbf{z}(t,\theta) = \int_{t_0}^{t} \mathbf{\Psi}(t,u) \begin{bmatrix} \mathbf{0} \\ \mathbf{p}(u) \end{bmatrix} du$

$a(t)$, $\mathbf{S}_{\mathbf{p}}(\omega) \longrightarrow \longrightarrow$ $\widehat{\mathbf{S}}_{\mathbf{z}}(t,\omega) = \mathbf{\Upsilon}(t,\omega) \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{\mathbf{p}} \end{bmatrix} \mathbf{\Upsilon}(t,\omega)^*$

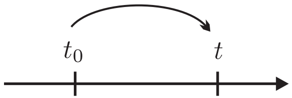
Evolutionary state transfer matrix

$$\mathbf{\Upsilon}(t,t_0,\omega) = \int_{t_0}^{t} a(u,\omega)\mathbf{\Psi}(t,u) \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} e^{\iota\omega u} du$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{G}(t,t_0,\omega) \\ \mathbf{0} & \partial_t \mathbf{G}(t,t_0,\omega) \end{bmatrix}$$

Efficient procedure : How to compute $\boldsymbol{\Upsilon}(t, t_0, \omega)$?

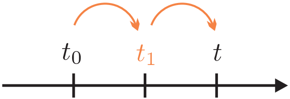
Semi-group property



$$\boldsymbol{\Psi}(t\,, t_0) \;=$$

Efficient procedure : How to compute $\boldsymbol{\Upsilon}(t, t_0, \omega)$?

Semi-group property



$$\boldsymbol{\Psi}(t\,, t_0) = \boldsymbol{\Psi}(t\,, t_1)\boldsymbol{\Psi}(t_1, t_0)$$

for LTI and LTV systems

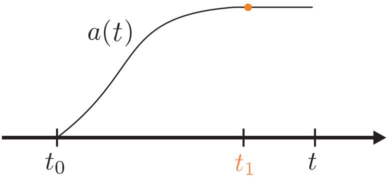
LTI system = Linear Time Invariant
LTV system = Linear Time Variant

Efficient procedure : How to compute $\boldsymbol{\Upsilon}(t, t_0, \omega)$?

Semi-group property

$$\boldsymbol{\Psi}(t\,, t_0) = \boldsymbol{\Psi}(t\,, t_1)\boldsymbol{\Psi}(t_1, t_0)$$

for LTI and LTV systems



Recurrence relation

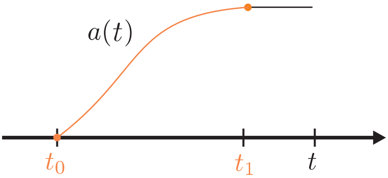
$$\boldsymbol{\Upsilon}(t, t_0, \omega) = \boldsymbol{\Upsilon}(t, t_1, \omega) + \boldsymbol{\Psi}(t, t_1)\boldsymbol{\Upsilon}(t_1, t_0, \omega)$$

Efficient procedure : How to compute $\boldsymbol{\Upsilon}(t, t_0, \omega)$?

Semi-group property

$$\boldsymbol{\Psi}(t\,, t_0) = \boldsymbol{\Psi}(t\,, t_1)\boldsymbol{\Psi}(t_1, t_0)$$

for LTI and LTV systems



Recurrence relation

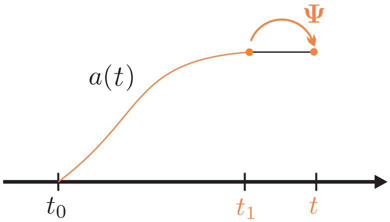
$$\boldsymbol{\Upsilon}(t, t_0, \omega) = \boldsymbol{\Upsilon}(t, t_1, \omega) + \boldsymbol{\Psi}(t, t_1)\underbrace{\boldsymbol{\Upsilon}(t_1, t_0, \omega)}_{\substack{\text{known} \\ t_0 \to t_1}}$$

Efficient procedure : How to compute $\boldsymbol{\Upsilon}(t, t_0, \omega)$?

Semi-group property

$$\boldsymbol{\Psi}(t\,,t_0) \;=\boldsymbol{\Psi}(t\,,t_1)\boldsymbol{\Psi}(t_1,t_0)$$

for LTI and LTV systems



Recurrence relation

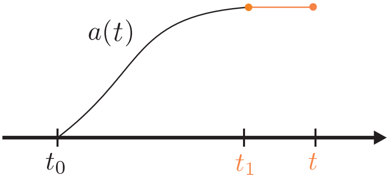
$$\boldsymbol{\Upsilon}(t,t_0,\omega) \quad = \quad \boldsymbol{\Upsilon}(t,t_1,\omega) + \underbrace{\boldsymbol{\Psi}(t,t_1)}_{\substack{\text{State transition} \\ t_1 \to t \\ \omega \text{ independent} \\ a(t) \text{ independent}}}\underbrace{\boldsymbol{\Upsilon}(t_1,t_0,\omega)}_{\substack{\text{known} \\ t_0 \to t_1}}$$

Efficient procedure : How to compute $\Upsilon(t, t_0, \omega)$?

Semi-group property

$$\Psi(t\,, t_0) = \Psi(t\,, t_1)\Psi(t_1, t_0)$$

for LTI and LTV systems



Recurrence relation

$$\Upsilon(t, t_0, \omega) = \Upsilon(t, t_1, \omega) + \Psi(t, t_1)\underbrace{\Upsilon(t_1, t_0, \omega)}_{\substack{\text{known} \\ t_0 \to t_1}}$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{G}(t, t_1, \omega) \\ \mathbf{0} & \partial_t \mathbf{G}(t, t_1, \omega) \end{bmatrix}$$

Convolution

$t_1 \to t$

State transition

$t_1 \to t$

ω independent

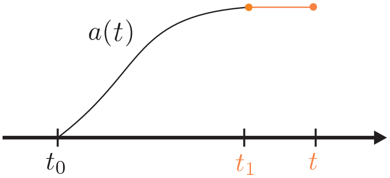
$a(t)$ independent

Efficient procedure : How to compute $\boldsymbol{\Upsilon}(t, t_0, \omega)$?

Semi-group property

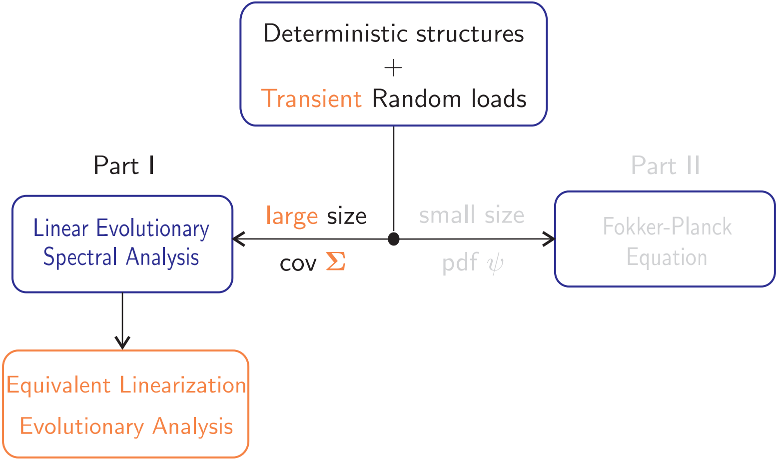
$$\boldsymbol{\Psi}(t\,, t_0) = \boldsymbol{\Psi}(t\,, t_1)\boldsymbol{\Psi}(t_1, t_0)$$

for LTI and LTV systems



Recurrence relation

$$\boldsymbol{\Upsilon}_N(t, t_0, \omega) = \underbrace{\boldsymbol{\Upsilon}_N(t, t_1, \omega)}_{\substack{\begin{bmatrix} \mathbf{0} & \mathbf{G}_N(t, t_1, \omega) \\ \mathbf{0} & \partial_t \mathbf{G}_N(t, t_1, \omega) \end{bmatrix} \\ \text{Convolution} \\ t_1 \rightarrow t}} + \underbrace{\boldsymbol{\Psi}(t, t_1)}_{\substack{\text{State transition} \\ t_1 \rightarrow t \\ \text{NO asymptotic approximation}}} \underbrace{\boldsymbol{\Upsilon}_N(t_1, t_0, \omega)}_{\substack{\text{known} \\ t_0 \rightarrow t_1}}$$


Deterministic structures
+
Transient Random loads
Part I
Part II
large size
small size
cov Σ
pdf ψ
Linear Evolutionary Spectral Analysis
Fokker-Planck Equation
Equivalent Linearization
Evolutionary Analysis

Equivalent linearization : basement

Nonlinear equation of motion

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} + \mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}) = \mathbf{f}(t, \theta)$$

$\mathbf{y}(t, \theta)$ is a Non-Gaussian process; covariance matrix $\mathbf{\Sigma_y}$?

Equivalent linearization : basement

Nonlinear equation of motion

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} + \mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}) = \mathbf{f}(t, \theta)$$

$\mathbf{y}(t, \theta)$ is a Non-Gaussian process; covariance matrix $\mathbf{\Sigma_y}$?

Gaussian Equivalent Linearization

$\mathbf{y}(t, \theta) \longrightarrow \mathbf{x}(t, \theta)$ is a Gaussian process

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{C}_{eq}\dot{\mathbf{x}} + \mathbf{K}_{eq}\mathbf{x} = \mathbf{f}(t, \theta)$$

Equivalent linearization : basement

Nonlinear equation of motion

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} + \mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}) = \mathbf{f}(t, \theta)$$

$\mathbf{y}(t, \theta)$ is a Non-Gaussian process; covariance matrix $\mathbf{\Sigma_y}$?

Gaussian Equivalent Linearization

$\mathbf{y}(t, \theta) \longrightarrow \mathbf{x}(t, \theta)$ is a Gaussian process

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{C}_{eq}\dot{\mathbf{x}} + \mathbf{K}_{eq}\mathbf{x} = \mathbf{f}(t, \theta)$$

How ?

minimize the error b/w $\mathbf{g}(\mathbf{x}, \dot{\mathbf{x}})$ and $\mathbf{C}_{eq}\dot{\mathbf{x}} + \mathbf{K}_{eq}\mathbf{x}$ in the mean squared sense

$$\mathbf{C}_{eq}(\mathbf{\Sigma_x}, \mathbf{\Sigma_{\dot{x}}}) \qquad \mathbf{K}_{eq}(\mathbf{\Sigma_x}, \mathbf{\Sigma_{\dot{x}}})$$

Covariance matrices $\mathbf{\Sigma_x}$ and $\mathbf{\Sigma_{\dot{x}}}$ are UNKNOWN

Equivalent linearization : example

Gaussian Equivalent Linearization

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{C} + \mathbf{C}_{eq}(\boldsymbol{\Sigma}_{\mathbf{x}}, \boldsymbol{\Sigma}_{\dot{\mathbf{x}}}))\,\dot{\mathbf{x}} + (\mathbf{K} + \mathbf{K}_{eq}(\boldsymbol{\Sigma}_{\mathbf{x}}, \boldsymbol{\Sigma}_{\dot{\mathbf{x}}}))\,\mathbf{x} = \mathbf{f}(t, \theta)$$

For instance

Nonlinear time-invariant		Linearized time-variant
$\ddot{y} + c\dot{y} + ky + \varepsilon y^3 = f(t,\theta)$	$\longrightarrow$	$\ddot{x} + c\dot{x} + kx + k_{eq}(t)x = f(t,\theta)$
ε constant in time		$k_{eq}(t) = 3\varepsilon\sigma_x^2(t)$ changes in time

stationary problem $\longrightarrow$ k_{eq} time-invariant

Equivalent linearization : example

Gaussian Equivalent Linearization

$$\mathbf{M}\ddot{\mathbf{x}} + \left(\mathbf{C} + \mathbf{C}_{eq}(\boldsymbol{\Sigma}_{\mathbf{x}}, \boldsymbol{\Sigma}_{\dot{\mathbf{x}}})\right)\dot{\mathbf{x}} + \left(\mathbf{K} + \mathbf{K}_{eq}(\boldsymbol{\Sigma}_{\mathbf{x}}, \boldsymbol{\Sigma}_{\dot{\mathbf{x}}})\right)\mathbf{x} = \mathbf{f}(t, \theta)$$

For instance

Nonlinear time-invariant | Linearized time-variant

$$\ddot{y} + c\dot{y} + ky + \varepsilon y^3 = f(t,\theta) \quad \longrightarrow \quad \ddot{x} + c\dot{x} + kx + k_{eq}(t)x = f(t,\theta)$$

ε constant in time | $k_{eq}(t) = 3\varepsilon\sigma_x^2(t)$ changes in time

Difficulty

$$\begin{matrix} \mathbf{C}_{eq}(t) \\ \mathbf{K}_{eq}(t) \end{matrix} \rightleftarrows \begin{matrix} \boldsymbol{\Sigma}_{\mathbf{x}}(t) \\ \boldsymbol{\Sigma}_{\dot{\mathbf{x}}}(t) \end{matrix}$$

nonlinear implicit relation

Linear Evolutionary Spectral Analysis

$$\text{LTI} \xrightarrow{\ ?\ } \text{LTV}$$

in an equivalent modal basis

Equivalent linearization : modal projection

Gaussian Equivalent Linearization

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{C} + \mathbf{C}_{eq}(\boldsymbol{\Sigma}_{\mathbf{x}}, \boldsymbol{\Sigma}_{\dot{\mathbf{x}}}))\,\dot{\mathbf{x}} + (\mathbf{K} + \mathbf{K}_{eq}(\boldsymbol{\Sigma}_{\mathbf{x}}, \boldsymbol{\Sigma}_{\dot{\mathbf{x}}}))\,\mathbf{x} = \mathbf{f}(t, \theta)$$

Projection in a generalized basis $\boldsymbol{\Phi}$ $\quad \mathbf{x} = \boldsymbol{\Phi}\mathbf{q}$

Eigenvalue problem $\quad (\mathbf{K} + \tilde{\mathbf{K}})\boldsymbol{\Phi} = \omega^2 \mathbf{M}\boldsymbol{\Phi}$

$\tilde{\mathbf{K}} = \mathbf{0}$ $\quad$ $\tilde{\mathbf{K}} \approx \mathbf{K}_{eq}$

Linear modes

$$\ddot{\mathbf{q}} + (\mathbf{D} + \mathbf{D}_{eq}(\boldsymbol{\Sigma}_{\mathbf{q}}, \boldsymbol{\Sigma}_{\dot{\mathbf{q}}}))\,\dot{\mathbf{q}} + (\boldsymbol{\Omega} + \boldsymbol{\Omega}_{eq}(\boldsymbol{\Sigma}_{\mathbf{q}}, \boldsymbol{\Sigma}_{\dot{\mathbf{q}}}))\,\mathbf{q} = \mathbf{p}(t, \theta)$$

- Linear equation of motion
- Nonlinear equivalent functions

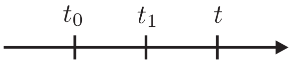
Equivalent linearization : generalized procedure

Equation of motion in a generalized basis

$$\ddot{\mathbf{q}} + \left(\mathbf{D} + \underbrace{\mathbf{D}_{eq}(\boldsymbol{\Sigma}_{\mathbf{q}}, \boldsymbol{\Sigma}_{\dot{\mathbf{q}}})}\right)\dot{\mathbf{q}} + \left(\boldsymbol{\Omega} + \underbrace{\boldsymbol{\Omega}_{eq}(\boldsymbol{\Sigma}_{\mathbf{q}}, \boldsymbol{\Sigma}_{\dot{\mathbf{q}}})}\right)\mathbf{q} = \mathbf{p}(t, \theta)$$

Time dependent in transient dynamics

Semi-group property



for LTI and LTV systems

$$\boldsymbol{\Psi}(t\,, t_0) = \boldsymbol{\Psi}(t\,, t_1)\boldsymbol{\Psi}(t_1, t_0)$$

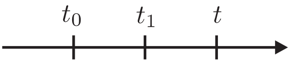
Equivalent linearization : generalized procedure

Equation of motion in a generalized basis

$$\ddot{\mathbf{q}} + \left(\mathbf{D} + \underbrace{\mathbf{D}_{eq}(\boldsymbol{\Sigma}_{\mathbf{q}}, \boldsymbol{\Sigma}_{\dot{\mathbf{q}}})}\right)\dot{\mathbf{q}} + \left(\boldsymbol{\Omega} + \underbrace{\boldsymbol{\Omega}_{eq}(\boldsymbol{\Sigma}_{\mathbf{q}}, \boldsymbol{\Sigma}_{\dot{\mathbf{q}}})}\right)\mathbf{q} = \mathbf{p}(t, \theta)$$

Time dependent in transient dynamics

Semi-group property



for LTI and LTV systems

$$\boldsymbol{\Psi}(t\,, t_0) = \boldsymbol{\Psi}(t\,, t_1)\boldsymbol{\Psi}(t_1, t_0)$$

Recurrence relation

$$\boldsymbol{\Upsilon}_N(t, t_0, \omega) = \boldsymbol{\Upsilon}_N(t, t_1, \omega) + \boldsymbol{\Psi}(t, t_1)\boldsymbol{\Upsilon}_N(t_1, t_0, \omega)$$

$$\cancel{\begin{bmatrix} \mathbf{0} & \mathbf{G}_N(t, t_1, \omega) \\ \mathbf{0} & \partial_t \mathbf{G}_N(t, t_1, \omega) \end{bmatrix}} \qquad \cancel{\mathbf{h}(t) = \int_{\mathbb{R}} \mathbf{H}(\omega) e^{\iota\omega t} d\omega}$$

LTV

Two timescales

Equation of motion in a generalized basis

$$\frac{d^2\mathbf{q}}{dt^2} + \left(\mathbf{D} + \mathbf{D}_{eq}(\mathbf{\Sigma}_{\mathbf{q}}, \mathbf{\Sigma}_{\dot{\mathbf{q}}})\right) \frac{d\mathbf{q}}{dt} + \left(\mathbf{\Omega} + \mathbf{\Omega}_{eq}(\mathbf{\Sigma}_{\mathbf{q}}, \mathbf{\Sigma}_{\dot{\mathbf{q}}})\right) \mathbf{q} = a(t)\mathbf{p}_s(t, \theta)$$

Two timescales

- Slow time t_s : envelope of the evolutionary process
- Fast time t_f : natural period of the structure

Two timescales

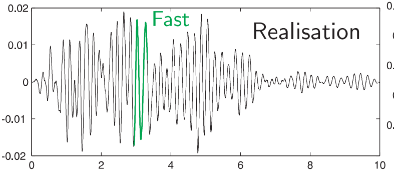
Equation of motion in a generalized basis

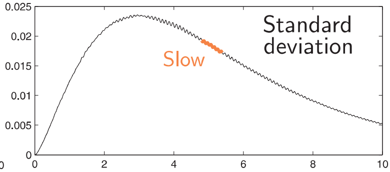
$$\frac{d^2\mathbf{q}}{dt^2} + \left(\mathbf{D} + \mathbf{D}_{eq}(\mathbf{\Sigma_q}, \mathbf{\Sigma_{\dot{q}}})\right)\frac{d\mathbf{q}}{dt} + \left(\mathbf{\Omega} + \mathbf{\Omega}_{eq}(\mathbf{\Sigma_q}, \mathbf{\Sigma_{\dot{q}}})\right)\mathbf{q} = a(t)\mathbf{p}_s(t,\theta)$$

Two timescales

- Slow time t_s : envelope of the evolutionary process
- Fast time t_f : natural period of the structure

For instance $\quad \ddot{y} + c\dot{y} + ky + k_{NL}y^3 = a(t)w(t,\theta)$ white noise





Two timescales

Two timescales

- Slow time t_s : envelope of the evolutionary process
- Fast time t_f : natural period of the structure

Flavour of demonstration $\left\{ \begin{array}{l} \mathbf{q}(t) \longrightarrow \mathbf{q}(t_s, t_f) \\ a(t) \longrightarrow a(t_s) \end{array} \right.$

Two timescales

Two timescales

- Slow time t_s : envelope of the evolutionary process
- Fast time t_f : natural period of the structure

Flavour of demonstration $\begin{cases} \mathbf{q}(t) \longrightarrow \mathbf{q}(t_s, t_f) \\ a(t) \longrightarrow a(t_s) \end{cases}$

1) Statistics evolve slowly $\boldsymbol{\Sigma}_{\mathbf{q}}(t_s), \boldsymbol{\Sigma}_{\dot{\mathbf{q}}}(t_s) \longrightarrow \mathbf{D}_{eq}(t_s), \boldsymbol{\Omega}_{eq}(t_s)$

2) $\dfrac{\partial^2 \mathbf{q}}{\partial t_f^2} + (\mathbf{D} + \mathbf{D}_{eq}(t_s)) \dfrac{\partial \mathbf{q}}{\partial t_f} + (\boldsymbol{\Omega} + \boldsymbol{\Omega}_{eq}(t_s))\, \mathbf{q} = a(t_s)\mathbf{p}_s(t_s, t_f, \theta)$

Two timescales

Two timescales

- Slow time t_s : envelope of the evolutionary process
- Fast time t_f : natural period of the structure

Flavour of demonstration
$$\begin{cases} \mathbf{q}(t) \longrightarrow \mathbf{q}(t_s, t_f) \\ a(t) \longrightarrow a(t_s) \end{cases}$$

1) Statistics evolve slowly $\quad \boldsymbol{\Sigma}_{\mathbf{q}}(t_s), \boldsymbol{\Sigma}_{\dot{\mathbf{q}}}(t_s) \longrightarrow \mathbf{D}_{eq}(t_s), \boldsymbol{\Omega}_{eq}(t_s)$

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LTV system $\longrightarrow$ piecewise-LTI system

$\mathbf{D}_{eq}(t), \boldsymbol{\Omega}_{eq}(t) \qquad \mathbf{D}_{eq}(t_s), \boldsymbol{\Omega}_{eq}(t_s)$

Generalized procedure in equivalent linearization

Evolutionary spectral analysis

$$\widehat{\mathbf{S}}_{\mathbf{z}}(t_k,\omega) = \boldsymbol{\Upsilon}(t_k,t_0,\omega)\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S_p} \end{bmatrix}\boldsymbol{\Upsilon}(t_k,t_0,\omega)^*$$

$$\longrightarrow \quad \boldsymbol{\Sigma}_{\mathbf{z}}(t_k) = \int_{\mathbb{R}} \widehat{\mathbf{S}}_{\mathbf{z}}(t_k,\omega)d\omega$$

Recurrence relation

$$\boldsymbol{\Upsilon}_{eq,N}(t_k,t_0,\omega) = \boldsymbol{\Upsilon}_{eq,N}(t_k,t_{k-1},\omega) + \boldsymbol{\Psi}(t_k,t_{k-1})\boldsymbol{\Upsilon}_{eq,N}(t_{k-1},t_0,\omega)$$

Generalized procedure in equivalent linearization

Evolutionary spectral analysis

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Recurrence relation

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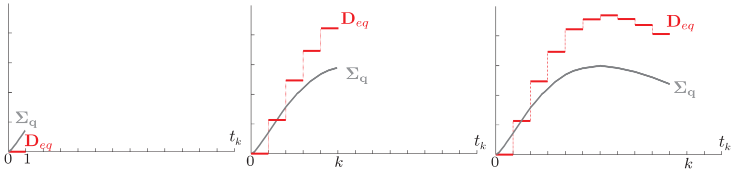
$$\downarrow$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{G}_N(t_k,t_{k-1},\omega) \\ \mathbf{0} & \partial_t \mathbf{G}_N(t_k,t_{k-1},\omega) \end{bmatrix} \longrightarrow$$

Asymptotic Expansion method

LTI on $[t_{k-1},t_k]$ with

$\mathbf{D}_{eq}(t_{k-1})$ and $\boldsymbol{\Omega}_{eq}(t_{k-1})$

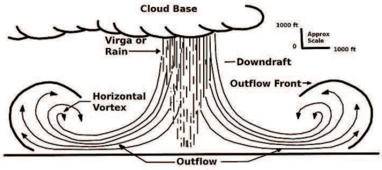


Tower subject to downburst

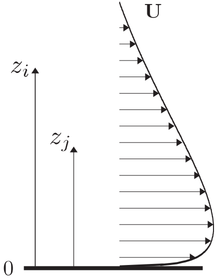
Downburst : a transient wind phenomenon

$$\mathbf{U}(t, \mathbf{z}) \quad + \quad \tilde{\mathbf{u}}(t, \mathbf{z})$$

mean wind — fluctuation



Mean profile

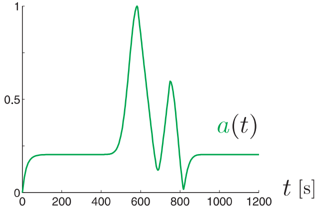


Wind psd

$$\mathbf{S}_{\mathbf{u}}(\omega, z_i, z_j) = \sqrt{S_{\tilde{u}}(\omega, z_i)\, S_{\tilde{u}}(\omega, z_j)}\,\Gamma(\omega, z_i, z_j)$$

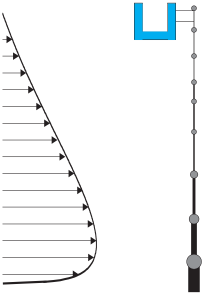
Coherence

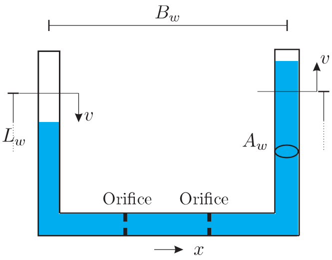
Time window



Tower subject to downburst

Tower damped by a TLCD





$$\rho_w A_w L_w \ddot{v} + \frac{1}{2}\rho_w A_w \delta\, |\dot{v}|\dot{v} + 2\rho_w A_w g v = -\rho_w A_w B_w \ddot{x}$$

$\downarrow$ Equivalent linearization

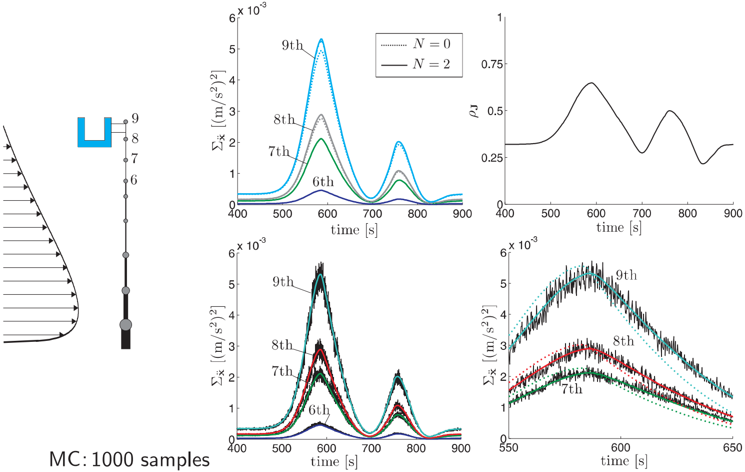
$$\rho_w A_w L_w \ddot{v} + \sqrt{\frac{2}{\pi}}\rho_w A_w \delta\, \sigma_{\dot{v}}\dot{v} + 2\rho_w A_w g v = -\rho_w A_w B_w \ddot{x}$$

Tower subject to downburst

Multiple scales approach

Natural frequency : 0.23 Hz

Time window : Δt =10s



MC: 1000 samples

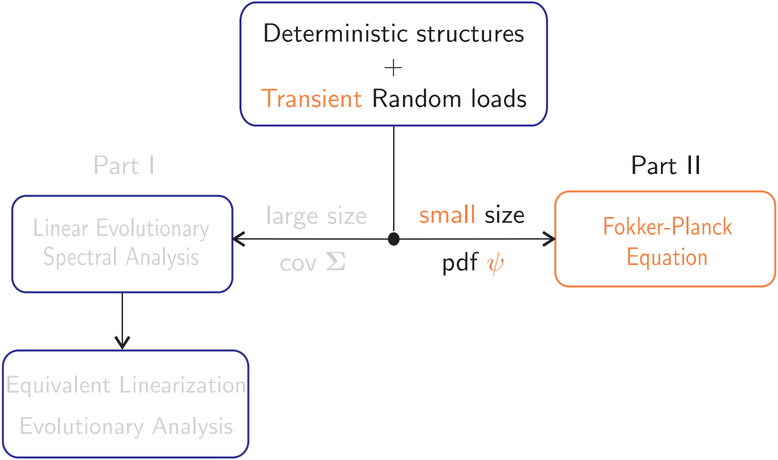
Conclusions

Goals

- Develop an efficient method for linear evolutionary spectral analysis
- Ability to deal with coherent random loads
- Extend this method to Gaussian equivalent linearization

Achievements

- Asymptotic expansions of $\mathbf{H}(\omega)$, $\mathbf{h}(t)$ and $\mathbf{G}(t,\omega)$
- Convergence criteria in frequency and time (improvements)
- Recurrence relation to compute $\boldsymbol{\Upsilon}(t, t_0, \omega)$ in a state space formalism
- Extension to equivalent linearization with a multiple scales approach
- Application in wind engineering : tower subject to downburst


Deterministic structures
+
Transient Random loads
Part I
Part II
Linear Evolutionary Spectral Analysis
large size
cov Σ
small size
pdf ψ
Fokker-Planck Equation
Equivalent Linearization
Evolutionary Analysis

Fokker-Planck-Kolmogorov Equation

Equation of motion

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} + \mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}) = a(t)\mathbf{w}(t)$$

$$\downarrow$$

$$\dot{\mathbf{z}}(t, \theta) = \mathbf{f}(t, \mathbf{z}) + \mathbf{b}(t, \mathbf{z})\mathbf{W}(t)$$

Fokker-Planck-Kolmogorov Equation

Equation of motion

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$$\downarrow$$

$$\dot{\mathbf{z}}(t, \theta) = \mathbf{f}(t, \mathbf{z}) + \mathbf{b}(t, \mathbf{z})\mathbf{W}(t)$$

FPK equation

$$\downarrow$$

$$\frac{\partial \psi}{\partial t} + \underbrace{\sum_{i=1}^{n} \frac{\partial}{\partial z_i} (f_i \psi)}_{\textit{convection}} = \underbrace{\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial z_i \partial z_j} (D_{ij} \psi)}_{\textit{diffusion}}$$

transport and conservation of the pdf $\psi(t, \mathbf{z})$

Fokker-Planck-Kolmogorov Equation

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transport and conservation of the pdf $\psi(t, \mathbf{z})$

Complementary conditions

$\psi_0(\mathbf{z})$	$\int_{\mathbb{R}} \psi(t, \mathbf{z}) d\mathbf{z} = 1$	$\psi(t, \mathbf{z}) > 0$
initial distribution	conservation	positivity

Motivations

Goal of this part

- solve numerically FPK equation
- for nonlinear systems and transient excitations
- apply to extreme value problems, tails of pdf

Which conditions must fulfill the FPK solver ?

- ensure the positivity and the conservation
- stability in the transient phase
- be able to deal with slightly to highly dispersed distributions
- extendable to high-dimensional systems

Method

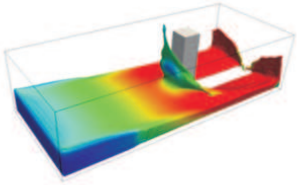
Smoothed **P**article **H**ydrodynamics

probability = hypothetical fluid

Generalities about SPH Method

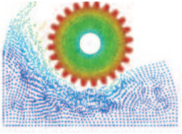
- Meshless method

  Particles are integration points



- Lagrangian formalism

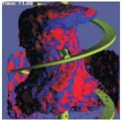
  Point of view of a non-fixed observator

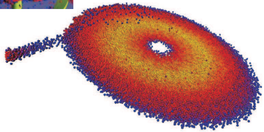


- Integral representation of a field $\phi(\mathbf{x})$

$$\phi(\mathbf{x}) = \int_{\mathbb{R}^n} \phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')d\mathbf{x}'$$

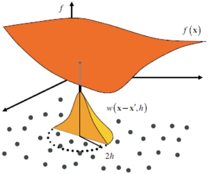
$\delta(.)$ a delta-Dirac function

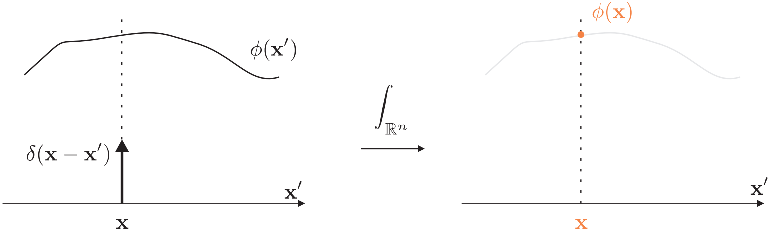




Integral representation of a field

$$\phi(\mathbf{x}) \quad = \int_{\mathbb{R}^n} \phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')d\mathbf{x}'$$





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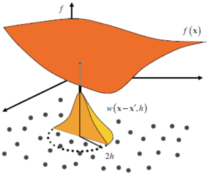
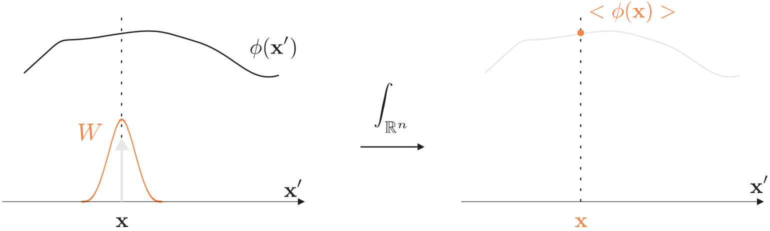
$$<\phi(\mathbf{x})> = \int_{\mathbb{R}^n} \phi(\mathbf{x}')W(|\mathbf{x}-\mathbf{x}'|, h)d\mathbf{x}'$$



with $W(., h)$ the kernel function and h the smoothing length



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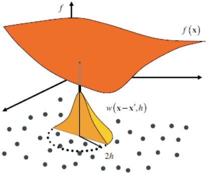


with $W(.,h)$ the kernel function and h the smoothing length

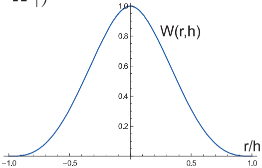
Kernel Function

δ-Dirac property $\lim_{h\to 0} W(|\mathbf{x}-\mathbf{x}'|,h) = \delta(|\mathbf{x}-\mathbf{x}'|)$

unity $\int_{\mathbb{R}^n} W(|\mathbf{x}-\mathbf{x}'|,h)d\mathbf{x}' = 1$

compact support, monotonic decay

symmetry, positivity and smoothness



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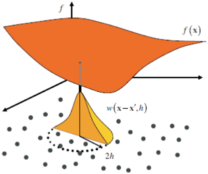

with $W(.,h)$ the kernel function and h the smoothing length

Discretization of the integral

$$< \phi(\mathbf{x}) >\approx \sum_{j=1}^{N_p} \phi(\mathbf{x}_j)W(|\mathbf{x}-\mathbf{x}_j|,h)\Delta V_j$$

number of particles

volume of particle j

Integral representation of a field

$$\phi(\mathbf{x}) = \int_{\mathbb{R}^n} \phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')d\mathbf{x}'$$

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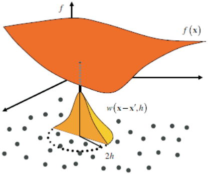

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$$< \phi(\mathbf{x}) > \approx \sum_{j=1}^{\overbrace{N_p}^{\text{number of particles}}} \phi(\mathbf{x}_j)W(|\mathbf{x}-\mathbf{x}_j|, h)\Delta V_j$$

volume of particle j

Discretization of the kernel approximation

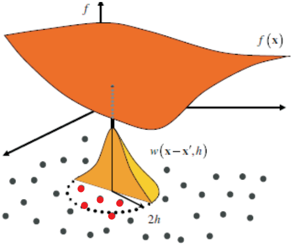
$$< \phi(\mathbf{x}_i) > = \sum_{j=1}^{N_p} \phi(\mathbf{x}_j)W(|\mathbf{x}_i-\mathbf{x}_j|, h)\Delta V_j$$

Particle Interaction

Kernel approximation

$$< \phi(\mathbf{x}_i) >= \sum_{j=1}^{N_p} \phi(\mathbf{x}_j) W(|\mathbf{x}_i - \mathbf{x}_j|, h) \Delta V_j$$

A particle i interacts with all the particles within its compact support



How to compute the interaction ?

Particle Interaction

Kernel approximation

$$< \phi(\mathbf{x}_i) >= \sum_{j=1}^{N_p} \phi(\mathbf{x}_j) W(|\mathbf{x}_i - \mathbf{x}_j|, h) \Delta V_j$$

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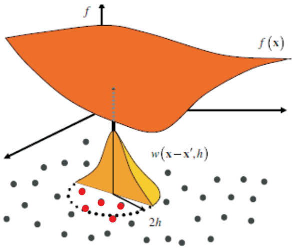
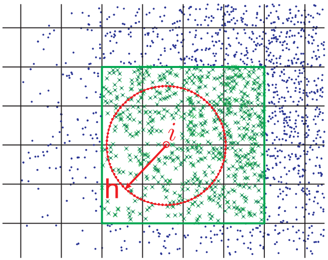


How to compute the interaction ?



1. The state space is divided into *cells*
2. The particles are sorted in these *cells*
3. For a particle i, a set of cells including its compact support is found
4. $< \phi_i >$ is built with particles contained in these cells

Lagrangian Paradigm

Eulerian formalism of conservation equation

$$\frac{\partial \psi}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial z_i} \left(f_i \psi - \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \left(D_{ij} \psi \right) \right) = 0$$

$$\frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v}\psi) = 0 \quad \text{with} \quad v_i = f_i - \sum_{j=1}^{n} \frac{\partial D_{ij}}{\partial z_j} - \sum_{j=1}^{n} D_{ij} \frac{1}{\psi} \frac{\partial \psi}{\partial z_j}$$

Particle velocity : convection and diffusion effects

Lagrangian Paradigm

Eulerian formalism of conservation equation

$$\frac{\partial \psi}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial z_i} \left(f_i \psi - \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \left(D_{ij} \psi \right) \right) = 0$$

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Particle velocity : convection and diffusion effects

Lagrangian formalism

particles = non-fixed observators

conservation of the total probability $\frac{D\psi}{Dt} + \psi \nabla \cdot \mathbf{v} = 0$ material derivative

transport equation for particles $\mathbf{v} = \frac{d\mathbf{X}}{dt}$ particle position

Lagrangian Paradigm

Eulerian formalism of conservation equation

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$$\frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{v} \psi) = 0 \quad \text{with} \quad v_i = f_i - \sum_{j=1}^{n} \frac{\partial D_{ij}}{\partial z_j} - \sum_{j=1}^{n} D_{ij} \frac{1}{\psi} \frac{\partial \psi}{\partial z_j}$$

Particle velocity : convection and diffusion effects

Lagrangian formalism

particles = non-fixed observators

conservation of the total probability $\quad \dfrac{D\psi}{Dt} + \psi \nabla \cdot \mathbf{v} = 0 \quad$ material derivative

transport equation for particles $\quad \mathbf{v} = \dfrac{d\mathbf{X}}{dt} \quad$ particle position

Conservation equation

The conservation equation of probability is similar to the conservation of mass in the Navier-Stokes equation

$$\xcancel{\frac{D\psi}{Dt} + \psi\nabla \cdot \mathbf{v} = 0}$$

we don't solve this equation

With SPH method, ψ is a *scalar field* that can be approximated by

$$< \psi(\mathbf{x}_i) >= \sum_{j=1}^{N_p} \psi_j W_{ij} \Delta V_j = \sum_{j=1}^{N_p} \mu_j W_{ij}$$

Conservation equation

The conservation equation of probability is similar
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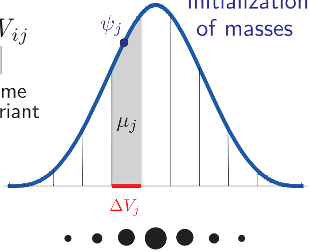
we don't solve this equation

With SPH method, ψ is a *scalar field* that can be approximated by

$$< \underbrace{\psi(\mathbf{x}_i)}_{\text{time variant}} >= \sum_{j=1}^{N_p}\psi_j W_{ij}\Delta V_j = \sum_{j=1}^{N_p}\overbrace{\mu_j}^{\text{constant ``mass''}}\underbrace{W_{ij}}_{\text{time variant}}$$

Initialization of masses



Advantages

- The positivity of the pdf is ensured
- The total probability is conserved
- No differential, nor algebraic equation, is solved

Transport equation

A particle i at a position $\mathbf{X}_i$ has a velocity $\mathbf{v}_i$ given by

$$\mathbf{v}_i = \underbrace{\mathbf{f}(\mathbf{X}_i)}_{\text{convection}} - \underbrace{\mathbf{D}\frac{< \nabla\psi(\mathbf{X}_i) >}{< \psi(\mathbf{X}_i) >}}_{\text{diffusion}}$$

gradient of the probability field

Transport equation of particle i

$$\frac{d\mathbf{X}_i}{dt} = \mathbf{v}_i \quad \longrightarrow \quad \underbrace{\mathbf{X}_i^{t+\Delta t}}_{\text{new position}} = \mathbf{X}_i^t + \mathbf{v}_i \Delta t$$

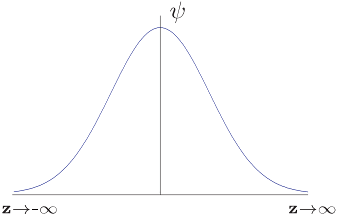
Resolution scheme

$$\mathbf{X}_i^t \longrightarrow r_{ij} \longrightarrow W_{ij} \longrightarrow < \psi_i^t > \longrightarrow \mathbf{v}_i^t \longrightarrow \mathbf{X}_i^{t+\Delta t}$$

Boundary conditions

Vanishing condition in the far-field

$$\lim_{\|\mathbf{z}\|\to\infty} \psi(t,\mathbf{z}) \to 0, \qquad \forall t \in \mathbb{R}^+$$



- No need for specific treatment
- Presence of low mass particles
- Instability in mesh-based methods

Other boundary conditions

Absorbing condition Periodic condition

Hysteretic Oscillator

Elastoplastic oscillator in 3D

Bouc-Wen model
$$\begin{cases} \ddot{q} + 2\xi\omega_0\dot{q} + \Phi = aW \\ \Phi = \omega_0^2\left(\alpha q + (1-\alpha) z\right) \\ \dot{z} = \dot{q}\left(A - \left(\beta \mathsf{sign}(\dot{q}z) + \gamma\right)|z|^n\right) \end{cases}$$

SPH solution

- SPH-FPK : 9261 particles
- Transience : a time window
- Simplification : W white noise
- $n = 1$, hist.variable z is bounded
- At t_0, particles are regularly spread
- Initial condition *quasi*-deterministic

Reliability problems

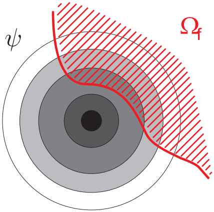
Probability of exceedance

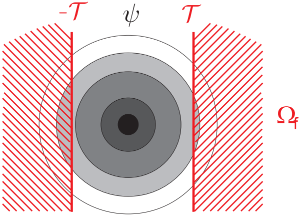
The mathematical formulation of a reliability problem consists in calculating the probability of exceedance P_f to be in Ω_f the subspace of failure

$$P_f(t) = \int_{\Omega_f} \psi(t, \mathbf{x}) d\mathbf{x} \qquad \longrightarrow \qquad < P_f(t) >= \sum_{j=1}^{N_p} \mu_j \mathbb{I}(\mathbf{X}_j(t) \in \Omega_f)$$

sum of the masses of the particles within Ω_f

For instance





Hysteretic Oscillator

Elastoplastic oscillator in 3D

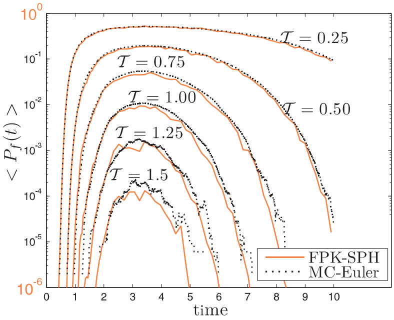
Bouc-Wen model
$$\begin{cases} \ddot{q} + 2\xi\omega_0\dot{q} + \Phi = aW \\ \Phi = \omega_0^2\left(\alpha q + (1-\alpha)z\right) \\ \dot{z} = \dot{q}\left(A - (\beta \mathsf{sign}(\dot{q}z) + \gamma)\,|z|^n\right) \end{cases}$$

Reliability problem

$$\Omega_{\mathcal{T}} = \{[q\ \dot{q}\ z] \in \mathbb{R}^3; |q| > \mathcal{T} > 0\}$$

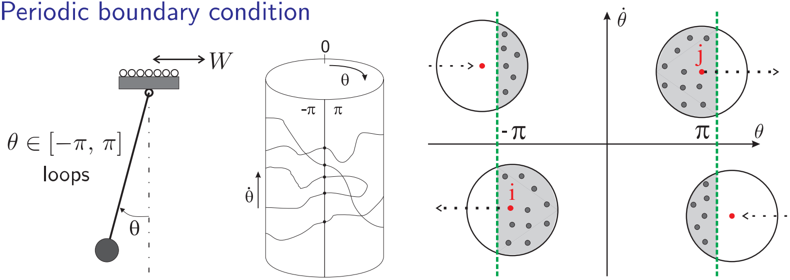
with $\mathcal{T}$ a threshold

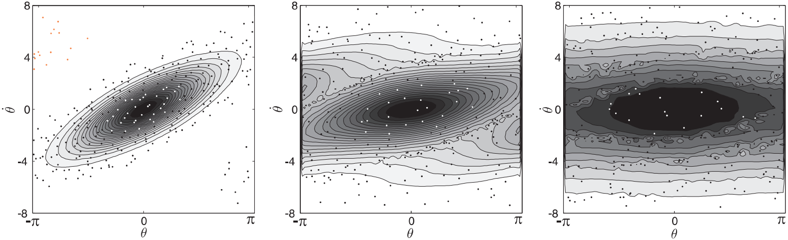
- SPH-FPK : 9261 particles
- Monte-Carlo : $5 \cdot 10^5$ samples
- 6 orders of magnitude are covered
- Time evolution well estimated



Randomly excited pendulum

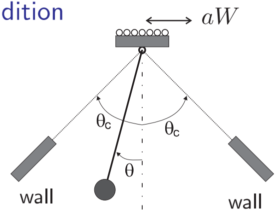
Periodic boundary condition





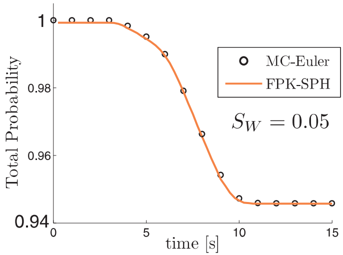
Randomly excited pendulum

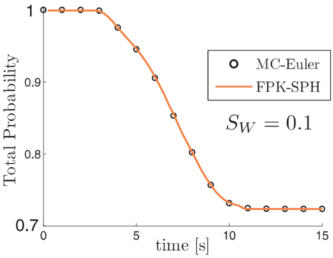
Absorbing boundary condition



$\theta \in [-\theta_c,\, \theta_c]$

Probability at t to not have hit the wall ?





Conclusions

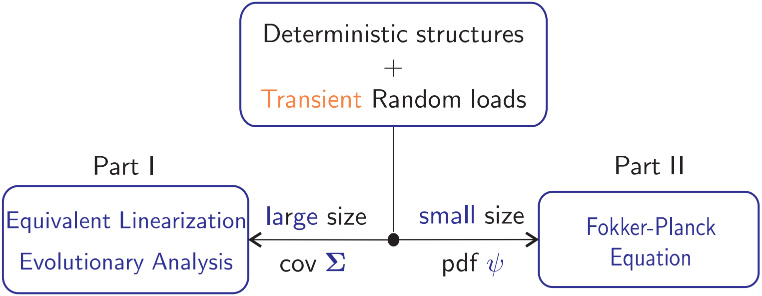
Main advantages of SPH method

- The positivity of the pdf is ensured
- There is no particular treatment for vanishing conditions
- The formalism is easily extended to high-dimension systems
- No algebraic equation is solved

Some limitations

- The stationary distribution cannot be directly computed
- The computation of interactions for large number of particles to improve

Summary



1. Asymptotic expansion method
2. Efficient generalized procedure
3. Multiple timescales approach

1. Smoothed Particle Hydrodynamics
2. Eulerian vs Lagrangian formalisms
3. Probability of exceedance