

New perspectives on probabilistic methods for nonlinear transient dynamics in civil engineering

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L'âme d'une cathédrale gémit dans l'effort vertical de la pierre.

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Summary

It is a common opinion that Monte Carlo method is considered as the paramount approach to solve transient random problems, in linear and nonlinear dynamics. Although the simulation approach can effectively be applied to a large range of problems, it is not necessarily the most efficient in all circumstances. The analysis of large-dimensional structures subject to random loadings, coherent in space and time, is a classical problem encountered in earthquake and wind engineering. In this case, the simulation of statistically consistent realizations of such random fields is a heavy computational operation in structural dynamics. Even for small random structures, the main drawback with the simulation method is the large number of simulations required to estimate low probabilities.

Based on these observations, this work is devoted to the development and the improvement of analytical and numerical methods to perform nonstationary probabilistic analysis and to compete against the crude simulation approach. Since the challenges are different, we separately deal with large and small problems.

First, we focus on the development of an efficient procedure for large structures to perform evolutionary spectral analysis, which is based on a time-frequency dual representation of nonstationary random processes. We aim at developing a method taking advantage of the usual spectral representation of coherent random loads. Then, we explore the solution of the Fokker-Planck Equation by a numerical method in order to accurately analyze small nonlinear problems.

For large-dimensional structures, we develop a method based on evolutionary spectrum for linear analysis and nonlinear analysis using Gaussian equivalent statistical linearization. The two main difficulties in evolutionary spectral analysis is the computation of the impulse response matrix of a structure and the convolution between this matrix and a time window.

To circumvent those difficulties, the equation of motion is first projected in an equivalent modal basis, in order to reduce the size of the system. However, the equivalent modal matrices may be not diagonal, such that a mechanical coupling occurs between the modes. For slight to moderate coupling, the out-of-diagonal elements can be seen as perturbations of a main diagonal matrix. Based on this assumption, we asymptotically expand the modal transfer matrix of the system around the one of a virtual decoupled system. The *asymptotic expansion-based method* circumvents these two main difficulties, by returning to linear algebra. As the modal transfer matrix of a structure and its impulse response matrix form a Fourier transform pair, we can advantageously find an asymptotic expansion of this matrix in a closed form. Afterward, the aforementioned convolution is analytically computed for some well-known time envelopes.

Based on this expansion, we develop an original and relevant formulation to perform

linear evolutionary spectral analysis. This formulation is then extended to nonlinear transient analysis through equivalent statistical linearization. This extension is formalized by a *multiple scales approach*, allowing to consider an equivalent system as piecewise time-invariant, since the equivalent properties evolve at slow timescale with the probabilistic characteristics of the response.

For small size structures, we explore the solution of the Fokker-Planck (FPK) equation with the Smoothed Particle Hydrodynamics (SPH) method. The FPK equation expresses the time evolution of the probability density function of a random state vector. This function is thus of paramount importance in reliability problems, since we can compute information about the tails.

The SPH method is a well-known mesh-free Lagrangian method, based on the integration representation of fields. It presents notable advantages to solve the FPK equation : (i) the conservation of total probability in the state space is explicitly written, (ii) no artifact is required to manage far-field boundary conditions , (iii) the positivity of the pdf is ensured and (iv) the extension to higher dimensional state space is straightforward. Furthermore, thanks to the moving particles, this method is adapted for a large kind of initial conditions, even slightly dispersed distributions. The FPK equation is solved without any *a priori* knowledge of the stationary distribution; just a precise representation of the initial distribution is required.

The method is applied to different problems of mechanical and civil engineering, as well as to more academic reliability problems.

Members of the jury

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Contents

1	Introduction	1
1.1	Structural dynamics in civil engineering	1
1.2	Motivations and scientific approach	6
1.3	Organization of the dissertation	7
2	Background and Methods in Probabilistic Analysis	9
2.1	Basics of stochastic calculus	10
2.1.1	Probability theory and probability space	10
2.1.2	Random variables and stochastic processes	11
2.1.3	Itô's calculus and stochastic differential equations	13
2.2	Spectral analysis and stochastic processes	15
2.2.1	Spectral representation of a stochastic process	15
2.2.2	Spectral analysis of linear systems	17
2.3	Fokker-Planck equation for diffusive processes	18
2.4	Monte Carlo simulation	20
2.5	Equivalent statistical linearization	22
2.6	Uncertain structures	24
2.7	Summary	26
I	Perturbation Methods in Nonlinear Evolutionary Spectral Analysis	27
3	Stationary Spectral Analysis	29
3.1	Context and propositions	30
3.2	Spectral strategy for equivalent statistical linearization	32
3.3	Asymptotic expansion of modal transfer matrix	33
3.3.1	About the choice of a modal basis	33
3.3.2	Modal coupling in dynamical systems	35
3.3.3	Derivation of the asymptotic formulation	36
3.3.4	Discussion on the convergence of the asymptotic expansion	40
3.3.5	Integration of the correction terms	41
3.4	Validation in linear spectral analysis	44
3.4.1	About the sources of non-proportional damping	44
3.4.2	Wind buffeting analysis of the Millau viaduct	44
3.5	Stationary equivalent statistical linearization	49
3.5.1	Newton's iterative procedure and asymptotic approach	49

3.5.2	Shear-type building under earthquake	51
3.6	Conclusions	54
4	Linear Evolutionary Spectral Analysis	57
4.1	Context and propositions	58
4.2	Spectral analysis of linear time-invariant systems	61
4.2.1	Expansion theorem and evolutionary spectrum	61
4.2.2	Evolutionary spectral analysis in physical coordinates	62
4.2.3	Evolutionary spectral analysis in state space	63
4.3	Asymptotic expansion in evolutionary spectral analysis	66
4.3.1	Derivation of an approximation for the impulse response matrix	66
4.3.2	Convergence of the series $\mathbf{h}_N(t)$	72
4.3.3	Analytical expressions for $\mathcal{I}_k^\alpha$ and $\mathcal{J}_k^\alpha$	73
4.4	Efficient formulation for numerical computation	76
4.5	Multi-span bridge under differential ground acceleration	78
4.6	Nongeometric spectral moments and first passage time	86
4.7	Linear systems driven by non-Gaussian excitation	91
4.8	Conclusions	99
5	Nonlinear Evolutionary Spectral Analysis	101
5.1	Context and propositions	102
5.2	Multiple scales approach in stochastic linearization	104
5.3	Evolutionary spectral analysis for equivalent LTV systems	108
5.3.1	Equivalent statistical linearization for LTV systems	108
5.3.2	Piecewise approximation of equivalent systems	109
5.3.3	Comments about the underlying assumptions	111
5.3.4	Application to a 2-DoF nonlinear system	112
5.4	Application in wind and earthquake engineering	114
5.4.1	Multi-span bridge subject to earthquake	114
5.4.2	Tower subject to downburst	118
5.5	Update of the equivalent modal basis	126
5.6	Conclusions	130
	Conclusion	133
II	A Lagrangian Method for Fokker-Planck-Kolmogorov Equation	135
6	Fokker-Planck Equation : Issues and Challenges	137
6.1	The Fokker-Planck-Kolmogorov equation	138
6.2	Resolution methods : an overview	139
6.3	Purpose of this part	142
7	SPH Method and Fokker-Planck Equation	145
7.1	Smoothed particle hydrodynamics	146
7.1.1	Integral representation of a field	146
7.1.2	Kernel function	147

7.2	SPH formulation of Fokker-Planck-Kolmogorov equation	148
7.2.1	Lagrangian formalism	149
7.2.2	Conservation equation	150
7.2.3	Estimation of statistical moments	150
7.2.4	Probability of exceedance	151
7.2.5	Transport equation	152
7.3	Computational aspects	153
7.3.1	Initialization of particle masses	153
7.3.2	Particle interaction and smoothing length	154
7.3.3	Boundary conditions	155
7.4	Description of the algorithm	157
8	Applications	159
8.1	Verification and validation	160
8.2	Nonlinear single DOF systems	162
8.2.1	Probability of exceedance of a nonlinear SDOF oscillator	162
8.2.2	Reduced pounding problem	163
8.3	Probability of exceedance of a hysteretic oscillator	166
8.4	Examples from other scientific disciplines	168
8.4.1	Randomly excited pendulum	168
8.4.2	Lorenz attractor	175
8.4.3	Stochastic Lotka-Volterra system	177
	Conclusion	181
III	Closure	183
	Appendices - Bibliography	189
A	Addendum to background in probabilistic analysis	189
A.1	Approximate Methods - Closure methods	189
B	Addendum to Part I	193
B.1	Extension of the background-resonant approximation	193
B.2	Linear evolutionary spectral analysis - Usefull constants	194
B.3	Illustration of the convergence of $\mathbf{h}_N$	194
C	Addendum to Part II	197
C.1	Finite-Element method for FPK equation	197
D	Curriculum vitae	199
	Bibliography	202

Chapter 1

Introduction

1.1 Structural dynamics in civil engineering

Structural dynamics

For years now, the improvement of material technology and the development of efficient numerical methods have given to engineers the opportunity to design slender and lighter structures, to bridge larger spans, to raise higher tops, to resist more extreme situations. However, overstepping the past limitations has led up engineers to consider new technological challenges and to understand new problematics and new subtleties. Furthermore, the economical environment has imposed to design offices and construction companies to insure themselves against inherent risks and natural hazards over a given period of time. The quantification of the risk exposed by the companies depends upon the possible or probable circumstances encountered by structures during their life.

This is a reason why the static “world” of civil engineering has been definitely dropped after the second world war. Previously, civil structures were generally assumed to respond in a static manner only. Thence, the effects of dynamical loadings were readily discarded assuming them as negligible, or taken into account by use of equivalent static loads. This philosophy is still present in the Eurocodes related to seismic or wind design, though it presents a lot of weaknesses from a physical point of view. We can point out some of them. In seismic design, the quasi-static approach is only pertinent for simple structures or buildings and the use of linear response spectrum to design nonlinear structures is definitely questionable. The quasi-static approach also forgets the specificity of transient dynamics, as it is the case with earthquakes. In wind engineering, on the other hand, the distribution of equivalent static loads is only pertinent for standard situations and are extended with difficulties to more complex structures like large roofs. Also shall we be concerned about the dubious use of gust factors and the perpetual assumption of stationary wind.

Actually, the crux of the problem is the design of large and unique structures, like bridges, large roofs or high-rise buildings, and the calling into question of usual assumptions. For this kind of structures, codes and norms provide limited answers (even disregarded) and engineers must revisit some theoretical statements leading to a correct modeling of the underlying phenomena. In this context, dynamical analysis is central for a proper handling of the time evolution of the loadings and the inertial forces, especially as structures become lighter, slender

and larger.

The ordinary differential equation governing the motion of a structure, can be written in a canonical form as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{g}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f}(t) \quad (\star)$$

with $\mathbf{x}(t)$ the vector of nodal displacements and with $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ the mass, damping and stiffness matrices, respectively. The vector $\mathbf{g}$ gathers nonlinear forces, while the vector $\mathbf{f}(t)$ is the vector of external loadings. This equation describes the dynamics of any *discrete* multi-degree-of-freedom (MDOF) structure. Thence, solving this equation is the underlying aim of any structural dynamics problem. Alternatively to $(\star)$, because the dimension of $\mathbf{x}$ may be large, a model reduction procedure may be applied, like the projection in modal basis.

Before highlighting the special features of $(\star)$ in our context, we would like to emphasize the difficulties to deal with dynamics in civil engineering. First of all, the physical dimensions of those problems may be impressive. For instance, the Millau viaduct, example in the following, is about 2.5km long and 350m above the bed of the valley it overshadows. Such dimensions are only encountered in civil engineering, never in mechanical engineering. Therefore, the modeling of a structure, e.g. with finite elements, requires a large amount of elements, without being able to go into the details of each connection. Secondly, the uniqueness of each major construction implies necessarily a particular approach, because no try, no past error and no prototype can be exploited to verify and to ensure the viability and the operability of a structure. The questions of safety and durability are of paramount importance and must be addressed as exhaustively as possible at a design step.

The purpose of this work falls within the environment of this equation, providing some further assumptions and specificities related to civil engineering applications. First, the nonlinear forces are not discarded. We consider previously $\mathbf{f}(t)$ as an external loading corresponding possibly to wind pressure on a building or ground acceleration at the foundation of a bridge. Because these two actions are not just exogenous, our purpose becomes more complex. Indeed, those excitations are affected by uncertainties. The random nature of these loadings invite us to leave the usual *deterministic description* of structural dynamics and to embrace a *non-deterministic paradigm*, taking into account all the sources of uncertainty in $(\star)$. Indeed, the computation of structural responses in a non-deterministic framework is necessary to quantify its resistance and serviceability in a risk assessment.

The purpose of this dissertation consists in developing and improving methods to compute the probabilistic response of structures subject to random loadings. Now we briefly highlight the sources of nonlinear behaviors and uncertainties in civil engineering.

Nonlinear behavior in civil engineering

The most suitable definition that can be given to *nonlinearity* is just to be the opposite to *linearity*: a system is thus said to be nonlinear, provided it is not linear. Therefore the superposition principle does not hold. In physical systems, the existence of nonlinear forces causes important difficulties. We limit ourselves by mentioning that nonlinear systems are prone to particular phenomena such as bifurcation, period-doubling or chaos. In spite of their great importance in system theory, those mathematical concepts are not central in this work. We must just point out that the superposition principle being broken, the Fourier analysis and

the classic modal projection cease to be valid. In random dynamics, the question is not only to superimpose responses, but random processes, making the problem much more complex from a statistical point of view.

What are the main sources of nonlinearity in civil engineering? First, large displacements in structures constitute the so-called *geometric* nonlinearities, provided the material remains elastic. More advanced are the *material* nonlinearities, namely plastic deformations or cracks. At the foundation level, the soil constitutive laws and the soil-water interaction lead also to consider complex behaviors, which are in practice simplified and modeled by nonlinear springs at this level.

In the context of structural dynamics under extreme natural loads, structural nonlinear behaviors must be taken into account. For instance, the plasticity is an advantageous source of energy dissipation, sought and controlled in earthquake engineering and risk mitigation. Some devices with nonlinear behavior may also be used in order to mitigate wind hazard, like tuned mass dampers or tuned liquid column dampers. In bridge engineering, viscous dampers with nonlinear constitutive laws are used to mitigate lateral vibrations of bridge deck. Those examples of nonlinearity will be consistently used in the illustrations of this manuscript.

Uncertainty in civil engineering

Before identifying the different origins of uncertainty in civil engineering structures, as exhaustively as possible, the meaning and the origin of the word *uncertainty* or *certainty* is first briefly discussed.

The word *certain* in both French and English comes from the old Latin *cerno*, meaning both *perceiving* and *deciding*. In Cicero's quotation "*ne nunc quidem oculis cernimus ea quae videmus*"¹, the Latin appeals to sensitive experience, while in "*amicus certus in re incerta cernitur*"² the classical author invokes the discriminating intelligence. Therefore, the etymology relates the word *certainty* to a decision in the judgment of *truth*, taking into account both intellectual arguments and empirical experiences. *Uncertainty* is thus not only related to a lack of knowledge.

The sources of uncertainty are generally sorted out into two main categories. First, the *endogenous* uncertainties are the ones inherent to the lack of knowledge about the studied structures. For instance, in a steel structure, the material properties as the Young modulus, the yield stress or the density are non-deterministic quantities. In the *semi-probabilistic* approach exposed in the Eurocodes, the yield stress is divided by a coefficient (greater than one) depending on the material in order to take into account the uncertainty on this property. Anticipating the following Chapter, we may say that in (★) the structural matrices and vectors are functions of uncertain variables. This source of uncertainty is not amply addressed in this work.

Our work is more focused on the second kind of uncertainty: the *exogenous* sources. They make reference to the effects of random excitations on the response of a structure. Hence, the uncertainty in the structural response may be due to uncertainty in the structure (left-hand side of (★)) and to randomness in the external loadings (right-hand side of (★)). The

¹ "Maintenant même ce ne sont pas nos yeux qui distinguent ce que nous voyons"

² "L'ami sûr se reconnaît dans les circonstances peu sûres"

methods used to perform dynamical analysis considering those different sources are completely different, though they are not incompatible fortunately.

Many loadings encountered in civil engineering may be considered as uncertain : wind blowing on a tower, earthquake shaking a bridge, crowd walking on a footbridge or waves impacting a jetty. In this work, we will focus on wind pressure and seismic waves. For wind loads, the uncertainty is such that if we consider two different locations on a same structure, even if the mean wind velocity is the same, the wind profile considering the turbulence will be different. Measured wind velocities will be neither perfectly correlated, neither totally independent. Though the wind is usually assumed to be a stationary loading, some aeolian phenomena do not have this property. For earthquakes, the uncertainty comes from the propagation of seismic waves through the different soil layers which are random media. Considering now an earthquake shaking a multi-span bridge, like for wind velocity, the ground acceleration measured on a pile is neither perfectly correlated, nor independent of the acceleration measured at another pile. This is an additional source of uncertainty that must be modeled.

Probabilistic analysis in nonlinear dynamics

Performing analysis in a non-deterministic framework is a complex topic, also from a theoretical point of view. Actually, a fallacious belief about uncertainties would be to directly associate them with probability. This is not necessarily the case ! There exist other families of methods to quantify and analyze uncertainties different from the probabilistic ones. Fuzzy arithmetic or data set structure are two different examples that can be applied in non-deterministic mechanics. However, in this work, we stay true to the probabilistic approach for both intellectual and empirical reasons. Our choice is motivated by this rigorous approach based on the measure theory, as explained in Chapter 2. Moreover, the relation between theory and experience is non-subjective. For instance, a buffeting analysis requires to know an estimation of the mean wind velocity on a given site. This velocity is not decided *in abstracto* because of a lack of knowledge, but it could be effectively related to real anemometer measurements.

The probabilistic analysis gathers all the numerical and analytical methods to compute the statistics of a structural response. The term statistics also gathers a large range of properties and characteristics, like the probability density function, the cumulants, the statistical moments or the power spectral density of a process. The probability density function is the more complete information about the statistical distribution of a structural response between different instants. Nonetheless, it is computed with difficulties, especially for nonlinear systems, a reason for rather computing some cumulants and moments.

Indeed, it is well-known that linear systems driven by Gaussian processes respond with Gaussian processes, while nonlinear systems respond with non-Gaussian ones in any case. We can draw a conceptual parallel between non-Gaussianity and nonlinearity, because a random variable is said to be non-Gaussian, provided it is not Gaussian. Furthermore, the linear combination of Gaussian variables remains Gaussian. This distinction is important, because Gaussian variables are characterized unequivocally by its first two moments, while the polymorphism of non-Gaussianity is an issue.

We can identify three notable families of methods used to perform stochastic analysis of nonlinear structures.

Based on the Itô calculus, the *Fokker-Planck-Kolmogorov equation* (FPK) is a deterministic convection-diffusion equation describing the time evolution of the probability density function (pdf) of a random state vector. This equation is applied to both linear and nonlinear systems and in both transient and stationary settings. Then comes *Monte Carlo simulation* (MC), an inductive method consisting in generating samples of excitations and endogenous variables and then to compute response samples with deterministic solvers (e.g. Runge-Kutta, Newmark). Finally, the *equivalent statistical linearizations* are *approximate* methods, replacing a nonlinear system by an equivalent linear one according to some statistical criteria. By the linearization approach, the linearized system takes advantage of all the properties of linearity, like the analysis in the frequency domain or in a modal basis. Nonetheless, the spectral methods cannot be applied in transient dynamics without some hypotheses. The extended spectral representation consistent with unsteady processes is usually known as *evolutionary spectra*. The mathematical statement of all those methods are given in Chapter 2.

One or another method may be used depending on different criteria, especially the size of the problem but also the required statistical data and whether the process is stationary, or not.

		FPK equation	MC simulation	Equ. Linearization
Size	small	possible	possible	<i>inappropriate</i>
	large	<i>forget it !</i>		pertinent
Transience		adapted	possible	challenging
Data	variance	ideal	possible	targeted
	pdf			<i>too approximate</i>

Table 1.1: Summary of the usual stochastic methods to perform nonlinear structural analysis and their domain of applications.

Table 1.1 highlights the relations between methods and criteria. First, we see that any method can be used to perform transient analysis. The size of the problem is a leading criterion, but it must be interpreted with regard to the target statistics, namely the mean, the standard deviation or the extremes. For small size systems (up to 2 degrees of freedom), the resolution of FPK equation gives the time evolution of the response pdf, the richest statistical characteristic. On the other hand, the equivalent linearization allows to deal with large size systems, provided we are just interested in estimating the first two cumulants of its random response. Table 1.1 also shows that Monte Carlo simulation can be used anyhow. At this stage, no consideration has been formulated about the efficiency of the method and about the nature of the loadings, except its transience. This is obviously too restrictive.

Based on these observations, we can now expose the ambitions of this work in the improvement of computational methods for transient probabilistic analysis of structures presenting nonlinear behaviors.

1.2 Motivations and scientific approach

From the previous Section, we know that Monte Carlo method is the most general one. It can be applied to any situation, no matter the size or the statistics. However, from the point of view of the efficiency, this method may lose its transversality and alternative techniques may find a new interest. By *efficiency*, we mean the capability to compute a statistical characteristic in a limited computational time. The efficiency is thus completely different from the possibilities expressed in Table 1.1.

In order to compete with the simulation approach, improvements may be sought for small and large systems respectively,

1. for the resolution of the FPK equation
2. in the developments of equivalent statistical linearization combined with evolutionary spectral analysis,

Both FPK equation and equivalent linearization are well-known methods, amply studied in the past and considered sometimes as old-fashioned. However, as said Marcel Duchamp, “*I force myself to contradict myself in order to avoid conforming to my own taste*”, inviting us to question our attainments and our acquaintance. We are thus not concerned about those opinions, because improvements for those methods may lead to make them more efficient and so usable by professional engineers and practitioners. It should be noticed that this is not the case yet, especially for the evolutionary spectral analysis, while the resolution of FPK equation remains a more exploratory topic.

Since the challenges are different at these two scales, we separately focus on each problematic, a reason for presenting the motivations and the scientific approach in two distinct parts.

Part I

For large structures subject to random loadings, the main drawback in Monte Carlo simulation is the generation of consistent realizations of the loadings. We pointed out previously in this Introduction that wind pressures or ground accelerations in a same event cannot be assumed to be independent or perfectly correlated. Actually, pressures and accelerations are *coherent random fields*, i.e. the randomness depends on both time and space. The generation of coherent samples of wind velocities or ground accelerations requires thus to cope with large matrices to take into account the whole coherence in the field. This operation is computationally intensive. Furthermore, those samples are mainly built up according to fields modeled in the frequency domain. Thence, this fact invites us to consider the frequency domain (or spectral) approach with more interest.

In a linear stationary analysis, the spectral approach is the key method. Indeed, the linearity allows to reduce the model by a projection within a linear modal basis. Then, the analysis can just be carried on in the frequency domain thanks to a Fourier transform. Since we work with nonlinear structures in a non-stationary setting, these advantages are not *a priori* preserved in our applications. Taking advantage of linearity and of the frequency domain, our motivation is to develop a method based on rigorous assumptions, that will be computationally efficient, as accurate as possible and applicable in civil engineering.

In civil engineering precisely, transient loadings are usually modeled by evolutionary spectra or evolutionary processes, i.e. an embedded stationary process modulated by a time envelope. Combining evolutionary spectral analysis with equivalent linearization will thus provide a formalism able to gather the aforementioned advantages. However, model reduction technique based on a linearized modal basis may be inefficient, because it should be updated many times throughout the resolution. This question is addressed in this work by relaxing some requirements on this basis and by using an *asymptotic expansion* method.

Actually, the cornerstone of this part is to use analytical methods, more precisely *perturbation approaches*, to circumvent computational drawbacks and theoretical weaknesses. The analytical developments must be seen as complementary tools to numerical programs, because intermediate closed-form expressions can be derived in order to improve the efficiency and the accuracy of the proposed method. Moreover, a *multiple scales approach* will allow to bridge the gap between the linear non-stationary analysis and the nonlinear one in civil engineering applications.

Part II

For small structures subject to random loadings, the main drawback with Monte Carlo methods is the large number of simulations required to estimate low probabilities. Because the FPK equation models the time evolution of the pdf, being able to solve this partial differential equation leads to accurately know high and low probabilities. However, the solution of FPK equation is really complex, a reason why we propose to use a novel numerical method to circumvent some of the well-known limitations.

In this work, we are mostly concerned by the drawbacks of existing numerical methods (anyhow analytical ones are too limited) to ensure the positivity of the pdf and the vanishing condition in the far field. Furthermore, in transient dynamics, the shape of the pdf can be highly distorted compared with the initial distribution, while covering large zones of the state space.

Motivated by these limitations, we propose to investigate a numerical method in a Lagrangian paradigm with the so-called Smoothed Particle Hydrodynamics (SPH) method. Namely, this method discretized any scalar field by a set of particles moving within the state space according to some governing laws. Besides the aforementioned difficulties, this method must be computationally competitive, efficient, accurate and stable. We do not pretend being able to solve a complete civil structure with the proposed method, but we humbly aim to develop a formalism enabling to deal with large systems. This part of the work is mainly *exploratory* with regard to classic problems encountered in civil engineering.

1.3 Organization of the dissertation

This manuscript is divided into two main parts : *Perturbation Methods in Nonlinear Evolutionary Spectral Analysis* **and** *A Lagrangian Method for Fokker-Planck Equation*. These parts are organized in such a manner that each part can be read completely independently from the other one. Before these two parts, the Chapter 2 summarizes the theoretical background necessary to understand our developments. It contains basic statements about the different

methods and some elements of bibliography. We now briefly expose the internal organization of each part.

The first part of this manuscript is divided into three chapters and a conclusion. The first Chapter *Stationary Spectral Analysis*, exposes the concept of *asymptotic expansion-based method* applied to the modal transfer matrix of a reduced linearized system in a stationary setting. This method is validated for the analysis of a large linear structure, then it is applied to the development of a second-order numerical solver to perform equivalent statistical linearization. This asymptotic method is then used in the second Chapter *Linear Evolutionary Spectral Analysis*, to circumvent some difficulties encountered in the computation of the impulse response matrix of a linear coupled structure, matrix required in the analysis with evolutionary processes. Finally, these first two chapters smoothly lead to the third one entitled *Nonlinear Evolutionary Spectral Analysis*, in which we demonstrate that an equivalent linearized system in a transient dynamics can be seen as a piecewise linear time-invariant system provided that some clearly separated timescales can be identified. In all those Chapters, applications in civil engineering are provided in order to illustrate and to validate the developments.

The second part of this manuscript is similarly divided into three chapters and a conclusion. The first Chapter, *Fokker-Planck Equation: Issues and Challenges*, briefly exposes the challenges and the issues in solving the Fokker-Planck equation with regard to existing methods. Afterward, the numerical method we propose to use is widely described and some particularities inherent to our problem are highlighted. Finally the last Chapter, entitled *Applications*, gathers a large range of illustrations in civil engineering and in other fields of science.

Chapter 2

Background and Methods in Probabilistic Analysis

Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

Bertrand Russell

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- 2.1 Basics of stochastic calculus
 - 2.2 Spectral analysis and stochastic processes
 - 2.3 Fokker-Planck equation for diffusive processes
 - 2.4 Monte Carlo simulation
 - 2.5 Equivalent statistical linearization
 - 2.6 Uncertain structures
 - 2.7 Summary
-

2.1 Basics of stochastic calculus

Since it pertains to well-established theoretical purposes, this introduction is amply inspired by trustworthy references like [Grig 02, Okse 03, Lin 04b, Le M 10, Kroe 11]. The objective of this Section is not only to recall the fundamental mathematical concepts that are used throughout this document, but also to serve as a general introduction of the nomenclature. This nomenclature is closely followed in the whole document so that the reader might want to come back to this Chapter occasionally if necessary during the reading to formally precise some concepts.

2.1.1 Probability theory and probability space

Probabilistic structural dynamics, the main topic of this work, is an application to the dynamics of structures of the theory of probability. The word *probability* comes from the old Latin *probabilitas*, meaning the verisimilitude or the plausibility of an event. In essence, a probable event shall be defined by opposition to the sureness of occurrence, by introducing the nuance between the impossibility and the certitude. Although this definition uses a description between two extremes, the theory of probability is built upon a strong mathematical foundation aiming at measuring the likelihood of an event. This introduction, with a view to contextualization of the topic, summarizes the philosophy of the probability theory.

The first reference to the *probabilitas* is found in a translation by Cicero of the Aristotle's *Topics* [Schn 76]. At that time, this vocable designates the plausibility of an opinion or an idea provided it is shared by a large number of people... a first view in the law of large numbers? The question asked by the *probabilitas* is an old philosophical and theological topic. Indeed, the *probabilism*, heir of the skeptical Academy of Plato, expresses the morality to follow an opinion in minority, provided it is probable. Far from that, Russell refuses to consider a well-established scientific results as absolutely true, but rather as sufficiently probable to undertake rational actions. These reflexions are yet far from the structural dynamics, but they show the complexity of understanding the conceptual notion of *probability*.

Generally, the modern concept of probability is supposed to have been introduced by the famous mathematicians Blaise Pascal and Pierre de Fermat in the 17th century, when they were investigating the *problem of points*. The paternity of the concept of probability remains an open question, but surely the main scientific contributions that makes the probability theory a new branch of mathematics are the works of Andreï Kolmogorov in 1933 [Kolm 33].

The theory proposed by Kolmogorov considers random experiments modeled by a probability space on which a probability measure is defined. A probability space is composed by a triplet $(\Theta, \mathfrak{F}, \mathbb{P})$, where

- Θ is the **sample space** : the set of all possible outcomes of the random experiment.
- $\mathfrak{F}$ is a non-empty collection of all subsets (called event) of Θ to which a probability may be assigned. Such a collection, called **σ -algebra** on Θ contains Θ itself, is closed under complements ($A \in \mathfrak{F} \Rightarrow \bar{A} \in \mathfrak{F}$) and is closed under countable union ($A_{i \in I} \in \mathfrak{F} \Rightarrow \bigcup_{i \in I} A_i \in \mathfrak{F}$).
- $\mathbb{P}$ is a **probability measure**, quantifying the likelihood of a random outcome.

The measure $\mathbb{P}$ of an event $A_i \in \mathfrak{F}$ is characterized by three axioms : the *non-negativity* of the measure $\mathbb{P}(A_i) \geq 0$, the *countable additivity* of the measure $\mathbb{P}(\bigcup_i A_i) = \sum_i \mathbb{P}(A_i)$ with $A_i \cap A_{j \neq i} = \emptyset$ and the *unity* of the total measure $\mathbb{P}(\Theta) = 1$. The first two axioms are necessarily required to define a measurable space. The last axiom makes all the specificity of the probability theory with regard to the theory of measure. As said in [McNe 05], “*Kolmogorov’s system translates our intuition about randomness into a concise, axiomatic language and clear rules*”. Indeed, this system has paved the way to the modern theory of probability and to the stochastic calculus hitherto.

The theory of probability, even considered as a mathematical theory, is not exempt from a part of interpretation. A probability may aim at characterizing a same event repeated many times or at measuring the verisimilitude of an event occurring one time in given physical conditions. The first approach, the *frequentist philosophy*, relates the probability theory and the basement of the *statistical methods* through the *law of large numbers*. The second interpretation, the *Bayesian approach*, is a measure of the “subjective uncertainty” [Le M 10] inherent to an event. An example of the Bayesian philosophy is the Bayesian model updating, in which few available experiments are used to fit model in order to reproduce real measurements or to identify properties of a model. Fortunately, this interpretation is not major in the presentation of the theory of probability and the stochastic calculus.

2.1.2 Random variables and stochastic processes

On the probability space $(\Theta, \mathfrak{F}, \mathbb{P})$, a function $X(\theta) : \Theta \mapsto \mathbb{R}$ is called a **$\mathbb{R}$ -valued random variable**. Hence, a random variable may be interpreted as a mapping of the sample space Θ into $\mathbb{R}$, or a map from $(\Theta, \mathfrak{F})$ to $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$, two measurable spaces. Here, the Borel σ -algebra $\mathfrak{B}(\mathbb{R})$ is the collection of all the open intervals in $\mathbb{R}$ (noted $\mathfrak{B}$ in the following). The probability measure of X , called the **distribution function**, is given by

$$\mathbb{Q}(B) = \mathbb{P}(\theta \in \Theta : X(\theta) \in B) \quad (2.1.1)$$

with $B \in \mathfrak{B}$ and $\mathbb{Q}$ the probability measure of $(\mathbb{R}, \mathfrak{B})$. Equation (2.1.1) may seem to be arcane for non-insiders. Actually, it just means that the probability measure in the space $(\mathbb{R}, \mathfrak{B})$ can be related by the inverse mapping of X to the probability measure of the original space $(\Theta, \mathfrak{F})$. The **cumulative distribution function (cdf)** of X , noted $F_X(x) : \mathbb{R} \mapsto [0, 1]$, is defined as

$$F_X(x) = \mathbb{Q}([-\infty, x]) = \mathbb{P}(\{\theta \in \Theta : X(\theta) \leq x\}) = \mathbb{P}(X(\theta) \leq x) \quad (2.1.2)$$

and is right-continuous and increasing. Equation (2.1.2) is a key point in the theory of integration over random variables. The **expectation operator** can thus be defined as

$$E[X] = \int_{\Theta} X(\theta) d\mathbb{P}(\theta), \quad (2.1.3)$$

which measures the probability weight related to the value $X(\theta)$ of the random variable. This operator is linear and monotonic (among others) and it allows to define an $\mathbb{L}_2$ -space gathering the integrable random variables, such that $\|X\|_2 = (E[X^2])^{1/2}$ exists and is bounded. The integrability of random variables can be checked for $\mathbb{L}_p$ -space with $p \in [1, \infty]$.

For absolutely continuous random variables, the expectation operation may be calculated by use of the **probability density function (pdf)** defined as

$$\psi_X(x) = \frac{dF_X(x)}{dx}, \quad \text{or,} \quad F_X(x) = \int_{-\infty}^x \psi_X(y)dy \quad (2.1.4)$$

such that, for any measurable function $g(X) : (\mathbb{R}, \mathfrak{B}) \mapsto (\mathbb{R}, \mathfrak{B})$, the expectation operator may read

$$E[g(X)] = \int_{\Theta} g(X(\theta))d\mathbb{P}(\theta) = \int_{\mathbb{R}} g(x)d\mathbb{Q}(x) = \int_{\mathbb{R}} g(x)dF_X(x) = \int_{\mathbb{R}} g(x)\psi_X(x)dx \quad (2.1.5)$$

Given the previous definition, there is clear evidence that an $\mathbb{R}^d$ -**valued random vector** is a function $\mathbf{X}(\theta) : \Theta \mapsto \mathbb{R}^d$ with d the dimension of the vector, i.e. a set of d random variables $X_i(\theta)$. The **joint cdf** is

$$F_{\mathbf{X}}(\mathbf{x}) = \mathbb{P}(X_i \leq x_i, i = 1, \dots, d) \quad (2.1.6)$$

with $\mathbf{x} = (x_1, \dots, x_d)$. Furthermore, a joint probability density function and a **marginal pdf** can be defined as

$$\psi_{\mathbf{X}}(x_1, \dots, x_d) = \frac{\partial^d F_{\mathbf{X}}(\mathbf{x})}{\partial x_1 \dots \partial x_d}, \quad \text{and} \quad \psi_{x_i}(x_i) = \int_{\mathbb{R}^{d-1}} \psi_{\mathbf{X}}(\mathbf{x})d\mathbf{x}_{|i} \quad (2.1.7)$$

respectively, with $\mathbf{x}_{|i}$ meaning all the components of $\mathbf{x}$ except the i -th one [Le M 10].

The concept of random variable is not sufficient to describe and model efficiently phenomena encountered in finance and science. Usually, a time or space dependence must be taken into account. So, the concept of random variable is naturally extended to the concept of **stochastic process**. Grigoriu considers the vector $\mathbf{X}$ as an $\mathbb{R}^d$ -**valued stochastic process** on a probability space $(\Theta, \mathfrak{F}, \mathbb{P})$, provided $\mathbf{X}(t, \theta) : T \times \Theta \mapsto \mathbb{R}^d$ is an $\mathbb{R}^d$ -valued random variable (i.e. $\mathbf{X}(t, \theta) \in \mathfrak{F}$) for each $t \in T (\subset \mathbb{R}^+)$. In addition to this “vertical” definition with t interpreted as a parameter, the “horizontal” view results from fixing the variable θ to obtain a **realization** (or a path) of the process.

Assuming now time and space dependencies, the process $\mathbf{X}(t, \mathbf{y}, \theta) : T \times \mathcal{Y} \times \Theta \mapsto \mathbb{R}^d$ is said to be a **stochastic field**. For instance, the vector $\mathbf{y} \in \mathcal{Y}$ gathers the coordinates of the structural nodes required to model the spatial coherence in a wind field.

All the concepts presented in the context of random variables are extended to stochastic processes. For instance, the cdf and pdf of random variables become time-dependent for stochastic processes. The next part of this Section is devoted to the definition of classes of stochastic processes. Far from being exhaustive, this list gathers the classes required to understand the mathematical developments explained in the following.

A stochastic process is said to be

- a **strongly stationary process**, if the distributions of the n -dimensional random vectors $(X(t_1), \dots, X(t_n))$ and $(X(t_1 + \Delta), \dots, X(t_n + \Delta))$ are identical, for any n , distinct times t_i and time shift Δ ;
- an **ergodic process**, if ensemble averages equal time averages, i.e. $E[g(X(t))] = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} g(X(s))ds$ almost surely, for any real $\mathbb{L}_1$ -function $g(x)$;

- a **Gaussian process**, if the distribution $\mathcal{N}(\mu_{\mathbf{X}}, \sigma_{\mathbf{X}})$ of any n -dimensional random vector $\mathbf{X} = (X(t_1), \dots, X(t_n))$ is Gaussian for any n ;
- a **Markov process**, if it is a one-step memory process, i.e. if for any $t \in T$ and $\Delta \geq 0$ the process satisfies the property $\mathbb{P}(X(t + \Delta) | \mathfrak{F}_t) = \mathbb{P}(X(t + \Delta) | X(t))$, where the filtration $\mathfrak{F}_t$ collects the history of the process until time t ;
- an **independent increment process**, if the random variables $X(t) - X(v)$ and $X(u) - X(s)$ are independent for $s \leq u \leq v \leq t$ (also valid for vector) ;
- an **orthogonal increment process**, if the real or complex-valued process $Z(t)$ is such that $E[(Z(u) - Z(s))(Z(t) - Z(v))^*] = 0$ for $s \leq u \leq v \leq t$ with $*$ denoting the conjugate transpose operator ;
- a **martingale**, if the process $X(t)$ defined on $\mathbb{L}_1(\Theta, \mathfrak{F}, \mathbb{P})$ is $\mathfrak{F}_t$ -adapted (i.e. $X(t) \in \mathfrak{F}_t$) and $E[X(t) | \mathfrak{F}_s] = X(s)$ with $\forall 0 \leq s \leq t$, in other words $X(t)$ is a *fair game*.

2.1.3 Itô's calculus and stochastic differential equations

The stochastic calculus is a branch of the theory of probability. After having defined the components of the stochastic calculus, the arising question pertains to their manipulations in the construction of a formal *stochastic analysis*. Indeed, the Itô calculus, the historical basics of stochastic calculus proposed by Itô in the 50's, starts with the definition of stochastic operations on a fundamental process, called the **Brownian motion** or the **Wiener process**.

The Brownian motion $B(t)$ is a stochastic process, almost surely continuous in time such that $B(0) = 0$ and the increments are *independent*, *stationary* and *Gaussian*, i.e. that $B(t) - B(s) = \mathcal{N}(0, t - s)$. It is generally understood as the continuous extension of a random walk. By this definition, this process is obviously a martingale, but it is nowhere differentiable and the variation between two times are unbounded. Therefore, the definition of a Riemann-Stieltjes integral for a stochastic process $X(t)$ on a Brownian motion, noted $\int_0^T X(t) dB(t)$, is senseless, because the quadratic variations of the Brownian motion tends to t and not to zero as expected for a deterministic integral. However, the Riemann-Stieltjes integral is well-defined for $X(t)$ a deterministic function, if the integral is interpreted path by path.

Using the Darboux sum, a stochastic integral of a random process on a Brownian motion can be defined as

$$\int_0^T X(t) dB(t) = \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} X(t_k^*) (B(t_k) - B(t_{k+1})) \quad (2.1.8)$$

with $0 = t_0 < \dots < t_n = T$ and where the process $X(t)$ is $\mathfrak{F}_{[0,T]}$ -adapted and $E[\int_0^T X^2(t) dt] < \infty$. The convergence is thus ensured in the mean squared sense. The choice of t_k^* is crucial in stochastic calculus. Indeed, the **Itô integral** is defined by $t_k^* = t_{k-1}$ [Ito 51a, Ito 51b], while the **Stratonovich integral** by $t_k^* = \frac{1}{2}(t_{k-1} + t_k)$ [Stra 63]. To ensure the distinction between these two operations, the Stratonovich integral is usually written as $\int_0^T X(t) \circ dB(t)$.

In classic analysis, this choice does not affect the result of the integral. Avoiding some developments, it can be shown that the Stratonovich paradigm preserves the Riemann-Stieltjes calculus, while the Itô integral ensures that Equation (2.1.8) is a martingale [Okse 03]. The

stochastic integrals can be generalized to martingales and semi-martingales, but it is beyond the scope of the present work.

A process $X(s, \theta) : [0, t] \times \Theta \mapsto \mathbb{R}^d$ on $(\Theta, \mathfrak{F}, \mathbb{P})$ is said to be an **Itô process** or a **diffusion process**, if it can be written in the form

$$X(t) = X(0) + \int_0^t a(s, \theta) ds + \int_0^t b(s, \theta) dB(s), \quad (2.1.9)$$

where $a(s, \theta)$ and $b(s, \theta)$ are $\mathfrak{F}_{[0, T]}$ -adapted, among others. Equation (2.1.9) is most of the time written in a differential form

$$dX(t, \theta) = a(t, \theta)dt + b(t, \theta)dB(t), \quad (2.1.10)$$

while the concept of *derivative* is senseless in stochastic calculus.

From the definition of an Itô process, the famous **Itô differential rule** (or **Itô's lemma**) can be introduced. If $g(t, x) : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$ is a deterministic function twice continuously differentiable in $\mathbb{R}^+ \times \mathbb{R}$, the process $Y(t) = g(t, X(t))$ is also an Itô process provided $X(t)$ is an Itô process. Furthermore, $Y(t)$ admits the chain rule

$$dY(t) = \left(\frac{\partial}{\partial t} g(t, X) \right) dt + \left(\frac{\partial}{\partial x} g(t, X) \right) dX + \frac{1}{2} \left(\frac{\partial^2}{\partial x^2} g(t, X) \right) dX^2 \quad (2.1.11)$$

with dX^2 given according to the rules $(dB)^2 = dt$ and $(dt)^2 = dB \cdot dt = 0$. The necessity in the chain rule to take into account the second-order increment of X comes from the fact that it contains first-order increments in time. With the Stratonovich integral, the classic chain rule of deterministic analysis is recovered. From (2.1.11), it becomes obvious that $Y(t)$ is a martingale, if the term proportional to dt is equal to zero (memoryless transformation [Grig 02]). The previous results are directly extended to n -dimensional space, such that for $\mathbf{X}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^n$ and

$$d\mathbf{X}(t, \theta) = \mathbf{a}(t, \theta)dt + \mathbf{b}(t, \theta)dB(t), \quad (2.1.12)$$

with $\mathbf{a}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^n$, $\mathbf{b}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^{n \times m}$ and $\mathbf{B}(t)$ an m -dimensional vector of Brownian motions, the Itô lemma reads

$$dY(t) = \left(\frac{\partial}{\partial t} g(t, \mathbf{X}) \right) dt + \sum_{i=1}^n \left(\frac{\partial}{\partial x_i} g(t, \mathbf{X}) \right) dX_i + \frac{1}{2} \sum_{i,j=1}^n \left(\frac{\partial^2}{\partial x_i \partial x_j} g(t, \mathbf{X}) \right) d(X_i X_j) \quad (2.1.13)$$

with $(dB_i \cdot dB_j)^2 = \delta_{ij}dt$ and δ_{ij} the Kronecker index.

In (2.1.12), the variable θ is used to define any source of uncertainty in the functions $\mathbf{a}$ and $\mathbf{b}$. However, a common case in mechanical engineering or in structural dynamics, is that $\mathbf{a}(t, \mathbf{x})$ and $\mathbf{b}(t, \mathbf{x})$ are two deterministic functions modeling a physical system with $\mathbf{x}$ the state variable of this system. Therefore, the uncertainty in the system originates from the external Brownian motions transforming the deterministic vector $\mathbf{x}(t)$ into a random vector $\mathbf{X}(t)$. Equation (2.1.12) becomes

$$d\mathbf{X}(t) = \mathbf{a}(t, \mathbf{X})dt + \mathbf{b}(t, \mathbf{X})dB(t) \quad (2.1.14)$$

and is thus called a **stochastic differential equation (sde)** in $\mathbf{X}(t)$. Uncertainties in the model may also be taken into account without limiting the randomness in $\mathbf{a}$ and $\mathbf{b}$ to the random vector $\mathbf{X}$, such that

$$d\mathbf{X} = \mathbf{a}(t, \theta, \mathbf{X})dt + \mathbf{b}(t, \theta, \mathbf{X})d\mathbf{B}(t). \quad (2.1.15)$$

The coefficient $\mathbf{a}$ is the **drift vector** expressing the convection in the system, while the matrix $\mathbf{b}$ is such that the product $\frac{1}{2}\mathbf{b}\mathbf{b}^T$ is called the **diffusion matrix**. Existence and uniqueness of the solution of (2.1.14) are amply exposed in [Okse 03]. The following Sections of this Chapter deals with the particularization of (2.1.14) from the model of structures under dynamical loading, to the models of random excitation in wind and earthquake engineering. Then, the different resolution techniques of these equations are exposed.

2.2 Spectral analysis and stochastic processes

2.2.1 Spectral representation of a stochastic process

A random process $\mathbf{X}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^n$ defined in $\mathbb{L}_2(\Theta, \mathfrak{F}, \mathbb{P})$ is said to be a real second-order process. The second moment properties are defined as the **mean vector** and the **covariance matrix**, i.e.

$$\boldsymbol{\mu}(t) = E[\mathbf{X}(t)], \quad \boldsymbol{\Sigma}(t, s) = E \left[(\mathbf{X}(t) - \boldsymbol{\mu}(t)) (\mathbf{X}(s) - \boldsymbol{\mu}(s))^T \right] \quad (2.2.1)$$

respectively, and the **correlation function**

$$\mathbf{R}(t, s) = E [\mathbf{X}(t)\mathbf{X}(s)^T] \quad (2.2.2)$$

with $t, s \in \mathbb{R}^+$. The assumption of second-order processes ensures the existence of these integrals, while it is insufficient for the existence of higher-order moments. As a particular case, a Gaussian process is a second-order process with well-defined and existing higher-order moments, which can be expressed in terms of the first two moments.

In Section 2.1.2, strongly stationary processes have been defined. For second moment properties, a less strict condition may be expressed such that the mean value is time invariant ($\boldsymbol{\mu}(t) = \boldsymbol{\mu}$) and the covariance matrix only depends on the time lag between t and s . A process satisfying this condition is said to be **weakly stationary**. Naturally, a weakly stationary process is not stationary in the strict sense, unless it is Gaussian.

The spectral distribution and the power spectral density are two alternative ways to characterize the second-order properties of such processes. Indeed, the *Bochner theorem* states that, since $\mathbf{R}(\tau)$ (with $\tau = t - s$) is a positive definite function, the correlation function has the frequency domain representation

$$(\mathbf{R}(\tau))_{ij} = \int_{\mathbb{R}} e^{i\omega\tau} d\mathcal{S}_{ij}(\omega), \quad (2.2.3)$$

where $\mathcal{S}_{ii}$, called the **spectral distribution** of $X_i(t)$, is a real-valued, increasing and bounded function, while $\mathcal{S}_{ij}$ is a complex-valued and bounded function, called the cross-spectral distribution between $X_i(t)$ and $X_j(t)$. The matrix gathering the elements $\mathcal{S}_{ij}$ is hermitian, such

that $\mathcal{S}_{ij} = \mathcal{S}_{ji}^*$. In the strict sense of Bochner's theorem, the increment $\mathcal{S}_{ij}$ is a probability measure or a spectral measure of the probability space.

If the functions $\mathcal{S}_{ij}$ are absolutely continuous, the Riemann-Stieltjes integral (2.2.3) admits the following expression

$$\mathbf{R}(\tau) = \int_{\mathbb{R}} e^{i\omega\tau} \mathbf{S}(\omega) d\omega \quad (2.2.4)$$

with $(\mathbf{S}(\omega))_{ij} = d\mathcal{S}_{ij}/d\omega$. The matrix $\mathbf{S}(\omega)$ gathering the elements $(\mathbf{S}(\omega))_{ij}$, also hermitian, is called the **power spectral density** matrix (psd); the elements $(\mathbf{S}(\omega))_{ii}$ are the unilateral-psd's and the elements $(\mathbf{S}(\omega))_{ij}$ are the cross-psd's.

Since $\mathbf{X}(t)$ is a real-valued and weakly stationary process, continuous in the mean squared sense for any time t , then a complex-valued process $\tilde{\mathbf{X}}(\omega)$ with orthogonal increments can be associated with $\mathbf{X}(t)$. They are related by the Fourier-Stieltjes integral

$$\mathbf{X}(t) = \int_{\mathbb{R}} e^{i\omega t} d\tilde{\mathbf{X}}(\omega) \quad (2.2.5)$$

existing at any time t in the mean squared sense too. The *Bochner theorem* emphasizes that a time process with the aforementioned properties can be decomposed into a summation of harmonics such that a random amplitude and a random phase given by $d\tilde{\mathbf{X}}(\omega)$ corresponds to every harmonic ω . The process $\tilde{\mathbf{X}}(\omega)$ defined in the frequency domain is said to be the **spectral process** associated with $\mathbf{X}(t)$. The spectral process has some noteworthy properties, e.g. $E[\tilde{\mathbf{X}}] = 0$. The definition of orthogonal increments is central in the characterization of spectral processes. An orthogonal increment process involves a more restrictive concept compared with an independent increment process. Applying the definition in Section 2.1.2, the increments of $\tilde{\mathbf{X}}(\omega)$ are said to be *orthogonal*, iif

$$E \left[d\tilde{\mathbf{X}}(\omega_1) d\tilde{\mathbf{X}}^*(\omega_2) \right] = \mathbf{S}_{\mathbf{X}}(\omega_1) \delta(\omega_1 - \omega_2) d\omega_1 d\omega_2 \quad (2.2.6)$$

with $\delta(\omega)$ the delta-Dirac function. Hence, the covariance function, for a weakly stationary process, reads

$$\mathbf{R}(\tau) = \int_{\mathbb{R}^2} e^{i\omega_1\tau} e^{i(\omega_1 - \omega_2)t} E \left[d\tilde{\mathbf{X}}(\omega_1) d\tilde{\mathbf{X}}^*(\omega_2) \right] \quad (2.2.7)$$

and the assumption of orthogonal increment leads to

$$\mathbf{R}_{\mathbf{X}}(\tau) = \int_{\mathbb{R}} e^{i\omega\tau} \mathbf{S}_{\mathbf{X}}(\omega) d\omega. \quad (2.2.8)$$

Equation (2.2.8) is the purpose of the *Wiener-Khintchine theorem*, which expresses that the covariance matrix of a weakly stationary process and the power spectral density of this process constitute a Fourier-pair. This theorem is thus a natural extension of the *Bochner theorem* according to the assumption of orthogonal increments.

2.2.2 Spectral analysis of linear systems

In linear system theory, the random vector $\mathbf{X}(t)$ can be seen as the response of a **linear time invariant system** subject to a random force vector, such that

$$\mathcal{A}(\mathbf{X}(t)) = d\mathbf{F}(t) \quad (2.2.9)$$

with $\mathcal{A}$ a stochastic differential operator and $d\mathbf{F}$ an increment of force. In civil and mechanical engineering, the differential operator $\mathcal{A}$ denotes basically the equation of motion. Since $\mathbf{X}(t)$ is the response of a linear system, the spectral representation allows to interpret it as the summation of different harmonics. These individual harmonics are the responses of the system subject to individual harmonic forces, such that

$$\mathbf{X}(t) = \int_{\mathbb{R}} e^{i\omega t} \mathbf{A}(\omega) d\tilde{\mathbf{F}}(\omega) \quad (2.2.10)$$

with $\tilde{\mathbf{F}}(\omega)$ the spectral process associated with a random force vector $\mathbf{F}(t)$, supposed to be weakly stationary with $\boldsymbol{\mu}_{\mathbf{F}} = \mathbf{0}$. The **transfer function** $\mathbf{A}(\omega)$ is the Fourier transform associated with the differential operator $\mathcal{A}$. The covariance matrix reads

$$\mathbf{R}_{\mathbf{X}}(\tau) = \int_{\mathbb{R}^2} e^{i\omega_1 \tau} e^{i(\omega_1 - \omega_2)t} \mathbf{A}(\omega_1) E \left[d\tilde{\mathbf{F}}(\omega_1) d\tilde{\mathbf{F}}^*(\omega_2) \right] \mathbf{A}^*(\omega_2) d\omega_1 d\omega_2 \quad (2.2.11)$$

and the assumption of orthogonal increment allows to write

$$\mathbf{R}_{\mathbf{X}}(\tau) = \int_{\mathbb{R}} e^{i\omega \tau} \mathbf{A}(\omega) \mathbf{S}_{\mathbf{F}}(\omega) \mathbf{A}^*(\omega) d\omega \quad (2.2.12)$$

with $E[d\tilde{\mathbf{F}}d\tilde{\mathbf{F}}^*] = \mathbf{S}_{\mathbf{F}}d\omega$ and $\mathbf{S}_{\mathbf{F}}$ the psd matrix of the force vector $\mathbf{F}$. A random vector is said to be **coherent**, if the out-of-diagonal elements of $\mathbf{S}_{\mathbf{F}}$ are not equal to zero. Comparing (2.2.8) with (2.2.12), the famous spectral relation can be extracted

$$\mathbf{S}_{\mathbf{X}}(\omega) = \mathbf{A}(\omega) \mathbf{S}_{\mathbf{F}}(\omega) \mathbf{A}^*(\omega). \quad (2.2.13)$$

Equation (2.2.13) is a well-known relation in stochastic spectral analysis; it relates the psd of the response of a linear time-invariant system to the psd of the external forces by left- and right-multiplications of this psd by the system transfer matrix.

All these developments establish the formal relation between the theory of stochastic processes and the spectral analysis. The spectral approach in stochastic analysis has the great advantage of releasing the analysis from the random experiment. Indeed, the random nature of the excitation and of the system responses is described in terms of energy distribution in the frequency domain. The recourse to trajectory simulations is avoided. As a linear system driven by Gaussian loading responds with Gaussian processes, the response of the system in a stationary setting is completely described by $\boldsymbol{\mu}_{\mathbf{X}}$ and $\boldsymbol{\Sigma}_{\mathbf{X}}(\tau)$. As the natural extension of the deterministic Fourier analysis to stochastic phenomena, the spectral approach is of course the most amply used probabilistic method, because of its simplicity and its capability to deal with large and complex structures.

As long as the system remains linear, the spectral approach may be applied to compute higher-order moments by means of **Volterra kernels**, interpreted as higher-order transfer functions. This question is briefly broached here, since details are given in Part I.

2.3 Fokker-Planck equation for diffusive processes

The previous Section has addressed the resolution of linear time-invariant systems subject to random loadings provided this loading is an orthogonal increment process. This case is too restrictive, yet. Indeed, the spectral approach is mainly limited to linear systems driven by Gaussian loadings, although some extends could be found for non-Gaussian loadings using Volterra series. Based on Itô's lemma, this Section briefly shows that a deterministic equation can be extracted from the sde (2.1.14) in order to compute the time evolution of the statistics of the random vector $\mathbf{X}(t)$. The system modeled by the sde may be linear or nonlinear, time-invariant or not.

The *Fokker-Planck equation*, named in honor of A. Fokker [Fokk 14] and M. Planck [Plan 17], is a deterministic partial differential equation. It describes via a convection-diffusion equation the time evolution of the pdf of a diffusion process conditioned by an (random) initial condition. A corollary of the Fokker-Planck equation is the (*backward*) *Kolmogorov equation* which describes the time evolution of this pdf, but in terms of the initial conditions. The Fokker-Planck equation is sometimes called the *forward Kolmogorov equation*. Because they describe the same phenomenon in the same manner, provided some minor adaptations, the equation is referred to as the **Fokker-Planck-Kolmogorov equation** in the following, noted **FPK equation**.

Some comments and explanations, inspired from [Risk 96, Grig 02, Lin 04b], are given in this Section. Considering $\mathbf{X}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^n$ a random vector on a probability space $(\Theta, \mathfrak{F}, \mathbb{P})$ satisfying the sde (2.1.14), the Itô lemma (2.1.13) may read

$$g(\mathbf{X}(t)) - g(\mathbf{X}(0)) = \int_0^t \sum_{i=1}^n \left(a_i(s, \mathbf{X}) \frac{\partial g(\mathbf{X})}{\partial x_i} \right) ds + \int_0^t \sum_{i,j=1}^n \left(D_{ij}(s, \mathbf{X}) \frac{\partial^2 g(\mathbf{X})}{\partial x_i \partial x_j} \right) ds \quad (2.3.1)$$

with a deterministic function $g(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ and $\mathbf{D}(t, \mathbf{x}) = \frac{1}{2} \mathbf{b} \mathbf{b}^T$. Both members are derived with respect to time and then the expectation is calculated, such as

$$\frac{\partial}{\partial t} E[g(\mathbf{X})] = \sum_{i=1}^n E \left[a_i(t, \mathbf{X}) \frac{\partial g(\mathbf{X})}{\partial x_i} \right] + \sum_{i,j=1}^n E \left[D_{ij}(t, \mathbf{X}) \frac{\partial^2 g(\mathbf{X})}{\partial x_i \partial x_j} \right]. \quad (2.3.2)$$

Equations (2.3.1) and (2.3.2) result from simplifications that are beyond the scope of this presentation¹. Developing the expectation operator in terms of the probability density function (2.1.5), integrations by parts allow to highlight the existing conditions on the drift vector $\mathbf{a}$ and the diffusion matrix $\mathbf{D}$,

$$\lim_{\|\mathbf{x}\| \rightarrow \infty} (a_i \psi(t, \mathbf{x} | \mathbf{x}_0)) = \lim_{\|\mathbf{x}\| \rightarrow \infty} \partial_{x_i} (D_{ij} \psi(t, \mathbf{x} | \mathbf{x}_0)) = \lim_{\|\mathbf{x}\| \rightarrow \infty} (D_{ij} \psi(t, \mathbf{x} | \mathbf{x}_0)) = 0 \quad (2.3.3)$$

for $i, j = 1, \dots, n$ and $\forall t \in \mathbb{R}^+$, with $\psi(t, \mathbf{x} | \mathbf{x}_0) : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^+$ the probability density function of $\mathbf{X}$ at time t , knowing the initial distribution $\psi(0, \mathbf{x}_0)$. The conditional probability is omitted here, as there is no ambiguity. The existence conditions (2.3.3) express actually

¹Actually, the elements dB_i ($i = 1, \dots, n$) in (2.1.14) are local martingales, so the expectation operator is equal to zero.

that the decay of the pdf defined on an unbounded state space must be greater than the functions in the drift vector and the convection matrix.

The integration by parts aims at extracting the function $g(\mathbf{x})$ out of the derivative operators in (2.3.2). After some manipulations, Equation (2.3.2) yields

$$\int_{\mathbb{R}^n} \left(\frac{\partial}{\partial t} \psi(t, \mathbf{x}) + \sum_{i=1}^n \frac{\partial}{\partial x_i} (a_i(t, \mathbf{x}) \psi(t, \mathbf{x})) - \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (D_{ij}(t, \mathbf{x}) \psi(t, \mathbf{x})) \right) g(\mathbf{x}) d\mathbf{x} = 0, \quad (2.3.4)$$

which should be satisfied independently of $g(\mathbf{x})$, so the factor in brackets is equal to zero. The Fokker-Planck equation is extracted from (2.3.4), such as

$$\frac{\partial}{\partial t} \psi(t, \mathbf{x}) + \sum_{i=1}^n \frac{\partial}{\partial x_i} (a_i(t, \mathbf{x}) \psi(t, \mathbf{x})) = \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (D_{ij}(t, \mathbf{x}) \psi(t, \mathbf{x})). \quad (2.3.5)$$

The deterministic nature of the FPK equation highlights the possibility to use non-stochastic techniques in order to solve random problems, under some assumptions.

The discussion in Section 2.1.2 between the Stratonovich and Itô integrals has here a great significance. Indeed, the demonstration proposed hereinabove is based on the Itô lemma, so on the Itô assumptions on the stochastic integral. As in (2.3.5), the FPK equation is written in the Itô paradigm. However, systems and structures are usually modeled by ordinary differential equations, i.e. in the Stratonovich sense. Adaptions are thus required to consistently jump from a paradigm to another. The *Wonk-Zakai correction term* [Wong 65] corrects some coefficients to ensure the similarity between these two approaches. In Itô's sense, Equation (2.3.5) reads

$$\frac{\partial}{\partial t} \psi(t, \mathbf{x}) + \sum_{i=1}^n \frac{\partial}{\partial x_i} (\tilde{a}_i(t, \mathbf{x}) \psi(t, \mathbf{x})) = \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (D_{ij}(t, \mathbf{x}) \psi(t, \mathbf{x})), \quad (2.3.6)$$

with $\tilde{a}_i = a_i + \frac{1}{2} \sum_{j,k} b_{jk} (\partial_{x_j} b_{ik})$. This later term, known as the Wonk-Zakai term, only affects the drift coefficients and it shows that the distinction between the two paradigms is noteworthy only if the matrix $\mathbf{b}$ defined in Section 2.1.2 depends on $\mathbf{x}$. This means that the distinction should be clearly made between the two stochastic integrals only in presence of some kind of parametric excitations.

Assuredly, the FPK equation has received a serious attention in the last decades, because the pdf is the most complete statistical description of a random vector. Knowing the transient pdf of a process opens a lot of perspectives. For instance, information about the tails is required in reliability problems [Kiur 96] and risk analysis [McNe 05]. However, some difficulties occurring in the resolution of the FPK equation may be emphasized without endless speeches and developments. Indeed, few analytical solutions exist for convection-diffusion equation, especially in transient regime. From a numerical point of view, the positivity of the pdf must be ensured and managed in high-dimensional and unbounded state space. This is called the *curse of dimensionality*. Furthermore, the model of filtered or coherent excitations with Itô's formalism requires additional equations in the state space and therefore increases the dimension of the whole space in which the FPK equation is solved.

These are some drawbacks of the deterministic approach via the FPK equation. However, the field of research remains challenging, provided the curse of dimensionality can be broken, the FPK equation is an efficient tool for problems in mechanics or civil engineering. Keeping cold head, the FPK equation is not a tool for analyzing random vibrations of large dimensional structures, efficient reduction methods must be applied to the model before solving this equation. Actually, a large part of this work has been devoted to the resolution of this equation taking into account its own particularities and some of the exposed drawbacks. All those details are deeply explored in Part II.

2.4 Monte Carlo simulation

The philosophy of Monte Carlo simulation is completely different from the two previous methods. Although the spectral approach and the FPK equation are applied to solve systems driven by uncertain loadings, nowhere the randomness inside the problem is used, e.g. through the concept of *random experiment*.

Another approach can consist in using consistent information about processes (e.g. psd and pdf) to generate samples in accordance with the spectral and the statistical properties [Kroe 11]. Driving samples input through a system leads to build up a set of random outputs. From this set, spectral and statistical properties are post-processed. The theoretical background of Monte Carlo simulation is mainly the *law of large numbers* [Grah 11]. From another viewpoint, Monte Carlo simulation is related to an *empirical* conception of stochastic problems: a random experiment, characterized by some properties, is repeated as many times as required according to some convergence criteria [Grah 11] and properties are determined by *induction*. The two methods previously exposed may be more considered as *deductive*.

Though it is based on basic concept, the application of Monte Carlo approach is closely tied to the improvement of computational power. In earthquake engineering, the method has been first applied by Shinozuka [Shin 72]. Since the method has been improved and amply applied in wind [DiPa 01, Cara 06] and earthquake engineering [Shin 91, Shin 96]. As it uses deterministic solvers (e.g. Newmark, Runge-Kutta), the Monte Carlo methods are considered as the only feasible method to perform random analysis of large-dimensional structures. This opinion is also shared for nonlinear and/or transient dynamics, because the FPK equation suffers from the curse of dimensionality and the spectral approach is bounded to linear(ized) systems. However, the use of Monte Carlo simulation requires useful precautions, because some drawbacks that can spoil the numerical efficiency must be pointed out.

As it is grounded on the law of large numbers, the convergence to low probability can be an issue, even if the computational efficiency has not ceased to increase these last decades. For instance, in reliability theory in which events with low probability of occurrence are considered, the number of samples increases inversely with the probability to measure. About the convergence in probability law, the computational burden may increase especially as the process is not ergodic, or more trivially non-stationary.

Then, the operation of sampling, i.e. the generation of samples of the coherent random vector $\mathbf{f}(t)$ knowing its psd $\mathbf{S}_{\mathbf{f}}(\omega)$ may also become an heavy computational task, especially for large-dimensional excitation fields. For Gaussian processes, the whole information is contained in the mean vector and the psd. For non-Gaussian processes, the distribution is characterized by more than the first two moments and by higher-order spectra [Sche 80]. However, in most

of the cases, non-Gaussian generators do not take into consideration the higher-order spectra. Generally, an approach consists in applying a nonlinear geometric transformation to Gaussian processes in order to create the non-Gaussianity in the process [Deod 01]. This viewpoint is the one also used to generate wind pressure : the wind is supposed to be a Gaussian process generated with regard to a given psd, then a nonlinear transformation is applied that makes the wind pressure highly skewed.

Generation of coherent random fields

The generation of Gaussian random processes from a coherent power spectral density is a key problem in structural dynamics. Different methods have been proposed, but two of them go down in memory. First, Shinozuka [Shin 72] uses a Cholesky decomposition of the cross-psd matrix at each frequency in order to determine an amplitude and a phase for each harmonic both frequency dependent, then a random signal is created by summing those harmonics with additional random phases. Another method, more described hereinafter is proposed by Di Paola [DiPa 01] for the digital simulation of wind field history, but it can be applied to seism as well. The method is here exposed in a general context.

Let consider an ergodic Gaussian random field $\mathbf{f}(t, \mathbf{y}, \theta) : \mathbb{R}^+ \times \mathcal{Y} \times \Theta \mapsto \mathbb{R}^n$ characterized by its cross-psd matrix $\mathbf{S}_{\mathbf{f}}(\omega) : \mathbb{R} \mapsto \mathbb{C}^{m \times n}$. The *spectral decomposition* of this matrix is given by

$$\mathbf{S}_{\mathbf{f}}(\omega) = \mathbf{\Xi}_{\mathbf{f}}(\omega) \mathbf{\Lambda}_{\mathbf{f}}^2(\omega) \mathbf{\Xi}_{\mathbf{f}}^*(\omega), \quad (2.4.1)$$

with $\mathbf{\Xi}_{\mathbf{f}}(\omega) : \mathbb{R} \mapsto \mathbb{C}^{n \times n}$ gathering the eigenvectors of $\mathbf{S}_{\mathbf{f}}(\omega)$ and the diagonal matrix $\mathbf{\Lambda}_{\mathbf{f}}^2(\omega) : \mathbb{R} \mapsto \mathbb{C}^{n \times n}$ the eigenvalues. All those matrices are ω -depending. The eigenvalue matrix can be decomposed into $\mathbf{\Lambda}_{\mathbf{f}}^2(\omega) = \mathbf{\Lambda}_{\mathbf{f}}(\omega) \mathbf{\Lambda}_{\mathbf{f}}^*(\omega)$, such that

$$\mathbf{S}_{\mathbf{f}}(\omega) = \mathbf{\Xi}_{\mathbf{f}}(\omega) \mathbf{\Lambda}_{\mathbf{f}}(\omega) \mathbf{\Lambda}_{\mathbf{f}}^*(\omega) \mathbf{\Xi}_{\mathbf{f}}^*(\omega). \quad (2.4.2)$$

The Fourier transform of the process $\mathbf{f}(t)$, noted $\mathbf{F}(\omega)$, can now be built up. First, we define $\hat{\mathbf{F}}(\omega)$, a vector gathering n Fourier transforms such that

$$\left(\hat{\mathbf{F}}(\omega) \right)_i = (\mathbf{\Lambda}_{\mathbf{f}}(\omega))_{ii} e^{i\theta_i(\omega)} \quad (2.4.3)$$

with $i = 1, \dots, n$. The vector $\theta_i(\omega)$ contains a set of random numbers uniformly distributed between 0 and 2π . The vector $\theta_i(\omega)$ is different for each i and numerically has the length of the vector sampling the frequency ω . The Fourier vector $\mathbf{F}(\omega)$ is then built as a linear combination of the vector $\hat{\mathbf{F}}(\omega)$ in the basis $\mathbf{\Xi}_{\mathbf{f}}(\omega)$, i.e.

$$\mathbf{F}(\omega) = \mathbf{\Xi}_{\mathbf{f}}(\omega) \hat{\mathbf{F}}(\omega). \quad (2.4.4)$$

Finally, the random vector $\mathbf{f}(t)$ results from the inverse Fourier transform of $\mathbf{F}(\omega)$.

2.5 Equivalent statistical linearization

To solve nonlinear systems subject to random excitations, many approximate methods have been developed : the averaging method [Grig 02], the equivalent statistical linearization [Soch 91], quadratization and cubicization [Span 03], non-Gaussian closures [Cran 85] (see Appendix A.1) are seemingly the most famous. Among them, the equivalent linearization, originally introduced by Botoon and Caughey [Caug 59a, Caug 59b] is frequently used for the analysis of high-dimensional nonlinear structures, as encountered in earthquake engineering [Cunh 94, Won 96, Esma 12] or in wind engineering [Xu 92, Gani 12, Guha 12]. The theoretical background and the different variants of this method are exposed, because they are relevant in the purpose of this work. Indeed, Part I is utterly devoted to the development of a new technique to perform equivalent statistical linearization of dynamical system in stationary and non-stationary settings.

As previously in Section 2.1.2, a n -dimensional random state space vector $\mathbf{Y}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^n$ satisfies the diffusion sde

$$d\mathbf{Y} = \mathbf{a}(t, \mathbf{Y})dt + d\mathbf{F}(t), \quad (2.5.1)$$

where $\mathbf{a}(t, \mathbf{y}) : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is basically a nonlinear function of $\mathbf{y}$ and $d\mathbf{F}(t)$ is the increment of a random time-dependent vector, i.e. only additive excitations are considered in the system. The force vector $\mathbf{F}(t)$ is supposed to be a Gaussian process (not necessarily stationary). The equivalent statistical linearization is a famous approximate method used to solve nonlinear sde's. It consists in replacing (2.5.1) by an equivalent linear sde, noted

$$d\mathbf{X} = (\mathbf{A}_{eq}(t)\mathbf{X} + \bar{\mathbf{a}}(t))dt + d\mathbf{F}(t), \quad (2.5.2)$$

with $\mathbf{A}_{eq}(t) : \mathbb{R}^+ \mapsto \mathbb{R}^{n \times n}$ and $\bar{\mathbf{a}}(t) : \mathbb{R}^+ \mapsto \mathbb{R}^n$, both only depending on time and $\mathbf{X}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^n$ is the equivalent random vector. The elements of $\bar{\mathbf{a}}$ and $\mathbf{A}_{eq}$ in (2.5.2) are determined in order that the diffusive process $\mathbf{X}(t)$ is an adequate approximation of the state vector $\mathbf{Y}(t)$, according to some statistical criteria. A statistical approximation is operated on the drift function $\mathbf{a}(t, \mathbf{X})$, consisting in minimizing $\mathcal{E}(t)$ the mean squared discrepancy between the approximate drift term and the exact one, i.e.

$$\mathcal{E}(t) = E \left[(\mathbf{a} - \mathbf{A}_{eq}\mathbf{X} - \bar{\mathbf{a}})^T (\mathbf{a} - \mathbf{A}_{eq}\mathbf{X} - \bar{\mathbf{a}}) \right]. \quad (2.5.3)$$

The minimization of the error with regard to the elements of $\bar{\mathbf{a}}$ and $\mathbf{A}_{eq}$ is obtained by imposing $\partial_{\bar{\mathbf{a}}_i} \mathcal{E} = \partial_{\mathbf{A}_{eq,ij}} \mathcal{E} = 0$ for every coefficient. After some algebra, the relation between $\mathbf{a}(t, \mathbf{X})$, $\mathbf{A}_{eq}(t)$ and $\bar{\mathbf{a}}$ may be summarized by

$$\begin{aligned} \bar{\mathbf{a}} &= E[\mathbf{a}] - \mathbf{A}_{eq}\boldsymbol{\mu}_{\mathbf{X}} \\ \boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{A}_{eq}^T &= E[\mathbf{X}\mathbf{a}^T] - \boldsymbol{\mu}_{\mathbf{X}}E[\mathbf{a}]^T, \end{aligned} \quad (2.5.4)$$

which leads *prima facie* to statistically satisfactory approximation of $\mathbf{a}$ via $\bar{\mathbf{a}}$ and $\mathbf{A}_{eq}$. The mean vector $\boldsymbol{\mu}_{\mathbf{X}}$ and the covariance matrix $\boldsymbol{\Sigma}_{\mathbf{X}}$ of the random vector $\mathbf{X}(t)$ are defined in (2.2.1). It is important to notice that at this stage no approximation and no assumption has been formulated. Indeed, as long as the probability density function of the random state vector $\mathbf{Y}(t)$ is known, Equations (2.4.3) and (2.5.4) are exact and will give unbiased estimations of

the first two statistical moments of $\mathbf{X}(t)$. This concept is known as the *true linearization* [Kozi 88]. However, the main purpose of the method does not rely on this concept, because both $\mathbf{Y}(t)$ and $\mathbf{X}(t)$ are unknown random processes.

The objective function given in (2.4.3) is the most popular one for its robustness and simplicity to find the Equations (2.5.4). However, the pertinence of this criterion for the linearization procedure has been addressed by different authors. For instance, Socha [Soch 91] proposes to minimize the squared difference between the pdf of the nonlinear set of equations and the pdf of the linearized one, by use of the respective FPK equation. This method is extended to high-dimensional systems with difficulty, because it requires high-dimensional integrations. Not far from that idea, Grigoriu [Grig 00] uses the weighted absolute difference between the two corresponding characteristic functions in order to perform the statistical linearization of a system driven by an α -Lévy white noise. Elishakoff [Elis 92] proposes to base the equivalent stochastic linearization on the potential energy of the system, while Elishakoff and Cai (in [Lin 04b]) introduce the concept of partial linearization in which the nonlinear restoring forces are kept and only the nonlinear damping forces are linearized. However, the criterion (2.4.3) remains the most popular, especially for large size systems.

In the statistical linearization method, assumptions on the statistical distribution of the response are formulated. The pioneers supposed that the equivalent distribution is Gaussian, so the responses of the equivalent system are Gaussian processes. The so-called **Gaussian equivalent linearization** expresses the properties of the equivalent linear system in terms of the mean vector and the covariance matrix of the system response. Nonetheless, statistical linearization methods have been extended to non-Gaussian processes, i.e. the expectation operations in Equations (2.5.4) are computed according to a non-Gaussian pdf. Indeed, the concept of true linearization emphasizes that the accuracy of an equivalent linearization depends on the difference between the assumed and the true probability distributions. Equivalent linearization has also been performed in the case that the excitation is a Poisson process [Grig 95, Prop 03a], but the reflexion is limited in this work to diffusion processes.

About **non-Gaussian linearization**, Chang [Chan 92] proposes a method based on a non-Gaussian pdf constructed as a weighted sum of undetermined Gaussian pdf. Just before, Pradlwarter [Prad 91] suggests a numerical method based on nonlinear transformations determined by use of the Fokker-Planck equation in order to estimate the non-Gaussian properties of the response. Those methods necessarily lead to the resolution of a possibly large set of nonlinear algebraic equations. More recently, Crandall [Cran 04] explores the use of non-Gaussian distributions by fixing a *shape* for the distribution and by adapting a *spread* parameter. Especially, he motivates the use of non-Gaussian distributions by the improvement of computing tools. However, his technique is applied with difficulty to large-scale structures subject to realistic loadings, even with the computational progress. His study is limited to SDOF systems, while the parametrization of high-dimensional pdf for coupled systems is an issue. Ricciardi [Ricc 07] also puts his touch about this topic. He proposes a two-step method: first a Gaussian equivalent linearization is performed, then this distribution is used in an A-type Graham Charlier expansion in order to refine the target pdf. This method is based on a Lyapunov equation resulting from the Itô lemma and so limited to white noise or filtered white noise excitations. With a view to generalization, a concern about this method could be that the A-type Graham Charlier series may lead to negative pdf. These works, quite recent actually, highlights the open view about this question, even though they are focused on stationary regime.

The non-Gaussian linearization has some drawbacks compared with Gaussian linearization. A classical result from probabilistic theory states that linear deterministic systems driven by Gaussian processes respond with Gaussian processes. The joint-pdf of the response is thus completely characterized by a mean vector and a covariance matrix. However, for nonlinear systems or in the case of non-Gaussian excitations, the determination of the system response is more complicated, partly due to the statistical polymorphism of a non-Gaussian process. Furthermore, the Gaussian equivalent linearization allows the use of the important properties of linearity between Gaussian variables. Analytical expressions for the equivalent coefficients of the linearized system can be readily found by use of the Kazakov formula [Kaza 65, Robe 99], i.e.

$$E[f(\mathbf{X})\mathbf{X}] = E[\mathbf{X}\mathbf{X}^T]E[\nabla_{\mathbf{X}}f(\mathbf{X})] \quad (2.5.5)$$

with $f(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ and $\mathbf{X}(t)$ an n -dimensional Gaussian random vector. By using this formula, some heavy integrals are avoided for usual kinds of nonlinear forces. The spectral analysis largely exposed in Section 2.2 can also be performed, allowing to deal efficiently with coherent random excitation fields [Robe 99].

In the opinion of the author, the goal of an approximate method is not to solve exactly an underlying sde, but to compute very efficiently and quite accurately some required statistical information and extract tendencies. Indeed, equivalent linearization are usually performed at a design stage, when the influence of many parameters is investigated. If an approximate method cannot be easily extended to high-dimensional problems or if the computational burden is prohibitive compared with Monte Carlo simulations, the approximate method loses its own purpose. Furthermore, the resolution of a large set of nonlinear algebraic equations is a new source of “uncertainties”, as amply explained in [Pres 07]. The results must be necessarily checked by stochastic simulations in order to assess their relevance. **Therefore, we feel more involved in improving strategies for Gaussian equivalent linearization than in developing non-Gaussian linearization.**

2.6 Uncertain structures

In the Introduction, we insist on the difference between exogenous and endogenous sources of uncertainty in civil structures. Hitherto we have focused on the exogenous source, namely the external loadings. We now briefly introduce the main methods to deal with uncertain structures.

On a probability space $(\Theta, \mathfrak{F}, \mathbb{P})$, let consider a system governed by an operator $\mathcal{M}$ and modeled by

$$\mathcal{M}(\mathbf{X}(t, \theta), \mathbf{d}(\theta)) = \mathbf{F}(t, \theta) \quad (2.6.1)$$

with $\mathbf{d}(\theta)$ a vector of random *data* and $\mathbf{F}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^n$ the vector of random loading. The arising question is thus to determine the probability distribution of $\mathbf{X}(t, \theta)$ knowing all the sources of uncertainty.

Among the previously presented methods, Monte Carlo simulation is the most readily adapted to solve (2.6.1). Indeed, samples of $\mathbf{d}(\theta)$ are generated, then Equation (2.6.1) is solved for each sample. Responses of the system are thus random processes conditioned on the random data space. We understand that the resolution of (2.6.1) for a sample of $\mathbf{d}$ can be performed by Monte Carlo simulation, by a spectral approach or by an approximate method.

Though Monte Carlo method is presented here as a transversal method as well, it is not necessarily the most efficient. Actually, its main limitation is its relatively low convergence rate [Le M 10]. Some *non-intrusive* techniques (as they use deterministic solvers) exist in order to speed up the convergence, as the Latine hypercube, the importance samplings or the collocation method [Le M 10]. The non-intrusive method can be efficiently combined with the spectral analysis or to approximate methods previously presented.

Contrary to those methods, the *intrusive* methods require an adaptation of the deterministic solver. For instance, the *Galerkin methods* are based on the resolution of (2.6.1) in terms of the spectral coefficients in the *polynomial chaos* representation of the solution. Even though those methods have better convergence properties, the development of adapted solvers is a drawback, especially for our purposes.

Finally, we must point out that the Fokker-Planck equation can be adapted as well to cope with random properties. As explained in [Grig 02], the random data are put into the random state vector of the governing sde by a *state augmentation procedure*. These new states do not have drift contribution, nor diffusion. For instance, this approach has been applied to the problem of random flutter instability in [Cara 10].

2.7 Summary

Figure 2.1 summarizes the main methods exposed in this Chapter. The Itô calculus or the theory of diffusion processes is the central theory of this work. We see that the first part is focused on the spectral analysis presented here for linear system and on the equivalent statistical linearization. The second part deals with the resolution of the Fokker-Planck equation. The Monte Carlo approach is the reference method in each part; the results obtained with the different methods developed in the following are consistently and systematically compared with Monte Carlo simulation to assess their accuracy and correctness.

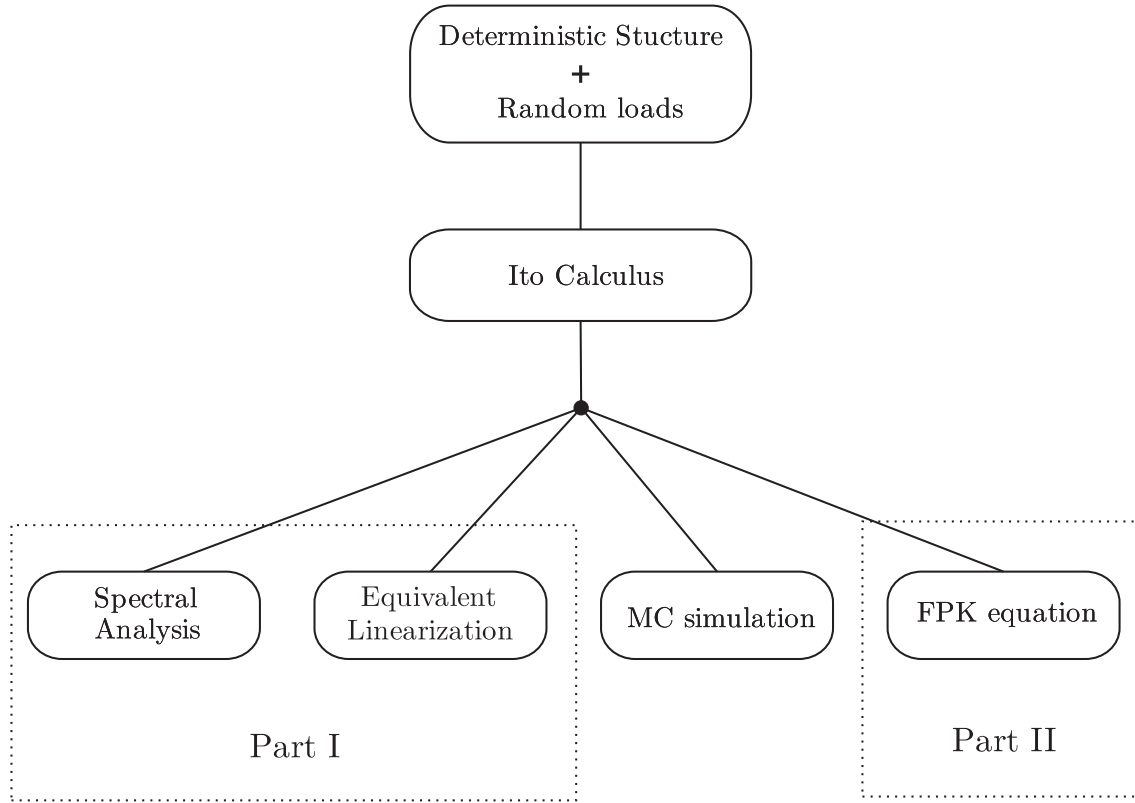


Figure 2.1: Stochastic methods to analyze deterministic structures driven by random loadings. Organization of the manuscript with respect to these methods.

Part I

Perturbation Methods in Nonlinear Evolutionary Spectral Analysis

Chapter 3

Stationary Spectral Analysis

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- 3.1 Context and propositions
 - 3.2 Spectral strategy for equivalent statistical linearization
 - 3.3 Asymptotic expansion of modal transfer matrix
 - 3.4 Validation in linear spectral analysis
 - 3.5 Stationary equivalent statistical linearization
 - 3.6 Conclusions
-

Purpose of the chapter

The concept of asymptotic expansion of a modal transfer matrix is developed, validated and applied in the context of equivalent statistical linearization in a stationary setting. Provided the structure is slightly to moderately coupled in a modal basis, the modal transfer matrix can be interpreted as the transfer matrix of a main decoupled system, corrected by higher-order correction terms.

The proposed method is formally developed in Section 3.3 in the context of linearized random systems. A convergence criterion for the expansion is also derived. The method is then applied in Section 3.4 to the analysis of a large-scale linear structure in presence of non-proportional damping. Different properties of the correction terms are emphasized in the light of this example. Finally, the expansion-based method is extended in Section 3.5 to equivalent statistical linearization by allowing the implementation of an efficient second-order solver. An application in earthquake engineering illustrates the interest of this solver compared with more versatile first-order methods.

The content of this Chapter was concisely presented in [Cano 12] and [Cano 14].

3.1 Context and propositions

The equivalent statistical linearization is an optimization problem that aims at approximating the second-order moments of the response of a structure subject to random loadings. The equivalent linearization method offers the opportunity to keep on working in the frequency domain for stationary processes, which enables to deal with complex and realistic loadings defined by high-dimensional power spectral density (psd) matrices as used to model coherent wind or seismic fields. In these cases, the equivalent linearization clearly outcores the exact approaches to the solution of nonlinear problems, especially in approximating the second-order moments at a reduced computational cost. Indeed, the use of state augmentation procedures in Itô formalism [Grig 02] to adequately color the excitation increases the computational burden. On the other hand, the synthetic generation of correlated samples is a heavy computational operation in the Monte Carlo approach.

Even though the system has been linearized (by a Gaussian linearization), the set of equations to calculate the response covariance matrix is nonlinear. The computational effort in this method results from the solution of a possibly large set of nonlinear algebraic equations. The usual technique to solve this set of equations is a *fixed-point algorithm* [Quar 07], as suggested by Roberts [Robe 99]. This method is simply implemented in any software. However, for moderate nonlinearities, the rate of convergence of this first-order accuracy method is slow. In some other cases, the fixed-point algorithm does not converge. Furthermore, for large-dimensional structures in a nodal basis, the computational effort may be prohibitive.

In this chapter, we propose to develop and apply a *perturbation approach* [Nayf 73, Hinc 91] to facilitate and improve the Gaussian equivalent linearization of nonlinear structures subject to stationary loadings. As smartly exposed by Hinch in the preface of his book [Hinc 91], perturbation methods can be seen as analytical methods used to find approximate solutions of problems in which some *small parameters* may be identified. The question of the approximation in solving mathematical problem is not a fundamental issue provided the error between the approximation and the true solution can be controlled and measured. From that point of view, analytical developments must not be naively opposed to numerical methods, but they must be understood as complementary in the construction of efficient solvers. Humbly, this is the philosophy of this part of the work. As already emphasized in the Introduction, this step is one of the construction blocks that will lead us to an efficient method for nonlinear transient analysis.

The idea to apply the perturbation theory in spectral analysis originates from the fact that, in a modal basis, most of the linear and linearized systems are slightly coupled. Thus, the generalized (modal) matrices of the system have the property of *diagonal dominance* for every frequency, i.e. the out-of-diagonal elements are small compared to the diagonal ones. The identification of small parameters is the first step in the application of perturbation theory. The second one consists in solving the problem by discarding those parameters. Finally, an asymptotic expansion in terms of the small parameters is introduced in the problem to calculate the correction terms and quantify the error. For this reason, the proposed method is referred to as the *asymptotic expansion-based method* throughout this first Part.

In the frequency domain, be it deterministic or stochastic, the analysis consists in a sequence of computations, i.e. matrix inversions, performed for a sequence of frequencies. As it requires to be repeated a large amount of times, any saving would naturally reduce the total computational burden. To avoid full matrix inversion to obtain the transfer matrix, Denoël

and Degée [Deno 09b] propose, in a deterministic framework, an *asymptotic expansion of the modal transfer matrix* assuming the relative smallness of out-of-diagonal elements with regard to diagonal ones. Even though the inversion of full matrices for a set of frequencies ceases to be a complex operation, the proposed method bypasses this operation and gives a theoretical explanation to some *too*-convenient hypotheses, like Lord Rayleigh’s decoupling assumption [Rayl 45].

In this Chapter, the proposed developments demonstrate that for slightly to moderately coupled nonlinear systems in a modal (or generalized) basis, the equivalent linearization can be seen as a convergent series of correction terms initiated with the stochastic response of a decoupled linear system. This work shows that the concept of asymptotic expansion of modal transfer matrix might be efficiently used to speed up the Gaussian equivalent linearization technique and improve the conditioning of the equation set, especially for large structures. Actually, the asymptotic approach offers the use of a second-order method, as the Newton-algorithm without increasing the computational burden compared to the traditional first-order fixed point algorithm [Quar 07]. The convergence is considerably improved and the computational effort is thus attractively reduced, while the method also offers much insight on how to physically interpret the moderate nonlinear coupling. It also opens perspectives for the use of efficient analytical methods for the integration of correction terms in a stationary setting, but it also makes possible new investigations in evolutionary spectral analysis. This well-known but rarely applied method, because of its numerical and computational cost, might become really attractive for practitioners thanks to the developed theory. This topic will be amply developed in Chapters 4 and 5.

Because simulation techniques or alternative exact approaches would equally perform in small-size structures with simple loadings, a specific attention in the developments is dedicated to large-scale structures subject to coherent and realistic excitation fields such as those encountered in wind engineering problems. The proposed method is capable of dealing with nonlinear conservative as well as dissipative forces, either affecting some degrees of freedom only, or more regularly distributed in the whole structure.

The validity of the proposed approach must not be only compared with the Monte Carlo simulations, but more pertinently with the formal statistical linearization. Indeed, the asymptotic expansion-based method gathers an error coming from the optimization problem and from the truncation of the series. These aspects are both investigated in this Part, but with regard to those facts.

In order to ground the proposed asymptotic expansion-based method, the first Section of this chapter is devoted to the statement of the stochastic linearization of the equation of motion. For possibly large-dimensional discrete systems, the use of a modal basis to reduce the number of unknowns is a delicate question to deal with, especially in presence of nonlinearities. A strategy is proposed to improve the computational method while keeping in mind the tricky paradigm in dealing with *equivalent* normal modes. In these different sections, the Gaussian statistical linearization is effectively grounded, before jumping into the heart of the method applied throughout this first Part.

3.2 Spectral strategy for equivalent statistical linearization

On a probability space $(\Theta, \mathfrak{F}, \mathbb{P})$, the equation of motion of a n -DOF nonlinear system is

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} + \mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}) = \mathbf{f}, \quad (3.2.1)$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the n -dimensional mass, damping and stiffness matrices of the system, respectively, $\mathbf{f}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^n$ is the vector of random exogenous Gaussian forces and the dot denotes the time derivative. The vector $\mathbf{y}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^n$ gathers the nodal displacements expected to be non-Gaussian processes due to the nonlinear forces in the vector function $\mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}) : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$. With this formalism, the equation of motion is split into four contributions : inertial forces, internal linear forces, internal nonlinear forces and exogenous random forces. Discarding the nonlinear forces $\mathbf{g}(\mathbf{y}, \dot{\mathbf{y}})$ in (3.2.1) leads to a linear governing equation, which is referred to as the *linear subsystem* in the sequel.

In the considered problems, the size n of the system is possibly large, and the exogenous forces are characterized by a psd matrix $\mathbf{S}_f(\omega)$ with possibly complex expressions as typically encountered in realistic wind turbulence models [Simi 96] or spatial coherence in seismic engineering [Kiur 92]. For the sake of clarity in the following analytical developments, only antisymmetric nonlinear forces and zero-mean excitation processes are considered. Otherwise, some minor modifications to the method must be operated to take into account the mean response and the non-centered statistical moments [Robe 99].

The stochastic linearization aims at replacing (3.2.1) by the equation of motion of a n -DOF equivalent linear system. The equivalent equation of motion reads

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{C} + \mathbf{C}_{eq})\dot{\mathbf{x}} + (\mathbf{K} + \mathbf{K}_{eq})\mathbf{x} = \mathbf{f}, \quad (3.2.2)$$

where $\mathbf{x}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^n$ and $\dot{\mathbf{x}}$ are the Gaussian nodal displacements and velocities of the equivalent linear system, respectively, and where $\mathbf{K}_{eq}$ and $\mathbf{C}_{eq}$ are the equivalent stiffness and damping matrices, respectively [Robe 99]. The probabilistic response of the system is thus completely characterized by the symmetric covariance matrices $\Sigma_{\mathbf{x}}$ and $\Sigma_{\dot{\mathbf{x}}}$, obtained by integration of the corresponding psd matrices, as in (2.2.8) by imposing $\tau = 0$,

$$\Sigma_{\mathbf{x}} = \int_{\mathbb{R}} \mathbf{S}_{\mathbf{x}} d\omega \quad \text{and} \quad \Sigma_{\dot{\mathbf{x}}} = \int_{\mathbb{R}} \mathbf{S}_{\dot{\mathbf{x}}} d\omega. \quad (3.2.3)$$

They are themselves obtained by left- and right-multiplication of $\mathbf{S}_f(\omega)$ by the nodal frequency response function of the system [Clou 93].

The equivalent stiffness and damping matrices in (3.2.2) are determined by minimizing the error function $\mathcal{E}$ [Grig 02], defined as

$$\mathcal{E} = E \left[(\mathbf{K}_{eq}\mathbf{x} + \mathbf{C}_{eq}\dot{\mathbf{x}} - \mathbf{g}(\mathbf{x}, \dot{\mathbf{x}})) (\mathbf{K}_{eq}\mathbf{x} + \mathbf{C}_{eq}\dot{\mathbf{x}} - \mathbf{g}(\mathbf{x}, \dot{\mathbf{x}}))^T \right] \quad (3.2.4)$$

with $E[\cdot]$ the expectation operator. Since the covariance between displacements and velocities is equal to zero in a stationary setting, i.e. $E[\mathbf{x}\dot{\mathbf{x}}^T] = \mathbf{0}$, the equivalent matrices $\mathbf{K}_{eq}$ and $\mathbf{C}_{eq}$ are respectively given by

$$\mathbf{K}_{eq} = \Sigma_{\mathbf{x}}^{-1} E[\mathbf{x}\mathbf{g}(\mathbf{x}, \dot{\mathbf{x}})^T] \quad \text{and} \quad \mathbf{C}_{eq} = \Sigma_{\dot{\mathbf{x}}}^{-1} E[\dot{\mathbf{x}}\mathbf{g}(\mathbf{x}, \dot{\mathbf{x}})^T]. \quad (3.2.5)$$

For usual nonlinear mappings $\mathbf{g}$, the expectations in the above equations may be expressed as a function of $\Sigma_{\mathbf{x}}$ and $\Sigma_{\dot{\mathbf{x}}}$, in which case (3.2.5) also reads

$$\mathbf{K}_{eq} = \mathcal{K}(\Sigma_{\mathbf{x}}, \Sigma_{\dot{\mathbf{x}}}) \quad \text{and} \quad \mathbf{C}_{eq} = \mathcal{C}(\Sigma_{\mathbf{x}}, \Sigma_{\dot{\mathbf{x}}}) \quad (3.2.6)$$

with $\mathcal{K}$ and $\mathcal{C}$ two operators depending on the kind of nonlinearity in $\mathbf{g}$. For instance, the statistical linearization of a nonlinear cubic restoring force κx^3 gives the equivalent restoring force $\kappa_{eq}x$ with the equivalent stiffness

$$\kappa_{eq} = E \left[\frac{d}{dx} (\kappa x^3) \right] = 3\kappa\sigma_x^2, \quad (3.2.7)$$

by use of the Kazakov formula (2.5.5). Other examples can be found in [Robe 99].

Equations (3.2.3) and (3.2.6) constitute a set of nonlinear algebraic equations that need to be solved for $\Sigma_{\mathbf{x}}$ and $\Sigma_{\dot{\mathbf{x}}}$. A classic iterative approach to solve it ([Robe 99], p. 140) is the *fixed-point procedure* which is briefly detailed here.

First, the unknown matrices $\mathbf{K}_{eq}$ and $\mathbf{C}_{eq}$ are initiated to zero, in which case the linear subsystem is analyzed (provided non-essential nonlinearities are considered). Then, knowing the $(k-1)$ -th step of the iterative procedure, the k -th approximations of $\Sigma_{\mathbf{x}}^{(k)}$ and $\Sigma_{\dot{\mathbf{x}}}^{(k)}$ are computed as

$$\Sigma_{\mathbf{x}}^{(k)} = \int_{\mathbb{R}} \mathbf{S}_{\mathbf{x}}(\omega; \mathbf{K}_{eq}^{(k-1)}, \mathbf{C}_{eq}^{(k-1)}) d\omega \quad \text{and} \quad \Sigma_{\dot{\mathbf{x}}}^{(k)} = \int_{\mathbb{R}} \mathbf{S}_{\dot{\mathbf{x}}}(\omega; \mathbf{K}_{eq}^{(k-1)}, \mathbf{C}_{eq}^{(k-1)}) d\omega, \quad (3.2.8)$$

with the $(k-1)$ -th approximations of the equivalent matrices calculated with (3.2.6), i.e.

$$\mathbf{K}_{eq}^{(k-1)} = \mathcal{K}(\Sigma_{\mathbf{x}}^{(k-1)}, \Sigma_{\dot{\mathbf{x}}}^{(k-1)}) \quad \text{and} \quad \mathbf{C}_{eq}^{(k-1)} = \mathcal{C}(\Sigma_{\mathbf{x}}^{(k-1)}, \Sigma_{\dot{\mathbf{x}}}^{(k-1)}). \quad (3.2.9)$$

This iterative scheme is repeated until a given tolerance is achieved. In large scale problems, this traditional algorithm is particularly involved, but it requires the costly construction and integration of a full matrix at each step. Furthermore, this algorithm behaves poorly in terms of convergence [Quar 07], especially in large-dimensional spaces. In this work, we provide a formulation that circumvents these two limitations.

3.3 Asymptotic expansion of modal transfer matrix

3.3.1 About the choice of a modal basis

The structural response of a given linear dynamical system can be computed using a limited number m of normal modes of vibration, with $m \ll n$ for large structures. The costly construction (and then integration) of $\mathbf{S}_{\mathbf{x}}$ and $\mathbf{S}_{\dot{\mathbf{x}}}$ is thus replaced by

$$\mathbf{S}_{\mathbf{x}} = \Phi \mathbf{S}_{\mathbf{q}} \Phi^T \quad \text{and} \quad \mathbf{S}_{\dot{\mathbf{x}}} = \omega^2 \Phi \mathbf{S}_{\mathbf{q}} \Phi^T, \quad (3.3.1)$$

where Φ is the $n \times m$ matrix collecting the normal modes and $\mathbf{S}_{\mathbf{q}}(\omega)$ represents the psd matrix of modal coordinates. In principle, the normal modes should result from the eigenproblem

$(\mathbf{K} + \mathbf{K}_{eq}) \Phi = \omega^2 \mathbf{M} \Phi$, in which case, they have to be updated at each iteration, as $\mathbf{K}_{eq}$ is initially unknown. The possible saving on the construction and multiplication of large matrices is thus shifted to repeated eigenvalue decompositions. The use of the modal basis related to matrices $\mathbf{K} + \mathbf{K}_{eq}$ and $\mathbf{M}$ is a natural reaction from linear dynamics aiming at uncoupling the resulting set of equations, on top of reducing the size of the problem. Because it is computationally expensive, this option is abandoned.

Instead, inspired by a similar approach within environments with moderate and non-essential nonlinear forces [Prad 03], it is suggested to work in the constant generalized basis associated with the eigenproblem

$$(\mathbf{K} + \tilde{\mathbf{K}}) \Phi = \omega^2 \mathbf{M} \Phi, \quad (3.3.2)$$

where $\tilde{\mathbf{K}}$ is an estimation of $\mathbf{K}_{eq}$ that accounts for the distribution of the nonlinear stiffness forces in the structure, in an equivalent manner, even if their intensities are *a priori* unknown. The matrix $\tilde{\mathbf{K}}$ can be updated, but not necessarily at each step of the linearization. The construction of this matrix is discussed later in the light of a relevant example. This new basis is normalized to have unit generalized masses, such that

$$\Phi^T \mathbf{M} \Phi = \mathbf{I}, \quad (3.3.3)$$

$$\Phi^T \mathbf{K} \Phi = \Omega \quad (3.3.4)$$

with $\mathbf{I}$ the $m \times m$ -identity matrix, and Ω a non-diagonal generalized stiffness matrix.

We consciously decide to call the basis Φ a *modal basis*, although we know that the adjective *modal* may be considered as improper. Actually, this basis is more judiciously interpreted as a *generalized basis* and Φ a set of *generalized shapes*. In the particular case of $\tilde{\mathbf{K}} = \mathbf{0}$, we recover the basis of the *linear normal modes*, in which both the mass and the stiffness matrices are diagonal. In order to not create confusion in the manuscript, since a matrix $\tilde{\mathbf{K}}$ is not necessarily considered in all the applications and developments, we decide to call Φ a modal basis, acquainted with this formal distinction.

Using the modal superposition principle [Clou 93], Equation (3.2.2) reads

$$\ddot{\mathbf{q}} + (\mathbf{D} + \mathbf{D}_{eq}) \dot{\mathbf{q}} + (\Omega + \Omega_{eq}) \mathbf{q} = \mathbf{p}, \quad (3.3.5)$$

where $\mathbf{q}(t)$ is the vector of modal coordinates ($\mathbf{x} = \Phi \mathbf{q}$) and $\mathbf{D} = \Phi^T \mathbf{C} \Phi$, $\Omega_{eq} = \Phi^T \mathbf{K}_{eq} \Phi$, $\mathbf{D}_{eq} = \Phi^T \mathbf{C}_{eq} \Phi$ and $\mathbf{p}(t) = \Phi^T \mathbf{f}(t)$. Even though the number of variables has been reduced by the modal projection, the equation of motion is not decoupled because the matrices in (3.3.5) are not necessarily diagonal, even Ω , quite the opposite actually. Therefore, the uncoupling advantage offered by the modal projection is lost. This issue may however be treated with existing solutions [Cano 12], as developed next.

The psd matrix of modal coordinates $\mathbf{S}_q(\omega)$ introduced in (3.3.1) is given by

$$\mathbf{S}_q = \mathbf{H} \mathbf{S}_p \mathbf{H}^*, \quad (3.3.6)$$

where the superscript $*$ denotes the hermitian operator. The modal transfer matrix of the equivalent linearized system $\mathbf{H}(\omega) : \mathbb{R} \mapsto \mathbb{C}^{m \times m}$ is defined as

$$\mathbf{H} = (-\omega^2 \mathbf{I} + \iota \omega (\mathbf{D} + \mathbf{D}_{eq}) + \mathbf{\Omega} + \mathbf{\Omega}_{eq})^{-1} \quad (3.3.7)$$

with $\iota = \sqrt{-1}$, and where the psd matrix of modal loadings is obtained as

$$\mathbf{S}_p = \mathbf{\Phi}^T \mathbf{S}_f \mathbf{\Phi} \quad (3.3.8)$$

with $\mathbf{S}_f(\omega)$ the (hermitian) cross-psd matrix of the nodal forces. The projection of the high-dimensional random excitation field $\mathbf{f}(t)$ in the modal basis constitutes the major computational cost (i.e. $\mathcal{O}(n^2)$), for the large structures encountered in civil engineering. The projection of this field in a fixed and energy-independent basis is another key point in our formulation, that makes it rational for large structures. The modal loading $\mathbf{S}_p$ is indeed established once and for all or intermittently updated, instead of being updated, together with the modal basis, at each iteration.

With this modal approach, the set of nonlinear equations (3.2.6) is not supplemented with the heavy estimation of the covariance matrices $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_{\dot{x}}$ as given in (3.2.3), but rather

$$\mathbf{\Sigma}_x = \mathbf{\Phi} \mathbf{\Sigma}_q \mathbf{\Phi}^T \quad \text{and} \quad \mathbf{\Sigma}_{\dot{x}} = \mathbf{\Phi} \mathbf{\Sigma}_{\dot{q}} \mathbf{\Phi}^T, \quad (3.3.9)$$

where the covariance matrices of the modal coordinates as well as their time derivatives are

$$\mathbf{\Sigma}_q = \int_{\mathbb{R}} \mathbf{S}_q d\omega \quad \text{and} \quad \mathbf{\Sigma}_{\dot{q}} = \int_{\mathbb{R}} \omega^2 \mathbf{S}_q d\omega. \quad (3.3.10)$$

The covariance matrices $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_{\dot{x}}$ are still functions of the *a priori* unknown equivalent matrices $\mathbf{K}_{eq}$ and $\mathbf{C}_{eq}$ through $\mathbf{D}_{eq}$ and $\mathbf{\Omega}_{eq}$ in (3.3.7) and the same fixed-point procedure may be used to solve the equation set (3.3.6)-(3.3.9) as well. Nevertheless, as the convergence of the fixed-point algorithm is sensitive to the Jacobian of operators $\mathcal{K}$ and $\mathcal{C}$, with known evidences of slow convergence [Quar 07], the use of a gradient-based method is promoted. At the expense of slightly heavier computational costs, this procedure offers a faster convergence and an increase of the convergence domain. This is formalized by transforming (3.3.10) into the canonical form

$$\mathbf{\Sigma}_q = \mathcal{F}_1(\mathbf{\Sigma}_q, \mathbf{\Sigma}_{\dot{q}}) \quad \text{and} \quad \mathbf{\Sigma}_{\dot{q}} = \mathcal{F}_2(\mathbf{\Sigma}_q, \mathbf{\Sigma}_{\dot{q}}). \quad (3.3.11)$$

With this formulation, the unknowns of the problem are the modal covariance matrices. The operators $\mathcal{F}_1$ and $\mathcal{F}_2$ correspond to the integral operation in (3.3.10), while $\mathbf{H}(\omega; \mathbf{\Omega}_{eq}, \mathbf{D}_{eq})$ is an implicit function of the unknown modal covariance matrices. One objective of this Chapter is to provide an efficient formulation for the solution of (3.3.11).

3.3.2 Modal coupling in dynamical systems

In the previous Section, it is emphasized that the modal basis $\mathbf{\Phi}$ does not necessarily lead to uncoupled equations after the modal projection. Now, some explanations are provided about the *linear modal coupling* and its meaning in structural analysis. The coupling occurring in purely nonlinear dynamics is not considered here [Kaha 96].

The external force vector $\mathbf{f}(t)$ is defined as a random vector and the probabilistic distributions of the forces on a structure (for both wind and earthquake) are not independent. The process $\mathbf{f}(t)$ is more judiciously defined as a random field (see Section 2.1.2), since it depends on both time and space. In spectral analysis, such a field is said to be *coherent*, i.e. the term

$(\mathbf{S}_f)_{ij}$ expresses the coherence between the loading $\mathbf{f}$ at the points i and j . The coherence is a spectral view on the *correlation* between the forces. From (3.3.3), the projection in this basis of the structural mass matrix is diagonal. If the matrix $\mathbf{S}_f$ does not have a structure comparable to the one of $\mathbf{M}$, the matrix $\mathbf{S}_p(\omega)$ is *a priori* not diagonal.

After the projection of the nodal forces, the spectral analysis is performed according to (3.3.6), rewritten here in an indicial form¹,

$$(\mathbf{S}_q)_{ij} = \sum_{k,l=1}^m (\mathbf{H})_{ki} (\mathbf{S}_p)_{kl} (\mathbf{H})_{lj}. \quad (3.3.12)$$

If the system is decoupled in the modal basis, the response in a mode can be seen as mechanically independent from the responses in the other modes. Thus, the modal transfer matrix $\mathbf{H}(\omega)$ is diagonal. In the opposite case, the modal coupling between two modes expresses that these two modes do not mechanically respond in an independent manner. We call this source of coupling, the *mechanical modal coupling*.

The correlation between the generalized forces does not induce necessarily coupling in the response, which is our first concern. Equation (3.3.8) is only a geometric operation, while the spectral analysis is a frequency dependent operation that can be interpreted as a filtration operation of the loading by a “large-scale” linear filter. In (3.3.12), the matrix $\mathbf{H}(\omega)$ amplifies the components of $\mathbf{S}_p(\omega)$ in the quasi-static domain (frequency range around zero). The non-diagonality of matrix $\mathbf{S}_p(\omega)$ is the source of the so-called *quasi-static modal coupling* [Deno 09a]. This name highlights the fact that this coupling only results from the modal projection, i.e. from the mode geometry without dynamical amplification.

A last source of modal coupling, so-called *dynamical modal coupling*, results from the possible clustering of natural frequencies. Indeed, if the modes i and j have close natural frequencies, the peaks of the functions $(\mathbf{H}(\omega))_{ii}$ and $(\mathbf{H}(\omega))_{jj}$ simultaneously amplify the component $(\mathbf{S}_f(\omega))_{ij}$ in the range of the i -th and j -th natural frequencies. The quasi-static and dynamical couplings are thus both dependent on the intensity of the coherence inside the random field $\mathbf{f}$, but in two different ways.

3.3.3 Derivation of the asymptotic formulation

The governing equations in the modal basis (3.3.5) are slightly to moderately coupled from a *mechanical* point of view, depending on the intensity of nonlinear forces in the global balance of forces, as a result of the projection into a fixed basis. In this work, we want to demonstrate the advantage of considering this kind of coupled set of modal equations as a perturbation of the uncoupled problem [Deno 09b, Cano 12]. In order to decouple the modal equation of motion of the equivalent linearized structure, the modal damping and stiffness matrices are respectively split into two contributions, as illustrated in Figure 3.1, such that

$$\mathbf{D} + \mathbf{D}_{eq} = \mathbf{D}_d + \mathbf{D}_o, \quad \text{and} \quad \mathbf{\Omega} + \mathbf{\Omega}_{eq} = \mathbf{\Omega}_d + \mathbf{\Omega}_o, \quad (3.3.13)$$

where $\mathbf{D}_d$ and $\mathbf{D}_o$ are built as $\mathbf{D}_d = (\mathbf{D} + \mathbf{D}_{eq}) \circ \mathbf{I}$ (diagonal elements) and $\mathbf{D}_o = \mathbf{D} + \mathbf{D}_{eq} - \mathbf{D}_d$ (out-of-diagonal elements) and where $\mathbf{\Omega}_d$ and $\mathbf{\Omega}_o$ are built as $\mathbf{\Omega}_d = (\mathbf{\Omega} + \mathbf{\Omega}_{eq}) \circ \mathbf{I}$ (diagonal

¹The expression $(\mathbf{H})_{ij}$ denotes an element of the matrix $\mathbf{H}$, while the symbol $\mathbf{H}_k$ designates a matrix.

elements) and $\mathbf{\Omega}_o = \mathbf{\Omega} + \mathbf{\Omega}_{eq} - \mathbf{\Omega}_d$ (out-of-diagonal elements), with $\circ$ the Hadamard product. With these notations, the modal transfer matrix defined in (3.3.7) becomes

$$\mathbf{H} = (\mathbf{J}_d + \mathbf{J}_o)^{-1}, \quad (3.3.14)$$

where

$$\mathbf{J}_d(\omega) = \mathbf{\Omega}_d - \omega^2 \mathbf{I} + i\omega \mathbf{D}_d, \quad \text{and} \quad \mathbf{J}_o(\omega) = \mathbf{\Omega}_o + i\omega \mathbf{D}_o \quad (3.3.15)$$

gather diagonal and out-of-diagonal elements, respectively. The matrices $\mathbf{J}_d(\omega)$ and $\mathbf{J}_o(\omega)$ gather elements coming from both the linear subsystem and the equivalent terms. The subscript d is there to recall that these matrices are *diagonal by construction*.

$$\begin{array}{ccc} \mathbf{D} + \mathbf{D}_{eq} & \mathbf{D}_d & \mathbf{D}_o \\ \left(\begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{array} \right) & = & \left(\begin{array}{ccc} \square & & \\ & \square & \\ & & \square \end{array} \right) + \left(\begin{array}{ccc} & \square & \square \\ \square & & \square \\ \square & \square & \end{array} \right) \\ & \text{Diagonal elements} & \text{Off-diagonal elements} \end{array}$$

Figure 3.1: Concept of diagonal dominance illustrated for the damping matrices. Computation of the related diagonal and out-of-diagonal matrices.

The diagonal modal transfer matrix of the virtually decoupled system is also introduced as $\mathbf{H}_d(\omega) = \mathbf{J}_d^{-1}(\omega)$. It corresponds to the modal transfer matrix that would be obtained if the coupling resulting from the projection in a perturbed modal space was neglected. In more concrete terms, we write

$$(\mathbf{H}_d(\omega))_i = \frac{1}{(\mathbf{\Omega}_d)_i - \omega^2 + i\omega (\mathbf{D}_d)_i}, \quad (3.3.16)$$

where the single subscript i (associated with the subscript d) designates the i -th term on the diagonal.

Substitution of (3.3.14) into (3.3.6) yields after some algebra

$$\mathbf{S}_q = (\mathbf{I} + \mathbf{H}_d \mathbf{J}_o)^{-1} \mathbf{S}_{qd} (\mathbf{I} + \mathbf{J}_o^* \mathbf{H}_d^*)^{-1}. \quad (3.3.17)$$

The *virtual* response of the decoupled system $\mathbf{S}_{qd}$, as would be obtained by only retaining diagonal elements in the modal matrices, is

$$\mathbf{S}_{qd} = \mathbf{H}_d \mathbf{S}_p \mathbf{H}_d^*. \quad (3.3.18)$$

Upon smallness conditions on the product $\mathbf{H}_d \mathbf{J}_o$ formally studied in the following, Equation (3.3.14) may be written

$$\mathbf{H}(\omega) = \left(\mathbf{I} + \sum_{i=1}^{\infty} (-\mathbf{H}_d \mathbf{J}_o)^i \right) \mathbf{H}_d. \quad (3.3.19)$$

We denote by **asymptotic expansion-based method**, all the developments based on (3.3.19). Equation (3.3.17) now reads

$$\mathbf{S}_{\mathbf{q}} = \left(\mathbf{I} + \sum_{i=1}^{\infty} (-\mathbf{H}_d \mathbf{J}_o)^i \right) \mathbf{S}_{\mathbf{q}_d} \left(\mathbf{I} + \sum_{j=1}^{\infty} (-\mathbf{J}_o^* \mathbf{H}_d^*)^j \right), \quad (3.3.20)$$

transforming thus the inverse of full matrices into convergent series. This product of series is written as a single series which is truncated, for practical needs, at an order N such that

$$\mathbf{S}_{\mathbf{q},N} = \mathbf{S}_{\mathbf{q}_d} + \sum_{k=1}^N \Delta \mathbf{S}_{\mathbf{q},k} \quad (3.3.21)$$

with $\Delta \mathbf{S}_{\mathbf{q},k}$ the k -th correction term added to the decoupled approximation $\mathbf{S}_{\mathbf{q}_d}$. Thence, provided a measure of $\mathbf{H}_d \mathbf{J}_o$ is of order ϵ (a small parameter), the same measure of the correction term $\Delta \mathbf{S}_{\mathbf{q},k}$ is of order ϵ^k . Working out the algebra yields

$$\Delta \mathbf{S}_{\mathbf{q},1}(\omega) = -\mathbf{H}_d \mathbf{J}_o \mathbf{S}_{\mathbf{q}_d} - \mathbf{S}_{\mathbf{q}_d} \mathbf{J}_o^* \mathbf{H}_d^* \quad (3.3.22)$$

for the first correction term and a recurrence relation

$$\Delta \mathbf{S}_{\mathbf{q},k}(\omega) = -(\mathbf{H}_d \mathbf{J}_o \Delta \mathbf{S}_{\mathbf{q},k-1} + \Delta \mathbf{S}_{\mathbf{q},k-1} \mathbf{J}_o^* \mathbf{H}_d^*) - \mathbf{H}_d \mathbf{J}_o \Delta \mathbf{S}_{\mathbf{q},k-2} \mathbf{J}_o^* \mathbf{H}_d^* \quad (3.3.23)$$

for $k \geq 2$ and $\Delta \mathbf{S}_{\mathbf{q},0} = \mathbf{S}_{\mathbf{q}_d}$. The correction terms do not require the inversion of any full matrix, because they are expressed in terms of the decoupled system.

Although it is expected to be *globally* less than the first correction $\Delta \mathbf{S}_{\mathbf{q},1}$, $\Delta \mathbf{S}_{\mathbf{q},2}$ is not systematically negligible compared with $\Delta \mathbf{S}_{\mathbf{q},1}$, because the elements of $\mathbf{S}_{\mathbf{q}_d}$ concerned with the successive corrections are not necessarily identical. This is demonstrated by splitting up the psd matrix of the generalized forces into its diagonal and out-of-diagonal elements

$$\mathbf{S}_{\mathbf{p}} = \mathbf{S}_{\mathbf{p}_d} + \mathbf{S}_{\mathbf{p}_o}, \quad (3.3.24)$$

i.e. into the unilateral spectral generalized forces $\mathbf{S}_{\mathbf{p}_d}(\omega)$ and the cross-spectral generalized forces $\mathbf{S}_{\mathbf{p}_o}(\omega)$. Insertion of (3.3.24) into (3.3.18) leads to

$$\mathbf{S}_{\mathbf{q}_d} = \mathbf{H}_d \mathbf{S}_{\mathbf{p}_d} \mathbf{H}_d^* + \mathbf{H}_d \mathbf{S}_{\mathbf{p}_o} \mathbf{H}_d^* \quad (3.3.25)$$

If the coherence between generalized forces is negligible, Equation (3.3.25) reduces to $\mathbf{S}_{\mathbf{q}_d} \simeq \mathbf{H}_d \mathbf{S}_{\mathbf{p}_d} \mathbf{H}_d^*$ and is therefore diagonal dominant. Consequently, $\Delta \mathbf{S}_{\mathbf{q},1}$ becomes a zero-diagonal matrix, as seen from (3.3.22), whereas $\Delta \mathbf{S}_{\mathbf{q},2}$ remains a full matrix. Thus, when the generalized forces are almost uncorrelated, the first correction term $\Delta \mathbf{S}_{\mathbf{q},1}$ brings no significant correction to the variance of modal coordinates, while $\Delta \mathbf{S}_{\mathbf{q},2}$ does.

Substitution of (3.3.21) into (3.3.10) provides the asymptotic expansion of the covariance matrices $\Sigma_{\mathbf{q}}$ and $\Sigma_{\dot{\mathbf{q}}}$ that read

$$\Sigma_{\mathbf{q},N} = \Sigma_{\mathbf{q}_d} + \sum_{k=1}^N \Delta \Sigma_{\mathbf{q},k} \quad \text{and} \quad \Sigma_{\dot{\mathbf{q}},N} = \Sigma_{\dot{\mathbf{q}}_d} + \sum_{k=1}^N \Delta \Sigma_{\dot{\mathbf{q}},k}. \quad (3.3.26)$$

The reference solutions of (3.3.11) will be denoted by $\Sigma_{\mathbf{q},\infty}$ and $\Sigma_{\dot{\mathbf{q}},\infty}$ in the sequel. The first correction terms for the modal covariance matrices defined in (3.3.10) read

$$\Delta\Sigma_{\mathbf{q},1} = -(\mathbf{L}_{\mathbf{q},1} + \mathbf{L}_{\mathbf{q},1}^*), \quad \Delta\Sigma_{\dot{\mathbf{q}},1} = -(\mathbf{L}_{\dot{\mathbf{q}},1} + \mathbf{L}_{\dot{\mathbf{q}},1}^*) \quad (3.3.27)$$

with

$$\mathbf{L}_{\mathbf{q},1} = \int_{\mathbb{R}} \mathbf{H}_d \mathbf{J}_o \mathbf{S}_{\mathbf{q}_d} d\omega, \quad \mathbf{L}_{\dot{\mathbf{q}},1} = \int_{\mathbb{R}} \omega^2 \mathbf{H}_d \mathbf{J}_o \mathbf{S}_{\mathbf{q}_d} d\omega. \quad (3.3.28)$$

Replacing $\mathbf{J}_o$ by the expression (3.3.15), the matrices $\mathbf{L}_{\mathbf{q},1}$ and $\mathbf{L}_{\dot{\mathbf{q}},1}$ are finally expressed as

$$(\mathbf{L}_{\mathbf{q},1})_{ij} = \sum_k^m (\Omega_o)_{ik} \mathcal{I}_{ijk,0} + \sum_k^m (\mathbf{D}_o)_{ik} \mathcal{I}_{ijk,1} \quad (3.3.29)$$

$$(\mathbf{L}_{\dot{\mathbf{q}},1})_{ij} = - \left(\sum_k^m (\Omega_o)_{ik} \mathcal{I}_{ijk,2} + \sum_k^m (\mathbf{D}_o)_{ik} \mathcal{I}_{ijk,3} \right) \quad (3.3.30)$$

with

$$\mathcal{I}_{ijk,\alpha} = \int_{\mathbb{R}} (\iota\omega)^\alpha (\mathbf{H}_d)_i (\mathbf{H}_d)_k (\mathbf{S}_{\mathbf{p}})_{kj} (\mathbf{H}_d^*)_j d\omega, \quad (3.3.31)$$

$\alpha = 0, \dots, 3$ and $i, j, k = 1, \dots, m$. The noteworthy advantage of this formulation is that the integral $\mathcal{I}_{ijk,\alpha}$ only depends upon the decoupled system. The out-of-diagonal elements of $\mathbf{J}_o$ are seen as higher-order perturbation sources of the *virtual* decoupled system. The reason why the derivation ends up with such a simple formulation is that the whole concept is based on *linear algebra*. Retrospectively, the matrix inversions in (3.3.17) are replaced by linear combinations, thanks to the asymptotic expansion, and any operation in the derivation is linear with respect to its arguments. The permutation of these linear operators precisely allows the expression of the response of the coupled systems as a perturbation of the response of the uncoupled system. From an algorithmic point of view, the integrals $\mathcal{I}_{ijk,\alpha}$ are promptly established at each iteration. As explained next, they are also stored in order to set up the Jacobian matrix used in the iterative resolution of the nonlinear equation set resulting from the equivalent linearization.

Similarly, the second correction terms for the modal covariance matrices read

$$\Delta\Sigma_{\mathbf{q},2} = (\mathbf{L}_{\mathbf{q},2,I} + \mathbf{L}_{\mathbf{q},2,I}^* + \mathbf{L}_{\mathbf{q},2,II}), \quad \Delta\Sigma_{\dot{\mathbf{q}},2} = (\mathbf{L}_{\dot{\mathbf{q}},2,I} + \mathbf{L}_{\dot{\mathbf{q}},2,I}^* + \mathbf{L}_{\dot{\mathbf{q}},2,II}) \quad (3.3.32)$$

with

$$\mathbf{L}_{\mathbf{q},2,I} = \int_{\mathbb{R}} (\mathbf{H}_d \mathbf{J}_o)^2 \mathbf{S}_{\mathbf{q}_d} d\omega, \quad \mathbf{L}_{\mathbf{q},2,II} = \int_{\mathbb{R}} \mathbf{H}_d \mathbf{J}_o \mathbf{S}_{\mathbf{q}_d} \mathbf{J}_o^* \mathbf{H}_d^* d\omega \quad (3.3.33)$$

$$\mathbf{L}_{\dot{\mathbf{q}},2,I} = \int_{\mathbb{R}} \omega^2 (\mathbf{H}_d \mathbf{J}_o)^2 \mathbf{S}_{\mathbf{q}_d} d\omega, \quad \mathbf{L}_{\dot{\mathbf{q}},2,II} = \int_{\mathbb{R}} \omega^2 \mathbf{H}_d \mathbf{J}_o \mathbf{S}_{\mathbf{q}_d} \mathbf{J}_o^* \mathbf{H}_d^* d\omega. \quad (3.3.34)$$

The computation of the second correction term is recommended but not mandatory for all terms. Usually the first correction is sufficient for the estimation of out-of-diagonal terms in $\Delta\Sigma_{\mathbf{q}}$ and $\Delta\Sigma_{\dot{\mathbf{q}}}$, whilst the second correction offers a better estimation of diagonal elements (variances). This general trend should be regarded with a discussion on the coherence in the generalized loading, as shown with (3.3.25).

3.3.4 Discussion on the convergence of the asymptotic expansion

The conditions upon which the expansion derived above shall be valid are now reviewed. A necessary and sufficient condition for the series expansions in (3.3.20) to converge is

$$\Gamma [\mathbf{H}_d(\omega)\mathbf{J}_o(\omega)] < 1, \quad \forall \omega \in \mathbb{R} \quad (3.3.35)$$

with $\Gamma [\mathbf{B}] = \max \{|\lambda_{\mathbf{B},i}|, i = 1, \dots, m\}$ the **spectral radius** of $\mathbf{B} \in \mathbb{C}^{m \times m}$ and $\lambda_{\mathbf{B},i}$ the i -th eigenvalue of $\mathbf{B}$ [Horn 13]. The spectral radius of $\mathbf{H}_d\mathbf{J}_o$ also defines the **index of diagonality** ρ of $\mathbf{J}_d + \mathbf{J}_o$ [Horn 13, Morz 09],

$$\Gamma [\mathbf{H}_d(\omega)\mathbf{J}_o(\omega)] = \rho [\mathbf{J}_d(\omega) + \mathbf{J}_o(\omega)], \quad (3.3.36)$$

as it measures the relative importance of the diagonal terms in $\mathbf{J}_d$ compared with the out-of-diagonal ones in $\mathbf{J}_o$.

The general condition formulated in (3.3.35) must be checked for all the $\omega \in \mathbb{R}$. It is relevant to first investigate the limit behaviors for ω tending to zero and to infinity, such that

$$\lim_{\omega \rightarrow 0} \mathbf{H}_d\mathbf{J}_o = \lim_{\omega \rightarrow 0} (\mathbf{\Omega}_d - \omega^2\mathbf{I} + i\omega\mathbf{D}_d)^{-1} \mathbf{J}_o = \mathbf{\Omega}_d^{-1}\mathbf{\Omega}_o, \quad (3.3.37)$$

$$\lim_{|\omega| \rightarrow +\infty} \mathbf{H}_d\mathbf{J}_o = \lim_{\omega \rightarrow +\infty} -\frac{1}{\omega^2}\mathbf{J}_o = \mathbf{0}. \quad (3.3.38)$$

These conditions are related to the quasi-static and the inertial regimes of the linear system, respectively. The smallness of the product $\mathbf{H}_d\mathbf{J}_o$ is ensured for high-frequency responses, while the smallness of $\mathbf{\Omega}_d^{-1}\mathbf{\Omega}_o$ is required for the quasi-static response. In this case, the spectral radius of $\mathbf{\Omega}_d^{-1}\mathbf{\Omega}_o$ is also the diagonality index of $\mathbf{\Omega} + \mathbf{\Omega}_{eq}$. If the spectral radius of $\mathbf{\Omega}_d^{-1}\mathbf{\Omega}_o$ is less than one, i.e.

$$\rho [\mathbf{\Omega} + \mathbf{\Omega}_{eq}] = \Gamma [\mathbf{\Omega}_d^{-1}\mathbf{\Omega}_o] < 1, \quad (3.3.39)$$

the matrix $\mathbf{\Omega} + \mathbf{\Omega}_{eq}$ is said to be diagonal dominant [Horn 13]. However, the criterion (3.3.39) is only sufficient for $\omega \rightarrow 0$.

Concerning the resonant regime occurring for ω equal to a natural frequency of the decoupled system, a straight criterion is not easily derived. And yet, the *Gerschgorin theorem* [Horn 13] allows to show that (i) all the Gerschgorin discs are concentric and centered on zero regardless of ω and that (ii) the largest disc is one of the discs calculated for ω equal to a natural frequency of the decoupled system.

Indeed, a subspace $\mathcal{D}$ of $\mathbb{C}$ in which all the eigenvalues of the matrices $\mathbf{H}_d(\omega)\mathbf{J}_o(\omega)$ are contained can be circumscribed. This subspace is the union of the so-called Gerschgorin discs defined as $\mathcal{D}_i((\mathbf{H}_d\mathbf{J}_o)_{ii}, R_i)$ with the radius $R_i(\omega) = \sum_{j=1, j \neq i}^m |(\mathbf{H}_d\mathbf{J}_o)_{ij}|$ and $(\mathbf{H}_d\mathbf{J}_o)_{ii}$ the center of the circle. The formal description of the disc $\mathcal{D}_i$ is such that

$$\mathcal{D}_i = \left\{ z \in \mathbb{C}, \omega \in \mathbb{R}, |z| \leq \sum_{j \neq i=1}^m |(\mathbf{H}_d)_i(\mathbf{J}_o)_{ij}| \right\}, \quad (3.3.40)$$

because the diagonal elements of $\mathbf{H}_d \mathbf{J}_o$ are equal to zero. Therefore, all the Gerschgorin discs are concentric, such that

$$\mathcal{D}(\omega) = \bigcup_{i=1}^m \mathcal{D}_i(0, R_i(\omega)) = \mathcal{D}\left(0, \max_{i=1, \dots, m} R_i(\omega)\right), \quad (3.3.41)$$

and the k -th eigenvalue $\lambda_k \in \mathcal{D}(\omega)$ with $k = 1, \dots, m$. The inequality defining the i -th disc can be rewritten and an upper bound can be calculated as

$$\begin{aligned} |z| &\leq |(\mathbf{H}_d)_i(\omega)| \sum_{j \neq i=1}^m \sqrt{(\boldsymbol{\Omega}_o)_{ij}^2 + \omega^2 (\mathbf{D}_o)_{ij}^2} \\ &< |(\mathbf{H}_d)_i(\omega_i)| \sum_{j \neq i=1}^m \sqrt{(\boldsymbol{\Omega}_o)_{ij}^2 + \omega_i^2 (\mathbf{D}_o)_{ij}^2} \end{aligned} \quad (3.3.42)$$

with $(\boldsymbol{\Omega}_d)_i = \omega_i^2$. Therefore, the largest eigenvalue of $\mathbf{H}_d \mathbf{J}_o$ can be potentially found in one of these discs. Indeed, in the inertial regime, the disc radius tends to zero ($R_i(\omega) \sim \mathcal{O}(\frac{1}{\omega})$ as $\omega \rightarrow \infty$), while in the quasi-static regime it is equivalent to (3.3.6). The maximum of $\Gamma[\mathbf{H}_d \mathbf{J}_o]$ is thus more judiciously sought in the biggest disc $\mathcal{D}(\omega)$, which is one of the discs $\mathcal{D}(\omega_i)$ with $i = 1, \dots, m$. This explanation has the drawback to not properly justify the convergence criterion by giving explicit expressions of the eigenvalues, but those expressions are quite hard to calculate, a reason for rather trying to bound them, at least. Then, the convergence of the series might be evaluated with the condition

$$\rho_{\mathbf{J}} = \max \{\Gamma_{d,i} = \Gamma[\mathbf{H}_d(\omega_i) \mathbf{J}_o(\omega_i)] , i = 1, \dots, m\} < 1. \quad (3.3.43)$$

The **convergence criterion** $\rho_{\mathbf{J}}$ is a key indicator in this part. It must be calculated in each application to check the validity of the expansion.

3.3.5 Integration of the correction terms

The proposed formulation relies on the estimation of the definite integrals $\mathcal{I}_{ijk,\alpha}$ and, to a lesser extent $\mathbf{L}_{\mathbf{q},2}$ and $\mathbf{L}_{\dot{\mathbf{q}},2}$, as defined previously. Taking into account the symmetry of the integrand, integration can be carried out on $\mathbb{R}^+$ with available numerical integration methods (rectangle rule, trapezoidal rule) [Pres 07]. The only subtlety in the estimation of these integrals is related to the peakiness of the integrand, which requires an adaptive integration scheme [Quar 07]. Actually a non-adaptive scheme, with a proper selection of the integration points, could also be developed, since the poles of the integrands are readily identified. This is consequence of the decoupling procedure, as the integrands are the mixed products of transfer functions of single degree-of-freedom (SDOF) systems. Although this numerical procedure is the standard application of the proposed method, by default, the estimation of these integrals may be simplified (e.g. in the case that many modes are kept in the analysis), in some cases.

δ -correlated processes

For some simple analytical models of random excitation, Jordan's lemma and Cauchy's residue theorem [Krey 06] may provide practical explicit expressions. For instance, if the excitations

are uncorrelated white noises, the psd matrix $\mathbf{S}_p$ is full of frequency-independent elements. This idea is inspired by well-known integrals (e.g. in the the CQC combination in earthquake engineering [Clou 93]), such as

$$\int_{\mathbb{R}} (\mathbf{H}_d)_i (\mathbf{S}_p)_{ij} (\mathbf{H}_d^*)_j d\omega = (\mathbf{S}_p)_{ij} \frac{4\pi(\xi_i\omega_i + \xi_j\omega_j)}{(\omega_i^2 - \omega_j^2)^2 + 4\omega_i\omega_j(\xi_i\omega_i + \xi_j\omega_j)(\xi_j\omega_i + \xi_i\omega_j)} \quad (3.3.44)$$

with $(\mathbf{\Omega}_d)_i = \omega_i^2$ and $(\mathbf{D}_d)_i = 2\xi_i\omega_i$ and may be extended to the computation of the integrals $\mathcal{I}_{ijk,\alpha}$.

Extension of the background-resonant approximation

Approximate results may also be found, if the loading acts on a different timescale than the structural response. Inspired by the background-resonant approximation introduced by Davenport [Dave 61, Dave 67] for SDOF systems, the method has been extended to other classes of problems [Deno 09a]. This approximation is a common method used in wind engineering to compute the integral of the product between the psd of a background excitation and the transfer function of a linear SDOF system in order to estimate the variance of the response. The philosophy of the method is based on the evident separation between two timescales in the process : (i) the quasi-static response of the system due to the low frequency content of the excitation and (ii) the resonant response due to dynamical amplification of the system at higher frequency.

This methodology is applied here in order to improve the computation of the integral $\mathcal{I}_{ijk,\alpha}$, since the excitation is a low frequency process like turbulent wind. The unique assumption that different frequency bands contribute independently to the total definite integral, may also be applied in a more general context to the determination of the integrals $\mathcal{I}_{ijk,\alpha}$. For instance, the approximate integrals

$$\mathcal{I}_{iii,0} \approx \frac{(\mathbf{\Sigma}_p)_{ii}}{\omega_i^6} + \frac{\pi}{4\xi_i\omega_i^5} \left((\mathbf{S}_p(\omega_i))_{ii} - \frac{\omega_i}{2} (\mathbf{S}'_p(\omega_i))_{ii} \right) \quad (3.3.45)$$

$$\mathcal{I}_{ii,0} \approx \frac{(\mathbf{\Sigma}_p)_{ji}}{\omega_i^4\omega_j^2} + \frac{\pi}{2\xi_i\omega_i^3} \Re \left((\mathbf{S}_p(\omega_i))_{ji} (\mathbf{H}_d(\omega_i))_j \right) \quad (3.3.46)$$

indicate that these integrals are replaced by simple function evaluations, which indeed offers a considerable computational speedup. The prime in (3.3.45) denotes here the derivative with respect to the pulsation ω . Equations (3.3.45) and (3.3.46) are obtained by an analytical technique exposed in Appendix B.1.

The formulas (3.3.45) and (3.3.46) are compared with numerical results in Figure 3.2. All the elements of $\mathbf{S}_p(\omega)$ (perfectly coherent) are supposed to be Ornstein-Uhlenbeck processes $S_{OU}(\omega)$ characterized by a drift coefficient a_{OU} and a unit variance, i.e.

$$S_{OU} = \frac{a_{OU}}{\pi} \frac{1}{a_{OU}^2 + \omega^2}. \quad (3.3.47)$$

The accuracy of (3.3.45) depends upon the separation between the background bandwidth and the resonant frequency, as illustrated in Figure 3.2-a. The higher the frequency ω_i , the

more significant the timescale separation, the better the approximation. For the coupled term $\mathcal{I}_{ij,0}$ illustrated in Figure 3.2-b, it is shown that the thickness of the peaks (influenced by the damping ratios) and the separation between ω_i and ω_j are central in the approximation (3.3.46). This relation is accurate, especially as the peaks are narrower and the three components well separated. More details about the criterion used to quantify the timescale separation are available in [Deno 09a].

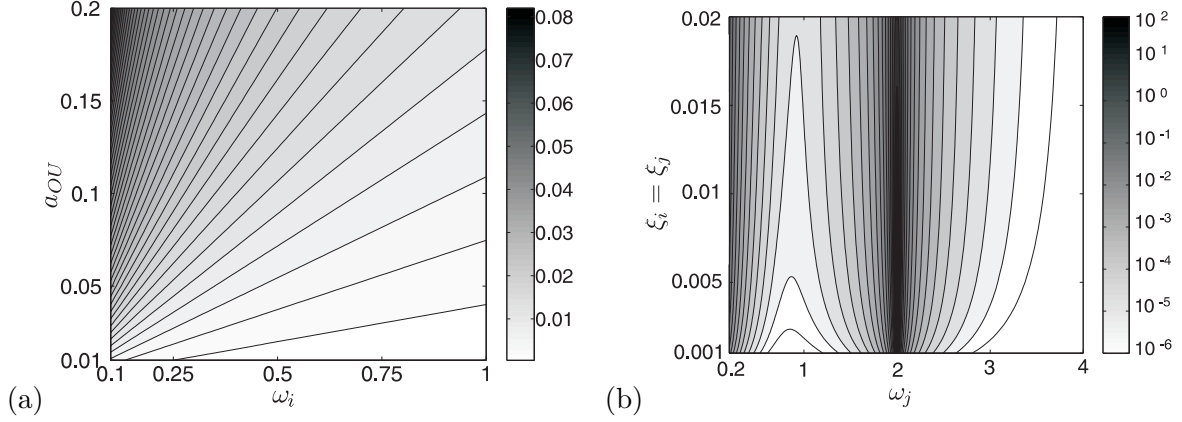


Figure 3.2: Relative errors between the numerical computation and the proposed approximation (a) for $\mathcal{I}_{iii,0}$ ($\xi_i = 0.01$) and (b) $\mathcal{I}_{ij,0}$ ($\omega_{d,i} = 2\text{rad/s}$, $a_{OU} = 0.02\text{rad/s}$), normalized with respect to $\mathcal{I}_{iii,0}$ numerically integrated.

3.4 Validation in linear spectral analysis

3.4.1 About the sources of non-proportional damping

Although the structural damping in a structure is commonly supposed to be proportional, i.e. a linear combination of the mass and stiffness matrices, there exists some cases where this assumption cannot be stated. For instance, viscous dampers (shock absorbers) or aerodynamic damping usually come along with their own damping models. Consideration of these models yields a damping matrix that does not necessarily offer proportionality. In these cases, the modal projection still enables to reduce the size of the system, as usually $m \ll n$, but cease to decouple the modal equations.

The decoupling approximation, as proposed by Lord Rayleigh [Rayl 45], consists in neglecting the out-of-diagonal elements of the modal damping matrix ($\mathbf{D}_o = \mathbf{0}$ in (3.3.13)). This approach is motivated by the smallness of the off-diagonal elements compared with the diagonal ones. Recently, some paradoxical results have been highlighted about this practiced assumption. Indeed, Morzfled *et al.* show [Morz 09] that the relative smallness between off-diagonal and diagonal elements is not sufficient to ensure small decoupling errors.

Other methods have been proposed to deal with non-proportional damping. The complex modal analysis, originally proposed by Foss [Foss 58], is an extension of the classical modal analysis to complex modes in a state space. This method allows to decouple the equations of motion in the state space of the system and therefore to perform deterministic or stochastic analyzes in time or frequency domains without assumption about damping matrix. Nevertheless, engineers and practitioners, especially in civil engineering, have never considered this approach as suitable, because the physical meaning of complex modes is interpreted with more difficulty. In some applications, the use of complex mode can make the resolution really inefficient [Prop 03b]. In an alternative approach, Ibrahimbegovic [Imbr 89] proposes a simple numerical algorithm considering off-diagonal damping forces as pseudo-forces applied to the uncoupled system. The asymptotic expansion-based method is here applied to the spectral analysis of a large-dimensional linear structure with non-proportional damping. Because the structure is linear, the modal basis is the one of the linear structure ($\tilde{\mathbf{K}} = \mathbf{0}$).

3.4.2 Wind buffeting analysis of the Millau viaduct

The wind velocity is usually supposed to be a Gaussian random variable characterized by a mean value U and its power spectral density S_u [Simi 96]. The effect of the mean loading is easily determined by a simple static (or quasi-static) analysis, while a stochastic analysis is required to evaluate the effects of turbulence. In a unidimensional turbulence model, the wind velocity and the applied forces are related by the squared relative velocity $V(t)$ between the wind $U + u(t)$ and the structure $\dot{x}(t)$,

$$f_{tot} = \frac{1}{2}\rho C_d A V^2 = \frac{1}{2}\rho C_d A (U + u - \dot{x})^2 \quad (3.4.1)$$

where ρ is the air density, C_d is the drag coefficient and A is the surface exposed to wind. Usually from (2.4.1) a simple linearized relation is adopted

$$f_{tot} \simeq C_a \left(\frac{U}{2} + u - \dot{x} \right) \quad (3.4.2)$$

where $C_a = \rho C_d A U$. The first term of (3.4.2) is the mean force (static analysis), the second term is the turbulent component of the force due to wind turbulence and the third term is the aerodynamic source of damping. Replacing (3.4.2) into the equation of motion and disregarding the mean force leads to

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}_s\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{C}_a(\mathbf{u} - \dot{\mathbf{x}}) \quad (3.4.3)$$

where $\mathbf{C}_s$ is the proportional structural damping matrix. Equation (3.4.3) is rewritten as

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{C}_s + \mathbf{C}_a)\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \quad (3.4.4)$$

where $\mathbf{f}(t) = \mathbf{C}_a\mathbf{u}(t)$ is the applied loading considering only the wind turbulence. The matrix $\mathbf{C}_a$ can be seen as a diagonal aerodynamic damping matrix, which elements are not necessarily equal, e.g. the mean velocity of wind or drag coefficients can vary along a structure. In this application, not only drag but also lift and moment coefficients contribute to the establishment of $\mathbf{C}_a$. These contributions are calculated similarly to (3.4.2), as explained in [Simi 96]. More details about the construction of buffeting wind forces are given in [Stro 10]. As previously, Equation (3.4.3) is projected in the modal basis,

$$\ddot{\mathbf{q}} + (\mathbf{D}_s + \mathbf{D}_a)\dot{\mathbf{q}} + \mathbf{\Omega}_d\mathbf{q} = \mathbf{p} \quad (3.4.5)$$

where $\mathbf{D}_s = \mathbf{\Phi}^T \mathbf{C}_s \mathbf{\Phi}$ is the structural modal damping matrix which is usually assumed to be diagonal and $\mathbf{D}_a = \mathbf{\Phi}^T \mathbf{C}_a \mathbf{\Phi}$ is the modal aerodynamic damping matrix which is no longer diagonal and generates modal coupling.

The concept of aerodynamic damping forces, summarized here for the purpose of clarity, is readily extended to the buffeting analysis of a spatial structure with drag, lift and moment aerodynamic coefficients, and in a 3-D wind-flow, i.e. with the three components of the wind turbulence [Sola 01]. Such a refined model was considered in the following illustration, which deals with a buffeting analysis of the Viaduct of Millau, see Figure 3.3. This seven-span cable-stayed bridge (about 2.5km long), crossing the Tarn valley about 350m above the bed of the river, is the highest bridge ever built in Europe. This bridge is also renowned for the launching technique used to erect the deck.

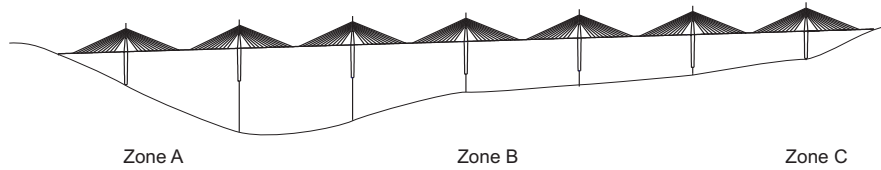


Figure 3.3: Sketch of the Viaduct of Millau.

The structure is modeled by means of a finite element software (FineLg, [Fine 03])². The model counts 1425 nodes and 2439 beam elements with 12 DOFs. The first forty modes are kept and have natural frequencies below 1Hz. The structural modal damping matrix is built to ensure a structural damping ratio of 0.3% in these forty modes. The characteristics of the wind were chosen in accordance with [Euro 91] and with on-site measurements. Three zones

²The author personally acknowledges the design office Greisch for providing the finite element model of the Viaduct.

are considered with distinct characteristics of mean velocity U and standard deviations σ_u , σ_v , σ_w (longitudinal, vertical and transversal turbulent wind) given in Table 3.1. Considering the aerodynamic damping, the diagonality index $\rho(\mathbf{D}) = \rho(\mathbf{D}_s + \mathbf{D}_a)$ is equal to 1.02, while the index of convergence $\rho_{\mathbf{J}}$ is equal to 0.58.

	U [m/s]	σ_u [m/s]	σ_v [m/s]	σ_w [m/s]
Zone A	38	6.5	6.5	4.5
Zone B	34	5.5	5.5	4.0
Zone C	36	5.5	5.5	4.0

Table 3.1: Millau Viaduct: main characteristics of the wind velocity from on-site measurements.

Exact variances of the modal coordinates and the related correlation coefficients are represented in Figure 3.4. The modal truncation is justified by the decrease of the variance of the modal coordinates as depicted in Figure 3.4-a. Concerning modes 17, 18 and 19, the vicinity of their natural frequencies (resp. 0.531Hz, 0.533Hz and 0.534Hz) and the similarities of mode shapes induce dynamic coupling and high modal correlations as shown in Figure 3.4-b. Also, coupling is noticed for modes 29, 30 and 31. Additional observations about the structural behavior of this structure are reported in [Deno 09a]. The covariance matrices $\Sigma_{\mathbf{q},\infty}$, $\Sigma_{\mathbf{q}_d}$, $\Sigma_{\mathbf{q},1}$ and $\Sigma_{\mathbf{q},2}$ are computed and the respective correlation matrices $\rho_{\mathbf{q},\infty}$, $\rho_{\mathbf{q}_d}$, $\rho_{\mathbf{q},1}$ and $\rho_{\mathbf{q},2}$ are deduced.

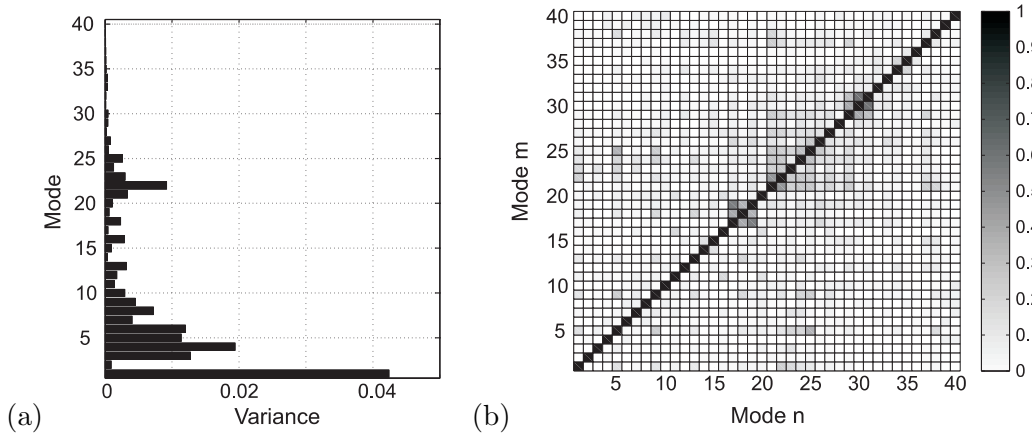


Figure 3.4: (a) Reference variances (diagonal of $\Sigma_{\mathbf{q},\infty}$) of the modal coordinates and (b) Reference correlation coefficients ($\rho_{\mathbf{q},\infty}$) of modal coordinates.

Figure 3.5 depicts the relative error on the variance of the modal coordinates obtained with the different approximations. For the three approximations, the errors are the most significant for the two groups of correlated modes (17, 18, 19 and 29, 30, 31). Figure 3.5-a indicates that neglecting modal coupling, i.e. neglecting off-diagonal terms, induces a maximum error of +45% on the variance of mode 19. The proposed method does not neglect this coupling and allows to reduce this error down to +10% (Figure 3.5-c). Comparison between Figure 3.5-a and -b shows the influence of the first correction term on the variances: the sign of the errors is changed but the order of magnitude is the same as before. With the proposed method

(Figure 3.5-c) and an extension to the second order, the errors are significantly reduced even if they remain maximum for the groups of coupled modes (+10%). Figure 3.5 also illustrates the alternate convergence towards the exact values.

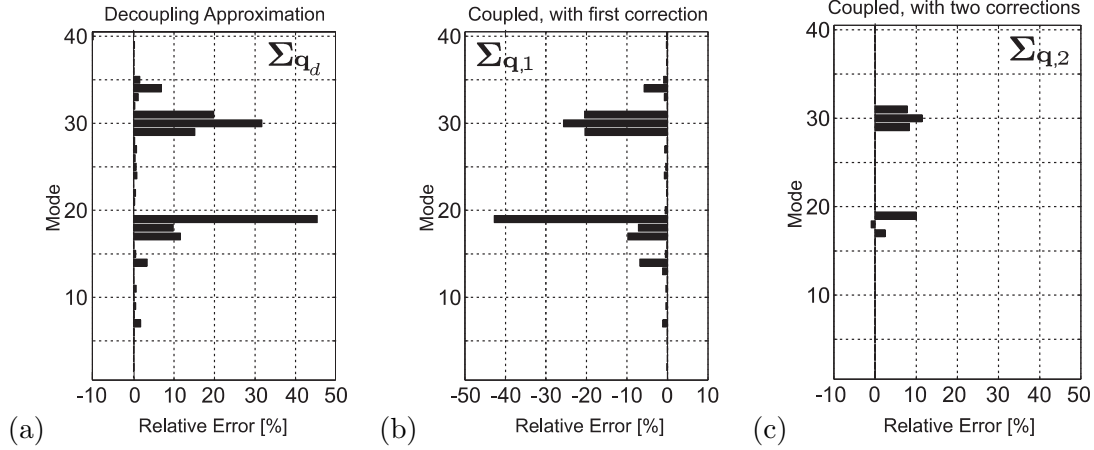


Figure 3.5: Relative error on the variance of the modal coordinates for different approximations (a) Σ_{q_d} , (b) $\Sigma_{q,1}$ and (c) $\Sigma_{q,2}$. Relative errors are expressed with respect to $\Sigma_{q,\infty}$.

In the event of negligible modal correlation, an SRSS combination (Square Root of the Sum of the Squares) of modal amplitudes would be used to calculate variances of structural responses. Because the proposed method reduces significantly the error on the modal amplitudes, the error in a SRSS combination will also be reduced. In this example however, the importance of modal correlation (see Figure 3.4-b) requires a CQC approach (Complete Quadratic Combination), at least in the establishment of structural responses that are concerned by the clustered modes 17, 18, 19 or 29, 30, 31.

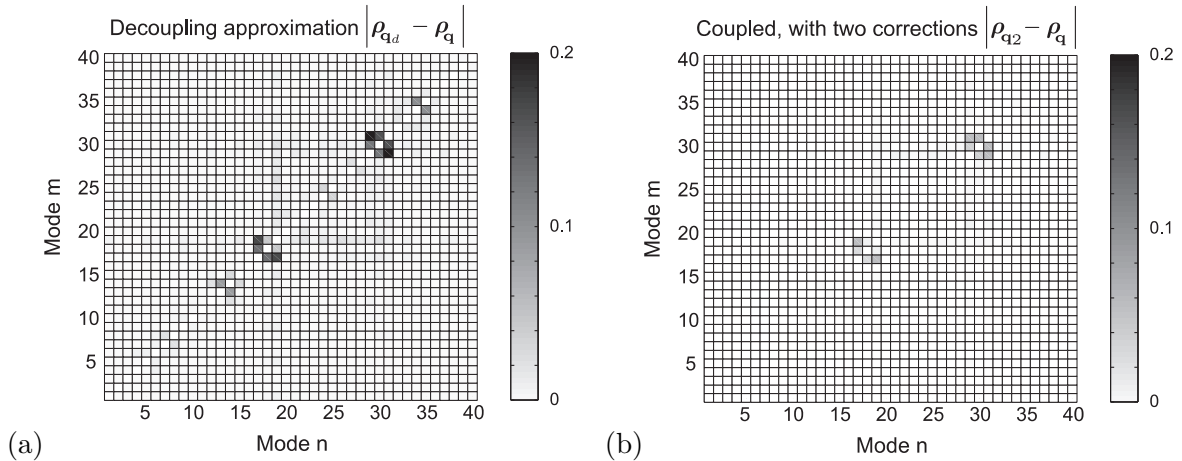


Figure 3.6: Differences between exact correlation matrix of modal coordinates $\rho_{q,\infty}$ and approximations (a) ρ_{q_d} and (b) $\rho_{q,2}$.

Figure 3.6 shows the errors on the correlation coefficients obtained with the decoupling approximation (in this case, the correlation of modal coordinates just results from the correlation

of generalized forces) and the proposed approximations. They are expressed as an absolute difference with exact coefficients. The decoupling approximation (Figure 3.6-a) provides important differences (up to 0.2), especially for the groups of correlated modes. The proposed method reduces significantly these differences down to 0.06 (Figure 3.6-b), precisely because the correlation coming from non-proportional damping is integrated in this approximation.

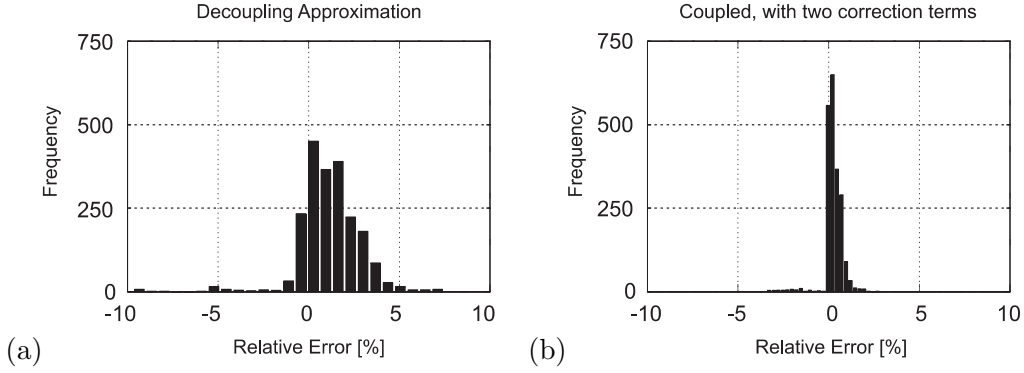


Figure 3.7: Histograms of the relative error on bending moments for approximations based on (a) $\Sigma_{\mathbf{q}_d}$ and (b) $\Sigma_{\mathbf{q},2}$. Both results are obtained with a complete quadratic combination (CQC).

The covariance matrices of bending moments in 2058 beam elements are calculated using a CQC approach with the full covariance matrices $\Sigma_{\mathbf{q},2}$ and $\Sigma_{\mathbf{q}_d}$. Therefore, both errors on variances and correlation coefficients of the modal amplitudes influence the variances of the bending moments. Figure 3.7 shows histograms of the relative errors on variances of bending moments with the two approximations. The proposed method provides a better estimation of the covariance of modal coordinates and therefore of the variances of the bending moment, as shown in Figure 3.7-b. For instance, more than 98% estimated errors are less than 1% (against only 50% errors for the decoupled approximation) and the most important error is -3.4% (against -9.7% for the decoupled approximation). The error is homogeneously reduced in the structure with the proposed method in spite of an index of diagonality close to 1. These results emphasize not only the importance of modal coupling resulting from non-proportional damping, but also the pertinence of the proposed method.

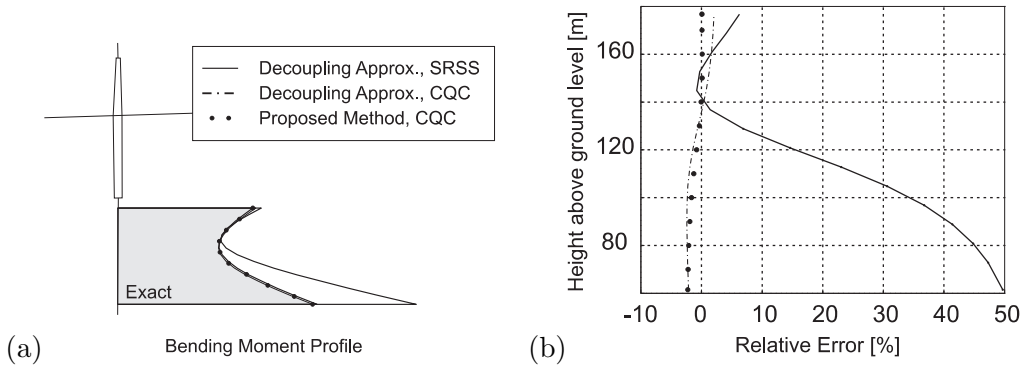


Figure 3.8: (a) Bending moment diagram and (b) relative errors on bending moment for different approximations in the third pile of Millau Viaduct.

As shown in Figure 3.8, the error on bending moment in the third pile (from the left in Figure 3.3) with the proposed method does not exceed 3%. The bending moment in this pile depends on the correlated modes, especially 17, 18 and 19, see [Deno 09a]. In this figure, the proposed method is compared with the amply used decoupling SRSS approximation (neither coupling nor correlation is taken into account) and with decoupling CQC approximation (no modal coupling is considered, but the modal correlation is taken into account). Figure 3.8-b shows that the common assumption consisting in neglecting coupling and correlation induces unacceptable errors (around 50%). The proposed method compares favorably with the other CQC approach, although both allow for a significant reduction of the discrepancy pertaining to the SRSS combination. The reasons why the decoupling CQC approximation provides acceptable results is that the errors due to coupling (overestimation, see Figure 3.5-a) and correlation (underestimation by 20%, see Figure 3.6-a) compensate for this particular application. This could not be the case for any other application. However, the proposed approach seeks to reduce them both and successively with the first two corrections to the covariance of modal coordinates. This makes it indisputably more robust and reliable. This example illustrates clearly the validity and the interest of the method.

3.5 Stationary equivalent statistical linearization

3.5.1 Newton's iterative procedure and asymptotic approach

In this Section, the application of Newton's algorithm to solve (3.3.11) is discussed, in the light of the asymptotic expansion formulation. As a major outcome, it will be observed that, again, the linear algebra that is deliberately kept in the former derivation, results in a particularly efficient formulation.

Because of the symmetry in the modal covariance matrices $\Sigma_{\mathbf{q}}$ and $\Sigma_{\dot{\mathbf{q}}}$, $m(m+1)$ unknown elements are identified in the set of equations (3.3.11). For the sake of simplicity in the notations, they are gathered in a vector of unknowns $\boldsymbol{\sigma}$, while operators $\mathcal{F}_1$ and $\mathcal{F}_2$ are rearranged into the vector function $\mathbf{s}$. The set of equations is finally written in the residual form

$$\mathbf{r}(\boldsymbol{\sigma}) = \boldsymbol{\sigma} - \mathbf{s}(\boldsymbol{\sigma}) = \mathbf{0}. \quad (3.5.1)$$

The equation in this set are nonlinear, involving hidden integrals. Newton's algorithm to solve (3.5.1) requires the computation of a Jacobian matrix $\mathbf{T}$, defined as

$$\mathbf{T}(\boldsymbol{\sigma}) = \nabla_{\boldsymbol{\sigma}} \mathbf{r}^T = \mathbf{I} - \nabla_{\boldsymbol{\sigma}} \mathbf{s}^T. \quad (3.5.2)$$

Starting from an initial guess $\boldsymbol{\sigma}^{(0)}$, which is the response of the linear subsystem in this case, a new iterate $\boldsymbol{\sigma}^{(k+1)}$ is obtained by

$$\boldsymbol{\sigma}^{(k+1)} = \boldsymbol{\sigma}^{(k)} - \mathbf{T}(\boldsymbol{\sigma}^{(k)})^{-T} \mathbf{r}(\boldsymbol{\sigma}^{(k)}), \quad k \geq 0 \quad (3.5.3)$$

and the procedure is repeated until a certain norm of $\mathbf{r}(\boldsymbol{\sigma})$ becomes less than a desired threshold.

The formal derivation of $\mathbf{T}$ in the general case is rather cumbersome. For instance, the construction of $\nabla_{\boldsymbol{\sigma}} \mathbf{s}^T$, in the upper half of the Jacobian matrix (i.e. related to $\mathcal{F}_1$), requires

the estimation of

$$\nabla_{\boldsymbol{\sigma}} \left(\int_{\mathbb{R}} \mathbf{H} \mathbf{S}_{\mathbf{p}} \mathbf{H}^* d\omega \right) = \int_{\mathbb{R}} (\nabla_{\boldsymbol{\sigma}} \mathbf{H}) \mathbf{S}_{\mathbf{p}} \mathbf{H}^* d\omega + \int_{\mathbb{R}} \mathbf{H} \mathbf{S}_{\mathbf{p}} (\nabla_{\boldsymbol{\sigma}} \mathbf{H}^*) d\omega, \quad (3.5.4)$$

where $\nabla_{\boldsymbol{\sigma}} \mathbf{H}$ further requires the application of the chain rule of differentiation, as $\mathbf{H}(\omega; \boldsymbol{\Omega}_{eq}, \mathbf{D}_{eq})$ defined with respect to $\boldsymbol{\Omega}_{eq}$ and $\mathbf{D}_{eq}$ is differentiated with respect to $\boldsymbol{\sigma}$. As a compensation, a finite difference method may be used to estimate the Jacobian matrix $\mathbf{T}(\boldsymbol{\sigma}^{(k)})$. However, this finite difference operation constitutes a heavy computational step in the method, precisely because the function estimation itself is already heavy computed. Indeed, for each perturbation of the vector $\boldsymbol{\sigma}^{(k)}$, the almost full modal transfer matrix must be inverted for every circular frequency and then the integration of the psd matrices $\mathbf{S}_{\mathbf{q}}$ and $\mathbf{S}_{\mathbf{q}}^*$ must be performed. The repeated inversions and integrations make this method *prima facie* quite inefficient, a reason why the fixed-point method is traditionally preferred.

Interestingly however, the previous asymptotic developments can be used to simplify the computation of the Jacobian matrix and to offer, with a much less expensive algorithm, the quality of a second-order method. According to the previous developments, the vector $\mathbf{s}(\boldsymbol{\sigma})$ is replaced by its first order asymptotic approximation in (3.5.1). In this case, the Jacobian matrix may be analytically and swiftly constructed. Indeed the construction of $\nabla_{\boldsymbol{\sigma}} \mathbf{s}^T$ in (3.5.4), requires the estimation of $\nabla_{\boldsymbol{\sigma}} \mathbf{H}$ which is now regarded as

$$\nabla_{\boldsymbol{\sigma}} \mathbf{H} \simeq \nabla_{\boldsymbol{\sigma}} ((\mathbf{I} - \mathbf{H}_d \mathbf{J}_o) \mathbf{H}_d) \simeq -\mathbf{H}_d (\nabla_{\boldsymbol{\sigma}} \boldsymbol{\Omega}_{eq} + \iota \omega \nabla_{\boldsymbol{\sigma}} \mathbf{D}_{eq}) \mathbf{H}_d, \quad (3.5.5)$$

where the two approximations correspond to first order developments. According to the approximation (3.5.5), Equation (3.5.4) reads

$$\nabla_{\boldsymbol{\sigma}} \left(\int_{\mathbb{R}} \mathbf{H} \mathbf{S}_{\mathbf{p}} \mathbf{H}^* d\omega \right) \simeq (\nabla_{\boldsymbol{\sigma}} \mathbf{L}_{\mathbf{q}}) + (\nabla_{\boldsymbol{\sigma}} \mathbf{L}_{\mathbf{q}}^*) \quad (3.5.6)$$

with

$$(\nabla_{\boldsymbol{\sigma}} \mathbf{L}_{\mathbf{q}})_{ij} = \sum_k^m (\nabla_{\boldsymbol{\sigma}} \boldsymbol{\Omega}_{eq})_{ik} \mathcal{I}_{ijk,0} + \sum_k^m (\nabla_{\boldsymbol{\sigma}} \mathbf{D}_{eq})_{ik} \mathcal{I}_{ijk,1} \quad (3.5.7)$$

and $\mathcal{I}_{ijk,\alpha}$ defined in (3.3.31). Similarly, the second half of the Jacobian matrix, which is related to operator $\mathcal{F}_2$, is filled with $\nabla_{\boldsymbol{\sigma}} \mathbf{L}_{\dot{\mathbf{q}}} + \nabla_{\boldsymbol{\sigma}} \mathbf{L}_{\dot{\mathbf{q}}}^*$ where

$$(\nabla_{\boldsymbol{\sigma}} \mathbf{L}_{\dot{\mathbf{q}}})_{ij} = \sum_k^m (\nabla_{\boldsymbol{\sigma}} \boldsymbol{\Omega}_{eq})_{ik} \mathcal{I}_{ijk,2} + \sum_k^m (\nabla_{\boldsymbol{\sigma}} \mathbf{D}_{eq})_{ik} \mathcal{I}_{ijk,3}. \quad (3.5.8)$$

A final substitution into (3.5.2) with the corresponding matrix-to-vector rearrangement provides an approximation of the Jacobian matrix, which is a cornerstone in this Newton algorithm. Indeed, the numerical construction of this matrix is effectively efficient, since (i) it mainly relies on the scalars $\mathcal{I}_{ijk,\alpha}$ which were computed in advance for the calculation of $\mathbf{r}(\boldsymbol{\sigma}^{(k)})$, (ii) no matrix is inverted and (iii) no additional integral is performed. Only the results computed for a given guess of $\boldsymbol{\sigma}$ in the iterative procedure are exploited. The gradients $\nabla_{\boldsymbol{\sigma}} \boldsymbol{\Omega}_{eq}$ and $\nabla_{\boldsymbol{\sigma}} \mathbf{D}_{eq}$ can be established analytically or with a finite difference scheme.

All in all, the derivation of the Jacobian matrix in this Section shows that, with the projection of the equation of motion into a suitable modal (generalized) basis, together with

the asymptotic expansion, it is possible to rapidly construct an accurate approximation of the Jacobian matrix. Compared to the finite difference estimation of the full residual vector, the proposed method efficiently concentrates the time-consuming and ill-conditioned operations (matrix inversion and integration) in analytical developments.

It is also remarkable that the proposed method does not require any user-tuned continuation to converge towards the solution. In the problems investigated by the author, among which a significant one is reported hereinafter (while others in [Cano 14]), a part of the solution of the linear subsystem is always used as initial guess $\sigma^{(0)}$. While the fixed-point algorithm is only able to handle limited nonlinearities in amplitude, the proposed algorithm smoothly converges towards the solution of the problem, without recourse to any relaxation nor backtracking artifacts.

3.5.2 Shear-type building under earthquake

The following example is inspired from [Lin 04b]. A multistorey shear-type building under a unilateral seismic excitation is considered. The structural model consists of N_s stories modeled by lumped masses m connected by clamped-clamped nonlinear beam elements. The hardening behavior of a steel beam can be taken into account for large elastic displacements by an additional cubic nonlinear stiffness as in [Ricc 07, Jin 12]. The structure is sketched in Figure 3.9-a and the equation of motion for the j -th storey is

$$m\ddot{Y}_j + k(Y_{j+1} - 2Y_j + Y_{j-1}) + g_j(Y_{j-1}, Y_j, Y_{j+1}) = -m\ddot{u}, \quad (3.5.9)$$

where g_j , i.e. the j -th component of the nonlinear force vector, is defined as

$$g_j(Y_{j-1}, Y_j, Y_{j+1}) = \kappa k \left((Y_j - Y_{j-1})^3 - (Y_{j+1} - Y_j)^3 \right) \quad (3.5.10)$$

with Y_j the relative horizontal displacement (with respect to the ground displacement) of that storey. At the first floor, Y_0 is set equal to zero, while at the last floor, the term $Y_{N_s+1} - Y_{N_s}$ is set equal to zero as well. In this illustration, the number of stories N_s is fixed equal to 10 and the characteristics of the structure are $m = 1290\text{tons}$, $k = 10^8\text{N/m}$. The damping terms result from a proportional structural damping imposed to 1% in the first two linear modes. The parameter κ , quantifying the intensity of the nonlinear forces, is adjustable in this example. The natural frequencies of the first five linear normal modes are 0.21, 0.62, 1.02, 1.40, 1.75Hz. The unilateral excitation field is modeled by a modified Kanai-Tajimi spectrum [Clou 93]

$$S_{\ddot{u}} = S_0 \frac{\left(1 + 4\xi_1^2 \left(\frac{\omega}{\omega_1}\right)^2\right)}{\left(1 - \left(\frac{\omega}{\omega_1}\right)^2\right)^2 + 4\xi_1^2 \left(\frac{\omega}{\omega_1}\right)^2} \frac{\left(\frac{\omega}{\omega_2}\right)^4}{\left(1 - \left(\frac{\omega}{\omega_2}\right)^2\right)^2 + 4\xi_2^2 \left(\frac{\omega}{\omega_2}\right)^2} \quad (3.5.11)$$

with $\omega_1 = 5\text{rad/s}$, $\xi_1 = 0.2$, $\omega_2 = 0.5\text{rad/s}$, $\xi_2 = 0.6$ and $S_0 = 0.03\text{m}^2/\text{s}^3$ (see Figure 3.9-b).

Since all the degrees of freedom are involved in the nonlinear behavior, this example is a pertinent application of our method. Furthermore, the classical fixed-point method might fail to solve this problem for moderate nonlinear behavior.

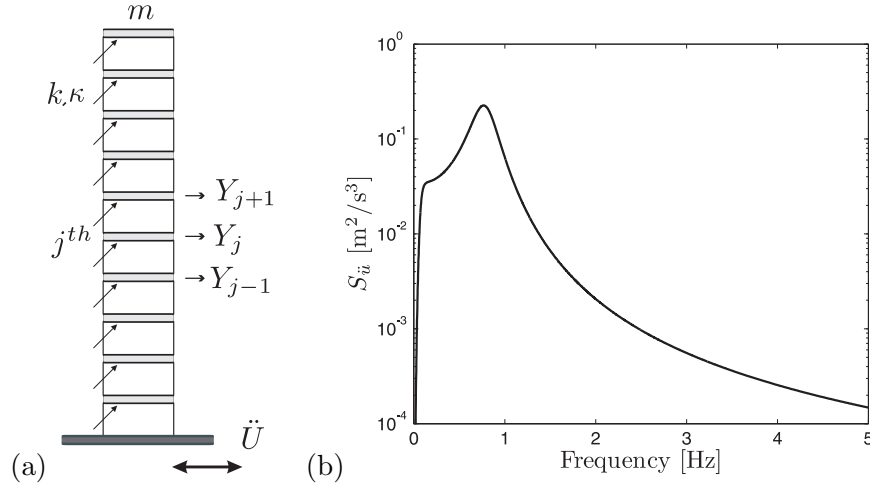


Figure 3.9: (a) Sketch of the shear-type building. (b) Kanai-Tajimi spectrum with the aforementioned parameters.

Indeed, in the nodal basis ($n = 10$), for a given guess of $\Sigma_{\mathbf{x}}^{(k)}$, the equivalent stiffness matrix $\mathbf{K}_{eq}$ may shift the natural frequencies in the high-frequency domain, for which the spectrum (3.5.11) is low. The updated response of the structure $\Sigma_{\mathbf{x}}^{(k+1)}$ is therefore reduced, and so is the following guess of $\mathbf{K}_{eq}$. Thus, the natural frequencies decrease and the risk here is to come back to the previous guess $\Sigma_{\mathbf{x}}^{(k)}$. The fixed-point iterative process is thus eventually trapped in oscillations between two states that are not solutions, as illustrated in Figure 3.10. This argument briefly and physically explains why the equation set can be ill-conditioned and why the convergence can be dramatically slow. Obviously, this fact corroborated by numerical experiments, shall not be encountered with white noise excitations, as commonly assumed in (much more) academic examples. This may justify why the fixed-point method has apparently never been criticized.

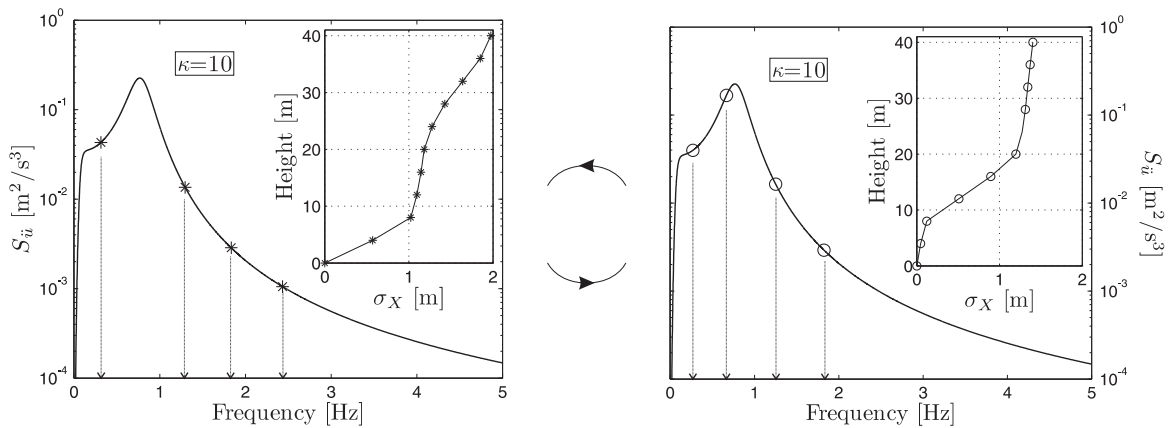


Figure 3.10: Non-convergence of the fixed-point algorithm. Oscillations occur between a state depicted in (*) (left) and a state depicted in (o) (right), although those states are not solutions. Both the standard deviations σ_X of the linearized lateral displacements and the related natural frequencies (on the Kanai-Tajimi spectrum) are illustrated.

In this context, the proposed method (with $N = 2$) shows its relevance, because the Newton algorithm is able to explore more efficiently the solution space. With the proposed developments, the problem consists in minimizing the residue function (3.5.1), as well as keeping the convergence criterion (3.3.35) as small as possible. This latter aspect is ensured by a proper selection of the equivalent matrix $\tilde{\mathbf{K}}$. Due to the nonlinear stiffness forces, the initial choice of the matrix $\tilde{\mathbf{K}}$ for the modal projection in (3.3.2) is important, because it conditions the convergence of the series. Computing $\mathcal{K}$ in (3.2.6) with $\mathbf{u}_0$ gives a crucial information about the distribution of the nonlinear forces and yet the amplitudes are roughly estimated. In this example, a good initial trial for $\tilde{\mathbf{K}}$ consists in computing $\mathcal{K}$ with a fraction of $\mathbf{u}_0$. This fraction can be interpreted as a ratio between the internal forces coming from the linear subsystem and from the equivalent linear one (in this problem the ratio $1/(1 + 3\kappa)$ is used according to (3.2.7)). In this manner, both linear and equivalent linear forces are taken into account with an adequate preponderance in the total stiffness matrix $\mathbf{K} + \tilde{\mathbf{K}}$. The matrix $\tilde{\mathbf{K}}$ may be updated, when a better assessment of $\tilde{\mathbf{u}}$ is known ($\tilde{\mathbf{K}} = \mathcal{K}(\tilde{\mathbf{u}})$) or when the convergence criterion is violated, in order to converge to a more accurate results. Here the proposed method reveals another valuable aspect, because the expansion-based method does not require a modal update at each iteration, as in a classical statistical linearization [Robe 99]. In fact, examples show that one or two updates are by far sufficient to offer an accurate result. The combination of a small convergence criterion and a small residual function ensures that the solution is the sought one.

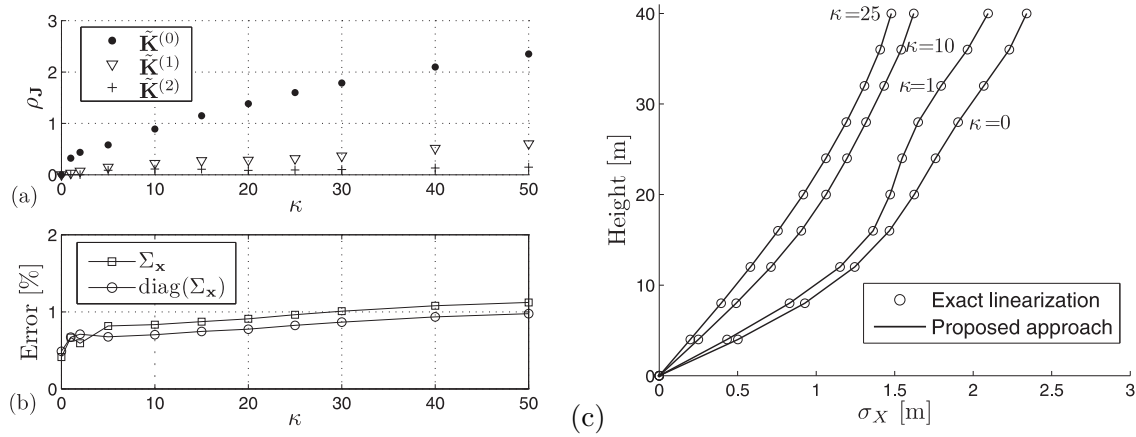


Figure 3.11: (a) Convergence criterion ρ_J for three iterations on $\tilde{\mathbf{K}}$. (b) Error [%] between the proposed method ($N = 2$) and the formal statistical linearization. (c) Standard deviation of the lateral displacements for different values of κ .

For all simulations, five modes are kept in the analysis and $\tilde{\mathbf{K}}$ is updated twice in order to satisfy the convergence criterion as illustrated in Figure 3.11-a. Even if ρ_J is greater than one with the first guess $\tilde{\mathbf{K}}^{(0)}$, it does not mean that the problem cannot be solved, but it shows that the solver does not converge to the good solution because the approximation of $\mathbf{s}(\mathbf{u})$ in (3.5.1) is not accurate enough. However, the developed strategy just requires few iterations on $\tilde{\mathbf{K}}$ before reaching a suitable guess, as illustrated in Figure 3.11-b, for strong nonlinearities. The comparison between Figure 3.11-a and -b shows that a small index ρ_J (with a small residue) is a proof of convergence for the linearization. The nonlinear forces become more important,

but a small index $\rho_{\mathbf{J}}$ ensures the converge independently of κ . Figure 3.11-c highlights the effects of the nonlinear forces on the structure through the standard deviation of the lateral displacements.

The number of iterations inside the Newton solver for all the simulations in Figure 3.11 does not exceed four and the computation time does not exceed 1s per iterate (in MatLab, our processor is an *Intel CORE i5*): the method is not extensively consuming in virtue of its second-order accuracy. Figure 3.11-b shows the accuracy of the method. The error on the diagonal terms and on the whole covariance matrix is at most 1% even for strong nonlinearities. Figure 3.11-c shows profiles of the standard deviation of the lateral displacements for different values of κ . The good agreement with the proposed method and the exact linearization is noteworthy.

3.6 Conclusions

The spectral analysis is here applied in the context of equivalent statistical linearization in a stationary setting. We aim at developing a method that can be used to analyze structures with nonlinear behaviors subject to coherent random loads. We base our approach on an asymptotic expansion of the modal transfer matrix related to linear or linearized structures.

This expansion is inspired by the usual smallness of the out-of-diagonal elements of the modal damping and stiffness matrices, compared with the diagonal ones. These diagonal matrices characterize a main decoupled structure, while the out-of-diagonal elements can be seen as perturbations. Thanks to the asymptotic expansion, we can see the modal transfer matrix of a structure as the modal transfer matrix of a main decoupled structure, corrected by additional terms estimating the modal coupling in the system. These correction terms are readily obtained by linear algebra, which leads to a smart and convenient formulation.

Evoking the Gerschgorin theorem, we derive a criterion of convergence, which measures the modal coupling in the structure. The criterion $\rho_{\mathbf{J}}$ must be less than one in order to ensure the convergence of the series and the validity of our approach.

As a verification, our asymptotic approach is applied to the wind buffeting analysis of the Millau Viaduct, assumed to linearly respond. Due to the aerodynamic damping forces, the damping model of the structure ceases to be proportional and the resulting coupling must be taken into account. This example highlights the alternate convergence of the expansion and the properties of the different correction terms, i.e. the mechanical coupling is essentially taken into account in the first correction term.

After this validation in linear structural analysis, we focus on equivalent statistical linearization. This linearization aims at calculating the first two statistical moments of a structural responses through an optimization problem. The original nonlinear equation of motion, with random forces, is thus replaced by a set of nonlinear algebraic equations. In spectral analysis, the fixed-point procedure, usually applied to solve this set, may fail in dealing with moderate nonlinearities and realistic excitation models. With our asymptotic expansion, we propose an efficient formulation of a second-order numerical solver to circumvent those limitations. Indeed, the computation of a Jacobian matrix is deeply improved and simplified by our asymptotic method, allowing the use of a gradient-based solver.

About the convergence in this context, this criterion $\rho_{\mathbf{J}}$ depends upon the considered modal (or generalized) basis, in which the linearized equations of motion are projected. It

may be required to take into account the linearized counterpart of nonlinear forces to evaluate their influence on the natural frequencies and to keep a slight modal coupling. Therefore, we introduce a stiffness matrix $\tilde{\mathbf{K}}$ to take into account the linearized stiffness forces in the computation of the modal basis. This matrix is a mathematical artifact to calculate a suitable linearized basis. In an iteration procedure, the matrix $\tilde{\mathbf{K}}$ and the modal basis are not necessarily updated at each step. The update of the generalized basis is discussed later in this Part.

Both the second-order solver and the modal basis updating procedure are applied in this Chapter to the equivalent linearization of a shear-type building subject to earthquake. This example also highlights the difficulty to apply a first-order solver, such as the fixed-point, to deal with realistic loadings. Indeed it results in a non-convergent solution, a consequence that the power spectral density is non-uniformly distributed over the frequency domain.

The problems encountered in a stationary setting are different from those in a nonstationary regime. Nonetheless, the asymptotic expansion-based method can be interestingly applied to transient linear analysis, as presented in the following Chapter.

Chapter 4

Linear Evolutionary Spectral Analysis

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- 4.1 Context and propositions
 - 4.2 Spectral analysis of linear time-invariant systems
 - 4.3 Asymptotic expansion in evolutionary spectral analysis
 - 4.4 Efficient formulation for numerical computation
 - 4.5 Multi-span bridge under differential ground acceleration
 - 4.6 Nongeometric spectral moments and first passage time
 - 4.7 Linear systems driven by non-Gaussian excitation
 - 4.8 Conclusions
-

Purpose of the chapter

In linear transient analysis, some methods require to compute the impulse response matrix of a structure. Since this matrix is determined with difficulty in practice other approaches are usually considered. In this context, the asymptotic expansion-based method developed in the previous Chapter may find an interesting application field. Actually, it allows to efficiently compute a closed-form expression of the impulse response matrix in a modal basis, key point in evolutionary approach.

The concept of evolutionary psd and the related spectral analysis are presented in Section 4.2. Then, the asymptotic expansion-based method is applied in Section 4.3. To improve the implementation of the proposed developments, an efficient computational procedure is described in Section 4.4. Finally, some reflexions about the concept of non-geometrical spectral moment and non-Gaussian evolutionary spectral analysis are also shared in Sections 4.6 and 4.7, respectively.

4.1 Context and propositions

In the previous Chapter, the presented spectral analysis only focuses on stationary processes. As said before, the wind is usually considered as a stationary loading, making the previous developments quite pertinent to take into account localized nonlinear behaviors or nonlinear damping. However, some concerns may be judiciously formulated about the use of stationary equivalent linearization to perform seismic analysis, as earthquakes are transient by nature. Furthermore, it would be fallacious to restrict wind analysis to stationary process. In civil engineering, some aeolian phenomena are also transient by nature, like the aeroelastic forces [Stro 10] and the flutter behavior [Jone 01]. In the case of wind buffeting analysis, *downburst* or *thunderstorm*, more detailed in the following, require indubitably to take into account the time evolution of the mean wind velocity, and so its influence on the intensity of turbulence.

This chapter may be interpreted as a generalization of the previous one to non-stationary analysis for linear systems and linear phenomena. Indeed, it deals with a specific family of transient processes largely encountered in civil engineering: the process derived from evolutionary psd. An **evolutionary power spectral density** characterizes an unsteady process by means of the psd of an **embedded stationary process** and a deterministic **time window**, as illustrated in Figure 4.1.

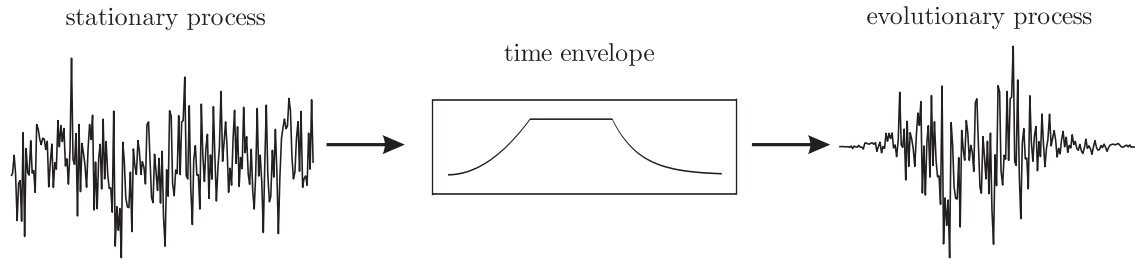


Figure 4.1: Evolutionary spectral analysis and evolutionary process: an embedded stationary random process is modulated by a time window in order to generate an evolutionary random process.

Therefore, the *evolutionary spectral analysis* may be understood as an extension of the spectral analysis presented before for steady processes, because the second approach should necessarily tend to the first one on large timescale provided the intensity envelope may remain constant in time. Although this topic is quite old, it remains an elegant formulation of transient phenomena, for which evolutionary spectral models are available, as for earthquakes. Indeed, the frequency content of seismic excitation is usually modeled by a Kanai-Tajimi spectrum or Clough-Penzien modified spectrum, while the time evolution is modeled by a deterministic time window. The same philosophy applies to transient gusty wind.

As source and ground, the *stricto sensu* evolutionary spectral analysis has been timidly initiated by Caughey and Stumpf [Caug 61], then formalized by Priestley [Pri 65, Pri 67]. The first pertinent applications to dynamical systems are due to Hammond for SDOF and MDOF systems [Hamm 68, Hamm 73]. Those contributions were only focused on linear dynamics of time-invariant systems. Even though the spectral analysis is a natural extension of Fourier analysis to random dynamics, the evolutionary spectral analysis expresses mathematically the transience of the energy contained in the different harmonics of an excitation,

and this description is thus less intuitive. The spectral analysis only requires to work in a frequency domain, while the evolutionary spectral analysis needs both time and frequency descriptions.

In a general theory, the time window may be time and frequency-dependent. In most cases, the time envelope is only time-dependent. Although the spectral approach is the most commonly used method to analyze large-dimensional linear structures subject to stationary loading, the evolutionary spectral analysis has not encountered a real enthusiasm in the fields of engineering.

Avoiding evolutionary spectral analysis, the analysis of structures subject to transient loadings may be performed by cumulant method (see Appendix A.1), with a limitation to Itô's processes and the difficulty to generate coherence in the excitation fields. Actually, the use of Monte Carlo method is in some cases the only feasible method to study transient dynamics of complex structures. Assuming a coherent seismic field, the computational burden is real, but limited by the duration of generated histories, not as long as for wind loads on structures. Furthermore, the time history of an earthquake is small compared with the duration of a storm. Indeed, an earthquake may last at most one minute, while a storm lasts about thirty minutes for the period of strong intensity. The strategy for time history generation is thus completely different in both situations. In seismic engineering, many earthquakes may be generated from a single embedded stationary signal, while for storms a single generation applies to less samples (for a given frequency resolution). This is partly due to the low frequency content of the wind and the computational limitation.

In order to be as exhaustive as possible, we cannot limit the time domain approach of transient dynamics to Monte Carlo simulation. At the end of the last millennium, To has developed the concept of stochastic integration scheme, based on the central difference [To 86] and on the Newmark method [To 92]. These methods have been recently improved by Tootk-aboni [Toot 10] and applied to non-white excitations. The method is constructed upon a general recursive state space formalism: the covariance matrix between displacements and velocities is updated at each time step by a recursive relation depending on the previous time steps, the system properties and the integration scheme. The method deals with general non-stationary random processes and with non-proportional damping. However, the method is only applied to small-scale problems and the correlation amongst the random excitations is never taken into account. Since many excitation models are characterized in the frequency domain, we want to go back to the spectral analysis.

First, the concept of pseudo-excitation has been proposed in stationary [Lin 92] and non-stationary [Lin 94] settings, in order to attach more importance to the dimension of the structure and to the coherence within the excitation field. The *pseudo-excitation method* consists in determining the random response of a structure driven by random loadings, as the response of this structure under a series of harmonic loads. Basically, the method uses the spectral decomposition, exposed in Section 2.4 for coherent field simulation. For stationary response, the linearity of the system readily leads to harmonic superposition; this method has been applied in wind [Xu 99] and earthquake [Lin 04a] engineering. For unsteady responses, the method leads to a superposition of Duhamel integrals [Clou 93] in a modal basis. However, this method is not able to deal with coupled systems, because its main drawback is hidden in the numerical computation of the impulse response matrix of mechanically coupled MDOF

systems. This limitation, encountered in evolutionary spectral analysis as well, may open a new perspective for the developments of Chapter 3. Instead of the Duhamel integral in the time domain, we propose to stay in the frequency domain with the evolutionary spectral analysis and to consider the time as a parameter.

For SDOF systems subject to transient loadings, the evolutionary spectral analysis is a very efficient tool, because closed-form solutions can be estimated, provided that the deterministic time window has a convenient shape [Gupt 00]. For most of those functions encountered in the literature, an instantaneous transfer function can be computed. The direct application of this result leads to classic decoupled modal analysis, for which each mode is considered independently as an SDOF system. A step beyond, Conte [Cont 96] enhances some closed-form expressions of the covariance matrix of the modal coordinates for a linear time-invariant system subject to white noises.

For MDOF systems, the main drawback of the evolutionary spectral analysis is in the computation of the impulse response matrix of the system. In a modal basis with perfectly decoupled coordinates, this problem is avoided. However, for structures with non-proportional damping, Lord Rayleigh's approximation may lead to poor results. Closed-form solutions or analytical results cannot be used. This is a reason why some authors forget the transient nature of the phenomena and apply directly the stationary theory. Provided the structure has a small memory time, i.e. the main structural modes are highly damped, this assumption is quite satisfactory and offers significant simplifications from a practical point of view. With the so-called **assumption of quasi-stationarity**, the transient response is seen as a time modulation of the stationary response. Nonetheless, we do not share this view because this assumption denatures the analysis, especially in the context of extreme value problems. Indeed, the past of the response conditions the statistics of extremes, especially for earthquakes because of their short duration.

Even in a modal basis, the modal coupling leads to consider full modal impulse response matrices, so the results from SDOF analysis cannot be used. The computation of this matrix in a transient analysis becomes so time consuming, even for linear systems, that Monte Carlo simulations are usually preferred. Nevertheless, the proposed expansion-based method may find here a possible and clever extension. It may offer new perspectives in evolutionary spectral analysis, for two main reasons: (i) the decoupled approximation in a suitable modal basis may be expressed as a linear combination of the transient modal responses, (ii) the computation of the correction terms is definitely simplified, since closed-form expressions can be computed after few analytical developments. The application of our developments to the resolution of extreme value problems can also be considered.

First of all, the spectral analysis of Linear Time-Invariant (LTI) systems is generalized to transient excitation by use of evolutionary processes. The application of the asymptotic expansion-based method is also proposed. Some computational aspects are also exposed in order to extend the method to any time window. The question of nonstationary spectral moments is also addressed in the context of a first passage problem. Some extensions of the method to non-Gaussian transient analysis are exposed in the last Section of this Chapter.

4.2 Spectral analysis of linear time-invariant systems

4.2.1 Expansion theorem and evolutionary spectrum

The concept of evolutionary psd is presented by considering the vector of the generalized forces. Like in a stationary setting, the vector of forces is a random vector such as $\mathbf{p}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^m$. A crucial assumption about the spectral distribution of the (modal) generalized forces, and thus the generalized forces, states that the vector of forces may be written in the form of a Fourier-Stieltjes integral, like

$$\mathbf{p}(t, \theta) = \int_{\mathbb{R}} e^{i\omega t} \mathbf{a}(t, \omega) d\tilde{\mathbf{p}}(\omega, \theta) \quad (4.2.1)$$

with $\mathbf{a}(t, \omega) : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}^{m \times m}$ a deterministic *time window* (also called *intensity function* or *time envelope*) The unsteady process is thus built by modulating a stationary process by an envelope depending on time (and possibly on frequency). In the particular case where $\mathbf{a}(t, \omega)$ is defined as a complex-valued matrix, it must be hermitian $\mathbf{a} = \mathbf{a}^*$. For sake of simplicity in the following developments, $\mathbf{a}(t, \omega)$ is supposed to be real. This formulation is used by Priestley on the basis of the developments of Bartlett ([Bart 56], p.143). In a particular case, but amply encountered, the scalar function $a(t, \omega)$ replaces the matrix $\mathbf{a}(t, \omega)$, supposing therefore that all the stationary forces on a structure are modulated by the same time window. Most of the time, this assumption is used in seismic engineering. In wind engineering, the use of a matrix $\mathbf{a}(t, \omega)$ allows to model wind direction evolutions during a storm, but its relevance only matters in presence of spatially coupled modes.

Without too many details, the validity of (4.2.1) comes from an **expansion theorem**: if the process $\mathbf{p}(t)$ admits the integral representation

$$E[\mathbf{p}(t_1)\mathbf{p}^*(t_2)] = \int_{\mathbb{R}^2} \phi(t_1, \omega_1)\phi^*(t_2, \omega_2) d\boldsymbol{\mu}(\omega_1, \omega_2) \quad (4.2.2)$$

with $t_1, t_2 \in \mathbb{R}^+$ and $\phi(t, \omega)$ a quadratically integrable function with respect to the measure $d\boldsymbol{\mu}(\omega_1, \omega_2)$ for all t , then there exists an expansion of $\mathbf{p}(t)$ in terms of the random increment $d\tilde{\mathbf{p}}(\omega)$ given by the following Riemann-Stieltjes integral

$$\mathbf{p}(t, \theta) = \int_{\mathbb{R}} \phi(t, \omega) d\tilde{\mathbf{p}}(\omega, \theta) \quad (4.2.3)$$

with $d\boldsymbol{\mu}(\omega_1, \omega_2) = E[d\tilde{\mathbf{p}}(\omega_1)d\tilde{\mathbf{p}}^*(\omega_2)]$. Immediately, the stationary case is recovered by setting $\phi(t, \omega) = e^{i\omega t}$, while the expression (4.2.1) is obtained by setting $\phi(t, \omega) = \mathbf{a}(t, \omega)e^{i\omega t}$ in (4.2.3). Assuming $\mathbf{p}(t)$ a zero-mean process, the covariance function of $\mathbf{p}(t)$, noted $\boldsymbol{\Sigma}_{\mathbf{p}}(t)$, is time-dependent. The stationary relation between the covariance matrix and the psd matrix is transposed into an evolutionary formalism and Equation (4.2.2) yields

$$\boldsymbol{\Sigma}_{\mathbf{p}}(t) = \int_{\mathbb{R}} \hat{\mathbf{S}}_{\mathbf{p}}(t, \omega) d\omega \quad (4.2.4)$$

with $\hat{\mathbf{S}}_{\mathbf{p}}(t, \omega)$ the evolutionary psd of $\mathbf{p}(t)$. The hat used to denote evolutionary spectrum stems from [Lin 04b] to avoid any possible confusion with the stationary case. Using the relation (2.2.6) in term of $d\tilde{\mathbf{p}}(\omega)$ in (4.2.4), the evolutionary psd $\hat{\mathbf{S}}_{\mathbf{p}}(t, \omega)$ is expressed in terms of the psd of the embedded stationary processes $\mathbf{S}_{\mathbf{p}}(\omega)$ and the time window, as

$$\widehat{\mathbf{S}}_{\mathbf{p}}(t, \omega) = \mathbf{a}(t, \omega) \mathbf{S}_{\mathbf{p}}(\omega) \mathbf{a}^*(t, \omega). \quad (4.2.5)$$

This last equation expresses the whole philosophy of the evolutionary spectral representation: an embedded stationary process mainly described by its cross-psd matrix and a modulation matrix introducing the time dependency in the problem. Here also, the hermitian assumption on $\mathbf{a}(t, \omega)$ originates from (4.2.5), in order to ensure $\widehat{\mathbf{S}}_{\mathbf{p}}(t, \omega)$ hermitian. In the following Section, the evolutionary spectrum are used in spectral analysis of linear time-invariant systems.

4.2.2 Evolutionary spectral analysis in physical coordinates

In this Section, the theory of linear time-invariant (LTI) system is exposed in presence of transient random excitation. The dynamic of the structure is modeled by (3.2.1) without nonlinear internal forces. The modal basis in this case gathers the linear normal modes of the linear substructure. From linear system theory [Clou 93], we know that the modal response function is expressed as the convolution integral between the impulse response matrix of modal coordinates and the vector of modal random forces, such that

$$\mathbf{q}(t, \theta) = \int_{t_0}^t \mathbf{h}(t-u) \mathbf{p}(u, \theta) du, \quad (4.2.6)$$

where $\mathbf{h}(t) : \mathbb{R}^+ \mapsto \mathbb{R}^{m \times m}$ denotes the **matrix of impulse response functions** and $\mathbf{p}(t)$ is the m -dimensional vector of random generalized forces as previously defined. Equation (4.2.6) is a classical result in structural dynamics. Hence, according to (4.2.1), the convolution integral (4.2.6) is rewritten as

$$\mathbf{q}(t, \theta) = \int_{\mathbb{R}} \mathbf{G}(t, t_0, \omega) d\tilde{\mathbf{p}}(\omega, \theta) \quad (4.2.7)$$

with $\mathbf{G}(t, t_0, \omega)$ the **evolutionary transfer matrix** defined as

$$\mathbf{G}(t, t_0, \omega) = \int_{t_0}^t \mathbf{h}(t-u) \mathbf{a}(u, \omega) e^{i\omega u} du. \quad (4.2.8)$$

The vocable used to denote the matrix $\mathbf{G}(t, t_0, \omega)$ has certainly a conceptual origin, because $\mathbf{a}(t, \omega)$ tends to $\mathbf{I}$ as $t \rightarrow \infty$, the matrix $\mathbf{G}(t, t_0, \omega)$ tends to the transfer matrix defined in (3.3.7). The evolutionary transfer function, as the convolution integral between the impulse response matrix $\mathbf{h}(u)$ and the time window, expresses the effects of the past intensities (of the embedded process) on the present state.

In the previous Section, the concept of *quasi-stationary analysis* has been introduced. With this hypothesis, the convolution integral is avoided and the intensity function is taken out of the integral, leading to the approximation

$$\mathbf{G}_{qs}(t, \omega) = \mathbf{H}(\omega) \mathbf{a}(t, \omega). \quad (4.2.9)$$

Results with this formulation are used and criticized in the illustrations of this Part.

For the sake of convenience in the developments ahead, we abandon the use of the second order differential equation in terms of the **modal coordinates** $\mathbf{q}(t)$ in favor of the first-order differential equation in terms of a **modal state vector** $\mathbf{z}(t)$. A physical argument leading to use this formalism is the correlation existing between displacements and velocities in transient dynamics, more efficiently grasped with this formalism. Nonetheless, Equation (4.2.8) remains central in the following developments.

4.2.3 Evolutionary spectral analysis in state space

The equation of motion (3.3.5) in a modal basis is now rewritten in a state space formalism

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B} \begin{pmatrix} \mathbf{0} \\ \mathbf{p}(t, \theta) \end{pmatrix}, \quad (4.2.10)$$

with $\mathbf{A}$ a $2m$ -dimensional matrix, constant in time, and $\mathbf{z}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^{2m \times 1}$ the modal state vector of the system, both defined as

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{\Omega} & -\mathbf{D} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \quad \text{and} \quad \mathbf{z} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}. \quad (4.2.11)$$

In this approach, we do not confuse (4.2.11) with the state space formalism of the *nodal* equation of motion (3.2.1), which leads to use complex normal modes [Foss 58]. As previously said, the complex modal analysis has the main drawback of dealing with possible huge matrices, as long as increasing the computational burden by adding the complex dimension in the problem. In (4.2.11), the modal basis is real-valued and results from the eigenproblem (3.3.2) with $\tilde{\mathbf{K}} = \mathbf{0}$.

The general solution of an LTI system is expressed as

$$\mathbf{z}(t, \theta) = \mathbf{\Psi}(t, t_0)\mathbf{z}(t_0) + \int_{t_0}^t \mathbf{\Psi}(t, u)\mathbf{B} \begin{pmatrix} \mathbf{0} \\ \mathbf{p}(u, \theta) \end{pmatrix} du \quad (4.2.12)$$

with $\mathbf{\Psi}(t, t_0) : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^{2m \times 2m}$ the **state transition matrix** of the system in the modal basis from a time t_0 to a time t and $\mathbf{z}(t_0)$ the initial condition ($t > t_0$).

The developments of Section 4.2.2 also apply to the state-space formalism of the equation of motion and a **state space evolutionary transfer matrix** $\mathbf{\Upsilon}(t, t_0, \omega) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{C}^{2m \times 2m}$ may be defined as

$$\mathbf{\Upsilon}(t, t_0, \omega) = \int_{t_0}^t \mathbf{\Psi}(t, u) \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{a}(u, \omega) \end{pmatrix} e^{j\omega u} du. \quad (4.2.13)$$

This evolutionary transfer matrix differs from $\mathbf{H}(\omega)$, as it is expressed in a state space formalism and it also depends on the initial time t_0 , but a relation can be easily established between $\mathbf{\Upsilon}(t, t_0, \omega)$ and $\mathbf{G}(t, t_0, \omega)$.

The objective is now to highlight, thanks to the previous developments, an evolutionary and spectral relation between the evolutionary psd of the input and the evolutionary psd of the output, similarly to the stationary case. The correlation matrix between the modal coordinates at two different instants t_1 and t_2 is expressed as

$$E[\mathbf{z}(t_1)\mathbf{z}^*(t_2)] = \int_{t_0}^{t_1} \int_{t_0}^{t_2} \boldsymbol{\Psi}(t_1, u_1) \mathbf{B} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & E[\mathbf{p}(u_1)\mathbf{p}^*(u_2)] \end{pmatrix} \mathbf{B}^* \boldsymbol{\Psi}^*(t_2, u_2) du_1 du_2, \quad (4.2.14)$$

where the initial condition is supposed to be at rest, i.e. $\mathbf{z}(t_0) = \mathbf{0}$, thus dropping the first term in (4.2.12). According to the assumption of orthogonal increments in the spectral process $\tilde{\mathbf{p}}(\omega)$, the relations (4.2.2) to (4.2.4) are used and the correlation function (4.2.14) now reads

$$E[\mathbf{z}(t_1)\mathbf{z}^*(t_2)] = \int_{\mathbb{R}} \boldsymbol{\Upsilon}(t_1, t_0, \omega) \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{\mathbf{p}}(\omega) \end{pmatrix} \boldsymbol{\Upsilon}^*(t_2, t_0, \omega) d\omega. \quad (4.2.15)$$

The variance of $\mathbf{z}(t)$ at a given time t is given by assuming that $t = t_1 = t_2$ in (4.2.15). Therefore, an evolutionary psd $\hat{\mathbf{S}}_{\mathbf{z}}(t, t_0, \omega)$ for the modal state vector can be defined, according to the same argument previously given,

$$\hat{\mathbf{S}}_{\mathbf{z}}(t, t_0, \omega) = \boldsymbol{\Upsilon}(t, t_0, \omega) \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{\mathbf{p}}(\omega) \end{pmatrix} \boldsymbol{\Upsilon}^*(t, t_0, \omega). \quad (4.2.16)$$

The covariance matrix of the modal coordinate depending on time is given by

$$\boldsymbol{\Sigma}_{\mathbf{z}}(t, t_0) = \int_{\mathbb{R}} \hat{\mathbf{S}}_{\mathbf{z}}(t, t_0, \omega) d\omega. \quad (4.2.17)$$

From (4.2.13) and (4.2.16), it is admitted that evolutionary spectral analysis for LTI systems, is a simple extension of the stationary spectral analysis.

To conclude this Section in an exhaustive manner, a method to calculate the state transition matrix $\boldsymbol{\Psi}(t, t_0)$ must be briefly explained. The state transition matrix can be defined as the unique solution of the partial differential equation

$$\frac{\partial}{\partial t} \boldsymbol{\Psi}(t, t_0) = \mathbf{A} \boldsymbol{\Psi}(t, t_0) \quad (4.2.18)$$

with $\boldsymbol{\Psi}(t_0, t_0) = \mathbf{I}$. So, the i -th column of the matrix $\boldsymbol{\Psi}(t, t_0)$ is the vector resulting from the homogeneous differential equation appearing in (4.2.18) with the i -th canonical vector of the state space as initial condition. According to this definition and because the properties of the system are supposed to be time independent [Hesp 09], the transition matrix $\boldsymbol{\Psi}(t_1, t_0)$ is given by

$$\boldsymbol{\Psi}(t_1, t_0) = \begin{pmatrix} \hat{\mathbf{h}}(t_1 - t_0) & \mathbf{h}(t_1 - t_0) \\ \partial_{t_1} \hat{\mathbf{h}}(t_1 - t_0) & \partial_{t_1} \mathbf{h}(t_1 - t_0) \end{pmatrix} = e^{\mathbf{A}(t_1 - t_0)}. \quad (4.2.19)$$

We recognize $\mathbf{h}(t)$ first introduced in (4.2.6), while $\hat{\mathbf{h}}(t)$ is the matrix gathering the vectors $\hat{\mathbf{h}}_i(t)$, i.e. the response of the system at a time t_1 , if a unit displacement is imposed at the i -th mode as initial condition. The matrix $\hat{\mathbf{h}}(t)$ does not need to be computed in the second term of (4.2.13) and so the state space evolutionary transfer matrix $\boldsymbol{\Upsilon}(t, t_0, \omega)$ simply reads

$$\boldsymbol{\Upsilon}(t, t_0, \omega) = \int_{t_0}^t \begin{pmatrix} \mathbf{0} & \mathbf{h}(t-u)\mathbf{a}(u, \omega) \\ \mathbf{0} & \partial_t \mathbf{h}(t-u)\mathbf{a}(u, \omega) \end{pmatrix} e^{i\omega u} du. \quad (4.2.20)$$

The association of those convolution integrals to the evolutionary transfer matrix yields

$$\mathbf{\Upsilon}(t, t_0, \omega) = \begin{pmatrix} \mathbf{0} & \mathbf{G}(t, t_0, \omega) \\ \mathbf{0} & \partial_t \mathbf{G}(t, t_0, \omega) \end{pmatrix} \quad (4.2.21)$$

with $\mathbf{h}(0) = \mathbf{0}$ and $\mathbf{G}(t, t_0, \omega)$ defined in (4.2.8). Equations (4.2.18) and (4.2.19) show that the state transition matrix is also equal to the exponential matrix of the matrix $\mathbf{A}$ multiplied by the time interval of transition. The exponential matrix, defined as

$$e^{\mathbf{A}(t_1 - t_0)} = \sum_{k=0}^{\infty} \frac{(t_1 - t_0)^k}{k!} \mathbf{A}^k, \quad (4.2.22)$$

can be conveniently and efficiently computed by a Padé approximation (as used in MatLab), without solving the initial value problem (4.2.18) to get $\hat{\mathbf{h}}(t)$ and $\mathbf{h}(t)$.

Now that the basics of the evolutionary spectral approach have been stated, some comments about this method may be more concretely formulated. Although the statements of this method are old and well-known, its application has never been really popular or common. Some details may be given in order to situate the contributions of this work.

According to Lin and Cai [Lin 04b], “*the time spent to determine the impulse response function is greater than what is required subsequently to calculate the evolutionary cross-spectra*”. They suggest the use of the inverse Fourier transform of the transfer matrix, such that

$$\mathbf{h}(t) = \int_{\mathbb{R}} \mathbf{H}(\omega) e^{i\omega t} d\omega \quad (4.2.23)$$

with $\mathbf{H}(\omega)$ defined in (3.3.7) for linear time-invariant structures. This general expression is not readily computed, even though it is a well-known inverse Fourier transform. For perfectly decoupled systems, $\mathbf{h}(t)$ is a diagonal matrix containing the impulse response function of each modal coordinate. The use of a closed-form expression for $\mathbf{h}(t)$ may considerably speed up the computation of (4.2.8). For coupled systems, like non-classically damped structures, the decoupled approach as proposed by Lord Rayleigh and summarized in Section 3.4.1 is definitely insufficient.

The efficient computation of (4.2.23) is the key to perform evolutionary spectral analysis. The developments realized in Section 3.3.3 find here a great interest. Indeed, the matrix $\mathbf{H}(\omega)$ is advantageously replaced by its asymptotic expansion (3.3.19), provided the out-of-diagonal matrices contain moderate elements. By means of linear algebra, we can develop $\mathbf{H}(\omega)$ in a series of terms, of which the closed form is known and easily integrated. All the analytical details are given in the next Section.

4.3 Asymptotic expansion in evolutionary spectral analysis

4.3.1 Derivation of an approximation for the impulse response matrix

In Chapter 3, the **asymptotic expansion-based method** takes advantage of the dynamical specificities of LTI structures in a modal basis. The same approach is used in this Section for the computation of (4.2.8) and (4.2.23). Hence, the matrix $\mathbf{h}(t)$ may be obtained by replacing $\mathbf{H}(\omega)$ in (4.2.23) by its asymptotic expansion (3.3.19). Thus, the **impulse response matrix** reads

$$\mathbf{h}(t) = \int_{\mathbb{R}} \mathbf{H}_d(\omega) e^{i\omega t} d\omega + \sum_{k=1}^{\infty} (-1)^k \int_{\mathbb{R}} (\mathbf{H}_d \mathbf{J}_o)^k \mathbf{H}_d e^{i\omega t} d\omega. \quad (4.3.1)$$

How might these developments help us in performing integral (4.2.23), because replacing (4.2.23) by (4.3.1) is not *prima facie* a significant progress? The dependence in ω is only contained in the elements of the matrix $\mathbf{H}_d(\omega)$. The higher-order correction terms have no intrinsic dependence on ω , they are just products of elements of $\mathbf{H}_d(\omega)$ besides ω itself (through $\mathbf{J}_o$). Yet, the poles of the functions (elements) in $\mathbf{H}_d(\omega)$ are easily extracted according to (3.3.16). The impulse response matrix may thus be interpreted as the impulse response matrix of a main decoupled system perturbed by higher-order correction terms,

$$\mathbf{h}(t) = \mathbf{h}_d(t) + \sum_{k=1}^{\infty} (-1)^k \Delta \mathbf{h}_k(t), \quad (4.3.2)$$

where the integral expressions of the correction terms of $\mathbf{h}(t)$ are

$$\mathbf{h}_d(t) = \int_{\mathbb{R}} \mathbf{H}_d(\omega) e^{i\omega t} d\omega, \quad \Delta \mathbf{h}_k(t) = \int_{\mathbb{R}} (\mathbf{H}_d \mathbf{J}_o)^k \mathbf{H}_d e^{i\omega t} d\omega. \quad (4.3.3)$$

The diagonal matrix $\mathbf{h}_d(t)$ is the impulse response matrix of the main decoupled system, i.e. the i -th element on the diagonal is the impulse response function of the i -th mode of the structure. Indeed, because of the decoupling assumption, the system is interpreted as responding independently in m modes. This matrix is obtained by neglecting the out-of-diagonal elements in $\mathbf{\Omega}$ and $\mathbf{D}$. The computation of the terms $\Delta \mathbf{h}_k(t)$ is discussed in the following, because some linear algebra may facilitate these integrals. As previously, the series (4.3.2) must be truncated at a reasonable order N to be implemented efficiently, so

$$\mathbf{h}_N(t) = \mathbf{h}_d(t) + \sum_{k=1}^N (-1)^k \Delta \mathbf{h}_k(t). \quad (4.3.4)$$

The convergence of the series (4.3.2) is naturally related to the convergence of the asymptotic expansion $\mathbf{H}_N(\omega)$. Provided the criterion $\rho_{\mathbf{J}}$ is less than one, the series $\mathbf{h}_N(t)$ is also convergent due to the linearity of the Fourier operator. However, the difference concerns the convergence rate with respect to time. Indeed, the truncation error between $\mathbf{h}(t)$ and $\mathbf{h}_N(t)$ is time dependent, i.e. the required number of correction terms N for a given accuracy is not identical at each time t . It is noteworthy that the shorter the time interval $[0, t]$, the smaller the number N . This convergence property will be of paramount importance in the development of an efficient evolutionary spectral method. Nevertheless, it is tediously demonstrated,

particularly because of the alternating series of matrices and the integral operation in (4.3.3). An heuristic demonstration and an explicit illustration are proposed in Section 4.3.2 and in Appendix B.3, respectively.

In Section 4.2.2, the fundamental relation of evolutionary spectral analysis (4.2.16) relates the evolutionary transfer function to the power spectral density of the embedded stationary process in the excitation. The goal of the method is to gather the time depending functions $\mathbf{h}(t)$ and $\mathbf{a}(t)$ in a same function $\mathbf{G}(t, t_0, \omega)$, the evolutionary transfer function. With the same philosophy, the asymptotic expansion of $\mathbf{h}(t)$ is used in order to find a suitable asymptotic expansion of this evolutionary transfer matrix. Introducing (4.3.4) into (4.2.8), the N -order asymptotic expansion $\mathbf{G}_N(t, t_0, \omega)$ is obtained as

$$\mathbf{G}_N(t, t_0, \omega) = \mathbf{G}_d(t, t_0, \omega) + \sum_{k=1}^N (-1)^k \Delta \mathbf{G}_k(t, t_0, \omega). \quad (4.3.5)$$

The leading order term of this expansion is a diagonal matrix containing the evolutionary transfer functions of the modes of the main decoupled system, and it is given by

$$\mathbf{G}_d(t, t_0, \omega) = \int_{t_0}^t \mathbf{h}_d(t-u) \mathbf{a}(u, \omega) e^{i\omega u} du, \quad (4.3.6)$$

while the higher-order correction terms are obtained by the convolution integral between the time window and the correction terms of the impulse response matrix

$$\Delta \mathbf{G}_k(t, t_0, \omega) = \int_{t_0}^t \Delta \mathbf{h}_k(t-u) \mathbf{a}(u, \omega) e^{i\omega u} du. \quad (4.3.7)$$

Equations (4.3.5)-(4.3.7) constitute the core of the method developed in this Chapter for LTI systems. The proposed method allows to avoid the computation of the inverse Fourier transform of the transfer matrix in order to get the impulse response matrix of the coupled system. For usual time envelopes, closed-form expressions can be found for the leading order and the correction terms. To the knowledge of the author, this method is the most efficient to deal with coupling in evolutionary spectral analysis, because the most important part of the computation is treated analytically. Now, the rest of this Section is devoted to the explicit calculation of the terms of the asymptotic expansion. Like the underlying philosophy in Chapter 3, the purpose of this Chapter is to show that **returning to linear algebra may lead to reliable results for the resolution of classic problems in engineering by contradicting the notorious idea that a transient problem can only be solved with simulations.**

For the sake of simplicity and clearness in the equations and developments, the matrix $\mathbf{a}(u, \omega)$ is replaced by the more suitable expression

$$\mathbf{a}(u, \omega) = a(u, \omega) \mathbf{I}, \quad (4.3.8)$$

which allows to consider only one real-valued function for all the time evolution processes. This is the case in most earthquake and engineering problems.

Decoupled approximation

Only keeping the leading order in the expansion (4.3.5), the covariance matrix of the modal coordinate reads

$$\Sigma_{\mathbf{q}_d}(t) = \int_{\mathbb{R}} \mathbf{G}_d(t, t_0, \omega) \mathbf{S}_p(\omega) \mathbf{G}_d^*(t, t_0, \omega) d\omega, \quad (4.3.9)$$

since it is the response of the main decoupled linear system, with $\mathbf{G}_d(t, t_0, \omega)$ defined in (4.3.6). At this stage, the coherence between the modal coordinates only results from the coherence in the excitation field $\mathbf{S}_p(\omega)$, as explained in Section 3.3.2. According to (4.2.23), the inverse Fourier transform of the decoupled transfer function of the system reads

$$(\mathbf{h}_d(t))_i = \frac{e^{-\xi_i \omega_i t}}{\omega_i \sqrt{1 - \xi_i^2}} \sin(\omega_i \sqrt{1 - \xi_i^2} t) = \frac{e^{\Omega_i^+ t} - e^{\Omega_i^- t}}{2i\omega_i \sqrt{1 - \xi_i^2}} \quad (4.3.10)$$

with

$$\Omega_i^\alpha = -\xi_i \omega_i + \alpha i \omega_i \sqrt{1 - \xi_i^2}, \quad (4.3.11)$$

$\alpha = \{+, -\}$, $i = 1, \dots, m$ and $(\mathbf{\Omega}_d)_i = \omega_i^2$ and $(\mathbf{D}_d)_i = 2\xi_i \omega_i$ as previously. The choice to express the impulse response function in terms of an imaginary exponential is a way to facilitate the presentation of former results. Introducing now (4.3.10) into (4.3.6), the diagonal elements of the evolutionary transfer function at a time t are given by

$$(\mathbf{G}_d(t, t_0, \omega))_i = \frac{\mathcal{I}_i^+(t, t_0, \omega) - \mathcal{I}_i^-(t, t_0, \omega)}{2i\omega_i \sqrt{1 - \xi_i^2}} \quad (4.3.12)$$

with the integral $\mathcal{I}_k^\alpha(t, t_0, \omega)$ defined as

$$\mathcal{I}_k^\alpha(t, t_0, \omega) = e^{\Omega_k^\alpha(\omega)t} \int_{t_0}^t a(u, \omega) e^{-\mathcal{C}_k^\alpha(\omega)u} du \quad (4.3.13)$$

with

$$\mathcal{C}_k^\alpha(\omega) = \Omega_k^\alpha - i\omega. \quad (4.3.14)$$

Definitely, the computation of the integral (4.3.13) depends upon the analytical form of the time envelope $a(t)$. Section 4.3.3 deals with this problem in depth. They might be readily obtained in closed form for usual time windows.

For slightly coupled systems, the computation of the decoupled approximation offers already a good insight into the behavior of the system, especially about the variance of $\mathbf{q}(t)$. For moderately coupled systems, the computation of higher-order correction terms is required and they are partly presented hereinafter.

First-order correction term

The first correction term of the evolutionary transfer matrix reads

$$\Delta \mathbf{G}_1(t, t_0, \omega) = \int_{t_0}^t a(u, \omega) \Delta \mathbf{h}_1(t - u) e^{i\omega u} du. \quad (4.3.15)$$

The term $\Delta \mathbf{h}_1(u)$ of the impulse response matrix must be computed. Some algebra in (4.3.3) with $k = 1$ emphasizes two integrals involving elements of $\mathbf{H}_d(\omega)$,

$$(\Delta \mathbf{h}_1(u))_{ij} = (\mathbf{\Omega}_o)_{ij} \int_{\mathbb{R}} (\mathbf{H}_d)_i (\mathbf{H}_d)_j e^{\iota \omega u} d\omega + (\mathbf{D}_o)_{ij} \int_{\mathbb{R}} \iota \omega (\mathbf{H}_d)_i (\mathbf{H}_d)_j e^{\iota \omega u} d\omega, \quad (4.3.16)$$

which is more conveniently formulated as

$$\Delta \mathbf{h}_1(u) = \mathbf{\Omega}_o \circ \Delta \mathbf{h}_{1,1}(u) + \mathbf{D}_o \circ \Delta \mathbf{h}_{1,2}(u) \quad (4.3.17)$$

with the operator $\circ$ denoting the Hadamard product. The matrices $\Delta \mathbf{h}_{1,1}$ and $\Delta \mathbf{h}_{1,2}$ are defined as

$$(\Delta \mathbf{h}_{1,L}(u))_{ij} = \int_{\mathbb{R}} (\iota \omega)^{L-1} (\mathbf{H}_d)_i (\mathbf{H}_d)_j e^{\iota \omega u} d\omega \quad (4.3.18)$$

with $L = 1, 2$ and $i, j = 1, \dots, m$.

The poles of the integrand $(\mathbf{H}_d)_i (\mathbf{H}_d)_j$ in (4.3.18) are readily obtained. Evoking Cauchy's theorem, the integral (4.3.18) becomes

$$(\Delta \mathbf{h}_{1,L}(u))_{ij} = (\mathbf{V}_{L,1})_{ij} \exp(\Omega_i^+ u) + (\mathbf{V}_{L,2})_{ij} \exp(\Omega_i^- u) + (\mathbf{V}_{L,3})_{ij} \exp(\Omega_j^+ u) + (\mathbf{V}_{L,4})_{ij} \exp(\Omega_j^- u) \quad (4.3.19)$$

for $i \neq j$. The case of $i = j$ is particular, as the multiplicity of the poles is changed and Equation (4.3.19) is not proper anymore and must be adapted. This case is not broached here, because the matrices $\mathbf{\Omega}_o$ and $\mathbf{D}_o$ have zero elements on their diagonal, so the computation of $(\Delta \mathbf{h}_{1,1})_{ii}$ and $(\Delta \mathbf{h}_{1,2})_{ii}$ is not necessary. The constant matrices $\mathbf{V}_{1,k}$ and $\mathbf{V}_{2,k}$ (with $k = 1, \dots, m$) are given by

$$\begin{aligned} (\mathbf{V}_{1,1})_{ij} &= \frac{-\iota}{2\omega_i \sqrt{1 - \xi_i^2 (\Omega_i^+ - \Omega_j^-)} (\Omega_i^+ - \Omega_j^+)}, & (\mathbf{V}_{2,1})_{ij} &= (\mathbf{V}_{1,1})_{ij} \Omega_i^+ \\ (\mathbf{V}_{1,2})_{ij} &= \frac{\iota}{2\omega_i \sqrt{1 - \xi_i^2 (\Omega_i^- - \Omega_j^-)} (\Omega_i^- - \Omega_j^+)}, & (\mathbf{V}_{2,2})_{ij} &= (\mathbf{V}_{1,2})_{ij} \Omega_i^- \\ (\mathbf{V}_{1,3})_{ij} &= \frac{-\iota}{2\omega_j \sqrt{1 - \xi_j^2 (\Omega_j^+ - \Omega_i^+)} (\Omega_j^+ - \Omega_i^-)}, & (\mathbf{V}_{2,3})_{ij} &= (\mathbf{V}_{1,3})_{ij} \Omega_j^+ \\ (\mathbf{V}_{1,4})_{ij} &= \frac{\iota}{2\omega_j \sqrt{1 - \xi_j^2 (\Omega_j^- - \Omega_i^+)} (\Omega_j^- - \Omega_i^-)}, & (\mathbf{V}_{2,4})_{ij} &= (\mathbf{V}_{1,4})_{ij} \Omega_j^- \end{aligned} \quad (4.3.20)$$

with Ω_i^α previously defined. Those constants are time-invariant and only depend on the projection in the modal basis. Consequently, for LTI systems, they must be calculated once and for all at the beginning of the analysis. The $m \times m$ -integrals (4.3.16) are thus transformed thanks to the asymptotic expansion-based method into a sum, for which only the computation of $2m^2$ constants is required. However, the major computational gain of the method is found in the convolution integral (4.3.15). Actually, the first-order correction term of the instantaneous transfer function $\mathbf{G}_N(t, t_0, \omega)$ now reads

$$\Delta \mathbf{G}_1(t, t_0, \omega) = \mathbf{\Omega}_o \circ \Delta \mathbf{G}_{1,1}(t, t_0, \omega) + \mathbf{D}_o \circ \Delta \mathbf{G}_{1,2}(t, t_0, \omega), \quad (4.3.21)$$

since the system is time-invariant and the matrices $\mathbf{\Omega}_o$ and $\mathbf{D}_o$ are time-independent. The terms $\Delta \mathbf{G}_{1,1}$ and $\Delta \mathbf{G}_{1,2}$, defined as

$$\Delta \mathbf{G}_{1,L}(t, t_0, \omega) = \int_{t_0}^t a(u, \omega) \Delta \mathbf{h}_{1,L}(t - u) e^{\iota \omega u} du \quad (4.3.22)$$

with $L = 1, 2$, are computed by use of the previously defined integral $\mathcal{I}_k^\alpha(t, t_0, \omega)$. Indeed, introducing (4.3.19) into (4.3.22) leads to

$$(\Delta \mathbf{G}_{1,L}(t, t_0, \omega))_{ij} = (\mathbf{V}_{L,1})_{ij} \mathcal{I}_i^+ + (\mathbf{V}_{L,2})_{ij} \mathcal{I}_i^- + (\mathbf{V}_{L,3})_{ij} \mathcal{I}_j^+ + (\mathbf{V}_{L,4})_{ij} \mathcal{I}_j^-. \quad (4.3.23)$$

The computation of this first correction term is rapid. It does not demand any more complicated operation than the computation of the constants $(\mathbf{V}_{L,\cdot})_{ij}$, which are time invariant. Actually, the integral $\mathcal{I}_k^\alpha(t, t_0, \omega)$ is already computed for the decoupled leading order.

The underlying philosophy of this work is here clearly highlighted. The application of analytical methods and linear algebra allows to avoid some heavy computational operations.

We do not seek to utterly apply analytical methods. However, the understanding of the governing equations and the identification of general properties in structures lead us to develop mathematical tools in order to extract blocks of elementary results. Although rough numerical methods can be applied for those kinds of problems, the main advantage in semi-analytical approach is to simplify them into more basic problems.

Second-order correction term

The second correction term of the evolutionary transfer function reads

$$\Delta \mathbf{G}_2(t, t_0, \omega) = \int_{t_0}^t a(u, \omega) \Delta \mathbf{h}_2(t - u) e^{\iota \omega u} du. \quad (4.3.24)$$

The second correction term of the modal impulse response matrix $\Delta \mathbf{h}_2(u)$ is given by

$$\begin{aligned} (\Delta \mathbf{h}_2(u))_{ij} &= \sum_{k=1}^M \int_{\mathbb{R}} (\mathbf{H}_d)_i (\mathbf{J}_o)_{ik} (\mathbf{H}_d)_k (\mathbf{J}_o)_{kj} (\mathbf{H}_d)_j e^{\iota \omega u} d\omega \\ &= \sum_{k=1}^M (\mathbf{\Omega}_o)_{ik} (\mathbf{\Omega}_o)_{kj} \int_{\mathbb{R}} (\mathbf{H}_d)_i (\mathbf{H}_d)_j (\mathbf{H}_d)_k e^{\iota \omega u} d\omega \\ &\quad + \sum_{k=1}^M (\mathbf{D}_o)_{ik} (\mathbf{D}_o)_{kj} \int_{\mathbb{R}} (-\omega^2) (\mathbf{H}_d)_i (\mathbf{H}_d)_j (\mathbf{H}_d)_k e^{\iota \omega u} d\omega \\ &\quad + \sum_{k=1}^M \left((\mathbf{\Omega}_o)_{ik} (\mathbf{D}_o)_{kj} + (\mathbf{D}_o)_{ik} (\mathbf{\Omega}_o)_{kj} \right) \int_{\mathbb{R}} \iota \omega (\mathbf{H}_d)_i (\mathbf{H}_d)_j (\mathbf{H}_d)_k e^{\iota \omega u} d\omega. \end{aligned} \quad (4.3.25)$$

As previously, for $i = k$ or $j = k$, the result is trivial and should not be computed. Therefore, only the cases $i = j \neq k$ and $i \neq j \neq k$ must be studied. In a more general way, contributions to the second correction term are defined as

$$(\Delta \mathbf{h}_{2,L}(u))_{ijk} = \int_{\mathbb{R}} (\iota\omega)^{L-1} (\mathbf{H}_d)_i (\mathbf{H}_d)_j (\mathbf{H}_d)_k e^{\iota\omega u} d\omega \quad (4.3.26)$$

with $L = 1, 2, 3$. For $i \neq j \neq k$, the three poles of the integrand in (4.3.26) are distinct from each other, while the i -th pole has a multiplicity of two for $i = j \neq k$. The Cauchy theorem applied to (4.3.26) leads to

$$\begin{aligned} (\Delta \mathbf{h}_{2,L})_{ijk} &= (\mathbf{V}'_{L,1})_{ijk} \exp(\Omega_i^+ u) + (\mathbf{V}'_{L,2})_{ijk} \exp(\Omega_i^- u) \\ &\quad + (\mathbf{V}'_{L,5})_{ijk} \exp(\Omega_k^+ u) + (\mathbf{V}'_{L,6})_{ijk} \exp(\Omega_k^- u) \\ &\quad + (\mathbf{V}'_{L,3})_{ijk} u^{\delta_{ij}} \exp(\Omega_j^+ u) + (\mathbf{V}'_{L,4})_{ijk} u^{\delta_{ij}} \exp(\Omega_j^- u) \end{aligned} \quad (4.3.27)$$

with δ_{ij} the Kronecker index. Equation (4.3.27) is an expression similar to (4.3.19) for the first order correction term. Even if the use of $u^{\delta_{ij}}$ may seem exotic, it is convenient to gather the analytical expression in a compact and tractable form. As previously, introducing (4.3.27) into (4.3.24) leads to

$$\begin{aligned} (\Delta \mathbf{G}_{2,L})_{ijk} &= (\mathbf{V}'_{L,1})_{ijk} \mathcal{I}_i^+ + (\mathbf{V}'_{L,2})_{ijk} \mathcal{I}_i^- + (\mathbf{V}'_{L,3})_{ijk} \mathcal{I}_j^+ \\ &\quad + (\mathbf{V}'_{L,4})_{ijk} \mathcal{I}_j^- + (\mathbf{V}'_{L,5})_{ijk} \mathcal{I}_k^+ + (\mathbf{V}'_{L,6})_{ijk} \mathcal{I}_k^- \end{aligned} \quad (4.3.28)$$

for $i \neq j \neq k$, because the system is time invariant. For $i = j \neq k$, the knowledge of the integral $\mathcal{I}_k^\alpha$ is not sufficient and a new integral must be defined as

$$\mathcal{J}_k^\alpha(t, t_0, \omega) = e^{\Omega_k^\alpha t} \int_{t_0}^t (t-u) a(u, \omega) e^{-C_k^\alpha u} du \quad (4.3.29)$$

in order to write

$$\begin{aligned} (\Delta \mathbf{G}_{2,L})_{iik} &= (\mathbf{V}'_{L,1})_{iik} \mathcal{I}_i^+ + (\mathbf{V}'_{L,2})_{iik} \mathcal{I}_i^- + (\mathbf{V}'_{L,3})_{iik} \mathcal{J}_i^+ \\ &\quad + (\mathbf{V}'_{L,4})_{iik} \mathcal{J}_i^- + (\mathbf{V}'_{L,5})_{iik} \mathcal{I}_k^+ + (\mathbf{V}'_{L,6})_{iik} \mathcal{I}_k^- \end{aligned} \quad (4.3.30)$$

with C_k^α previously defined. Hence, the second correction term of the evolutionary transfer function reads

$$\begin{aligned} (\Delta \mathbf{G}_2)_{ij} &= \sum_{k=1}^M (\boldsymbol{\Omega}_o)_{ik} (\boldsymbol{\Omega}_o)_{kj} (\Delta \mathbf{G}_{2,1})_{ijk} + \sum_{k=1}^M (\mathbf{D}_o)_{ik} (\mathbf{D}_o)_{kj} (\Delta \mathbf{G}_{2,3})_{ijk} \\ &\quad + \sum_{k=1}^M \left((\boldsymbol{\Omega}_o)_{ik} (\mathbf{D}_o)_{kj} + (\mathbf{D}_o)_{ik} (\boldsymbol{\Omega}_o)_{kj} \right) (\Delta \mathbf{G}_{2,2})_{ijk}. \end{aligned} \quad (4.3.31)$$

A short analysis of $\mathcal{J}_k^\alpha(t, t_0, \omega)$ highlights possible simplifications of that integral, such that

$$\mathcal{J}_k^\alpha(t, t_0, \omega) = t \mathcal{I}_k^\alpha(t, t_0, \omega) - e^{\Omega_k^\alpha t} \int_{t_0}^t u a(u, \omega) e^{-C_k^\alpha u} du \quad (4.3.32)$$

after some algebra. The constants $\mathbf{V}'_{L,\cdot}$ are given in Appendix B.2.

Derivative of the evolutionary transfer matrix

From (4.3.12), (4.3.20) and (4.3.30), the N -order expansion of $\partial_t \mathbf{G}(t, t_0, \omega)$ required in (4.2.21) is directly computed by replacing in those equations, the terms $\mathcal{I}_k^\alpha$ and $\mathcal{J}_k^\alpha$ by their time derivatives, which are easily computed. Analytically, they are given by

$$\frac{d}{dt} \mathcal{I}_k^\alpha = \Omega_k^\alpha \mathcal{I}_k^\alpha + a(t, \omega) e^{i\omega t} \quad \text{and} \quad \frac{d}{dt} \mathcal{J}_k^\alpha = \mathcal{I}_k^\alpha + \Omega_k^\alpha \mathcal{J}_k^\alpha, \quad (4.3.33)$$

while they can also be numerically computed with a finite difference scheme. However, the time step must be adapted according to the natural frequencies of the structure. This fact may require really small steps, while the purpose of this chapter was to keep the time as an unconstrained parameter.

4.3.2 Convergence of the series $\mathbf{h}_N(t)$

In Section 3.3.4, the convergence of the asymptotic expansion $\mathbf{H}_N(\omega)$ is largely addressed. It is emphasized that the convergence criterion $\rho_{\mathbf{J}}$ must be less than one to ensure the convergence of the series. In (4.2.23), the inverse Fourier transform of the expansion $\mathbf{H}_N(\omega)$ is calculated to obtain an asymptotic expansion of the impulse response matrix $\mathbf{h}_N(t)$ of the structure, in the same modal basis.

We are now concerned about the convergence of $\mathbf{h}_N(t)$. The criterion $\rho_{\mathbf{J}} < 1$ ensures that the series $\mathbf{H}_N(\omega)$ converges. Therefore, the series $\mathbf{h}_N(t)$ is also convergent, because of the linearity of the Fourier operator. As experienced by the author, the computation of correction terms for a non-convergent series $\mathbf{H}_N(\omega)$ may have for consequences that the elements of the impulse response matrix do not start at zero as expected. Naturally, this criterion is a convergence criterion on the long term (since $\mathbf{H}_N(\omega)$ is related to the stationary response of the structure in the frequency domain). The matrix $\mathbf{h}_N(t)$ features a transient regime before this stationary response. Numerical experiments have shown that the convergence criterion might be relaxed over the short term, i.e. in the transient regime. For short time intervals $[0, t]$, the number of correction terms N required for a certain accuracy is thus smaller, than for the stationary solution.

We illustrate this statement with the particular case $\mathbf{J}_o = i\omega \mathbf{D}_o$. Thus, the term $\Delta \mathbf{h}_k(t)$ reads

$$\Delta \mathbf{h}_k(t) = \int_{\mathbb{R}} (i\omega)^k (\mathbf{H}_d \mathbf{D}_o)^k \mathbf{H}_d e^{i\omega t} d\omega. \quad (4.3.34)$$

We propose to set $\mathbf{H}_d(\omega) = \bar{H}(\omega) \mathbf{I}$ with

$$\bar{H}(\omega) = \frac{1}{(\omega_0^2 - \omega^2) + (2i\xi\omega_0\omega)}, \quad (4.3.35)$$

assuming thus, for the sake of simplicity, that all the modes have the same natural frequency and the same damping ratio. Indeed, the integral (4.3.35) may thus be written

$$\Delta \mathbf{h}_k(t) = \bar{h}_k(t) \mathbf{D}_o^k \quad (4.3.36)$$

with the time-dependent function

$$\hbar_k(t) = \int_{\mathbb{R}} (\iota\omega)^k \bar{H}(\omega)^{k+1} e^{\iota\omega t} d\omega. \quad (4.3.37)$$

We notice that the integral (4.3.37) exists and is well-defined for $k \in \mathbb{N}$. Evoking Cauchy's theorem, we can compute closed-form expressions for (4.3.37). Nonetheless, we are mainly interested in the behavior of those coefficients $\hbar_k$ around $t = 0$. Thus we only provide the first terms of their Taylor expansion, such that

$$\begin{aligned} \hbar_1(t) &= \frac{1}{\omega_0^2} \left(\pi(t\omega_0)^2 - \frac{4}{3} \pi \xi(t\omega_0)^3 \right) + \mathcal{O}(t^4) \\ \hbar_k(t) &= \frac{2\pi t^{k+1}}{k+1!} + \mathcal{O}(t^{k+2}) \end{aligned} \quad (4.3.38)$$

Provided the product $(t\omega_0)$ is small, the correction given by $\hbar_k$ becomes negligible. This short demonstration heuristically explains the reason why the number of required terms in $\mathbf{h}_N(t)$ is limited on short time intervals. This property will be of paramount importance in the following developments.

4.3.3 Analytical expressions for $\mathcal{I}_k^\alpha$ and $\mathcal{J}_k^\alpha$

The expressions of $\mathcal{I}_k^\alpha(t, t_0, \omega)$ and $\mathcal{J}_k^\alpha(t, t_0, \omega)$ may be obtained for many analytical expressions of the time envelope $a(t, \omega)$. The factor $e^{\iota\omega u}$ in (4.3.13) and (4.3.32) contains a range of harmonic that must be taken into account as accurately as possible. For instance, a loss of precision in a numerical representation of this factor may lead to poor approximation of (4.3.18), an integral in the frequency domain. The possibility to avoid numerical integration will be received by programmers and practitioners as a great advantage for the computational cost and the accuracy of the method, because the question of the convergence of the integrals is discarded. Conceptually, the reason for simplifying the problem posed in both frequency and time domains is pursued in this chapter by trying to avoid crude numerical integration in both domains.

First, two antiderivatives $\mathcal{P}_k^\alpha(u, \omega) : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{C}$ and $\mathcal{Q}_k^\alpha(u, \omega) : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{C}$ are defined as

$$\mathcal{P}_k^\alpha(u, \omega) = \int a(u, \omega) e^{-C_k^\alpha(\omega)u} du \quad (4.3.39)$$

$$\mathcal{Q}_k^\alpha(u, \omega) = \int u a(u, \omega) e^{-C_k^\alpha(\omega)u} du \quad (4.3.40)$$

in order to compute the integrals $\mathcal{I}_k^\alpha(t, t_0, \omega)$ and $\mathcal{J}_k^\alpha(t, t_0, \omega)$, such that

$$\begin{aligned} \mathcal{I}_k^\alpha(t, t_0, \omega) &= e^{\Omega_k^\alpha t} \int_{t_0}^t a(u, \omega) e^{-C_k^\alpha u} du \\ &= e^{\Omega_k^\alpha t} [\mathcal{P}_k^\alpha(t, \omega) - \mathcal{P}_k^\alpha(t_0, \omega)], \end{aligned} \quad (4.3.41)$$

$$\begin{aligned}
\mathcal{J}_k^\alpha(t, t_0, \omega) &= e^{\Omega_k^\alpha t} \int_{t_0}^t (t-u) a(u, \omega) e^{-\mathcal{C}_k^\alpha u} du \\
&= t\mathcal{I}_k^\alpha(t, t_0, \omega) - e^{\Omega_k^\alpha t} [\mathcal{Q}_k^\alpha(t, \omega) - \mathcal{Q}_k^\alpha(t_0, \omega)].
\end{aligned} \tag{4.3.42}$$

Elegant analytical expressions for $\mathcal{P}_k^\alpha$ and $\mathcal{Q}_k^\alpha$ can be found for some well-known and largely-used time envelopes $a(t, \omega)$. The technique used in (4.3.41) and (4.3.42) works with the ease of simplicity. Before giving some results, two families of spectra must be defined: the **separable spectra** are distinct from the **non-separable spectra**.

A spectrum $\hat{S}(t, \omega)$ is said to be separable, if the frequency and the time dependencies are gathered in two separate functions, i.e.

$$\hat{S}(t, \omega) = |a(t)|^2 S(\omega). \tag{4.3.43}$$

In this case, the time envelope is a deterministic function and the spectrum $S(\omega)$ is a Kanai-Tajimi (modified or not) spectrum, for instance. This definition applies also to seismic field.

For instance, the famous **Shinozuka window** reads

$$a(t) = c \left(e^{-b_1 t} - e^{-b_2 t} \right) \mathbb{H}(t) \tag{4.3.44}$$

with $\mathbb{H}(t)$ the Heaviside function. The exponential terms and the continuous formulation make Shinozuka's window useful to compute the previous integrals. Closed-form solutions of the primitive integrals are

$$\mathcal{P}_k^\alpha(u, \omega) = ce^{-\mathcal{C}_k^\alpha u} \left(\frac{e^{-b_2 u}}{b_2 + \mathcal{C}_k^\alpha} - \frac{e^{-b_1 u}}{b_1 + \mathcal{C}_k^\alpha} \right) \tag{4.3.45}$$

$$\mathcal{Q}_k^\alpha(u, \omega) = ce^{-\mathcal{C}_k^\alpha u} \left(\frac{e^{-b_1 u} (1 + (b_1 + \mathcal{C}_k^\alpha)u)}{(b_1 + \mathcal{C}_k^\alpha)^2} - \frac{e^{-b_2 u} (1 + (b_2 + \mathcal{C}_k^\alpha)u)}{(b_2 + \mathcal{C}_k^\alpha)^2} \right). \tag{4.3.46}$$

Now, the **Jenning window** is considered, as it expresses intuitively the three phases of an earthquake (build-up, plateau and decay). This piecewise window models each phase by distinct analytical functions, as

$$a(t) = \begin{cases} \left(\frac{t}{t_1} \right)^2 & , 0 < t < t_1 \\ 1 & , t_1 < t < t_2 \\ e^{\gamma(t_2-t)} & , t > t_2 \end{cases} \tag{4.3.47}$$

The main disadvantage of this window is that the integration must be performed interval by interval. The primitive integrals are

$$\mathcal{P}_k^\alpha(u) = -e^{-\mathcal{C}_k^\alpha u} \begin{cases} \frac{2 + \mathcal{C}_k^\alpha u (2 + \mathcal{C}_k^\alpha u)}{t_1^2 (\mathcal{C}_k^\alpha)^3} & , 0 < u < t_1 \\ \frac{1}{\mathcal{C}_k^\alpha} & , t_1 < u < t_2 \\ \frac{e^{\gamma(t_2-u)}}{\mathcal{C}_k^\alpha + \gamma} & , u > t_2 \end{cases} \tag{4.3.48}$$

$$\mathcal{Q}_k^\alpha(u) = -e^{-C_k^\alpha u} \begin{cases} \frac{6+C_k^\alpha u(6+C_k^\alpha u(3+C_k^\alpha u))}{t_1^2(C_k^\alpha)^4} & , 0 < u < t_1 \\ \frac{1+uC_k^\alpha}{(C_k^\alpha)^2} & , t_1 < u < t_2 \\ \frac{e^{\gamma(t_2-u)}}{(C_k^\alpha+\gamma)^2} (1+u(C_k^\alpha+\gamma)) & , u > t_2 \end{cases} \quad (4.3.49)$$

These two time envelopes are depicted in Figure 4.2-a. Many other functions exist in the literature ; most of them can be easily integrated according to (4.3.39) and (4.3.40). Hence, the method can be applied to a large range of envelopes.

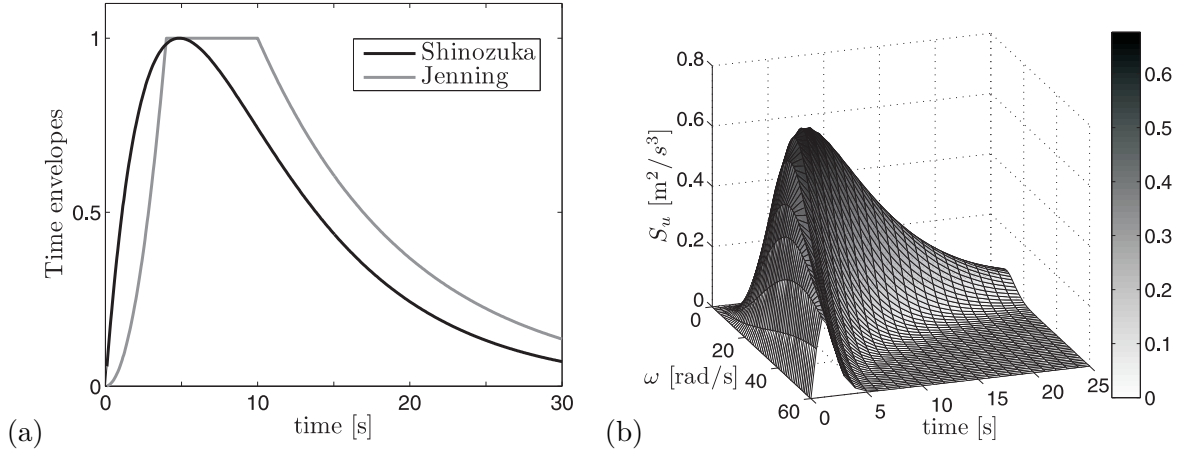


Figure 4.2: Separable spectrum (a) : Jennings's and Shinozuka's time windows. Non-separable spectrum (b) : Liu's spectrum with $\gamma(\omega) = \beta(\omega) = \left(\frac{\omega}{15\pi}\right)^2$, $\bar{S} = 1\text{m}^2/\text{s}^3$ and $b = 0.2\text{Hz}$.

In the second family of spectrum, the time window is frequency dependent. Indeed, the separable spectrum are not able to precisely model some phenomena occurring during an earthquake: the frequency content evolves in time, but it is not necessarily homothetic between different instants. Liu [Liu 70] and then Spanos [Span 83], propose to use a non-separable evolutionary spectrum, the so-called **Liu spectrum**, modeled as

$$\hat{S}_{\text{Liu}}(t, \omega) = \bar{S} \gamma(\omega) t^2 e^{-bt - \beta(\omega)t} \mathbb{H}(t) \quad (4.3.50)$$

with $\bar{S}$ the excitation amplitude. This spectrum is depicted in Figure 4.2-b. Equation (4.3.50) is an elegant way to gather in a simple formulation different existing models for non-separable (or not) earthquake evolutionary spectrum. The spectrum (4.3.50) enables to represent two main features of a seismic excitation as the *accumulation* phase and the *extinction* one [Span 83]. Furthermore, for realistic earthquakes, the dominant frequency range is decreasing with time [Span 83, Koug 13], the time-frequency dependence must be taken into account.

Even though the function is not separable in the strict sense, Equation (4.3.50) is split into two main factors: a purely frequency dependent factor and a mixed time-frequency dependent factor. A time window can thus be extracted such that

$$a(t, \omega) = t^2 e^{-bt - \beta(\omega)t}, \quad (4.3.51)$$

while the factor $\bar{S}\gamma(\omega)$ may be interpreted as the psd of an equivalent non-physical stationary process. The primitive integrals $\mathcal{P}_k^\alpha$ and $\mathcal{Q}_k^\alpha$ are given by

$$\mathcal{P}_k^\alpha(u, \omega) = \int u^2 e^{-\phi_k^\alpha(\omega)u} du = -\frac{e^{-\phi_k^\alpha u} (1 + (\phi_k^\alpha u + 1)^2)}{(\phi_k^\alpha)^3} \quad (4.3.52)$$

$$\mathcal{Q}_k^\alpha(u, \omega) = \int u^3 e^{-\phi_k^\alpha(\omega)u} du = -\frac{e^{-\phi_k^\alpha u} (6 + 3\phi_k^\alpha u + (\phi_k^\alpha u + 1)^3)}{(\phi_k^\alpha)^4} \quad (4.3.53)$$

with $\phi_k^\alpha(\omega) = b + \beta(\omega) + \mathcal{C}_k^\alpha(\omega)$. This example highlights that the proposed developments are not limited to time windows only depending on time. The generation method based on spectral decomposition [Shin 91], similar to the method described in Section 2.4, cannot be applied to that kind of spectra, more advanced generation techniques must be used to create the dependency between time and frequency in the seismic signal.

As the results of this Section show, the time in the analysis may be effectively seen as a parameter, independently of the past in the computation. This goal is achieved at this stage for some well-know and often used time envelopes. However, we cannot be just satisfied with these closed-form expressions. Actually, from a numerical point of view, the computation of the functions $\mathcal{P}(t, \omega)$ and $\mathcal{Q}(t, \omega)$ for a range of frequency ω may become time-consuming, depending on their complexity and on the number of required elementary operations. In other words, computing closed-form solutions is a first step, but it is not a guarantee of efficiency, as experienced by the author. Therefore, we need to generalize our method to any intensity function with a more convenient form, and perhaps more elementary. In the following Section, an efficient approach is proposed in order to improve the numerical implementation of our method. Although the evolutionary transfer matrix is developed in terms of physical (modal) coordinates, the state space formalism with the functions $\Psi(t, t_0)$ and $\Upsilon(t, t_0, \omega)$ will be of paramount importance.

4.4 Efficient formulation for numerical computation

In the preceding Section, all the functions are expressed as function of two times, both t_0 and t . For instance, Equation (4.2.8) highlights that $\mathbf{G}(t, t_0, \omega)$ requires to compute the convolution integral between $\mathbf{h}(t)$ and $a(t, \omega)e^{i\omega u}$ from an initial time t_0 to an instant t . This integral means that the whole history of the response must be known until time t to perform the integration. This formulation of the problem suits clearly well the aforementioned time windows, because closed-form expressions of primitive integrals may be found. Although the Shinozuka, Liu and Jennings time windows are common in civil engineering, a more general approach may be implemented knowing the time window of the evolutionary random process. Indeed, the previous envelopes are from classical applications in earthquake engineering. In wind engineering and downburst models, the time modulation results from a more complex relation between translation wind velocity and radial wind velocity [Holm 00], even if Chen uses analytical time windows to study transient wind on building [Chen 08].

In this Section, a strategy based on recurrence formula is derived from the elementary properties of LTI systems. This strategy will be of paramount importance in the following

Chapter in which nonlinear forces will be considered as evolving linearized forces. Indeed, the properties used in the following, called the *semi-group property* is not only a property of LTI systems, but of linear systems in general.

First, the whole time interval is divided into $M + 1$ non-overlapping intervals, such as

$$t_0 = 0 < \dots < t_k < \dots t_M \leq t \leq t_{M+1}, \quad (4.4.1)$$

with $k, M \in \mathbb{N}_0$ and with possible different lengths. Assuming piecewise continuity of the functions involved in the state space evolutionary transfer matrix $\Upsilon(t_0, t, \omega)$, the integral over the time domain $[t_0, t]$ may be rewritten as

$$\Upsilon(t, t_0, \omega) = \sum_{k=1}^M \int_{t_{k-1}}^{t_k} a(u, \omega) \Psi(t, u) \mathbf{B} e^{\iota \omega u} du + \int_{t_M}^t a(u, \omega) \Psi(t, u) \mathbf{B} e^{\iota \omega u} du \quad (4.4.2)$$

The well-known property of the transition matrix $\Psi(t, t_0)$ is called the **semi-group property** [Hesp 09], expressing that the transition between two instants t_0 and t is equivalent to a first transition from t_0 to an intermediate time t_1 , then a second transition from t_1 to t , i.e.

$$\Psi(t, t_0) = \Psi(t, t_1) \Psi(t_1, t_0). \quad (4.4.3)$$

This useful property allows to make the integral (4.4.2) independent of t , such that

$$\Upsilon(t, t_0, \omega) = \sum_{k=1}^M \Psi(t, t_k) \int_{t_{k-1}}^{t_k} a(u) \Psi(t_k, u) \mathbf{B} e^{\iota \omega u} du + \int_{t_M}^t a(u, \omega) \Psi(t, u) \mathbf{B} e^{\iota \omega u} du \quad (4.4.4)$$

In this last equation, we identify the last term as the state evolutionary transfer matrix in the interval $[t_M, t]$, i.e.

$$\Upsilon(t, t_M, \omega) = \int_{t_M}^t a(u) \Psi(t, u) \mathbf{B} e^{\iota \omega u} du. \quad (4.4.5)$$

Then, we use the *semi-group property* between the three instants t , t_M and t_k , i.e. $\Psi(t, t_k) = \Psi(t, t_M) \Psi(t_M, t_k)$, in the summation (4.4.4). Therefore, we can highlight a convenient **recurrence relation** for the computation of $\Upsilon(t, t_0, \omega)$, i.e.

$$\Upsilon(t, t_0, \omega) = \Psi(t, t_{M-1}) \Upsilon(t_M, t_0, \omega) + \Upsilon(t, t_M, \omega). \quad (4.4.6)$$

The physical meaning of the two terms in (4.4.6) is noteworthy. The first term expresses that the past of the stochastic system up to the time t_M is readily projected at the time t by the transition matrix between these two instants. The second term models the linear response of the system for the loads appearing during the interval $[t_M, t]$. This recurrence formula is of paramount importance from a computational point of view, because it highlights that a transition from one state to another requires only to compute information during this interval, the past being summarized in the function $\Upsilon(t_M, t_0, \omega)$. Equation (4.4.5) may also read

$$\Upsilon(t, t_M, \omega) = \begin{pmatrix} \mathbf{0} & \mathbf{G}(t, t_M, \omega) \\ \mathbf{0} & \partial_t \mathbf{G}(t, t_M, \omega) \end{pmatrix} \quad (4.4.7)$$

by use of the evolutionary transfer matrix and its time derivative computed by use of the integrals $\mathcal{I}_k^\alpha(t, t_M, \omega)$ and $\mathcal{J}_k^\alpha(t, t_M, \omega)$ previously defined.

In order not to bound these developments to the windows of Section 4.3.3, a generalized procedure may be imagined to perform those integrals with a limited numerical burden. Actually, this approach has been imagined to become able to deal with transient wind loads and with the time evolution proposed by [Holm 00]¹. It is proposed to sample the time window and to interpolate it linearly between the sample points. The integrals $\mathcal{I}_k^\alpha(t, t_M, \omega)$ and $\mathcal{J}_k^\alpha(t, t_M, \omega)$ may read

$$\mathcal{I}_k^\alpha(t, t_M, \omega) = e^{\Omega_k^\alpha t} \int_{t_M}^t a_M(u) e^{-\mathcal{C}_k^\alpha(\omega)u} du, \quad (4.4.8)$$

$$\mathcal{J}_k^\alpha(t, t_M, \omega) = e^{\Omega_k^\alpha t} \int_{t_M}^t (t - u) a_M(u) e^{-\mathcal{C}_k^\alpha(\omega)u} du \quad (4.4.9)$$

with $a_{M+1}(u)$ a linear interpolation of $a(u)$ between $a(t_M)$ and $a(t_{M+1})$, such that

$$a_{M+1}(u) = \frac{a(t_{M+1}) - a(t_M)}{t_{M+1} - t_M} (u - t_M) + a(t_M). \quad (4.4.10)$$

In [Sun 89], the intensity function is replaced by a staircase function. However, the linear interpolation allows to keep the continuity and to have a better accuracy over the whole function for a similar computational effort compared with Sun's approximation. Even if it is trivial, the following primitive integral is central in that case,

$$\int u^j e^{-\mathcal{C}_k^\alpha(\omega)u} du = \frac{e^{-\mathcal{C}_k^\alpha(\omega)u}}{-\mathcal{C}_k^\alpha(\omega)} u^j + \frac{j}{\mathcal{C}_k^\alpha(\omega)} \int u^{j-1} e^{-\mathcal{C}_k^\alpha(\omega)u} du \quad (4.4.11)$$

with $j \in \mathbb{N}$. These results and the procedure described in this Section allow to efficiently and generally implement the evolutionary spectral analysis for a large family of windows.

The sampling of the interval $[0, t]$ proposed in (4.4.1) should naturally take into account the shape of the time window in order to reproduce properly the true one. This method advantageously allows the use of different time steps in the simulation. For instance, for window presenting a large *plateau*, few (even a single one) steps may be considered. The time intervals may be chosen regardless of the dynamics of the system to capture the evolution of the window.

4.5 Multi-span bridge under differential ground acceleration

An application in earthquake engineering is now proposed. It is inspired from [Pero 90], in which the evolutionary spectral analysis is used to compute the response of a multi-span bridge subject to earthquake modeled as a coherent random field. Some assumptions formulated in [Pero 90] may be overtaken by the theoretical developments of this work. Indeed, the structure is supposed to be linear and the damping proportional according to Rayleigh assumption. In this case, the example illustrates the developments of this Chapter limited to the main decoupled approximation.

¹An application is proposed in the next Chapter with this window.

Model of coherence in earthquake engineering

This application allows to illustrate the purpose of this work on a large structure subject to coherent excitation field. As explained in Chapter 2, the concept of random field is distinct from a random process, since the former one depends on both time and space. During an earthquake, the coherence mainly originates from the propagation of the seismic waves taking into account their direction of propagation. For long structures, assuming fully coherent ground motion is phenomenologically fallacious and may lead to response overestimation. On the other hand, considering the acceleration on the structure as purely non-coherent may lead to underestimate the response. Some details can now be given about classic coherence functions used to model the spatial variation of seismic ground motions without taking the risk of excessive details, while extensive discussions can be found in [Kaha 96, Zerv 02].

Although the wind coherence function is mainly considered as a real-valued function, in earthquake engineering this is not the case due to the concept of wave transportation. It should be pointed out that this assumption in wind engineering is not systematically encountered in practice, but this discussion is beyond the scope of this work. The psd matrix of seismic ground acceleration can be modeled by

$$(\mathbf{S}_{\ddot{u}}(\omega))_{ij} = S_{\ddot{u}}(\omega) |\Gamma_{ij}(\omega, d_{ij})| \exp\left(-i\frac{\omega d_{ij}}{V}\right) \quad (4.5.1)$$

with $S_{\ddot{u}}(\omega)$ the unilateral psd of ground acceleration (e.g. the Kanai-Tajimi spectrum in (3.5.11)), d_{ij} the distance between the points i and j and V the *apparent wave velocity* (between 1 and 4km/s), i.e. the velocity detected by the structure. The imaginary exponential in (4.5.1) models the *wave passage effect*, i.e. the delay between wave arrival times at different points. The norm of the function $\Gamma_{ij}(\omega, d_{ij})$, usually referred to as the *lagged coherency* [Zerv 02] or the *incoherency* [Kiur 96], expresses the linear relation between two acceleration signals. It is expected that for a short separation distance between two points and a low frequency content, the signal measured at those points are almost similar, so the coherence function tends to one. In the opposite case, the coherence function tends to zero. As suggested in [Kiur 96], parameters can be added in the model to take into account possible site effects.

Before addressing any mathematical formulation, it should be noticed that the modeling of coherence function in earthquake engineering is achieved with difficulty, because it requires dense installation of instruments, explaining the scarcity of the recorded data. Even if it was not the first one, the project SMART-1 (Strong Motion ARray in Taiwan), located in Lotung and initiated in 1980, has provided valuable data for small and large magnitude earthquakes, and more precisely for modeling the intrinsic coherence. Since the 70's, many empirical models have been proposed in order to judiciously capture the incoherency [Luco 86, Some 91]; all of them are based on an exponential decay. However, Hariachandran and Vanmarcke [Hari 86] proposed the following expression based on the SMART-1 data,

$$|\Gamma_{ij}(\omega, d_{ij})| = A \exp\left(\frac{-2d_{ij}}{\alpha\vartheta(\omega)}(1 - A + \alpha A)\right) + (1 - A) \exp\left(\frac{-2d_{ij}}{\vartheta(\omega)}(1 - A + \alpha A)\right) \quad (4.5.2)$$

with

$$\vartheta(\omega) = k_\vartheta \left(1 + \left| \frac{\omega}{\omega_\vartheta} \right|^b \right)^{-\frac{1}{2}} \quad (4.5.3)$$

and $A = 0.736$, $\alpha = 0.147$, $\omega_\vartheta = 6.85 \text{ rad/s}$ and $b = 2.78$. The length k_ϑ (between 5 and 10km) may be adjusted to modify the coherence function. This model, used in this application, is also applied in [Pero 90]. In the following, the apparent velocity is fixed at 1km/s and the length k_ϑ at 5km. Synthetic ground accelerations can also be generated by numerically computing the wave transportation from the seismic source through the different ground levels [Shin 99]; statistics may then be computed on these samples.

Structural dynamics under differential ground motion

Because the seismic excitation is due to ground motion, the excitation comes from the foundation of the structure. Therefore, we do not write directly (3.2.1), because the forces on the structure do not act directly. Furthermore, due to the differential seismic excitation, the equation of motion is not given by (3.5.9), in which all the supports of the structure are excited by the same ground acceleration.

To derive the equation of motion in our situation, we must split the vector of nodal displacement $\mathbf{x}(t)$ into two vectors: a first vector $\mathbf{x}_S(t)$ containing the DOF of the structure and a vector $\mathbf{u}_g$ the ones of the foundation (i.e. the fixed DOFs). The equation of motion (3.2.1) thus reads

$$\begin{pmatrix} \mathbf{M} & \mathbf{M}_c \\ \mathbf{M}_c^T & \mathbf{M}_g \end{pmatrix} \begin{pmatrix} \ddot{\mathbf{x}}_S \\ \ddot{\mathbf{u}}_g \end{pmatrix} + \begin{pmatrix} \mathbf{C} & \mathbf{C}_c \\ \mathbf{C}_c^T & \mathbf{C}_g \end{pmatrix} \begin{pmatrix} \dot{\mathbf{x}}_S \\ \dot{\mathbf{u}}_g \end{pmatrix} + \begin{pmatrix} \mathbf{K} & \mathbf{K}_c \\ \mathbf{K}_c^T & \mathbf{K}_g \end{pmatrix} \begin{pmatrix} \mathbf{x}_S \\ \mathbf{u}_g \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{f}_g \end{pmatrix} \quad (4.5.4)$$

with $\mathbf{M}_S$, $\mathbf{M}_c$ and $\mathbf{M}_g$ the blocks of the global mass matrix. The same division is applied to the damping and the stiffness matrices. The vector $\mathbf{f}_g(t)$ gathers virtual forces inducing the accelerations $\ddot{\mathbf{u}}(t)$ at the ground level. Then, the first line of (4.5.4) is recast into

$$\mathbf{M}\ddot{\mathbf{x}}_S + \mathbf{C}\dot{\mathbf{x}}_S + \mathbf{K}\mathbf{x}_S = -\mathbf{M}_c\ddot{\mathbf{u}}_g - \mathbf{C}_c\dot{\mathbf{u}}_g - \mathbf{K}_c\mathbf{u}_g. \quad (4.5.5)$$

Since the ground acceleration $\mathbf{u}_g(t)$ is supposed to be a vector of Gaussian processes with zero mean, the vector $\mathbf{x}_S(t)$ gathers also zero mean processes, because the structure is supposed to start from rest. Without any loss of generality, the vector $\mathbf{x}_S(t)$ can be decomposed into a pseudo-static (or background) component and a dynamic (or resonant) one, such that

$$\mathbf{x}_S = \mathbf{x}_B + \mathbf{x}_R. \quad (4.5.6)$$

The background component $\mathbf{x}_B(t)$ results from a static analysis

$$\mathbf{x}_B = \mathbf{R}\mathbf{u}_g \quad (4.5.7)$$

with

$$\mathbf{R} = -\mathbf{K}^{-1}\mathbf{K}_c. \quad (4.5.8)$$

Substitution of (4.5.6) into (4.5.5) yields

$$\mathbf{M}\ddot{\mathbf{x}}_R + \mathbf{C}\dot{\mathbf{x}}_R + \mathbf{K}\mathbf{x}_R = -(\mathbf{M}\mathbf{R} + \mathbf{M}_c)\ddot{\mathbf{u}}_g - (\mathbf{C}\mathbf{R} + \mathbf{C}_c)\dot{\mathbf{u}}_g. \quad (4.5.9)$$

We could admit, as previously, that $\mathbf{M}_c$ is negligible with respect to $\mathbf{M}_R$. In typical applications, we expect $\mathbf{M}_R$ to be fully populated, while $\mathbf{M}_c$ has non-zero entries only for structural DOFs coupled with supports. Several authors ([Clou 93, Hari 96]) claim that the term related to the damping forces in (4.5.9) may be neglected, and so do we in the following. Thus, the remaining equation to solve is

$$\mathbf{M}\ddot{\mathbf{x}}_R + \mathbf{C}\dot{\mathbf{x}}_R + \mathbf{K}\mathbf{x}_R = -(\mathbf{M}_R + \mathbf{M}_c)\ddot{\mathbf{u}}_g. \quad (4.5.10)$$

Equations (4.5.7) and (4.5.10) highlight that the pseudo-static response is trivially obtained in the nodal basis and that the effective loading on the resonant contribution is high-passed filtered. Considering the set of linear normal modes of vibration Φ , the projection of (4.5.10) into the modal space reads

$$\ddot{\mathbf{q}}_R + \mathbf{D}\dot{\mathbf{q}}_R + \mathbf{\Omega}\mathbf{q}_R = -\Phi^T(\mathbf{M}_R + \mathbf{M}_c)\ddot{\mathbf{u}}_g \quad (4.5.11)$$

with the notation similar to (3.3.5) and with $\mathbf{x}_R = \Phi\mathbf{q}_R$.

In a stochastic framework now, the resolution of the equation set (4.5.7) and (4.5.10) can be performed with the method developed in this Chapter. The random transient pseudo-static response is readily given by

$$\hat{\mathbf{S}}_{\mathbf{x}_B}(t, \omega) = \mathbf{R}\hat{\mathbf{S}}_{\mathbf{u}_g}(t, \omega)\mathbf{R}^T. \quad (4.5.12)$$

It should be noticed that for evolutionary spectrum $\hat{\mathbf{S}}_{\ddot{\mathbf{u}}_g} \neq \omega^4\hat{\mathbf{S}}_{\mathbf{u}_g}$, due to (4.2.1). In the following of this application, we only focus on the resonant response, a reason for dropping the index R in (4.5.11), as no confusion is possible.

Characteristics of the multi-span bridge

The bridge, considered in this example and depicted in Figure 4.3, counts 20 spans of 40m long and the piles are 20m high. The characteristics of the deck are $E = 40\text{GPa}$, $I = 40\text{m}^4$, $A = 6\text{m}^2$ and the ones of the piles are $E = 30\text{GPa}$, $I = 9\text{m}^4$, $A = 14\text{m}^2$. The density of both elements is 3000kg/m^3 . Perotti's problem is now slightly modified. The connections between the piles and the deck are relaxed and the damping devices are set up on different connections in order to mitigate seismic vibration. By this way, a mechanical coupling is created within the structure. The connection stiffness is equal to 100MN/m . Based on these structural properties, a FE model is built up with 5 elements for both piles and spans. The damping matrix of the initial structure is constructed by imposing the damping ratio equal to 1.0% in the first and third modes. The first five natural frequencies of this structure are 1.28, 1.29, 1.30, 1.33 and 1.40Hz. The shape of the first mode is also shown in Figure 4.3.

About the excitation, the parameters of the Kanai-Tajimi spectrum ($S_{\ddot{u}}$ in (4.5.1)) are $S_0 = 3 \cdot 10^{-3}\text{m}^2\text{s}^{-3}$, $\omega_1 = 2.5\pi\text{rad/s}$, $\xi_1 = 0.2$, $\omega_2 = 0.5\text{rad/s}$, $\xi_2 = 0.6$ and Jennings's window (4.3.47) is considered with $t_1 = 3\text{s}$, $t_2 = 10\text{s}$ and $\gamma = 1\text{Hz}$.

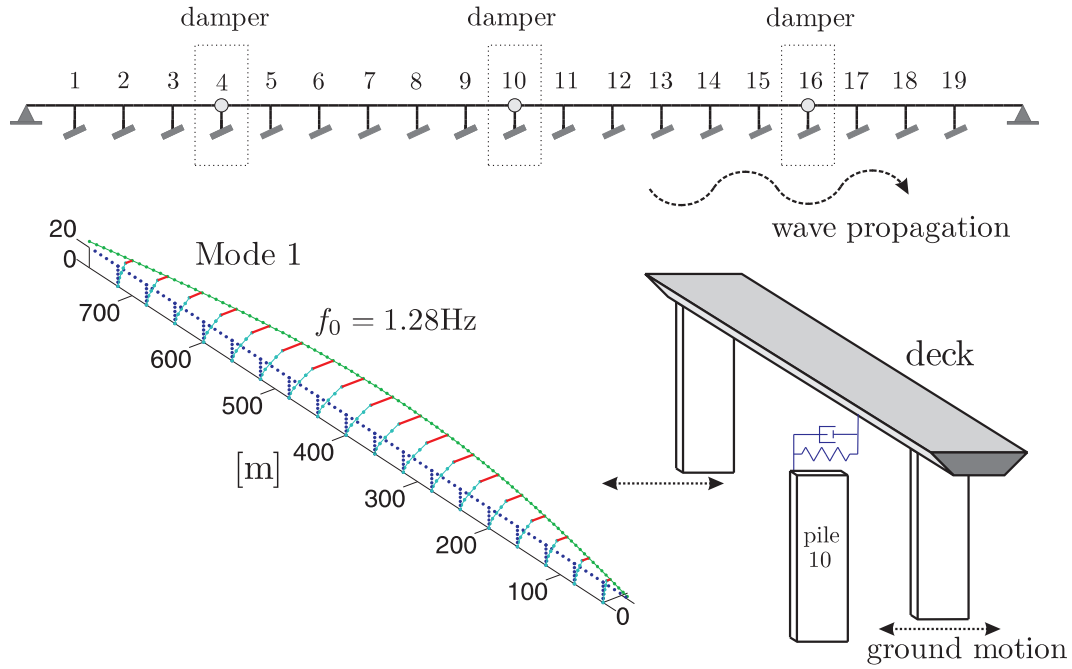


Figure 4.3: Multiple-span bridge subject to coherent earthquake. Dampers are set on the top of piles 4, 10 and 16. Sketch of a damped connection at pile 10. Shape of the first mode.

The structure is first analyzed without damping device, then a linear coupling is investigated by assuming a linear behavior of the dampers. From the example presented in Section 3.5, we know that the validity of the approach mainly depends on the projection of the non-linear forces in the modal basis. Hence, in order to show a case where more correction terms are involved in the computation, the dampers are supposed to be set on the connections 4, 10 (mid-length of the bridge) and 16, leading to a non-classically damped structure.

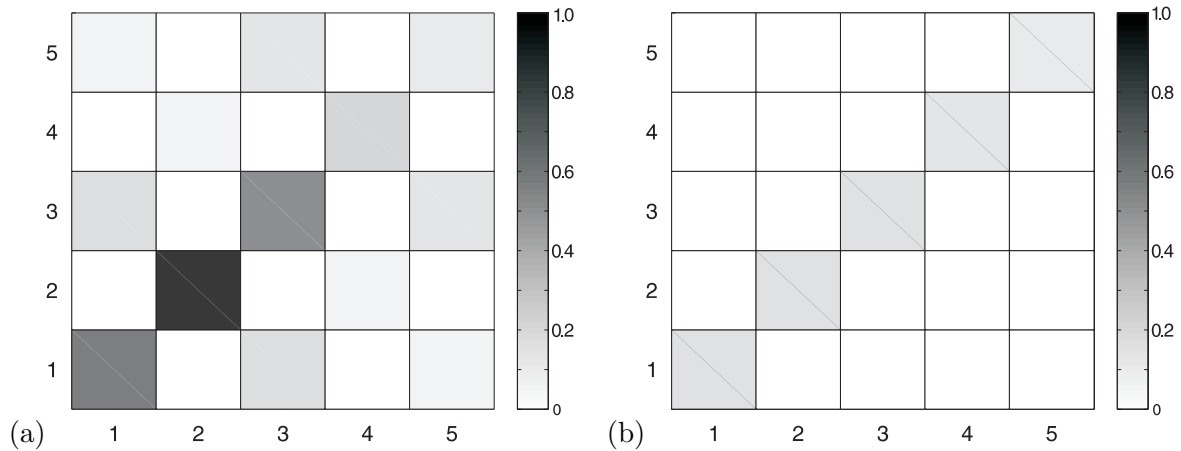


Figure 4.4: Multi-span bridge under earthquake. Absolute value of the covariance matrices of the generalized forces Σ_p in $[\text{kN}^2]$ for coherent (a) and non-coherent (b) excitations

Initial structure and coherence model

In the initial structure, the response is linear and the structural damping is built up to be proportional. This case allows to illustrate explicitly, the effect of coherence within the earthquake on the modal response of the structure. Figure 4.4 highlights the influence of the coherence on the generalized forces. With coherence the second mode is mainly excited, while the first five modes are similarly excited without coherence. Neglecting the coherence underestimates the forces on the structure.

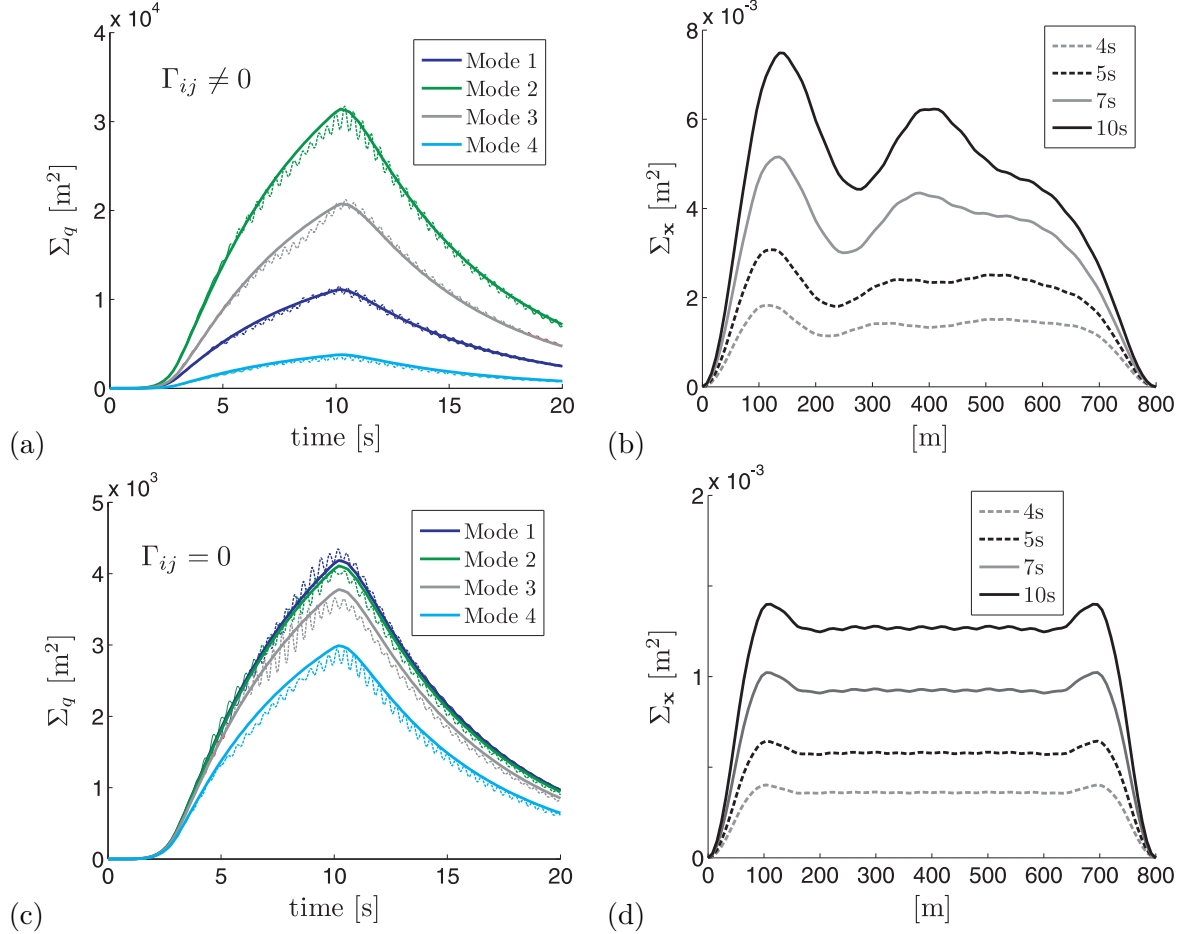


Figure 4.5: Multi-span bridge under (a,b) coherent and (c,d) non-coherent earthquake. (a,c) Variance of the first four modes with respect to time. Comparison between equivalent linearization and Monte Carlo simulation (thin dotted line). (b,d) Variance of the deck displacements at four different times.

Figure 4.5-a shows the time evolution of the variance of the first four modes under coherent earthquake, while Figure 4.5-c depicts these variances for a purely uncorrelated excitation, which is equivalent to an infinite apparent velocity. In the present problem, the response is deeply influenced by the coherence function: the modal amplitudes are more important for a coherent loading and the intensities among the modal amplitude is completely changed. Indeed, the loss of coherence along the bridge reduces the response in the first mode compared

with the second one. The results obtained with the asymptotic expansion-based method are in accordance with the Monte Carlo simulations (1000 samples). The loss of coherence is also noteworthy in comparing Figures 4.5-b and -d. In Figure 4.5-b we show clearly the seismic wave propagation along the structure leading up to non-uniform response in displacement. This is not the case in Figure 4.5-d.

Structure with linear dampers

Now the dampers with linear constitutive law are supposed to be set up on the structure at the connections 4, 10 and 16. In this application, the structure behaves linearly and the analysis is performed according to the *efficient approach* exposed in Section 4.4. The time window of 20s is sampled with a 0.5s time step and the cross-psd of the excitation is sampled with 5000 points essentially gathered around the natural frequencies. The results are validated by comparison with 10000 digital simulations computed with a time step of 2.4ms. The embedded stationary process is simulated on a 320s time interval before being divided in 20s earthquakes, so the frequency resolution is about 3mHz. Two different values of the damper constant C_D are studied.

In Figure 4.6, the convergence of the truncated series is investigated. For $C_D = 5\text{MN/s/m}$ (i.e. $\rho_J = 0.36$), the modal responses obtained with N equal to 0 and 2 are perfectly superimposed and they are not distinguished. For $C_D = 20\text{MN/s/m}$ (i.e. $\rho_J = 0.74$), the modal responses obtained with the decoupled approximation ($N = 0$), is not accurate enough in this case. Figure 4.6-a also shows the good agreement between the numerical simulations and the proposed method. From a computational point of view, a single time step with the proposed method is computed in about 0.5s.

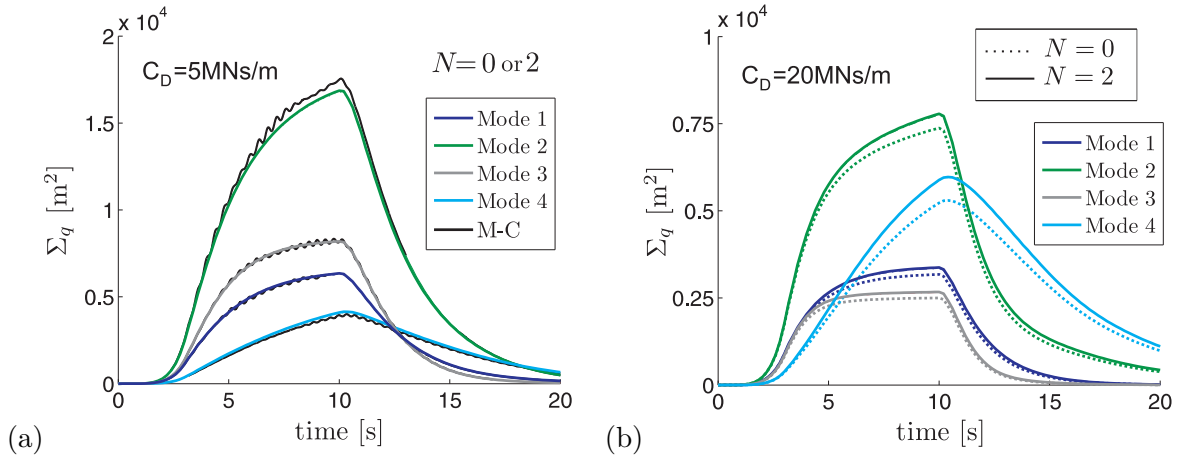


Figure 4.6: Multi-span bridge under coherent earthquake. Variances of the modal responses. (a) $C_D = 5\text{MN/s/m}$ and $\rho_J = 0.36$. Results obtained with $N = 0$ and $N = 2$ are perfectly superimposed. Comparison with Monte Carlo simulation (black curves). (b) $C_D = 20\text{MN/s/m}$ and $\rho_J = 0.74$. Results obtained with $N = 0$ and $N = 2$ cease to be superimposed.

In Figure 4.6, we see that the asymptotic expansion converges quite fast, in spite of a large index of convergence. This is partly explained in Section 4.3.2, in which we show that the convergence of the series $\mathbf{h}_N(t)$ is improved on shorter time interval, provided the series $\mathbf{H}_N(\omega)$

is convergent. The convergence is thus different in stationary and nonstationary settings. To highlight this difference, we propose to compare the stationary approach of Chapter 3 with respect to the method of this Chapter.

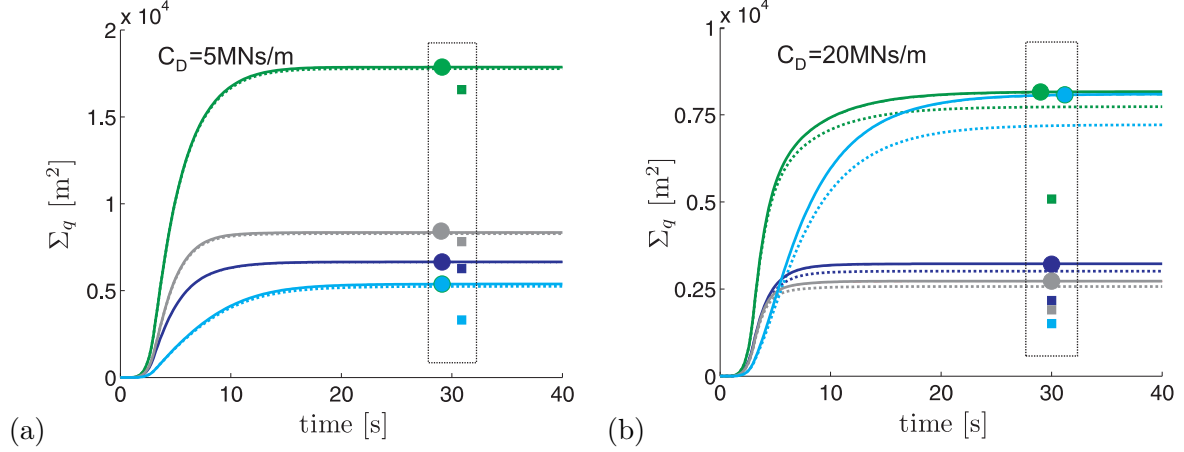


Figure 4.7: Multi-span bridge under coherent earthquake. Variances of the modal responses. The legend is similar to Figure 4.6. (a) $C_D = 5 \text{ MNs/m}$. Results obtained with $N = 0$ and $N = 2$ are perfectly superimposed. (b) $C_D = 20 \text{ MNs/m}$. Results obtained with $N = 0$ and $N = 2$ cease to be superimposed. Results from stationary spectral analysis shown with markers: Σ_{q_d} ($\blacksquare$), i.e. decoupled approximation, and $\Sigma_{q,\infty}$ ($\bullet$).

The Jennings window is therefore modified with $\gamma = 0$, such that the system evolves to its steady-state response. The same two damping coefficients C_D are chosen in order to compare different convergence criteria. In a nonstationary setting, the truncation order N is equal to 0 and 2. As in Figure 4.7-a, the three curves are virtually superimposed even for $\rho_J = 0.36$, but in Figure 4.7-b the decoupled approximation differs from the case $N = 2$. In Figure 4.7, the reference stationary response $\Sigma_{q,\infty}$ ($\bullet$) and the decoupled approximation Σ_{q_d} ($\blacksquare$) are depicted around the time $t = 30 \text{ s}$. As expected, the difference between these two responses is more important in Figure 4.7-b, because the convergence criterion is higher with $\rho_J = 0.74$.

The comparison between the curves and the markers illustrates explicitly our purpose. The transient decoupled approximation (dotted line) is much closer to the target Σ_q ($\bullet$) than the stationary decoupled one ($\blacksquare$), while the transient coupled approximation $N = 2$ meets the target response. We attribute the origin of this difference to the *efficient formulation* of Section 4.4. Indeed, the recurrence formula (4.4.6), recalled here

$$\Upsilon(t, t_0, \omega) = \Psi(t, t_M) \Upsilon(t_M, t_0, \omega) + \Upsilon(t, t_M, \omega),$$

presents two main advantages with regard to the analytical expressions calculated in Section 4.3.3.

First, the state transition matrix $\Psi(t, t_M)$ is not frequency dependent and can be computed efficiently by a Padé approximation of the exponential matrix taking into account the modal coupling. Therefore, the asymptotic approximation of the modal coupling is only considered on the interval $[t_M, t]$. Thus, the asymptotic approximation is not exactly decoupled all along the interval $[t_0, t]$, but only for the computation of $\Upsilon(t, t_M, \omega)$, the coupling is taken into account through $\Psi(t, t_M)$ in the projection of the past $[t_0, t_M]$ to the instant t .

Secondly, the modal coupling computed with the asymptotic expansion-based method is taken into account on the interval $[t_M, t]$. Provided this interval is not too long compared with the characteristic times of the structure (related to the natural frequencies), a single correction term is sufficient.

4.6 Nongeometric spectral moments and first passage time

In this Section, the author would like to broach the subject of *evolutionary spectral characteristics* derived from the spectral functions of evolutionary random processes. A significant advantage of the spectral analysis compared with Monte Carlo simulations pertains to the computation of extreme values. Indeed, for unsteady Gaussian processes, the related spectral moments allow thus to compute extreme value problems, while the Monte Carlo approach requires a lot of samples, moreover the ergodicity principle does not apply.

First, the spectral characteristics and the spectral moments for real-valued nonstationary processes was a straightforward extension of the *geometric* definition given by Vanmarcke [Vanm 72]. On a probability space $(\Theta, \mathfrak{F}, \mathbb{P})$, for a $\mathbb{L}_2$ -stationary process $X_s(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}$, the stationary spectral moments λ_n reads

$$\lambda_n = \int_{\mathbb{R}} |\omega|^n S_{X_s X_s}(\omega) d\omega \quad (4.6.1)$$

with $S_{X_s X_s}(\omega)$ the psd of $X_s(t)$. From these spectral moments, some parameters are computed, as the *central frequency* $\omega_c(t)$ and the *bandwidth parameter* $e(t)$ of the process. In a stationary setting, the first parameter expresses the center of mass of the one-sided psd of the process, while the second one may be seen as the inverse of the gyration radius of the same psd around the central frequency. Therefore, we understand the description *more geometrico* of those parameters for the spectral content of a random process.

However, the moments (4.6.1) may be unbounded for nonstationary processes, what makes this extension questionable (even wrong) from the mathematical point of view. Based on this major drawback, Di Paola [DiPa 85] has introduced the concept of *nongeometric spectral moments* applied to nonstationary processes, defined by use of the modulation of the Hilbert transform of the embedded stationary process. Di Paola's work has then been applied by Michaelov [Mich 99] in order to define nongeometric spectral characteristics. Nevertheless, this geometrical view ceases to be valid for unsteady processes, but Di Paola's definition allows them to be interpreted as *instantaneous* parameters.

In the works of Barbato [Barb 08, Barb 10], the nongeometric spectral characteristics are extended to complex-valued random processes in order to apply these concepts to complex modal analysis. In those papers, some analytical expressions for $\omega_c(t)$ and $e(t)$ are given for systems subject to modulated white noise excitations.

In this Section, we want to emphasize that our method can be used to solve such probabilistic problems, in a competitive manner with Monte Carlo simulation, without resorting to complex modal analysis.

Considering a real-valued evolutionary random process $X(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}$ defined on $\mathbb{L}_2(\Theta, \mathfrak{F}, \mathbb{P})$, the definitions of the first three nongeometric spectral moments [DiPa 85, Mich 99] are

$$\begin{aligned}
c_{XX}(t) &= \int_{\mathbb{R}} \hat{S}_{XX}(t, \omega) d\omega = \sigma_X^2(t), & c_{\dot{X}\dot{X}}(t) &= \int_{\mathbb{R}} \hat{S}_{\dot{X}\dot{X}}(t, \omega) d\omega = \sigma_{\dot{X}}^2(t), \\
c_{X\dot{X}}(t) &= -2\iota \int_0^\infty \hat{S}_{X\dot{X}}(t, \omega) d\omega \neq \sigma_{X\dot{X}}(t).
\end{aligned} \tag{4.6.2}$$

The process $X(t)$ can be interpreted as a response inherent to the structure (e.g. internal bending moments in a critical section, absolute or relative displacements) computed with the statistics of both $\mathbf{q}$ and $\dot{\mathbf{q}}$, i.e. the covariance matrix $\Sigma_{\mathbf{z}}$. Indubitably, the definitions (4.6.2) bear a striking resemblance to the corresponding stationary definition (4.6.1), except that the relation between the psd is not straightforward. Equation (4.6.1) does not apply in evolutionary spectral analysis, because the second moment λ_2 is not equal to the variance of the velocity in this case, as shown by [Mich 99]. Actually, the definition of $c_{X\dot{X}}$ may seem quite unexpected. This spectral characteristic is such that

$$c_{X\dot{X}} = \sigma_{X\dot{Y}}(t) - \iota \sigma_{X\dot{X}}(t) \tag{4.6.3}$$

with $Y(t)$ defined as the modulation of the Hilbert transform of the embedded process $X_s(t)$ of $X(t)$. The difference between $c_{X\dot{X}}(t)$ and $\sigma_{X\dot{X}}(t)$ comes from the integration domain: the domain being limited to $\mathbb{R}^+$, the imaginary part of the integral is not equal to zero.

According to definitions (4.6.2), the central frequency $\omega_c(t)$ and the bandwidth parameter $e_X(t)$ used in reliability analysis to calculate the peak distribution or the time of first-passage, can be calculated as

$$\omega_c(t) = \frac{\Re[c_{X\dot{X}}]}{c_{XX}}, \quad e_X(t) = \left(1 - \frac{\Re[c_{X\dot{X}}]^2}{c_{XX} c_{\dot{X}\dot{X}}}\right)^{\frac{1}{2}}. \tag{4.6.4}$$

The spectral moments may be used in a large range of extreme value problems or in reliability analysis, such as the **first-passage reliability problem** [Ghaz 12]. Considering a well-defined reliability domain, the stochastic problem of first-passage aims at finding the probability distribution of the *instants* at which a process reaches the reliability domain for the first time. In this work, it is supposed that the reliability domain is determined by extreme states. Hence, the initial configuration of the structure may be supposed to be in the reliability domain. Many instances of this problem may be provided, as it is a classic problem addressed by many authors for years in the different fields of science [Sieg 50, Spen 93, Preu 94, Musc 05], e.g. the probability of failure, the probability of pounding between adjacent buildings. The first-passage time is also called the *first hitting time*, a member of the family of the *exit times*, well-known in economy and finance.

In probability theory, provided the reliability domain may be interpreted as a threshold ℓ , i.e. a rigid barrier, the time of first passage t_{fp} may be defined, in a probabilistic framework, as

$$t_{fp} = \inf [t : X(t) = \ell]. \tag{4.6.5}$$

This is called a *single-barrier* first passage problem. A double-barrier reliability problem is defined in the same way, but with the absolute value of $X(t)$.

The first passage time problem usually consists in finding the probability distribution of t_{fp} , a random variable. The probability of failure at time t is usually expressed as [Preu 94, Tuba 12, Ghaz 12]

$$\mathbb{P}_f(t) = 1 - \exp \left(- \int_0^t \chi_\ell(\tau) d\tau \right), \quad (4.6.6)$$

with the *hazard function* $\chi_\ell(t)$ a real-valued function depending on $e(t)$ and $\nu_X^+(t, \ell)$, the mean out-crossing rate function of $X(t)$ with a positive slope ($\dot{X}(t) > 0$). This rate is calculated as

$$\nu_X^+(t, \ell) = \int_0^{+\infty} \dot{x} \psi_{X\dot{X}}(t, \ell, \dot{x}) d\dot{x}. \quad (4.6.7)$$

Equation (4.6.7) is referred to as *Rice's formula*. According to [Shin 88], the up-crossing rate of a threshold ℓ is given by

$$\nu_X^+(t, \ell) = \frac{\sqrt{1 - \rho_{X\dot{X}}^2}}{2\pi} \frac{\sigma_{\dot{X}}}{\sigma_X} \exp \left(\frac{-\kappa_1^2}{\rho_{X\dot{X}}^2} \right) \{1 + \sqrt{\pi} \kappa_1 \exp(\kappa_1^2) \operatorname{erfc}(\kappa_1)\} \quad (4.6.8)$$

with $\rho_{X\dot{X}}(t)$ the correlation coefficient between X and $\dot{X}$, $\operatorname{erfc}(\cdot)$ the complementary error function and the dimensionless constant

$$\kappa_1 = \frac{\ell \rho_{X\dot{X}}}{\sigma_X \sqrt{2(1 - \rho_{X\dot{X}}^2)}}. \quad (4.6.9)$$

Knowing a closed-form expression for $\nu_X^+(t, \ell)$, the function $\chi_\ell(t)$ can be expressed as

$$\chi_\ell(t) = \nu_X^+(t, \ell) \frac{1 - \exp \left(-\sqrt{2\pi} e_X(t) \frac{\ell}{\sigma_X(t)} \right)}{1 - \exp \left(-\frac{\ell^2}{2\sigma_X^2(t)} \right)} \quad (4.6.10)$$

with the assumption of a Markov process for the maxima, the so-called *Vanmarcke formula* [Vanm 72]. If the function $\chi_\ell(t)$ equals $\nu_X^+(t, \ell)$, the extrema are supposed to occur independently from each other, like a Poisson process. This last assumption is relatively frequent in practice for its simplicity, in spite of rough approximation. For a double barrier reliability problem, the hazard function reads

$$\chi_\ell(t) = 2\nu_X^+(t, \ell) \frac{1 - \exp \left(-\sqrt{\frac{\pi}{2}} e_X(t) \frac{\ell}{\sigma_X(t)} \right)}{1 - \exp \left(-\frac{\ell^2}{2\sigma_X^2(t)} \right)}. \quad (4.6.11)$$

An exponent 1.2 over the bandwidth parameter $e_X(t)$ in (4.6.10) and (4.6.11) was used by Vanmarcke himself in order to get a better agreement between his formula and simulations [Vanm 72]. This case is referred to as the *modified Vanmarcke formula* in the sequel.

Application

In order to illustrate the applicability of our method, we propose to go back to the example given in Section 3.5.2, i.e. the shear-type building under earthquake. The structure is supposed to be linear ($\kappa = 0$), the inter-storey stiffness k is fixed at 1GN/m and the damping ratio is equal to 2% for the first two modes. By construction, this structure is decoupled in its modal basis ($\tilde{\mathbf{K}} = \mathbf{0}$) in a first case. In a second case, three linear viscous dampers ($C_D = 16 \cdot 10^6 \text{Ns/m}$) are set up at the first three levels of the structure in order to mitigate vibrations. The structure is thus non-classically damped. The damping ratios of the first three modes become 0.04, 0.05 and 0.06; the convergence criterion is $\rho_{\mathbf{J}} = 0.44$.

We recognize the academical nature of this example, though it is basically comparable to the benchmark given in [Barb 11]. We propose to compute the probability of first passage of the *top of the building*, i.e. $t_{fp} = \inf [t : |X(t)| = \ell]$ with ℓ equal to both 1.35m and 1.3635m. These two values have been chosen really close in order to highlight the great sensitivity on ℓ of the relation (4.6.8). In this example we consider Shinozuka's envelope (4.3.44) with $b_1 = 0.045\pi$, $b_2 = 0.050\pi$ and a maximum value equal to one.

For the resolution with the proposed asymptotic expansion-based method, the time step is equal to 0.5s (it is not illustrated here, but a time step of 0.1s was also considered without any difference). The resolution of the problem is carried out with 5 modes and takes about 20s cpu time. For the decoupled structure, 128000 simulations have been run ($\Delta t = 0.006\text{s}$) and 10^6 samples ($\Delta t = 0.02\text{s}$) for the non-classically damped structure. The main results about the first passage time are given in Figures 4.8 and 4.9.

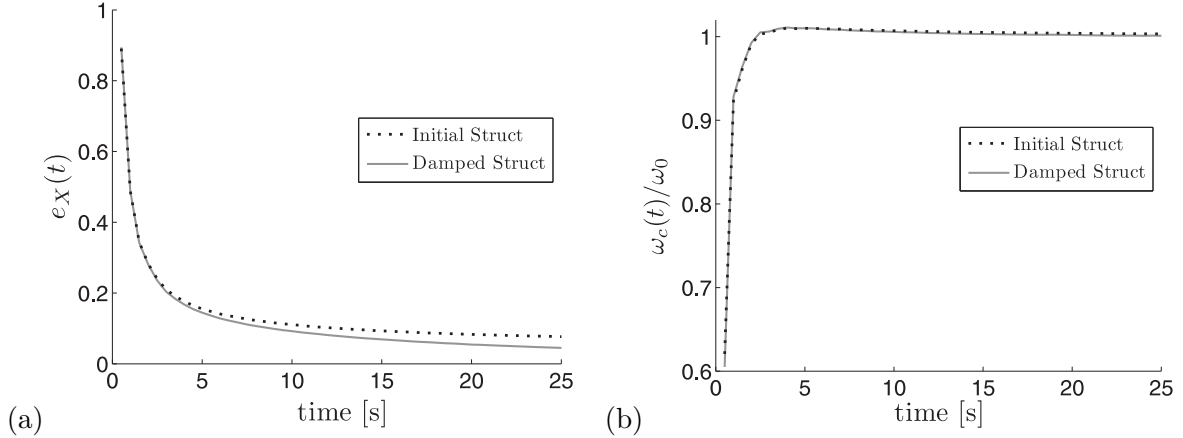


Figure 4.8: Non-geometrical spectral parameters. (a) bandwidth parameter $e_X(t)$ and (b) normalized central frequency for both initial (classically damped) and (non-classically) damped structure; ω_0 denotes the first natural pulsation of the structure.

In Figure 4.8, we show the time evolution of the bandwidth parameter $e_X(t)$ and the central pulsation $\omega_c(t)$ for both situations. The bandwidth parameter decreases with time. In the strong phase of the earthquake, ω_c is equal to the first natural pulsation of the structure and e_X is less than 0.1. A small bandwidth parameter e_X ensures a limited dispersion for the spectral content of $X(t)$ around the central frequency. In a stationary setting, we know that the smaller this parameter, the better the Vanmarcke approximation [Vanm 72]. By considering these parameters as instantaneous, we may expect a good agreement between the

spectral moment approach and Monte Carlo simulation.

In Figure 4.9, we compare different approximations for two different thresholds according to the two configurations. First, the comparison between Figures 4.9-a and -c shows that the effect of the dampers is to reduce by a factor 10 the probability of overcrossing the reliability domain. In Figures 4.9-a and -b, we can see the poor results obtained with Rice's formula. Roughly speaking, Rice's formula overestimates by a factor 4 the target probability (the reference being the probability obtained by simulation). Indeed, the assumption of independent crossings is not justified for short transient phenomenon. In Figures 4.9-c and -d, this formula overestimates the target by a factor 2.5. This reduction from a factor 4 to 2.5 can be explained by the Poisson assumption, a bit more suitable for high thresholds, and thus smaller overcrossing probability.

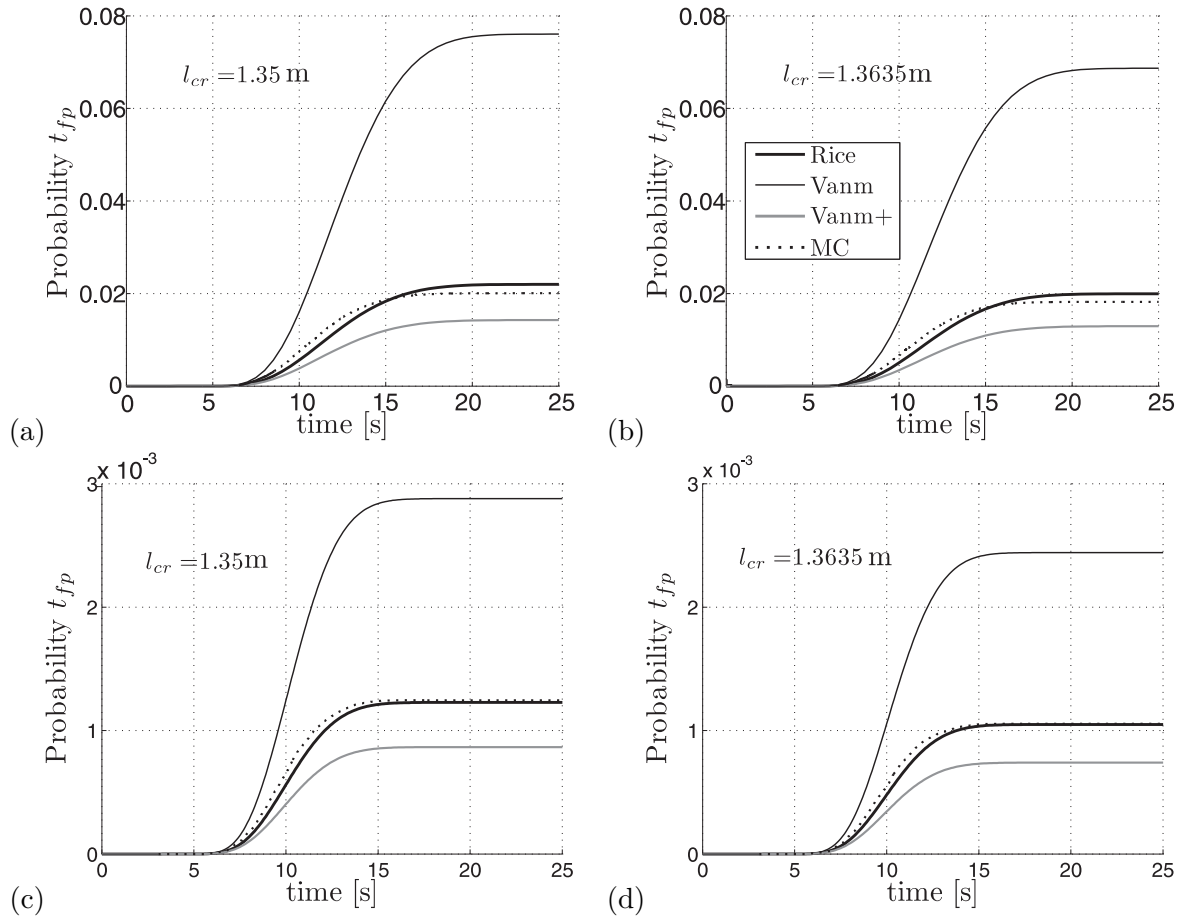


Figure 4.9: Non-geometrical spectral parameters. Application of the asymptotic expansion based method to a first passage problem for two different thresholds l_{cr} . (a,b) Classically damped structure. (c,d) Non-classically damped structure.

We shall notice the good agreement in the four cases between Monte Carlo simulation and the Vanmarcke formula. In this case, the Markovian property for the crossings is much more suitable than the Poisson approximation. In this context, we must point out the dramatic underestimation obtained with the modified Vanmarcke formula. We cannot justify

theoretically the exponent suggested by Vanmarcke, especially for nonstationary processes. In the opinion of the author, it is safer in this kind of problems to use the original Vanmarcke formula, but with precaution, because of the parameter sensitivity in extreme value problems. As illustrated by comparing Figures 4.9-a and -b together, or Figures 4.9-c and -d, the threshold increased by 1% modifies the target probability of about 10%. Even if this modification is not notable in this range of probability, it may lead to wrong interpretations and comparisons, abundant in the literature. Without details, we also would like to recall the importance of time steps in extreme value simulations. Monte Carlo simulation is neither exempt from precautions: an approximate digitalization may lead to poor results.

4.7 Linear systems driven by non-Gaussian excitation

Although the spectral analysis has been presented in this Chapter as limited to Gaussian excitations, the effects of non-Gaussian forces on linear systems in a transient framework can also be dealt with an evolutionary spectral analysis. Indeed, in a spectral approach, the definition of higher-order spectra is basically required as the natural extension of the evolutionary spectrum in (4.2.1). Broaching such a question at this stage opens a *Pandora's box*. The non-Gaussianity in transient loading has been addressed by different authors, but especially in the case of Poisson processes or impulse trains [DiPa 96]. In earthquake engineering, the spectral approach is amply used, but the characterization of a ground acceleration model by non-Gaussian properties is not an easy task. For instance, Kafali and Grigoriu [Kafa 06] propose to use a Student *t-distribution*. However, as the evolutionary spectral analysis remains an open question, the development of an efficient higher-order spectral analysis also remains open, especially for linear systems [Unsa 03] and non-Poisson processes.

For large linear structures, like bridges or buildings, the use of spectral approach can effectively compete with Monte Carlo simulations, also in nonstationary and non-Gaussian analysis. This section is not pretending to develop new non-Gaussian models for transient excitation. Independently of the excitation, provided their spectral characteristics are known, the proposed developments can be applied.

According to the Wiener-Kitchine theorem (Section 2.2), the power-spectral density of a process is the Fourier transform of the correlation function of the process. As a formal extension of this definition to higher-order spectral theory, the *evolutionary bispectrum* is defined as the bidimensional Fourier transform of the *bicorrelation function*

$$E \left[(\mathbf{q}(t_1))_i (\mathbf{q}(t_2))_j (\mathbf{q}(t_3))_k \right] = E \left[\sum_{l_1, l_2, l_3}^m \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} (\mathbf{h}(t_1 - u_1))_{il_1} (\mathbf{h}(t_2 - u_2))_{jl_2} (\mathbf{h}(t_3 - u_3))_{kl_3} (\mathbf{p}(u_1))_{l_1} (\mathbf{p}(u_2))_{l_2} (\mathbf{p}(u_3))_{l_3} du_1 du_2 du_3 \right]. \quad (4.7.1)$$

As previously introduced in Section 2.2, the correlation function captures the statistical information contained between two different instants of two components of a random vector. The bicorrelation tensor is an extension of the correlation matrix definition to third dimension statistics. Indeed, the bicorrelation function captures the statistical information contained in signals between three instants. For weakly stationary processes, only two time lags are required to model the aforementioned tensor. In the case of Gaussian processes, the

bicorrelation function is only dependent on the first and second order statistical moments, so the computation of a bicorrelation is straightforward. The purpose of bicorrelation function (or tricorrelation function and so on) is the first measure of the non-Gaussianity within a random process.

Few details are now given to highlight the possibilities offered by the asymptotic expansion-based method to perform a third-order spectral analysis in a transient regime. The extension theorem ensures that the vector of generalized forces $\mathbf{p}(t)$ admits the Riemann-Stieljes representation (4.2.1). Furthermore, the modal evolutionary transfer matrix (4.2.8), together with (4.7.1), leads to

$$E \left[(\mathbf{q}(t_1))_i (\mathbf{q}(t_2))_j (\mathbf{q}(t_3))_k \right] = \sum_{l_1, l_2, l_3}^m \iiint_{\mathbb{R}^3} (\mathbf{G}(t, \omega_1))_{il_1} (\mathbf{G}(t, \omega_2))_{jl_2} (\bar{\mathbf{G}}(t_3, \omega_3))_{kl_3} E \left[(d\tilde{\mathbf{p}}(\omega_1))_{l_1} (d\tilde{\mathbf{p}}(\omega_2))_{l_2} (d\tilde{\mathbf{p}}(\omega_3))_{l_3} \right]. \quad (4.7.2)$$

The assumption of orthogonal increments formulated about $\mathbf{p}(t)$ allows to define an adapted measure on the probability space, as in Section 2.2, such that

$$\begin{aligned} E \left[(d\tilde{\mathbf{p}}(\omega_1))_{l_1} (d\tilde{\mathbf{p}}(\omega_2))_{l_2} (d\tilde{\mathbf{p}}(\omega_3))_{l_3} \right] &= (\mathbf{B}_{\mathbf{p}}(\omega_1, \omega_2, \omega_3))_{l_1 l_2 l_3} \delta(\omega_1 + \omega_2 + \omega_3) d\omega_1 d\omega_2 d\omega_3 \\ &= (\mathbf{B}_{\mathbf{p}}(\omega_1, \omega_2))_{l_1 l_2 l_3} d\omega_1 d\omega_2 \end{aligned} \quad (4.7.3)$$

with $\mathbf{B}(\omega_1, \omega_2)$ the bispectrum (a third-order tensor) of the stationary process sustaining the transient excitation. The δ -Dirac function comes from the hypothesis of stationarity formulated on the stationary process embedded in $\mathbf{p}(t)$ [Unsa 03, Deno 05].

The bicorrelation at a given instant t allows to compute the time evolution of the skewness of the modal coordinates, such that

$$\begin{aligned} E[(\mathbf{q})_i (\mathbf{q})_j (\mathbf{q})_k (t)] &= \sum_{l_1, l_2, l_3}^m \iint_{\mathbb{R}^2} (\mathbf{G}(t, \omega_1))_{il_1} (\mathbf{G}(t, \omega_2))_{jl_2} \\ &\quad (\bar{\mathbf{G}}(t, \omega_1 + \omega_2))_{kl_3} (\mathbf{B}_{\mathbf{p}}(\omega_1, \omega_2))_{l_1 l_2 l_3} d\omega_1 d\omega_2 \end{aligned} \quad (4.7.4)$$

as $(\mathbf{G}(t, -\omega))_{ij} = (\bar{\mathbf{G}}(t, \omega))_{ij}$. A particular tensor can be identified from (4.7.4). Indeed, in the theory of Volterra series [Sche 80], the product of transfer functions in (4.7.4) is called the **second evolutionary Volterra kernel**. By a semantic shift, the vocable instantaneous or evolutionary is assigned to the Volterra kernel. The evolutionary transfer matrix (4.2.8) is thus the **first evolutionary Volterra kernel**.

As previously, the crude computation of the evolutionary transfer matrix is not an easy task due to the convolution integral in (4.2.8). Although the elements of $\mathbf{G}(t, \omega)$ may be efficiently computed for coupled or uncoupled systems with the proposed asymptotic expansion-based method, a formal asymptotic expansion of the Volterra kernel is expressed with difficulty. At this stage, we keep on working with the asymptotic expansion of the evolutionary transfer matrix, since using a N -order asymptotic expansion of $\mathbf{G}(t, \omega)$ ensures that at least the N -order approximation of the bicorrelation function is achieved.

Equation (4.7.4) becomes

$$E \left[(\mathbf{q})_i (\mathbf{q})_j (\mathbf{q})_k \right]_N = \sum_{l_1, l_2, l_3}^m \iint_{\mathbb{R}^2} (\mathbf{G}_N(t, \omega_1))_{il_1} (\mathbf{G}_N(t, \omega_2))_{jl_2} (\bar{\mathbf{G}}_N(t, \omega_1 + \omega_2))_{kl_3} (\mathbf{B}_p(\omega_1, \omega_2))_{l_1 l_2 l_3} d\omega_1 d\omega_2. \quad (4.7.5)$$

Equation (4.7.5) highlights an important contribution of this thesis. Thanks to the asymptotic expansion based-method, the integration on the time dimension is performed by analytical methods. The time variable in (4.7.5) may be seen as a parameter. As previously, those results constitute a major gain of this method, because it deals efficiently with coupling in dynamical systems at a low computational cost. The computation of higher-order moments requires to compute higher-order Volterra kernels, but this extension is beyond the scope of this research. The methodology is straightly extended, yet.

At this stage, the big issue is therefore in the computation of the bispectrum (and higher-order spectra) of the generalized forces, but it is not inherent to the proposed method. The Volterra series can be used in nonlinear dynamics, but it requires a non-reasonable computational effort. To conclude this digression about non-Gaussian response of linear systems, an academic application is proposed to validate the developments with some digital simulations.

Application

In order to illustrate and validate the former developments in a non-Gaussian framework, the system depicted in Figure 4.10 is considered. The mass, damping and stiffness matrices are given by

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}, \quad \mathbf{C} = 4\pi \begin{pmatrix} \xi + \zeta & -\zeta \\ -\zeta & \xi\sqrt{\mu} + \zeta \end{pmatrix}, \quad \mathbf{K} = 4\pi^2 \begin{pmatrix} 1 + \kappa & -\kappa \\ -\kappa & 1 + \kappa \end{pmatrix}, \quad (4.7.6)$$

where ξ , ζ , μ and κ are parameters of the studied system. For $\xi = 0.5\%$, $\zeta = 5\%$, $\mu = 0.8$ and $\kappa = 0.1$, the index of convergence is $\rho_{\mathbf{J}}$ is equal to 0.6 highlighting a strong coupling in the system after projection within the modal basis. The natural frequencies are 6.5 and 7.5 rad/s. The damping model of the system is thus non-proportional provided κ not too high.

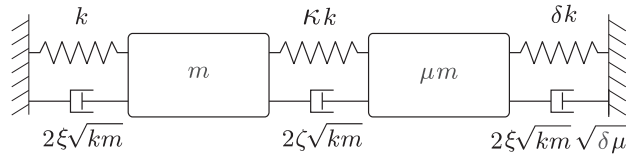


Figure 4.10: Sketch of a 2-DOF dynamical system

The modal equation of motion reads

$$\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{\Omega}_d\mathbf{q} = a(t) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \quad (4.7.7)$$

with $a(t)$ a Shinozuka time window ($b_1 = 0.1$ and $b_2 = 0.3$) and $p_i(t)$ a nonlinear function of $u_i(t)$, an Ornstein-Uhlenbeck process (O-U), such that

$$p_i(t) = (U_0 + u_i(t))^2 \quad (4.7.8)$$

with $i = 1, 2$. The psd of these O-U processes, supposed to be independent, is given by

$$S_u(\omega) = \frac{\alpha}{\pi} \frac{\sigma_u^2}{\alpha^2 + \omega^2}. \quad (4.7.9)$$

Hence, the lack of coherence between the forces in a modal basis allows to focus on the coupling in the response only coming from the system.

The bicornelation function of the process u_i is given by the expectation of a product of u_i evaluated at three different instants, i.e.

$$E \left[(U_0^2 + u_i(t))^2 (U_0^2 + u_i(t + \tau_1))^2 (U_0^2 + u_i(t + \tau_2))^2 \right], \quad (4.7.10)$$

then the bispectrum is obtained by a double-Fourier transform of the previous equation (a function of τ_1 and τ_2). As a result of the assumption on the independence between the O-U processes, the psd of $\mathbf{p}(t)$ is given by

$$(\mathbf{S}_{\mathbf{p}}(\omega))_{i_1 i_2} = S_p(\omega) \delta_{i_1 i_2} = \frac{4\alpha}{\pi} \frac{(U_0^2 \sigma_u^2 + \sigma_u^4)}{\alpha^2 + \omega^2} \delta_{i_1 i_2}, \quad (4.7.11)$$

where the effects of the non-Gaussian contributions on the spectrum are taken into account. Then, the bispectrum of $\mathbf{p}(t)$ is such that

$$\left(\tilde{\mathbf{B}}_{\mathbf{p}}(\omega_1, \omega_2) \right)_{i_1 i_2 i_3} = B_p(\omega_1, \omega_2) \delta_{i_1 i_2} \delta_{i_2 i_3} \quad (4.7.12)$$

with

$$B_p(\omega_1, \omega_2) = \frac{8\alpha^2}{\pi^2} \left(\frac{U_0^2 \sigma_u^4 (3\alpha^2 + \omega_1^2 + \omega_2^2 + (\omega_1 + \omega_2)^2)}{(\alpha^2 + \omega_1^2)(\alpha^2 + \omega_2^2)(\alpha^2 + (\omega_1 + \omega_2)^2)} + \frac{2\sigma_u^6 (12\alpha^2 + \omega_1^2 + \omega_2^2 + \omega_1 \omega_2)}{(4\alpha^2 + \omega_1^2)(4\alpha^2 + \omega_2^2)(4\alpha^2 + (\omega_1 + \omega_2)^2)} \right). \quad (4.7.13)$$

First, in this example, the parameters U_0 , α and σ_u are set to 10, 0.5 and $0.5U_0$, respectively. The spectrum $S_p(\omega)$ and the bispectrum $B_p(\omega_1, \omega_2)$ are depicted in Figure 4.11-a and -b, respectively. With these values, the excitation is essentially a background process. In that case, it is well-known [Grig 88, Grig 02] that the background component creates the non-Gaussianity in the response, while the resonant response, resulting from the amplification of the resonant peaks, generates Gaussian contributions.

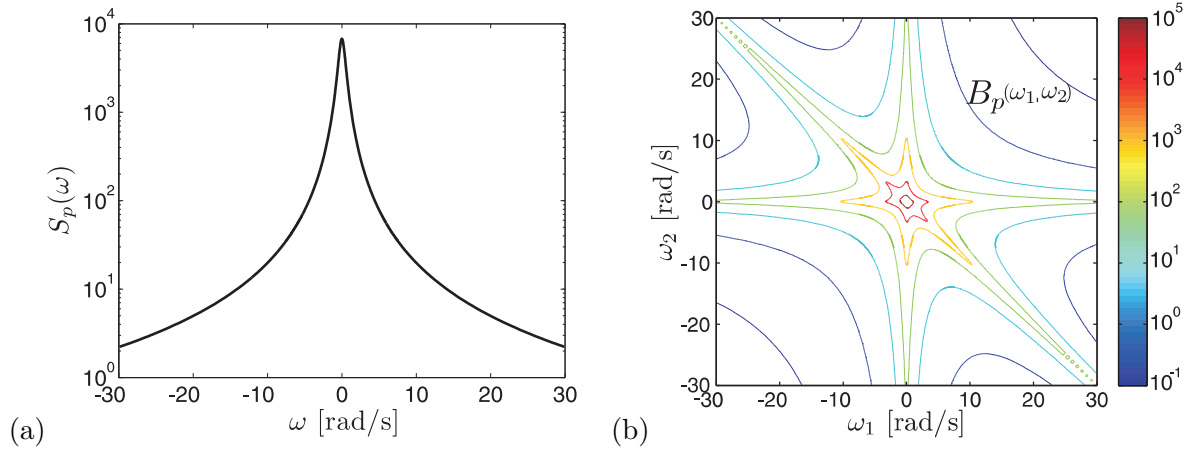


Figure 4.11: Power spectral density (a) and bispectrum (b) of the stationary process $p = (U_0 + u)^2$ with U_0 equal to 10 and $u(t)$ an Ornstein-Uhlenbeck process ($\alpha = 0.5$, $\sigma_u/U_0 = 0.5$).

The establishment of those relations require many developments, but the Gaussianity of $u(t)$ deeply simplifies the analytical manipulations. As shown in [Deno 05], the exact bispectrum related to (4.7.13) counts also other terms, but they do not contribute to the computation of the third-order cumulant, and are therefore disregarded in this example. Indeed, the main target of a third-order spectral analysis is to compute the skewness coefficient which is defined in terms of the third-order cumulants. Introducing the bispectrum in (4.7.5), the time evolution of this cumulant reads

$$\tilde{E} \left[(\mathbf{q})_i (\mathbf{q})_j (\mathbf{q})_k \right] = \sum_{l=1}^m \iint_{\mathbb{R}^2} \left(\tilde{\mathbf{B}}_{\mathbf{p}}(\omega_1, \omega_2) \right)_{ll} (\mathbf{G}(t, \omega_1))_{il} (\mathbf{G}(t, \omega_2))_{jl} (\bar{\mathbf{G}}(t, \omega_1 + \omega_2))_{kl} d\omega_1 d\omega_2 \quad (4.7.14)$$

with $\tilde{E}[\cdot]$ designating the expectation operator for the central moments. We use the analytical expressions developed in Section 4.3.3.

The evolutionary transfer function and the evolutionary second Volterra kernel related to the first normal mode are shown in Figure 4.12 at different time steps. The oscillations in the transfer function are of paramount importance, because they result from the convolution operation and the imaginary exponential in (4.2.8). The same behavior can be observed in the second kernel. The results shown in Figure 4.12 only depend on the system properties and the time window, independently of the underlying process actually. The band around the natural frequency becomes narrower, when the time passes. For the second kernel, the position of six peaks are identified : if ω_1 or ω_2 are set equal to zero, two peaks are identified at the first natural frequency, while the two last peaks come from the summation $\omega_1 + \omega_2$ and the conjugate term in (4.7.14).

Figures 4.13 and 4.14 show the results in terms of the second- and third-order cumulants for the modal coordinates of the system with the aforementioned values of parameters. In Figure 4.13, the convergence of the asymptotic expansion is checked. Since the convergence criterion is relatively high ($\rho_{\mathbf{J}} = 0.6$), at least two terms in the series are required, as we use the results of Section 4.3.3. As expected, the decoupled approximation is not able to compute

the correlation between the two modal coordinates, because the generalized forces are not coherent. Figure 4.13-b shows that the second-order moments are well estimated with two correction terms. The Monte Carlo simulations are performed: Ornstein-Uhlenbeck processes are generated with an ARMA filter, then a linear Newmark algorithm (0.25,0.5) is used to simulate the dynamics of this 2-DOF system.

Figure 4.14 compares marginal and cross third-order moments with two runs of Monte Carlo simulations with different number of samples. The convergence of the simulations is slower on the cross moments, for which 50000 simulations have not yet converged. A rapid estimation of the maximum skewness coefficient is about 0.7, highlighting an antisymmetry in the marginal pdf of the modal coordinates. About the computational time, a single time step (the time is here a parameter) in the spectral approach takes about 7s of cpu time. Although this time may be seen as prohibitive on a 2-DOF system, no optimization of the integration has been performed. Indeed, the integral (4.7.14) is operated with a rectangle method in the plane (ω_1, ω_2) . Since peaks are identified in the second-order Kernel and the excitation is mainly a background process, simple procedure can be imagined to refine those functions only where it is required. Extension of the background-resonant as exposed in Section 3.3.5 can be imagined, too.

To conclude this application, the parameters of the excitation are modified: α is now equal to 10. The excitation is not anymore a purely background excitation. Although the convergence of the second order moment is assessed in Figure 4.15-a, the convergence of the third-order moment is much slower. Runs of 10^6 and $5 \cdot 10^6$ simulations have been performed, but the convergence of the moment $\tilde{E}[q_1^3]$ is slow. These results highlight the precautions that must be used in the crude application of Monte Carlo simulations. Nevertheless, the maximum skewness coefficient is about 0.01, so the distribution is almost symmetric (and almost Gaussian by experience). This fact is well-known for such a small skewness. As explained previously, the non-Gaussian component in the response comes mainly from the background component. The lower α , the higher this component.

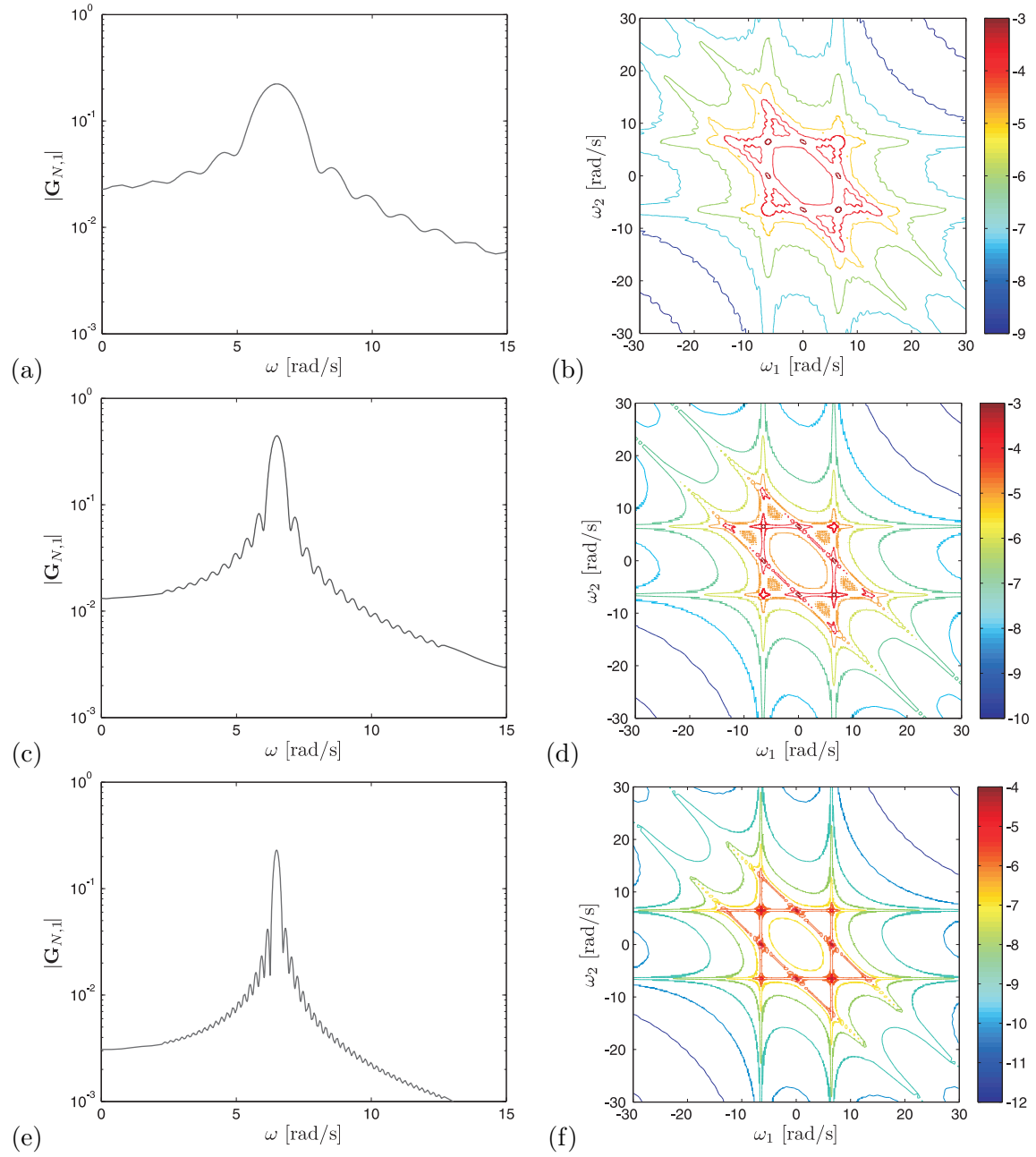


Figure 4.12: N -order transfer matrix $\mathbf{G}_N$ (left column) and N -order second Volterra kernel (right column) of the first mode of system depicted in Figure (4.10) at different time steps : $t = 5$ s (a-b), $t = 15$ s (c-d) and $t = 30$ s (e-f) with $N = 2$.

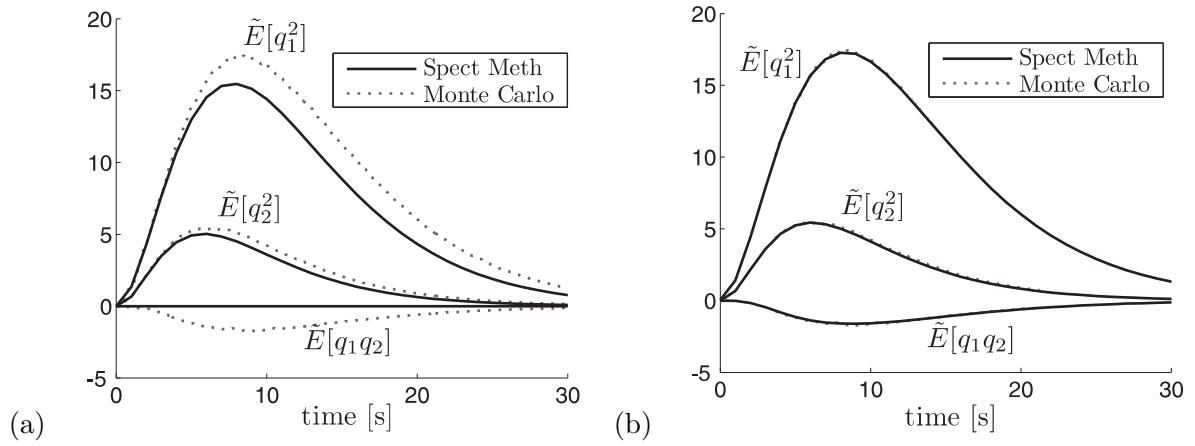


Figure 4.13: Second-order cumulants of the system response depicted in Figure (4.10). Comparison between the spectral approach (bold line) (a) $N = 0$ and (b) $N = 2$ with Monte Carlo simulation (10^4 samples).

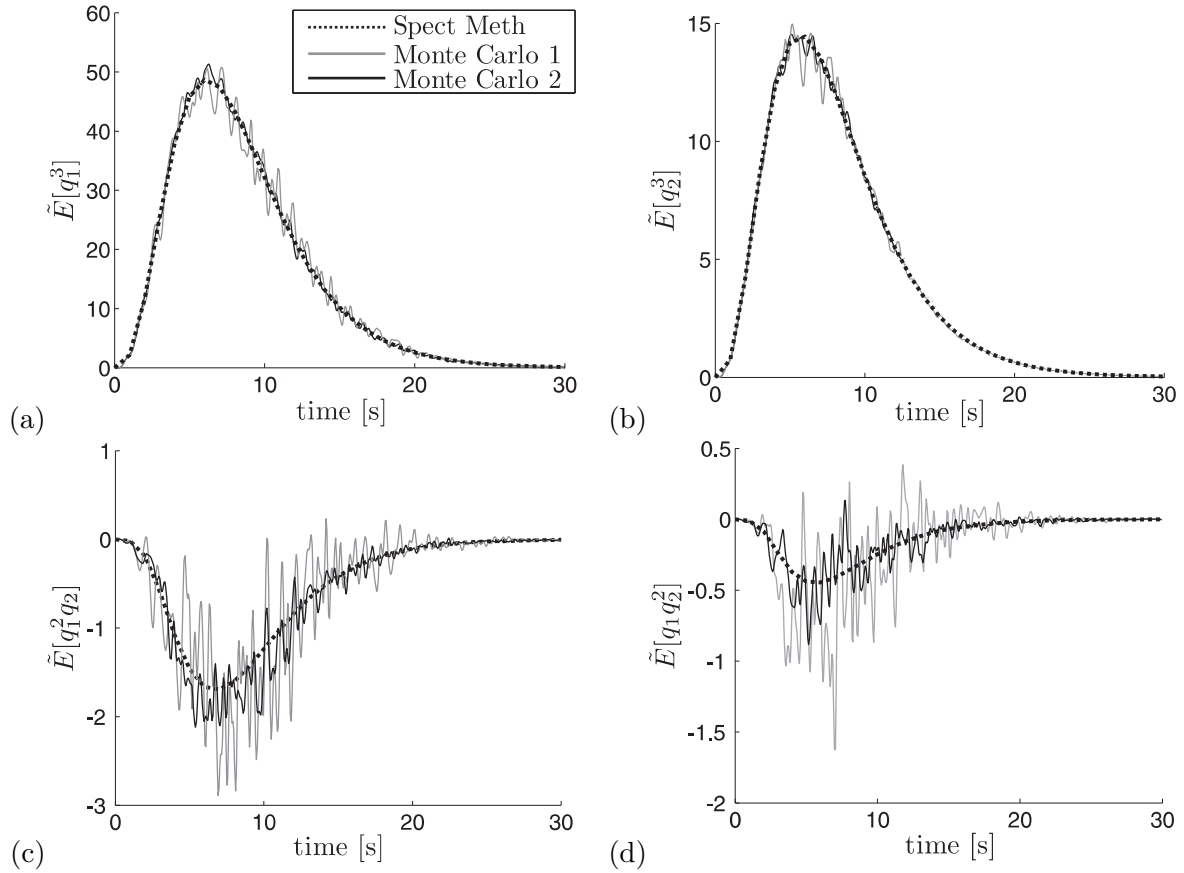


Figure 4.14: Third-order cumulants of the response of the system depicted in Figure (4.10). Comparison between the spectral approach with $N = 2$ and Monte Carlo simulations for two different numbers of sample (MC 1 with 10^4 samples and MC 2 $5 \cdot 10^4$ samples).

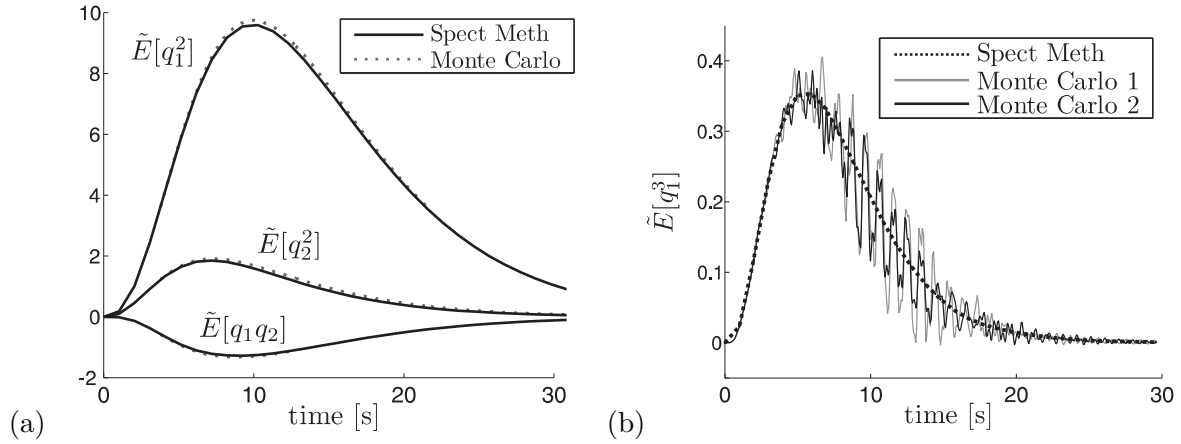


Figure 4.15: Cumulants of the non-Gaussian response. Random response of the 2-DOF system in Figure (4.10) subject to an excitation characterized by the parameters $U_0 = 10$, $\alpha = 10$ and $\sigma_u/U_0 = 0.2$. (a) Second cumulants : comparison between the spectral approach with $N = 2$ and Monte Carlo simulations with 10^5 samples. (b) Third cumulant : comparison between the spectral approach with $N = 2$ and Monte Carlo simulations for two different numbers of samples (MC 1 with 10^5 samples and MC 2 $5 \cdot 10^5$ samples).

4.8 Conclusions

Throughout this Chapter, we have been concerned about evolutionary spectral analysis for *linear* structures. In civil engineering, evolutionary spectra are common models for transient phenomena: an embedded stationary process, with a well-known spectral representation, is modulated by a time envelope. This time-frequency dual description of a random process allows to take advantage of the classic spectral analysis, exposed in Chapter 2. The crux of the problem in evolutionary spectral analysis is the computation of the impulse response matrix of the structure.

Thanks to the asymptotic expansion-based method developed in Chapter 3, we are able to efficiently compute an asymptotic approximation of the impulse response matrix. Hence, its computation, sometimes considered as intractable or computationally prohibitive, ceases to be a drawback. The correction terms in the expansion allow to take into account the modal coupling. For some common and well-known time envelopes, closed-form expressions can be derived after few analytical developments. In this approach, the time is seen as a parameter and the covariance of the response is thus readily computed by a numerical integration over the frequency domain.

Nevertheless, we do not want to limit our method to envelopes with some simple analytical expressions. Thus, a procedure is developed to deal with different windows. Thanks to the *semi-group property* inherent to linear systems (time-variant or -invariant), we have found a recurrence relation to build up the *state space evolutionary transfer matrix* of the modal coordinates. This relation, demonstrated in Section 4.4, is central in this work for two main reasons. First, by use of linear piecewise descriptions, we are able to deal with a large set of time windows. Secondly, we take advantage of the convergence of the asymptotic expansion of the impulse response matrix, improved on short time intervals. Even if the time ceases to

be considered as a parameter, this provides a general and easily-implemented procedure to perform evolutionary spectral analysis. This method has been applied to the study of a non-classically damped bridge subject to nonstationary coherent ground motion. The accuracy and the computational efficiency have been emphasized.

We want to insist that returning to linear algebra may lead to reliable results for the resolution of civil engineering problems, by contradicting the notorious idea that a transient problem can only be solved with simulations. For this reason, we hope that our method will be considered with interest by practitioners in the future, independently of its extension to nonlinear structures, another objective of this work, discussed in the following Chapter.

Due to the linearity of the structure, we may apply the asymptotic expansion-based method to two other structural engineering problems.

Provided the excitations are Gaussian processes, the structure responds with Gaussian processes as well. Therefore, we can use the non-geometrical spectral moments to solve some probabilistic problems, as the *first passage time*. We have seen that our method, together with these spectral moments, is able to match quite well the Monte Carlo results, for classically and non-classically damped structures. We know now that our method may be applied to such problems, without necessarily using complex modal analysis. Naturally, these spectral moments remain an approximate tool and they must be used carefully. The original Vanmarcke formula gives very good results, while the Rice formula remains excessively safe. Some authors suggest the use of the modified Vanmarcke formula, but with dubious evidences with regard to our illustration. Nevertheless, we are convinced that improvements and limitations for this approach could be further investigated in the future. The literature does not propose today a regular analysis of the different approximations, grounded on mathematical evidences.

The linear assumption on the structural behavior also gives novel perspectives in high-order evolutionary spectral analysis. Indeed, we may use the asymptotic expansion-based method to estimate higher-order Volterra kernels. They allow to compute higher-order moments of non-Gaussian structural responses. We provide an academic example to illustrate our developments about this topic. At this stage, the method being validated, we think that it could be applied to more realistic structures. However, improvements must be formally performed on the numerical (or analytical) integration of higher-order spectra. Due to their analytical properties, we could be able to spread out more efficiently the distribution of the integration points. Nevertheless, the characterization of real or realistic bispectra, e.g. in wind engineering, remains an issue.

In the following Chapter, we return to the asymptotic expansion-based method for Gaussian equivalent linearization. We aim to show that the developments for linear evolutionary spectral analysis may be applied to equivalent linearization in a nonstationary setting.

Chapter 5

Nonlinear Evolutionary Spectral Analysis

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- 5.1 Context and propositions
 - 5.2 Multiple scales approach in stochastic linearization
 - 5.3 Evolutionary spectral analysis for equivalent LTV systems
 - 5.4 Application in wind and earthquake engineering
 - 5.5 Update of the equivalent modal basis
 - 5.6 Conclusions
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Purpose of the chapter

This chapter aims at efficiently and rigorously unifying the equivalent statistical linearization presented in Chapter 3 and the evolutionary spectral analysis developed in Chapter 4. Based on a multiple scales approach, we demonstrate that a linearized structure, basically a statistical time-variant system, can be regarded as a piecewise linear time-invariant structure under the assumptions given in Section 5.2.

This method is then applied to two main examples in wind and earthquake engineering. Finally, Section 5.5 presents a statistically consistent procedure to adapt a modal (or generalized) basis throughout a transient solution, stumbling block of equivalent linearization for large structures in transient analysis.

5.1 Context and propositions

In Chapter 3, we expose the concept of asymptotic expansion of modal transfer matrix to perform linear spectral analysis and equivalent statistical linearization in a stationary setting. Then, in Chapter 4, this asymptotic method is used in linear evolutionary spectral analysis. The concept is now extended to nonlinear and nonstationary problems with the main objective to efficiently apply statistical linearization to large structures subject to coherent loadings. Thanks to the previous developments, this objective is achieved in this Chapter. However, the extension is not straightforward and some justifications are required in order to define a practical application field for the asymptotic expansion-based method in equivalent statistical linearization.

Although we deal with Gaussian equivalent linearization, we would like to give some comments about the possible relation between this Chapter and Section 4.7 coping with non-Gaussian loadings. Since the random response of a nonlinear system is non-Gaussian, it could be worth considering higher-order spectral analyses to perform equivalent linearization. Nevertheless, we keep on working in the context of Gaussian equivalent linearization. Indeed, though some comments are already given in Section 2.5, we reaffirm our severe opinions about the non-Gaussian linearization. The stochastic linearization is not simplified by the calculation of higher-order (cross and marginal) moments, since the moment truncation is still a problem, the number of moments to compute is *a priori* unknown and their number dramatically increases with this truncation. Furthermore, the computation of the equivalent matrices is deeply simplified by the Gaussian process assumption. Practitioners are aware that stochastic linearization is a simplified method, tarnished by an error inherent to the assumption of Gaussianity. Nevertheless, the trend on the parameters is usually well estimated at a low computational cost, as illustrated in this Chapter.

The search for suitable techniques to perform transient analysis has been carried out for a while. Proppe *et al.* [Prop 03b] show that Monte Carlo simulation and the equivalent linearization are the only two *feasible* methods to perform stochastic analysis of large-dimensional structures, especially in a nonstationary framework. Although the crude Monte Carlo simulations present few perspectives of improvement for structures driven by random forces, the field of nonstationary equivalent linearization has been the object of research and publications in the last thirty years. By presenting some methods, their concepts, advantages and drawbacks, we emphasize why our method is innovative and relevant.

First of all, we want to investigate whether linear methods can be extended to equivalent linearization. The *pseudo-excitation algorithm* briefly exposed in Section 4.1 is limited to uncoupled systems and the use of Duhamel integral, even piecewisely, is not feasible. According to Tootkaboni [Toot 10], the *direct integration scheme*, exposed in Section 4.1 as well, can be extended to nonlinear systems by considering them as piecewise linear. This idea is definitely pertinent and must be extended to piecewise linearized systems. Hence, though the original system is nonlinear and time-invariant, the linearized system can be seen as a piecewise linear time-variant system. This concept, roughly drawn here, is amply exposed in this Chapter with other assumptions.

Evolutionary power spectrum is not the unique way to deal with unsteady processes. In [Naso 00], an evolutionary wavelet spectrum is used to quantify how a process varies locally over time and frequency. For the statistical linearization approach, Spanos [Span 12] uses this

concept to obtain the excitation-response relation based on the orthogonality properties of the harmonic wavelet representation. The equivalent stiffness and damping elements depend on the response, defined itself on a specific frequency band and time window. An iterative procedure is then involved to determine the optimal representation of the response spectrum.

The question of nonstationary equivalent linearization has been also addressed wisely by the team of Innsbrück led by Pr. Schuëller in [Prad 03, Sche 04, Sche 05]. They develop a method based on the Karhunen-Loève (K-L) representation of the random processes involved in the dynamics of structure. The K-L expansion is a series aiming at representing a stochastic process according to a spectral decomposition of its correlation function [Le M 10]. Knowing the covariance matrix of the excitation (measured or assumed), the K-L expansion of the excitation is built. Then, the K-L expansion of the state space vector (displacements, velocities and history variables) of the structure is plugged into the linearized equation of motion. This equation is thus integrated to find a finite set of deterministic K-L vectors. The task also requires to adapt the linearized functions of the nonlinear forces. The reduction model proposed in [Sche 05] makes use of normal modes of the linear subsystem, as explained in Chapter 3 of this manuscript, especially because the nonlinear forces are non-essential and localized. We must point out the main advantage of the K-L decomposition: it can be used efficiently when measured data are available [Smyt 02]. However, as mentioned in [Sche 04], they are rare and scattered, a reason why the use of modulated Kanai-Tajimi spectrum remains definitely pertinent. A main drawback of this method is its difficulty to deal with large-dimensional coherent fields. Nonetheless, the philosophy to develop a method able to deal with large-dimensional systems encountered throughout these works is of great interest.

Actually, the possibility to apply a stochastic linearization method to large-dimensional structures subject to coherent random loading is a possible and reasonable argument to evaluate the pertinence of a proposed method. It is also an argument to discard many methods encountered in the literature.

The concept of evolutionary spectral analysis has not been effectively applied in equivalent statistical linearization, mainly for computational reasons. Indeed, some operations in evolutionary spectral analysis are implemented with difficulty. For instance, Kougioumtzoglou and Spanos [Koug 13] qualify the convolution integral between the impulse response matrix and the time window as “intractable” (see (4.2.8)). From Chapter 4, we know that the asymptotic expansion-based method can be efficiently used to compute modal impulse response function, even for coupled systems. However, these authors highlight the particular case for which the timescale of the deterministic envelope and the ones of the system are clearly separated, so the convolution integral can be deeply simplified, even avoided in some circumstances.

This assumption leads to the *quasi-stationary* approach (see (4.2.10)). The evolution of the structural properties is put aside and the time is seen as a parameter. The equivalent statistical linearization is thus similar to the problem addressed in Chapter 3. In Section 4.1, we say our critical and severe opinion about this restrictive assumption for linear systems. In an equivalent linearization context, the separation of the timescales does not theoretically lead to the quasi-steady assumption, but actually to a *multiple scales* formulation of the problem, in which the equivalent properties evolve at a different timescale than the ones of the structure.

Now that the context is outlined, we are ready to assemble the developments of Chapters 3 and 4. First, the multiple scales formulation for the proposed approach is rigorously justified in

Section 5.2. Since an unsteady linearized system is Linear Time-Variant (LTV), the theoretical background is briefly exposed. Then, we introduce the concept of piecewise LTI system, together with the asymptotic expansion-based method. In Section 5.5, the concept of modal basis adaptation throughout a transient solution is presented.

5.2 Multiple scales approach in stochastic linearization

On a probability space $(\Theta, \mathfrak{F}, \mathbb{P})$, we consider the random vector of modal coordinates $\mathbf{q}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^m$. According to (3.3.5), the time evolution of the vector $\mathbf{q}(t)$ of modal coordinates is modeled by the linearized modal equation of motion

$$\ddot{\mathbf{q}} + (\mathbf{D} + \mathbf{D}_{eq}(\Sigma_{\mathbf{q}}(t), \Sigma_{\dot{\mathbf{q}}}(t))) \dot{\mathbf{q}} + (\Omega + \Omega_{eq}(\Sigma_{\mathbf{q}}(t), \Sigma_{\dot{\mathbf{q}}}(t))) \mathbf{q} = a(t)\mathbf{w}(t) \quad (5.2.1)$$

with $\mathbf{w}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^m$ a random excitation vector and $a(t)$ a deterministic envelope. For the sake of simplicity and clarity, the vector $\mathbf{w}(t)$ is supposed to be full of δ -correlated processes, i.e. white noises. This vector may be possibly colored in order to model more realistic excitations. As in (4.2.11) and (4.2.12), Equation (5.2.1) is recast into a state space formalism

$$\dot{\mathbf{z}} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -(\Omega + \Omega_{eq}) & -(\mathbf{D} + \mathbf{D}_{eq}) \end{pmatrix} \mathbf{z} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & a(t)\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{w}(t) \end{pmatrix} \quad (5.2.2)$$

with $\mathbf{A}_{eq}$ the equivalent convection matrix, readily identified as the equivalent statistical extension of (4.2.11).

Since the system is linearized, a Lyapunov equation [Preu 94] in terms of the covariance matrix $\Sigma_{\mathbf{z}}$ can be written after application of Itô's lemma. It describes the time evolution of the second-order moments of the modal state vector $\mathbf{z}$ as in (4.2.11), such that

$$\dot{\Sigma}_{\mathbf{z}} = \mathbf{A}_{eq}\Sigma_{\mathbf{z}} + \Sigma_{\mathbf{z}}\mathbf{A}_{eq}^T + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & a(t)^2\mathbf{S}_{\mathbf{W}} \end{pmatrix} \quad \text{and} \quad \Sigma_{\mathbf{z}}(t) = E \begin{pmatrix} \mathbf{q}\mathbf{q}^T & \mathbf{q}\dot{\mathbf{q}}^T \\ \dot{\mathbf{q}}\mathbf{q}^T & \dot{\mathbf{q}}\dot{\mathbf{q}}^T \end{pmatrix} \quad (5.2.3)$$

with $\mathbf{S}_{\mathbf{W}}$ the matrix of the white noise intensities.

The equation of motion highlights different behaviors characterized by different timescales. The application of the multiple scales technique [Nayf 73, Hinc 91], amply used for decades in mechanics [Luon 99, Emam 04, Deno 10], looks perfectly appropriate. An analysis of the different regimes is mandatory at this step. A convenient way to sort the timescales is to introduce a distinction between t_1 the *fast timescale* related to both the system natural periods and the frequency content of the excitation and t_2 the *slow timescale* related to the envelope and the memory time of the system. The first timescale is related to high frequency contents of the mechanical system, while the second one to the *statistical timescale*. Indeed, the memory time of the system and the deterministic envelope condition the evolution of the statistical moments in the system.

Fast timescale

Now, a *small* parameter ε is introduced to compare the times t_1 and t_2 with respect to the physical time t , such that

$$\begin{cases} t_1 &= t \\ t_2 &= \varepsilon t \end{cases}, \quad (5.2.4)$$

and the time t becomes a function of t_1 and t_2 . Therefore, ordinary differential equations in time become partial differential equations, because

$$\frac{d \cdot}{dt} = \frac{dt_1}{dt} \frac{\partial \cdot}{\partial t_1} + \frac{dt_2}{dt} \frac{\partial \cdot}{\partial t_2} = \frac{\partial \cdot}{\partial t_1} + \varepsilon \frac{\partial \cdot}{\partial t_2}. \quad (5.2.5)$$

Thus, the modal equation of motion (5.2.1) reads

$$\frac{\partial^2 \mathbf{q}}{\partial t_1^2} + 2\varepsilon \frac{\partial^2 \mathbf{q}}{\partial t_1 \partial t_2} + \varepsilon^2 \frac{\partial^2 \mathbf{q}}{\partial t_2^2} + (\mathbf{D} + \mathbf{D}_{eq}) \left(\frac{\partial \mathbf{q}}{\partial t_1} + \varepsilon \frac{\partial \mathbf{q}}{\partial t_2} \right) + (\boldsymbol{\Omega} + \boldsymbol{\Omega}_{eq}) \mathbf{q} = a(t_2) (\mathbf{w}(t_1) + \varepsilon \mathbf{w}(t_2)). \quad (5.2.6)$$

The white noises in $\mathbf{w}(t)$ are split into two different terms by using the artifact that they can be expressed more conveniently (in an “engineering” framework) as the time derivative of a Brownian motion. Similarly, it consists in using the *fractal property* of the Wiener process. The choice to assume $a(t)$ only dependent on t_2 (and not on t_1) is inspired by the physics of the phenomena. For instance, the duration of an earthquake is usually longer than the fundamental period of a structure.

In a classical multiple scales technique, the function $\mathbf{q}$, initially a function of time t , becomes a function of both t_1 and t_2 . A Poincaré expansion of $\mathbf{q}$, defined as

$$\mathbf{q}(t_1, t_2) = \mathbf{q}_0(t_1, t_2) + \varepsilon \mathbf{q}_1(t_1, t_2) + \varepsilon^2 \mathbf{q}_2(t_1, t_2) + \dots \quad (5.2.7)$$

is introduced in (5.2.6). The leading order equation, i.e. the equation extracted from the terms of order $\varepsilon^0 = 1$, is given by

$$\frac{\partial^2 \mathbf{q}_0}{\partial t_1^2} + (\mathbf{D} + \mathbf{D}_{eq}(\boldsymbol{\Sigma}_{\mathbf{z}})) \frac{\partial \mathbf{q}_0}{\partial t_1} + (\boldsymbol{\Omega} + \boldsymbol{\Omega}_{eq}(\boldsymbol{\Sigma}_{\mathbf{z}})) \mathbf{q}_0 = a(t_2) \mathbf{w}_\theta(t_1) \quad (5.2.8)$$

and the equation at order ε^1 is

$$\frac{\partial^2 \mathbf{q}_1}{\partial t_1^2} + 2 \frac{\partial^2 \mathbf{q}_0}{\partial t_1 \partial t_2} + (\mathbf{D} + \mathbf{D}_{eq}(\boldsymbol{\Sigma}_{\mathbf{z}})) \left(\frac{\partial \mathbf{q}_1}{\partial t_1} + \frac{\partial \mathbf{q}_0}{\partial t_2} \right) + (\boldsymbol{\Omega} + \boldsymbol{\Omega}_{eq}(\boldsymbol{\Sigma}_{\mathbf{z}})) \mathbf{q}_1 = a(t_2) \mathbf{w}(t_2) \quad (5.2.9)$$

with $\mathbf{q}_0$ and $\mathbf{q}_1$ both depending on t_1 and t_2 . Calculating the ε -asymptotic expansion of the covariance matrix

$$\boldsymbol{\Sigma}_{\mathbf{z}}(t_1, t_2) = \boldsymbol{\Sigma}_{\mathbf{z},0}(t_1, t_2) + \varepsilon \boldsymbol{\Sigma}_{\mathbf{z},1}(t_1, t_2) + \varepsilon^2 \boldsymbol{\Sigma}_{\mathbf{z},2}(t_1, t_2) + \dots, \quad (5.2.10)$$

by means of (5.2.7) gives the leading order term of (5.2.10)

$$\boldsymbol{\Sigma}_{\mathbf{z},0}(t_1, t_2) = E \begin{pmatrix} \mathbf{q}_0 \mathbf{q}_0^T & \mathbf{q}_0 \left(\frac{\partial \mathbf{q}_0}{\partial t_1} \right)^T \\ \left(\frac{\partial \mathbf{q}_0}{\partial t_1} \right) \mathbf{q}_0^T & \left(\frac{\partial \mathbf{q}_0}{\partial t_1} \right) \left(\frac{\partial \mathbf{q}_0}{\partial t_1} \right)^T \end{pmatrix}. \quad (5.2.11)$$

Slow timescale

From the physics of the underlying phenomena, it is expected that the second-order moment of the system evolves according to the slow timescale t_2 . This fact is emphasized by introducing in the Lyapunov equation (5.2.3), the same timescales while modifying the role of ε in (5.2.4). Indeed, this substitution consists in reducing the slow timescale to capture its influence on the fast dynamics. In the context of the Lyapunov covariance equation, the fast timescale is stretched to be of the same order of magnitude as the slow one. Equation (5.2.4) may be rewritten as

$$\begin{cases} t_1 &= t/\varepsilon \\ t_2 &= t \end{cases} \quad (5.2.12)$$

and the ratio between t_2 and t_1 is still ε in both (5.2.4) and (5.2.12). This artifact is a key principle of the method, because it allows to separately focus on the fast dynamics by invoking (5.2.4) or on the slow dynamics with (5.2.12), without changing the physical meaning of the time variables. From another point of view, this technique is also used to analyze separate asymptotic behaviors in a system.

The differential operator (5.2.5) now reads

$$\frac{d\cdot}{dt} = \frac{dt_1}{dt} \frac{\partial \cdot}{\partial t_1} + \frac{dt_2}{dt} \frac{\partial \cdot}{\partial t_2} = \frac{1}{\varepsilon} \frac{\partial \cdot}{\partial t_1} + \frac{\partial \cdot}{\partial t_2} \quad (5.2.13)$$

and the Lyapunov equation (5.2.3) yields

$$\frac{\partial \Sigma_{\mathbf{z}}}{\partial t_1} + \varepsilon \frac{\partial \Sigma_{\mathbf{z}}}{\partial t_2} = \varepsilon \mathbf{A}_{eq} \Sigma_{\mathbf{z}} + \varepsilon \Sigma_{\mathbf{z}} \mathbf{A}_{eq}^T + \varepsilon \begin{pmatrix} 0 & 0 \\ 0 & a(t_2)^2 \mathbf{S}_{\mathbf{w}} \end{pmatrix} \quad (5.2.14)$$

with the term coming from the excitation and the white noises only dependent on t_2 . From (5.2.14), the leading-order equation is given by

$$\frac{\partial \Sigma_{\mathbf{z},0}}{\partial t_1} = \mathbf{0} \quad (5.2.15)$$

and therefore it does not depend on the fast timescale t_1 . This result is the key point of the proposed demonstration, because it means that **the covariance matrix, through its leading order in a formal asymptotic expansion, only depends on t_2 and so are the equivalent matrices** in (5.2.1). This last equation now reads

$$\frac{\partial^2 \mathbf{q}_0}{\partial t_1^2} + (\mathbf{D} + \mathbf{D}_{eq}(\Sigma_{\mathbf{q},0}(t_2), \Sigma_{\dot{\mathbf{q}},0}(t_2))) \frac{\partial \mathbf{q}_0}{\partial t_1} + (\mathbf{\Omega} + \mathbf{\Omega}_{eq}(\Sigma_{\mathbf{q},0}(t_2), \Sigma_{\dot{\mathbf{q}},0}(t_2))) \mathbf{q}_0 = a(t_2) \mathbf{w}(t_1), \quad (5.2.16)$$

or more conveniently,

$$\frac{\partial^2 \mathbf{q}_0}{\partial t_1^2} + (\mathbf{D} + \mathbf{D}_{eq}(t_2)) \frac{\partial \mathbf{q}_0}{\partial t_1} + (\mathbf{\Omega} + \mathbf{\Omega}_{eq}(t_2)) \mathbf{q}_0 = a(t_2) \mathbf{w}(t_1). \quad (5.2.17)$$

Equation (5.2.17) is the leading order component of the modal equation of motion taking into account the physical separation between the structural and the statistical times.

For the sake of completeness, it shall be pointed out that the methodology of multiple scales may be applied in the case of non-zero mean processes. In this case, the main conclusion is equivalent to the one drawn after considering the Lyapunov equation (5.2.3) : if two timescales can be highlighted in the problem, the mean vector of the state space vector evolves according to the slow time t_2 , like the covariance matrix. It must be emphasized that the mean response of the system is more rigorously calculated in the *nodal* basis ($\mathbf{x} = \Phi \mathbf{q}$). The mean vectors $\boldsymbol{\mu}_{\mathbf{x}}$ and $\boldsymbol{\mu}_{\dot{\mathbf{x}}}$ are calculated by solving the following differential equation

$$\frac{d}{dt} \begin{pmatrix} \boldsymbol{\mu}_{\mathbf{x}} \\ \boldsymbol{\mu}_{\dot{\mathbf{x}}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}(\mathbf{K} + \mathbf{K}_{eq}) & -\mathbf{M}^{-1}(\mathbf{C} + \mathbf{C}_{eq}) \end{pmatrix} \begin{pmatrix} \boldsymbol{\mu}_{\mathbf{x}} \\ \boldsymbol{\mu}_{\dot{\mathbf{x}}} \end{pmatrix} + a(t) \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \bar{\mathbf{f}} \end{pmatrix} \quad (5.2.18)$$

with $\bar{\mathbf{f}}$ the mean value of the embedded stationary process $\mathbf{f}(t)$ in (5.2.18). The same mathematical artifact about the timescales leads to $\partial_{t_1} \boldsymbol{\mu}_{\mathbf{x},0} = \partial_{t_1} \boldsymbol{\mu}_{\dot{\mathbf{x}},0} = \mathbf{0}$, i.e. the leading order term of the mean vector does not evolve according to the fast time.

Multiple scales approach and quasi-stationary assumption

Assuming that the leading order is exclusively considered (for sufficiently small values of ε , this approximation is sufficient), the Fourier transform in terms of the fast timescale (the only one relevant for this operation) of the right-hand side of (5.2.17) leads to

$$(-\omega^2 \mathbf{I} + i\omega (\mathbf{D} + \mathbf{D}_{eq}(t_2)) + (\boldsymbol{\Omega} + \boldsymbol{\Omega}_{eq}(t_2))) \mathbf{Q}_0(\omega, t_2) \quad (5.2.19)$$

with $\mathbf{Q}_0(\omega, t_2)$ the Fourier transform of $\mathbf{q}_0$ with respect to t_1 , which highlights an instantaneous transfer function of the system at the leading order of the fast timescale. The t_2 -dependence of $\mathbf{Q}_0$ emphasizes that the frequency distribution of the energy in the process $\mathbf{q}_0$ is time and frequency dependent, as expected in evolutionary spectral analysis.

According to (5.2.19), the *quasi-stationary* philosophy considers a single timescale by annihilating the influence of the past. The dependence on t_2 is simply omitted according to the assumption that “*the system natural period is small compared to the duration of the evolutionary excitation* [the quasi-stationary approach] *can yield satisfactory results*” [Koug 13]. However, this assumption does not lead to the quasi-stationary approach, but to the multiple scales philosophy. Indeed, the validity of the quasi-stationary approach should take into account the time memory of statistically equivalent properties. More prosaically, this is the capability of the system to “forget the past”. The proposed method offers another insight into the problem and a more rigorous approach to take into account the time-evolution of the system, provided that distinct timescales may be *a priori* identified. Indeed, the separation between the two timescales does not indicate that the past may be omitted or that its influence is negligible, but that the functions only depending on t_2 are evolving much slower than $\mathbf{q}_0$. Thanks to the multiple scales demonstration, it becomes natural that the time evolution of the equivalent matrices $\boldsymbol{\Omega}_{eq}$ and $\mathbf{D}_{eq}$ may be approximated on time intervals larger than the period of the system. In other words, these matrices *may* be supposed to be constant on intervals including several periods of vibration.

5.3 Evolutionary spectral analysis for equivalent LTV systems

5.3.1 Equivalent statistical linearization for LTV systems

In (5.2.1), the equivalent structural matrices depend on the covariance response matrix which is time dependent. The time evolution of the statistical properties of the system must be taken into account in the computation of the equivalent matrices. Equation (5.2.1) models a linear or linearized system, but the time-depending parameters of the system are such that the relation (4.2.23), between the impulse response matrix and the transfer matrix of the system, cannot be used anymore. Indeed, for linear time-invariant systems, their response only depends on the lag between the considered time and the time at which the force is applied. In the case of Linear Time-Variant systems, the instant at which the load is applied is relevant by itself.

The theory of LTV systems has not been effectively applied in the transient stochastic linearization method, because the formal developments of this theory lead to intricate equations. However, the results obtained before about the different timescales in the system, associated with the practical method to compute impulse response matrix of slightly (even moderately) coupled systems, may offer novel perspectives with respect to other methods proposed in the literature.

First of all, the equation set (3.3.6)-(3.3.9) in a stationary setting and in a modal basis, is now updated to take into account the time-variability of the system, such that

$$\Sigma_{\mathbf{q}}(t) = \int_{\mathbb{R}} \widehat{\mathbf{S}}_{\mathbf{q}}(t, \omega) d\omega \quad \text{and} \quad \Sigma_{\dot{\mathbf{q}}}(t) = \int_{\mathbb{R}} \widehat{\mathbf{S}}_{\dot{\mathbf{q}}}(t, \omega) d\omega \quad (5.3.1)$$

with $\widehat{\mathbf{S}}_{\mathbf{q}}(t, \omega)$ and $\widehat{\mathbf{S}}_{\dot{\mathbf{q}}}(t, \omega)$ the evolutionary power spectral densities of the modal coordinate vector and the related velocity vector, and

$$\mathbf{K}_{eq}(t) = \mathcal{K}(\Sigma_{\mathbf{xx}}(t), \Sigma_{\dot{\mathbf{x}}\dot{\mathbf{x}}}(t), \Sigma_{\mathbf{x}\dot{\mathbf{x}}}(t)), \quad \text{and} \quad \mathbf{C}_{eq}(t) = \mathcal{C}(\Sigma_{\mathbf{xx}}(t), \Sigma_{\dot{\mathbf{x}}\dot{\mathbf{x}}}(t), \Sigma_{\mathbf{x}\dot{\mathbf{x}}}(t)). \quad (5.3.2)$$

As in (5.2.1), the equivalent matrices are thus implicitly time-dependent [Robe 99]. Once again, the Kazakov derivative can be used to compute the equivalent structure, since the response of the equivalent system remains Gaussian at any time.

For convenience, in (5.2.1) we consider that the random excitation is a vector of uncorrelated white noises, which is restrictive anyway. The objective is now to develop a more general purpose. In a state formalism similar to (5.2.2), the equation of motion (5.2.1) is readily given by

$$\dot{\mathbf{z}} = \mathbf{A}_{eq}(t)\mathbf{z} + \begin{pmatrix} \mathbf{0} \\ \mathbf{p}(t) \end{pmatrix} \quad (5.3.3)$$

with $\mathbf{p}(t)$ a vector gathering evolutionary random processes with *orthogonal increments*. The general solution of an LTV system is expressed as

$$\mathbf{z}(t, \theta) = \Psi_{eq}(t, t_0)\mathbf{z}(t_0) + \int_{t_0}^t \Psi_{eq}(t, u) \begin{pmatrix} \mathbf{0} \\ \mathbf{p}(u, \theta) \end{pmatrix} du \quad (5.3.4)$$

with $\Psi_{eq}(t, t_0)$ the **equivalent state transition matrix** from a time t_0 to a time t . This equation is identical to (4.2.12) with the noteworthy difference that this matrix is here defined

for the equivalent system. As done before for LTI systems, an **equivalent state space evolutionary transfer matrix** $\Upsilon_{eq}(t, t_0, \omega) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{C}^{2m \times 2m}$ (with $t > t_0$) is defined as

$$\Upsilon_{eq}(t, t_0, \omega) = \int_{t_0}^t a(u, \omega) e^{i\omega u} \Psi_{eq}(t, u) \mathbf{B} du \quad (5.3.5)$$

The spectral relation (2.2.12) becomes

$$\Sigma_z(t) = \int_{\mathbb{R}} \Upsilon_{eq}(t, t_0, \omega) \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_p(\omega) \end{pmatrix} \Upsilon_{eq}^*(t, t_0, \omega) d\omega, \quad (5.3.6)$$

which is similar in form and substance to (4.2.17).

5.3.2 Piecewise approximation of equivalent systems

All those developments do not simplify *prima facie* the transient statistical linearization. Before applying the LTV theory, different possibilities have been tried in order to capture as precisely as possible the transient dynamics, while conserving the philosophy and the efficiency of the LTI approach. A first trial was to consider that the equivalent matrices were constant on $[0, t]$ with the value at t , so the integrals could be directly computed. In spite of good results in the “build-up” regime, the system could be notably overdamped in the strong period of excitation. Other approximations have been tried in order to replace the time-dependent parameters by suitable values on the intervals : the mean value on the interval of the equivalent matrices or a balanced mean value of those matrices. However, all the approaches aiming at using an adapted LTI theory have been discarded, because those approximations led to very poor results and dubious evidences.

Our objective remains to get results as close as possible to those obtained with Monte Carlo simulations, considered as our reference, in the different periods of a transient response. The elegance of analytical developments might not be spoiled in rough assumptions. Now, some motivated hypotheses are used in order to make the LTV theory easier to apply to the problem exposed in this Section.

According to the developments carried out in Section 5.2 and the conclusion of the demonstration, **the equivalent damping and stiffness matrices evolve with the statistical time** (or slow timescale of the system). **Based on this fact, the interval $[0, t]$ can be divided with regard to information about the time envelope and the memory time of the system.** Then, the continuous representation of the equivalent matrices can be approximated by a piecewise constant function, i.e.

$$\mathbf{K}_{eq}(t) \approx \sum_{k=1}^{M+1} \mathbf{K}_{eq}(\bar{t}_k) \mathbb{I}(t \in [t_{k-1}, t_k]), \quad (5.3.7)$$

with $\bar{t}_k \in [t_{k-1}, t_k]$ ($k = 1, \dots, M+1$) and $\mathbb{I}(\mathcal{A})$ the indicator function of a set $\mathcal{A}$. The same approximation is applied to $\mathbf{C}_{eq}(t)$ and $\mathbf{A}_{eq}(t)$. The integration interval is divided into $M+1$ non-overlapping intervals, such that

$$t_0 = 0 < \dots < t_k < \dots < t_M \leq t \leq t_{M+1}. \quad (5.3.8)$$

Using the piecewise approximation (5.3.7) of the equivalent matrices, the transient transfer function $\Upsilon_{eq}(t_{M+1}, t_0, \omega)$ now reads

$$\Upsilon_{eq}(t_{M+1}, t_0, \omega) = \sum_{k=1}^{M+1} \int_{t_{k-1}}^{t_k} a(u, \omega) \Psi_{eq,k}(t_{M+1}, u) \mathbf{B} e^{\omega u} du, \quad (5.3.9)$$

provided the system response evolves continuously on $[0, t]$. The matrices $\Psi_{eq,k}(t_2, t_1)$ with $t_1, t_2 \in [t_{k-1}, t_k]$ form a family of $M + 1$ state transition matrices, each one valid on the k -th interval calculated with the equivalent properties on this interval.

Back to Section 4.2.2, the transition matrix can be defined as the unique solution of the partial differential equation

$$\frac{\partial}{\partial t} \Psi(t, t_0) = \mathbf{A}_{eq}(t) \Psi(t, t_0) \quad (5.3.10)$$

with $\Psi(t_0, t_0) = \mathbf{I}$. According to this definition and since the properties of the system are supposed to be constant on an interval $[t_{k-1}, t_k]$, the transition matrix $\Psi_{eq}(t_2, t_1)$ is given, for $t_1, t_2, \bar{t}_k \in [t_{k-1}, t_k]$, by

$$\Psi_{eq,k}(t_2, t_1) = e^{\mathbf{A}_{eq}(\bar{t}_k)(t_2 - t_1)}. \quad (5.3.11)$$

The *semi-group property* of the transition matrices $\Psi_{eq,k}(t_2, t_1)$ applies to the piecewise approximation, as well as the developments exposed in Section 4.4. The equivalent state transfer matrix reads

$$\Upsilon_{eq}(t, t_0, \omega) = \Psi_{eq,M}(t, t_M) \Upsilon_{eq}(t_M, t_0, \omega) + \Upsilon_{eq}(t, t_M, \omega), \quad (5.3.12)$$

where

$$\Upsilon_{eq}(t, t_M, \omega) = \int_{t_M}^t a(u) \Psi_{eq,M+1}(t, u) \mathbf{B} e^{\omega u} du \quad (5.3.13)$$

and $t \in [t_M, t_{M+1}]$. Equation (5.3.12), similar to (4.4.6) in the context of equivalent statistical linearization, is a fundamental result of this work. Thanks to the multiple scales approach, we demonstrate that the efficient formulation developed in Section 4.4 can be applied in this nonlinear context. The main difference is that the equivalent matrices are modified on each interval. **The asymptotic expansion-based method, initially developed in a stationary setting, then extended to linear evolutionary analysis, is now definitely applicable to nonstationary equivalent statistical linearization.**

The last equation (5.3.13) may also be rewritten as

$$\Upsilon_{eq}(t, t_M, \omega) = \begin{pmatrix} \mathbf{0} & \mathbf{G}_{eq}(t, t_M, \omega) \\ \mathbf{0} & \partial_t \mathbf{G}_{eq}(t, t_M, \omega) \end{pmatrix}, \quad (5.3.14)$$

where $\mathbf{G}_{eq}(t, t_{k-1}, \omega)$ ($t \in [t_{k-1}, t_k]$) is the equivalent evolutionary transfer matrix of $\mathbf{q}(t)$ on the interval $[t_{k-1}, t_k]$ with the equivalent properties constant on this interval.

With this approach, the convolution integral leading to $\mathbf{G}_{eq}$ and $\partial_t \mathbf{G}_{eq}$ is performed at a negligible computational cost by the proposed asymptotic expansion-based method. In (5.3.14), these two matrices may be replaced by their asymptotic expansion $\mathbf{G}_{eq,N}$ and $\partial_t \mathbf{G}_{eq,N}$

defined in (4.3.5). That is performed for each of the $M + 1$ time intervals. For instance, assuming $t_1, t_0 \in [t_{k-1}, t_k]$, the main decoupled term is given by

$$(\mathbf{G}_{eq,d}(t_2, t_1, \omega))_i = \frac{\mathcal{I}_i^+(t_2, t_1, \omega) - \mathcal{I}_i^-(t_2, t_1, \omega)}{2i\omega_{d,i}} \quad (5.3.15)$$

with the integrals $\mathcal{I}_i^\alpha(t_1, t_0, \omega)$ defined in (4.3.13). This is also applied to higher-order correction terms computed interval per interval.

5.3.3 Comments about the underlying assumptions

Some questions are risen at this step with regard to the developments of this chapter : (i) how to determine a suitable *a priori* division of the interval $[0, t]$, (ii) how to choose the time $\bar{t}_k$ in (5.3.11), (iii) is the analytical integration necessary? All these questions precisely pertain to the stochastic linearization method in a transient regime and they are addressed in the sequel.

(i) The division of the time interval $[0, t]$ depends upon the different timescales involved in the system. Increasing the damping in a structure leads to decrease its memory time. For highly to moderately damped structures, the statistics of the response is thus mainly conditioned by the time window which must not be limited to the duration of the action. For instance, in the Jennings window, the three phases (build up, plateau and decrease) of the earthquake must be considered individually. A small natural period compared with a phase of the time envelope and the memory time of the system allows to consider large time steps in the resolution. In transient wind problems, the time window evolves much slower than seismic windows, while the structure periods are about few seconds, justifying thus the smallness of ε in these applications. For instance, Holmes' window used to model downburst has a strong period of 10 minutes, while the duration of earthquake exceeds rarely one minute.

These considerations allow to work with larger time steps in order to improve the numerical efficiency of the method. However, from the previous developments, the time interval used for the piecewise approximation of the equivalent matrices may be of the order of the fast timescale t_1 in (4.5.8). Any well modeled adaptation of the structure over a slow time interval is identically modeled over smaller time interval.

The convergence in the problem is such that the refinement of the division of $[0, t]$ does not necessarily yield to a better result. This fact is illustrated in the following examples.

(ii) Inside the k -th time interval $[t_{k-1}, t_k]$, the time $\bar{t}_k$ can be efficiently chosen as the lower bound of the interval. The discussion on this choice pertains to a balance between the computational burden and the accuracy of the implementation. Because of the causality principle, at the stage t_{k-1} no information about the property on the k -th interval is known. Hence, the best approximation for (5.3.7) is made with the covariance matrix $\mathbf{\Sigma}_z$ at t_{k-1} . Otherwise, a solver must be used to find a better value of the parameters to minimize a residue equation (like exposed in Chapter 3). Instead of using a solver, taking into consideration the convergence of the piecewise approximation, it will be sufficient to improve the time division of the whole interval.

(iii) We understand that the asymptotic expansion-based method conveniently associates analytical developments and numerical methods. The efficiency of this semi-analytical ap-

proach is deliberately tackled at this stage by comparing it with a purely numerical evolutionary spectral analysis. Considering that the system is effectively decoupled in the modal basis, the evolutionary transfer matrix must be computed according to (5.3.15). The numerical computation of the integrals $\mathcal{I}_i^\alpha$ requires a small time step in order to capture the timescales of the system and the range of frequencies in the term $e^{i\omega t}$. The numerical integration is therefore definitely inefficient. Cutting the chase, this semi-analytical approach allows to circumvent heavy computational tasks.

5.3.4 Application to a 2-DoF nonlinear system

In this Section, a 2-DOF nonlinear system is considered. Although this example may seem theoretical, it is pertinent to illustrate the proposed LTV-method. Actually, the developments of this chapter are inspired by the reading of [Koug 13], in which this example is discussed. It allows a simple comparison between the proposed approach and the one of [Koug 13], with an insight on the computational burden.

The equation of motion is

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} + \mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}) = a(t)p(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (5.3.16)$$

with the corresponding matrices

$$\mathbf{M} = m_s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{C} = c_s \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{K} = k_s \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad (5.3.17)$$

and the vector of nonlinear internal forces

$$\mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}) = \begin{pmatrix} \varepsilon_1 k_s y_1^3 + \varepsilon_2 c_s \dot{y}_1^3 \\ \varepsilon_1 k_s (y_2 - y_1)^3 + \varepsilon_2 c_s (\dot{y}_2 - \dot{y}_1)^3 \end{pmatrix}. \quad (5.3.18)$$

This system presents nonlinear behaviors in both stiffness and damping. The exogenous forces $p(t)$ are given by a Kanai-Tajimi excitation ($\omega_1 = 4\text{rad/s}$, $\omega_2 = 15\text{rad/s}$, $\xi_1 = 0.5$, $\xi_2 = 1$ and S_0 adjustable) modulated by a Shinozuka time window ($b_1 = 0.1$ and $b_2 = 0.3$). The same process is applied to both DOFs. For the numerical application, the parameters m_s , k_s and c_s equal 1, 1600 and 4, respectively. The parameters ε_1 and ε_2 are equal to 2 and 0.5, respectively.

The natural frequencies of the linear *subsystem* are 24.7 and 64.7 rad/s. In this example, the index of convergence in the modal basis of the linear subsystem ($\mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}) = \mathbf{0}$) evolves in time, but remains sufficiently small to limit the asymptotic expansion to the main decoupled approximation. We use the analytical expression of Section 4.3.3 between 0 and t . This fact is illustrated hereinafter. Therefore, the natural periods of the system are 0.1 and 0.25s, i.e. small compared with a 20s earthquake, and the multiple scales approach can be applied with confidence. The length of the time intervals in (5.3.8) is chosen equal to 0.5s, but its influence is then investigated. This length is justified by the timescales in the problem, i.e. twice the larger natural period of the linear subsystem.

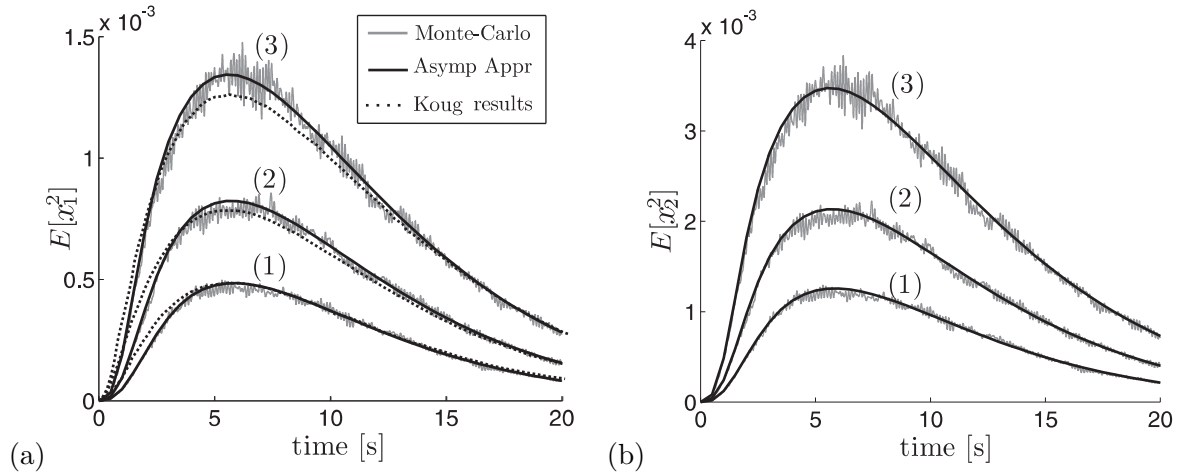


Figure 5.1: Kougioumtzoglou and Spanos' example for S_0 equal to 25 (1), 50 (2) and 100 (3). Time evolution of the variance of (a) x_1 and (b) x_2 . Comparison with Monte Carlo simulation (1280 samples) and Kougioumtzoglou's approach.

Figures 5.1-a and -b compare the results of the equivalent linearization with the proposed method and Monte Carlo simulation (1280 samples). Different intensities have been tested, i.e. S_0 is equal to 25, 50 and 100. The equivalent linearization matches the simulations for the different intensities in the different phases of the excitation. Figures 5.1-a also compares the proposed method with the quasi-stationary approach given by [Koug 13], which is not able to exactly follow the different phases of the response of the system. The computational cost is about 0.1s cpu time per time step; it is not an issue. Only the decoupled approximation is calculated in this application, because the index of convergence remains sufficiently small to avoid the computation of higher-order correction terms as illustrated in Figure 5.2-b. The modal damping ratio taking into account both the linear and the linearized damping forces is shown in Figure 5.2-a. This picture highlights the importance of the nonlinear damping forces.

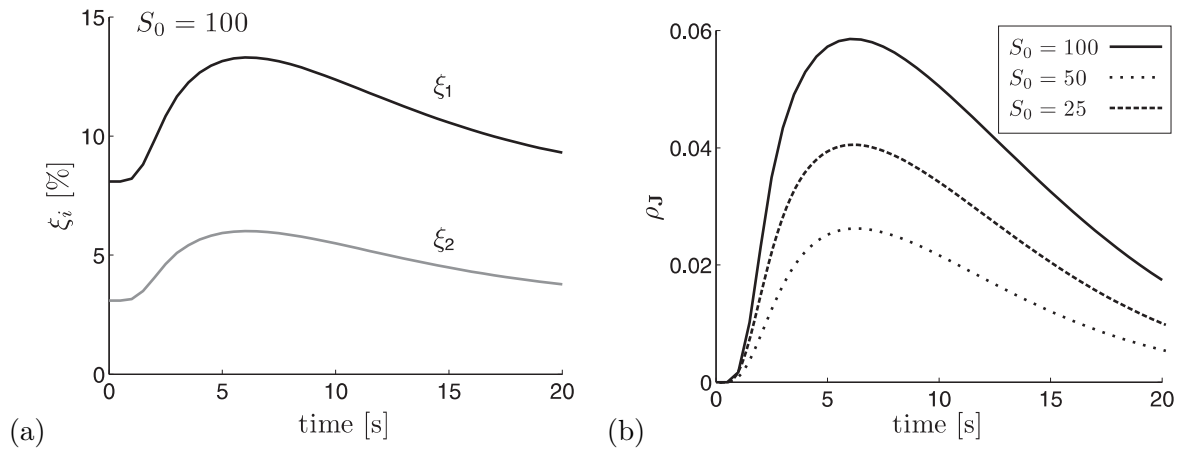


Figure 5.2: Kougioumtzoglou and Spanos' example. Time evolutions of (a) the equivalent damping ratios ($S_0 = 100$) and (b) the convergence criterion ρ_J for different S_0 .

A question arising at this stage pertains to the length of the time intervals. The time step $\Delta t = 0.5\text{s}$ has been chosen with regard to the parameters of the linear subsystem. Figure 5.3 shows the time evolutions of the second-order moment of q_1 computed with different time steps. For a time step of 0.2s the results do not change compared to the initial choice of 0.5s . This comparison emphasizes the convergence of the equivalent properties of the system with respect to (5.3.7). However, the use of a larger time step, i.e. $\Delta t = 2\text{s}$ completely degrades the quality of the results. The research of an optimum is important, because the whole linearization and the multiple scales assumption takes its advantage in the definition of larger time steps, comparing with the time step of the time domain approach.

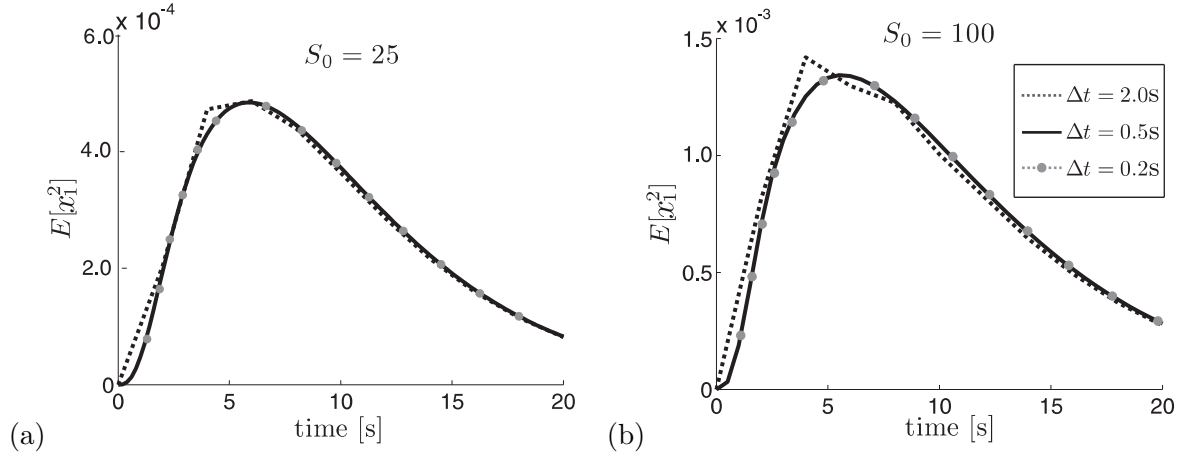


Figure 5.3: Kougioumtzoglou and Spanos' example. Influence of the time step for (a) $S_0 = 25$ and (b) $S_0 = 100$.

5.4 Application in wind and earthquake engineering

5.4.1 Multi-span bridge subject to earthquake

To first illustrate the multiple scales approach on a realistic example with coherent excitation field, we propose to come back to the example developed in Section 4.5, namely the multi-span bridge subject to coherent ground motion. The characteristics of the excitation and the structure are similar in this application. The behavior of the dampers, supposed to be linear in Section 4.5, is here considered as nonlinear, with the classic velocity-force law

$$F_D(v) = C_D \text{sign}(v) |v|^\alpha \quad (5.4.1)$$

with v the relative velocity between the top of the pile and the deck and C_D the damping coefficient. The exponent α is generally less than one in order to mitigate vibration at limited velocity. We consider three different laws (5.4.1) : (i) $\alpha = 0.75$, $C_D = 5 \cdot 10^6 \text{N}(\text{s}/\text{m})^\alpha$; (ii) $\alpha = 0.50$, $C_D = 3 \cdot 10^6 \text{N}(\text{s}/\text{m})^\alpha$ and (iii) $\alpha = 0.25$, $C_D = 10^6 \text{N}(\text{s}/\text{m})^\alpha$.

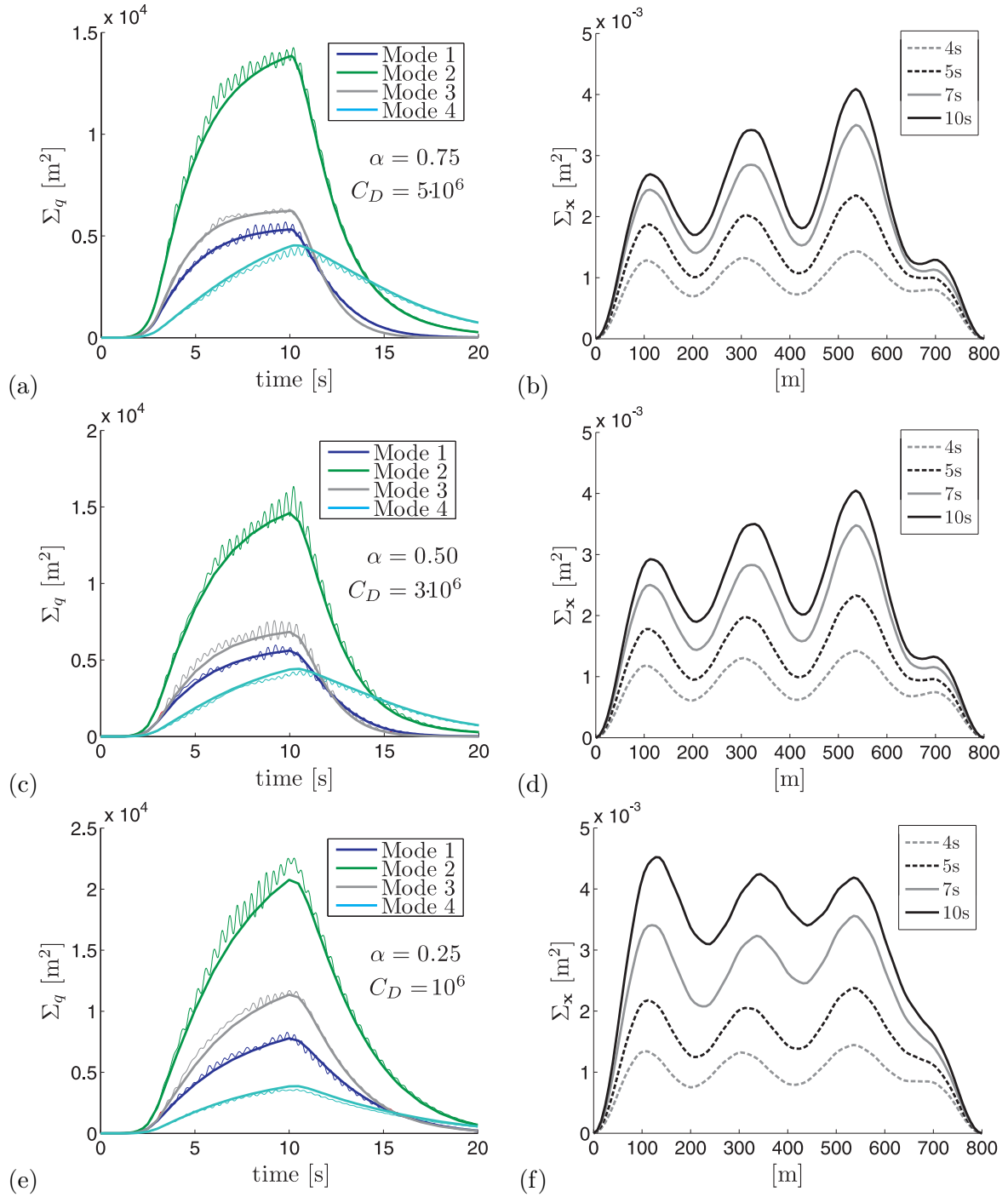


Figure 5.4: Multi-span bridge under coherent earthquake for different nonlinear damping laws (a,b) $\alpha = 0.75$, (c,d) $\alpha = 0.50$ and (e,f) $\alpha = 0.25$. The unity of C_D is N(s/m) ^{α} . (a,c,e) Variance of the first four modal coordinates with respect to time. Comparison between equivalent linearization (bold) and Monte Carlo simulation (thin) . (b,d,f) Variance of the deck displacements at four different times.

This damping law is non-differentiable for $v = 0$, meaning that the initial increment of the damping force, for a structure starting vibrating from rest, is infinite. The Kazakov formula (2.5.5) directly emphasizes an ill-conditioned linearization, especially in a transient regime. To circumvent this difficulty, we replace the force (5.4.1) by a linear law in $v \in [0, v_0]$ with $v_0 = 0.01\text{m/s}$. Naturally, this modification is considered in the Monte Carlo simulations as well.

As previously, the time window is sampled with a step of 0.5s, slightly less than the fundamental period of the bridge. This choice is a compromise between the characteristic times of the structure and a correct representation of the time envelope $a(t)$. The structure is projected in the modal basis related to the linear subsystem ($\tilde{\mathbf{K}} = \mathbf{0}$). The cross-psd of the excitation is sampled with 5000 points mainly gathered around the first five natural frequencies. In our method, the truncation order N is equal to 2. Though the convergence of the results has been compared with $N = 1$, it is not illustrated here. We think the comparison with Monte Carlo simulation ($\Delta t = 0.01\text{s}$, $\Delta f = 1\text{mHz}$ and 3200 samples) is sufficient to assess the convergence for the concerned reader.

The main results obtained for this example are plotted in Figure 5.4 for three different coefficients α . In Figures 5.4-a, -c and -e, we notice the good agreement of our method with Monte Carlo simulations. In Figure 5.4-e, the results obtained with our method match less well the simulations, because of the higher intensity of the nonlinear forces. We do not forget that the equivalent statistical linearization is an approximate method all the more accurate that the nonlinear forces are small. By comparing Figures 5.4-b, -d and -f, the effect of the nonlinear dampers is to significantly limit the vibrations of the deck. The dampers introduce a strong connection between the top of the piles and the deck.

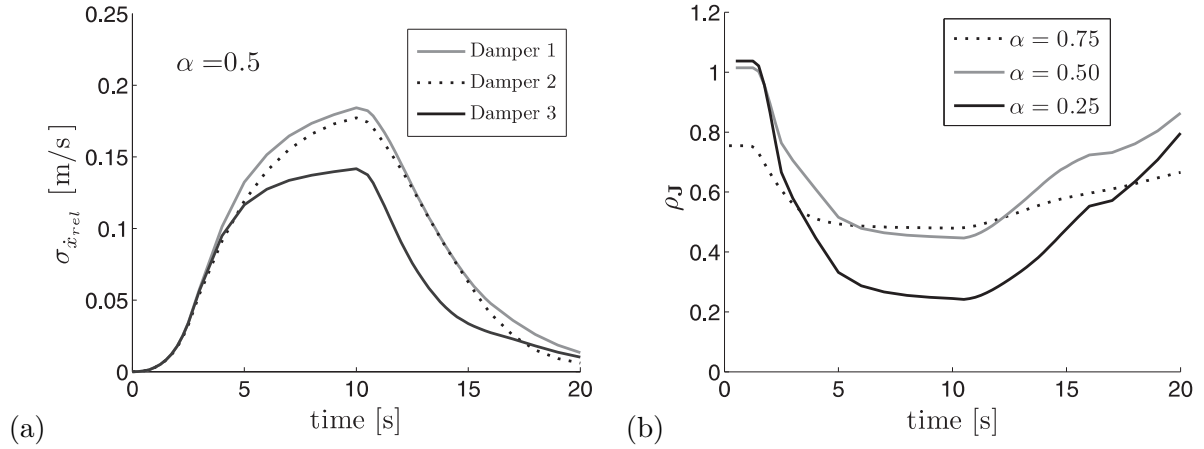


Figure 5.5: Multi-span bridge under coherent earthquake with $\alpha = 0.25$ and $C_D = 5 \cdot 10^6 \text{N(s/m)}^\alpha$. (a) Standard deviation of the relative velocity between the top of the piles and the deck at the three damped connections. (b) Time evolution of the convergence criterion ρ_J for the three different values of α .

In order to quantify the intensity of the nonlinear forces, the standard deviation of the relative velocity at the three connections for $\alpha = 0.50$ is shown in Figure 5.5-a. This figure shows that the linear behavior up to $v = v_0$ is not significant in the problem. Figure 5.5-b emphasizes that the convergence criterion ρ_J of the series is high, but less than one throughout

the solution.

In Figure 5.6, we emphasize a particular case, in which the damping coefficient C_D is equal to $5 \cdot 10^6 \text{N(s/m)}^\alpha$ for $\alpha = 0.25$. In this situation, the bridge response is highly nonlinear. By comparing the results we obtain, shown in Figure 5.6-a with $m = 7$, with Monte Carlo simulations, we notice a quite good agreement, in spite of a significant nonlinear behavior. Nevertheless, we cannot rush headlong into asserting the efficiency of our method in dealing with high nonlinearities. Indeed, the convergence criterion $\rho_{\mathbf{J}}$ is greater than one throughout the solution, as emphasized in Figure 5.6-c, indicating that the modal transfer matrix with seven modes is not well represented by the asymptotic expansion.

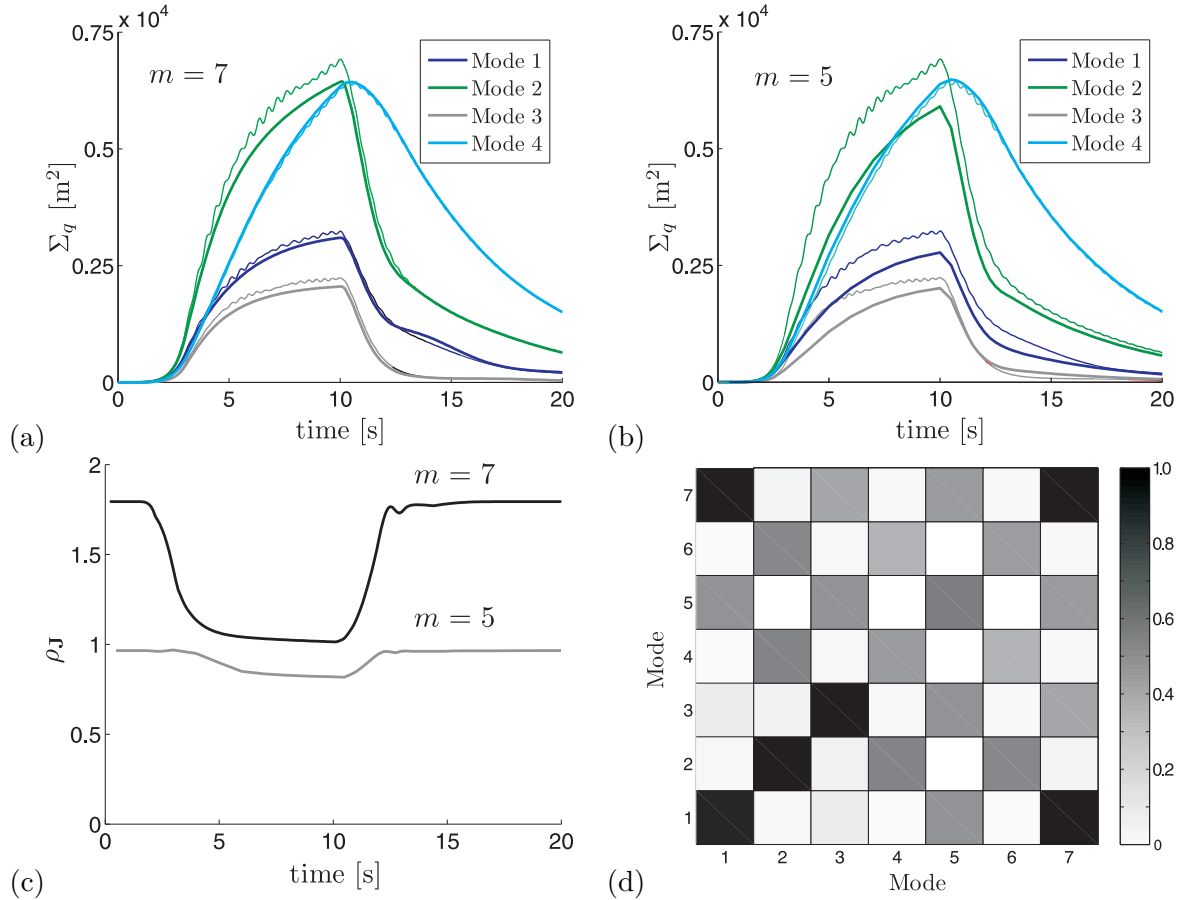


Figure 5.6: Multi-span bridge under coherent earthquake. Variances of the modal coordinates for (a) $m = 7$ and (b) $m = 5$. (c) Time evolution of the convergence criterion $\rho_{\mathbf{J}}$. Modal damping matrix $\mathbf{D} + \mathbf{D}_{eq}$ (linear + equivalent damping matrices) at $t = 10\text{s}$ [Ns/m].

Since the criterion $\rho_{\mathbf{J}}$ is related to the *modal* evolutionary transfer matrix, it also depends on the number of modes we keep in the analysis. Figure 5.6-b shows the variance of the modal coordinates for five modes ($m = 5$). These results do not match Monte Carlo simulation as well as for $m = 7$, although the criterion $\rho_{\mathbf{J}}$ is less than one as shown in Figure 5.6-c. Indeed, the modal damping matrix $\mathbf{D} + \mathbf{D}_{eq}$ (at $t = 10\text{s}$) is depicted in Figure 5.6-d to highlight the significant mechanical coupling existing between the first and the seventh modes. These

observations emphasize the dependence of $\rho_{\mathbf{J}}$ on the modal truncation. Nevertheless, we do not explain the so good agreement between Monte Carlo simulation and our method for $m = 7$.

Actually, we have investigated different explanations and the following one seems coherent with our developments, according to our opinion. We might justify these observations by understanding the essence of the criterion $\rho_{\mathbf{J}}$. This criterion is actually extremely severe, because it ensures the convergence of the matrices $\mathbf{H}_N(\omega)$ and $\mathbf{G}_N(t, t_0, \omega)$, for all $\omega \in \mathbb{R}$ and all the modes. Therefore, a criterion greater than one mainly indicates a non-convergence around the peaks of $\mathbf{H}_d(\omega)$. Although the series $\mathbf{H}_N(\omega)$ does not converge for $\rho_{\mathbf{J}} > 1$, it is possible that, at short time, the series $\mathbf{h}_N(t)$ converges. This statement is not demonstrated and constitute an extension of the reflexions shared in Section 4.3.2. Further investigations could more deeply focus on the effect of the inverse Fourier transform of $\mathbf{H}_N(\omega)$ on the convergence of $\mathbf{h}_N(t)$. Nevertheless, in the sequel, we keep in mind that the criterion $\rho_{\mathbf{J}}$ remains a trustworthy indicator.

5.4.2 Tower subject to downburst

In wind buffeting analysis, the wind velocity is usually assumed to be a stationary Gaussian process. Then, the wind pressure are also assumed to be stationary Gaussian processes obtained by a linear transformation of the velocity (see Section 3.4 and the application on the Millau Viaduct). The random characteristics of the wind velocity summarized in a psd (turbulence length, turbulence intensity, mean velocity) are generally estimated by an average of 10-minute samples on the measured velocities. The spatial coherence between different points is a quantity estimated with difficulty, but the exponential decay proposed by Davenport [Dave 61] is widely used. For instance, this model is quite realistic for the measured pressures on a wind ward building face. However, the stationary paradigm assumed in wind suffers from statistical inconsistency, especially for properties extracted from a long-length sample assuming ergodic processes. In practice, ergodicity cannot be applied because of the time evolution of the wind properties.

In substance, some phenomena well-known in wind engineering fail to be physically modeled with the stationary assumption. Tornadoes and downbursts are two notable examples. A downburst is a sudden fall of a column of air sinking from the higher atmospheric level that spreads out in all directions after hitting the ground. According to [Holm 00], such an event normally occurs during thunderstorm. Contrarily, tornadoes are rotating columns of air, joining the earth and the cloud stratum. It should be pointed out that Jone's assumption [Jone 01] consisting in decoupling the buffeting forces from the aeroelastic ones ceases to be valid for transient wind loading, because all the phenomena are nonstationary and coupled.

In particular, some pertinent theories have been developed in the last decade to apply more realistic models of transient wind velocity. For downburst, Holmes and Oliver [Holm 00] propose an empirical model based on the *translation velocity* of the storm and the *radial velocity* of the air. This empirical approach allows to fit interestingly some measured cases as shown in [Chen 07]. More precisely, Holmes' model depends upon an *observation point* compared with the trajectory of the storm supposed to be a straight line. This model is now briefly exposed, according to Figure 5.7. Then, a factual application is proposed, in which the modal coupling is a dominating fact.

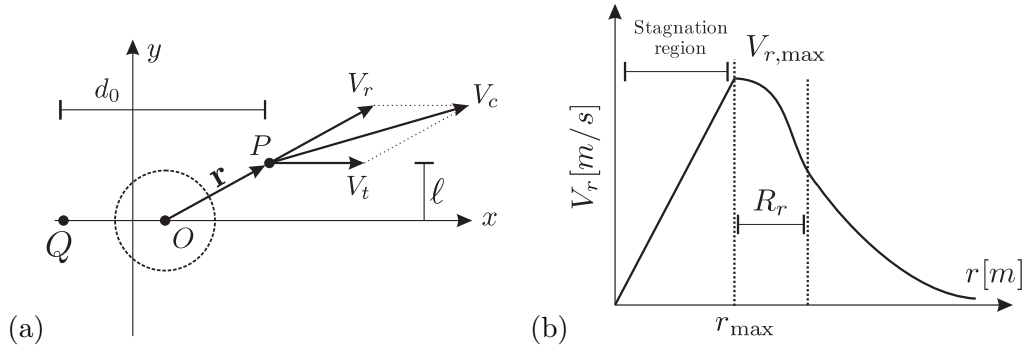


Figure 5.7: Downburst model. (a) Vector combination of radial and translation velocities. (b) Profile of radial wind velocity.

First, we assume that the storm moves along a straight line ($y = 0$) with a *translation velocity* V_t which is constant, as shown in Figure 5.7-a. The offset distance between this line and the observation point P is noted ℓ . The center of the storm, so-called *stagnation point*, is denoted by O and located at a distance $V_t t$ from Q . Like in [Chen 04], a point Q (d_0 is the distance between Q and P along the x -axis) locates where the center of the storm starts its path. At an instant t , the vector $\mathbf{r}(t)$ separating the center O and the observation point P is $\mathbf{r} = (d_0 - V_t t, \ell)$, while the radial velocity induced by the divergence of the air flux is given by

$$\mathbf{V}_r = \frac{\mathbf{r}}{\|\mathbf{r}\|} V_r(\|\mathbf{r}\|) \quad \text{and} \quad V_r(r) = V_{r,\max} \begin{cases} \frac{r}{r_{\max}}, & 0 < r < r_{\max} \\ \exp\left(-\frac{(r-r_{\max})^2}{R_r^2}\right), & r \geq r_{\max} \end{cases}, \quad (5.4.2)$$

where $V_{r,\max}$ is the maximum radial velocity, $r_{\max}$ is the distance from the *stagnation point* and R_r is a scaling length. The function $V_r(r)$ for the velocity distribution, shown in Figure 5.7-b, has been documented by Poreh and Cermak [Pore 59]. This function models the divergent nature of the event: at the center ($r = 0$), the wind velocity is zero, then it increases linearly on a short distance, before decreasing smoothly. This model integrates the axisymmetric distribution of the radial velocity around the storm center.

Consequently, the time envelope is determined by

$$a(t) = \frac{\|\mathbf{V}_r + \mathbf{V}_t\|}{\max_{t \in [0, T]} [\|\mathbf{V}_r + \mathbf{V}_t\|]} \quad (5.4.3)$$

with $\mathbf{V}_t = (V_t, 0)$ and T the duration of the phenomenon. In this application, we choose $V_t = 12\text{m/s}$, $r_{\max} = 1000\text{m}$, $V_{r,\max} = 47\text{m/s}$, $R_r = 700\text{m}$, $d_0 = -8000\text{m}$ and $\ell = 150\text{m}$, as depicted in Figure 5.8-a.

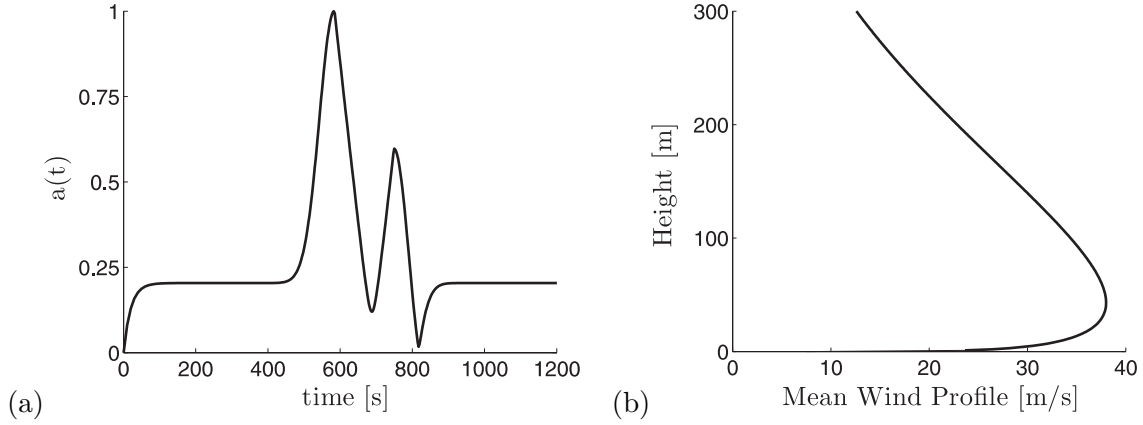


Figure 5.8: Downburst model. (a) Holmes' time window. (b) Vertical mean wind profile.

A downburst also presents a mean wind vertical profile different from the classic wind profile. Indeed, in stationary buffeting analysis according to European Standard [Euro 91] for instance, the mean wind profile $\mathbf{V}_t$ is assumed to increase with height according to a logarithmic-like function. For a downburst, the mean wind profile starts from zero, reaches a maximum before asymptotically decreasing to zero, as depicted in Figure 5.8-b. Different empirical descriptions of this profile have been proposed by several authors as Osegura [Oseg 88] and Vicroy [Vicy 92], but we use in this application the one proposed in [Wood 01],

$$V(z) = 1.55 \left(\frac{z}{\delta} \right)^{1/6} \left(1 - \operatorname{erf} \left(0.7 \frac{z}{\delta} \right) \right) V_{\max} \quad (5.4.4)$$

with z the vertical coordinate. Therefore, the evolutionary mean wind speed reads $\bar{U}(t, z) = a(t)V(z)$. In the following, the parameter δ is chosen equal to 234.4m, while $V_{\max}$ depends on $V_{r,\max}$ and V_t .

The mean wind velocity being determined, the fluctuating random contribution must be adequately modeled. Few data are available about the turbulence in downburst, so it is common practice to use any available psd of wind speeds. In this application, we use the von Karman psd defined as

$$S_u(f, z) = \frac{4L_u(z)}{V(z)} \frac{\sigma_u^2}{\left(1 + 70.78 \left(\frac{fL_u(z)}{V(z)} \right)^2 \right)^{5/6}} \quad (5.4.5)$$

with $\sigma_u = I_u V(z)$ the standard deviation of the wind turbulence, $I_u(z)$ the turbulence intensity ($I_u(z) = I_{u,10} \left(\frac{10}{z} \right)^{1/6}$), $L_u(z)$ the turbulence length ($L_u(z) = 300 \left(\frac{\max(z, 2)}{200} \right)^{0.52}$) determined according to [Euro 91] for a class 2 event. In the following, the turbulence intensity $I_{u,10}$ at 10m high is equal to 10%.

In order to build up a realistic and coherent random wind field, a coherence function is considered. In wind engineering, the model of common practice is the exponential decay [Simi 96], such that

$$(\mathbf{S}_u(\omega))_{ij} = \sqrt{S_u(f, z_i) S_u(f, z_j)} \Gamma(f, z_i, z_j) \quad (5.4.6)$$

with the coherence function given by

$$\Gamma(f, z_i, z_j) = \exp \left(-\frac{C_r f |z_i - z_j|}{\frac{1}{2} (V(z_i) + V(z_j))} \right), \quad (5.4.7)$$

C_r is equal to 7 in this application (as in [Xu 92], though this value is more related to undisturbed oncoming flow). The randomly fluctuating component of the wind now reads $\hat{u}(t, z) = a(t)u(t, z)$ with the distinction between $u(t, z)$ the stationary process characterized by $\mathbf{S}_u$ and the evolutionary process $\hat{u}(t, z)$.

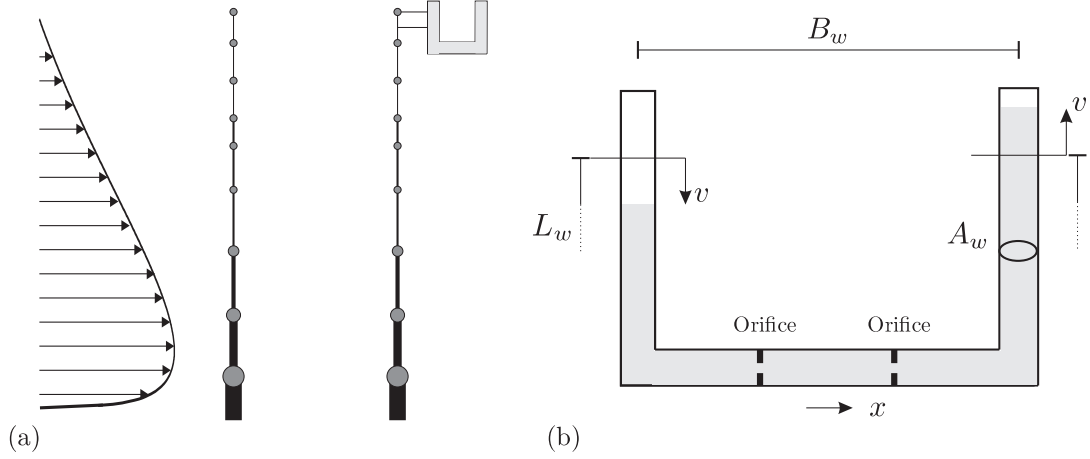


Figure 5.9: (a) Tower subject to downburst with a TLCD set on the top. (b) Sketch of a TLCD.

After this brief introduction to downburst phenomenon and related models, the transient coherent wind model is applied to a realistic structure inspired from [Xu 92, Chen 04]. The considered structure is the TV tower counting nine floors, depicted in Figure 5.9-a. The characteristics of inertia EI_i , wind surface A_i , mass m_i and height h_i are gathered in Table 5.1. The drag coefficient is equal to 0.7. A finite element model is built based on these characteristics. Each inter-storey is divided into four elements in order to better estimate the coherence between the different points. The damping ratio of the initial structure is set equal to 0.5% for the first and third modes. The natural frequencies of the first five structural modes are 0.23Hz, 0.35Hz, 0.92Hz, 1.39 Hz and 2.12Hz, respectively. Their mode shapes are depicted in Figure 5.10-a.

Storey i	1st	2nd	3rd	4th	5th	6th	7th	8th	9th
h_i [m]	20.00	28.00	28.00	27.25	19.25	12.50	17.75	17.75	14.25
EI_i [10^8kNm^2]	145.35	47.23	25.47	7.41	4.13	1.06	0.27	0.02	0.01
m_i [tons]	6134	3853	2578	3032	692	85	72	51	23
A_i [m^2]	489	460	320	474	125	75	49	24	10

Table 5.1: Characteristics of the TV tower.

On top of this structure, a Tuned Liquid Column Damper (TLCD) is set up in order to mitigate the vibrations of the tower. A TLCD is a nonlinear damping device, which controls the vibrations of a structure with the motion of liquid mass in a tube-like container [Xu 92], as depicted in Figure 5.9-b. The damping forces result from the head loss due to the presence of orifices in the container. The governing equation of the liquid elevation v due to the horizontal displacement x of the device [Saok 88] is a source of nonlinear damping, modeled by

$$\rho_w A_w L_w \ddot{v} + \frac{1}{2} \rho_w A_w \delta |\dot{v}| \dot{v} + 2 \rho_w A_w g v = -\rho_w A_w B_w \ddot{x}, \quad (5.4.8)$$

where ρ_w , A_w , B_w and L_w are the density, the cross-sectional area, the width and the length of the liquid column, respectively, while δ , g are the head loss coefficient and the acceleration of gravity, respectively. Applying the statistical linearization, the equivalent equation to (5.4.8) reads

$$\rho_w A_w L_w \ddot{v} + \sqrt{\frac{2}{\pi}} \rho_w A_w \delta \sigma_{\dot{v}} \dot{v} + 2 \rho_w A_w g v = -\rho_w A_w B_w \ddot{x}. \quad (5.4.9)$$

The container is only connected in mass with the top of the tower. The term $\rho_w A_w L_w$ is added to the lumped mass at the tower top and the coupling between this mass and the TLCD is modeled by the term $\rho_w A_w B_w$ in the mass matrix (see [Xu 92]). The mitigation of vibration with TLCD is developed in [Bale 95, Won 96, Gao 97] and the stochastic linearization gives good agreement with Monte Carlo simulations, because experiments have demonstrated that nonlinearity at the orifice remains small. In this application, we do not pretend to investigate extensively the behavior of that kind of damping devices. However, we propose to use it because its behavior is intrinsically nonlinear. The parameters ρ_w , A_w , L_w , B_w and δ are chosen equal to 1000kg/m³, 1.18m², 12.73m, 0.9 L_w , and 200, respectively. Due to the installation of the TLCD, the natural frequencies of the first five structural modes become 0.17Hz, 0.24Hz, 0.36Hz, 0.92 Hz, 1.38Hz, respectively. Their shapes are depicted in Figure 5.10-b.

In the following, the computational procedure exposed in Section 4.4 is used. The storm is simulated over a 1200s time window, the time step is equal to 10s for the first 400s and in the interval between 1000s and 1200s, it is reduced to 5s between these two intervals.

Figure 5.10 compares the response of the structure with and without TLCD. The effect of the TLCD is mainly to split the first natural frequency of the initial structure. Figures 5.10-c and -d show that the vibrations remain mainly in a global quart-sine mode, both in the initial and damped structures (mode 1 and mode 2, respectively). For the acceleration response, more modes are involved in the dynamics of the structure. The convergence on acceleration justifies the basis truncation at 5 modes. In Figures 5.10-c and -d, we show that the presence of the damper increases the response in the higher modes compared to the second global mode.

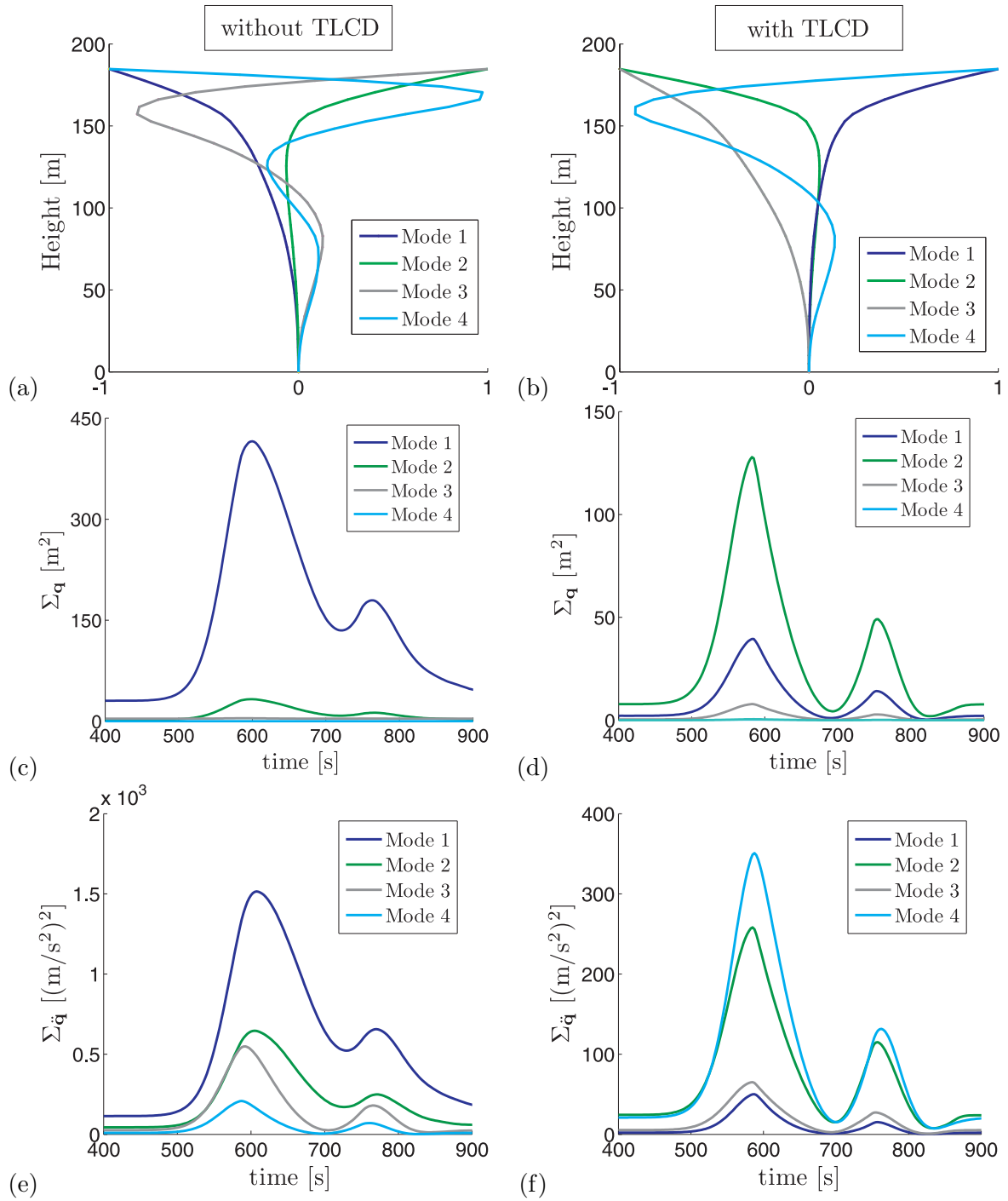


Figure 5.10: Tower subject to downburst without TLCD (a,c,d) and with TLCD (b,d,f). (a,b) Mode shapes of the structure. (c,d) Modal response in displacement. (e,f) Modal response in acceleration.

In Figure 5.11, the effect of the damper is illustrated : the responses in acceleration of the last four floors of the tower are reduced by a factor five. For the initial structure in Figure 5.11-a, the truncation order is $N = 0$. The convergence is investigated on the structure with the TLCD in Figure 5.11-b: the results obtained with $N = 0$ are in good agreement with $N = 2$ even a small difference around the first peak, while the results obtained with $N = 1$ and $N = 2$ are superimposed, confirming the convergence of the approximation.

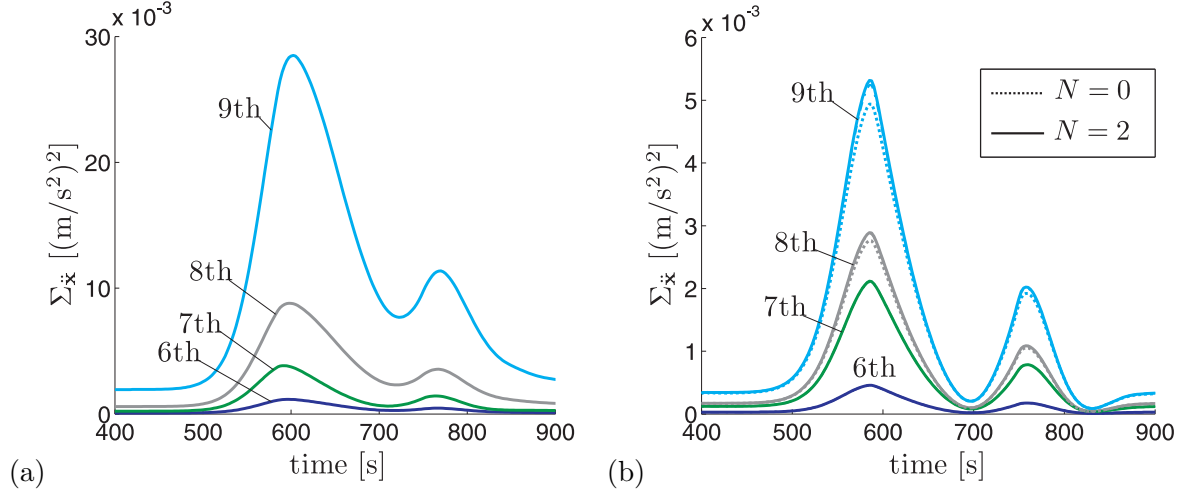


Figure 5.11: Tower subject to downburst without TLCD (a) and with TLCD (b). Time evolution of the four last floor displacements. Influence of the truncation order.

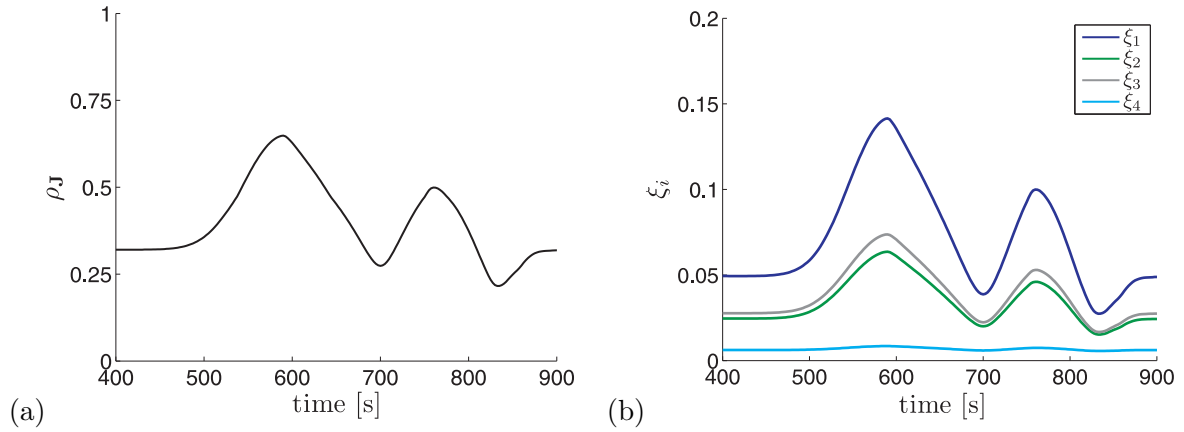


Figure 5.12: Tower subject to downburst. Time evolutions of (a) the convergence criterion ρ_J and (b) the modal damping ratios.

Figure 5.12-a shows the time evolution of the convergence criterion ρ_J , highlighting the strong coupling. With TLCD, the variance of the accelerations is divided by six. From these two figures, it is noteworthy that the damper also reduces the duration of the maximum response. In the initial structure, the effects of the first peak are still affecting the response on the second one. This is explained by the increase of the equivalent damping ratios as shown in Figure 5.12-b. The damping ratio ξ_2 at time 600s is at least multiplied by a factor 10

compared to the initial structure. The damping ratio ξ_1 is more related to a local vibration mode of the top of the tower.

From a computational point of view, the resolution with the proposed approach has required of 80s of cpu time (Intel Core i5) with five modes ($m = 5$) and one correction term ($N = 1$). Table 5.2 compares different total cpu times required for different computations depending on the number of selected modes m and the truncation order N . Those times are given, not as absolute references, but just to show that the developed method is absolutely not time consuming compared to Monte Carlo simulations (about one hour of computation for a resolution in a modal basis and 1000 samples). Furthermore, no specific optimization of the MatLab routines has been performed. For instance, we work in a pulsation domain over $\mathbb{R}$ (for theoretical reasons), while a representation over $\mathbb{R}^+$ is sufficient and reduce the number of integration points. From Table 5.2, it seems that the truncation order is not a consuming operation. Anyhow, the more consuming operation remains the computation of the integrals $\mathcal{I}_k^\alpha$, $\mathcal{J}_k^\alpha$ and their derivatives.

	$N = 0$	$N = 1$	$N = 2$
$m = 5$	78	80s	94s
$m = 7$	115s	127s	167s

Table 5.2: Tower subject to downburst. Computation time for different truncation orders and number of modes.

To conclude this application in transient wind engineering, the asymptotic expansion-based method and the stochastic linearization, more generally, are validated with Monte Carlo simulations (600 samples) in Figure 5.13. The transient equivalent linearization computed with our approach is in really good agreement with the simulations, validating the multiple scales assumptions, the use of equivalent linearization for TLCD and the efficient computation procedure exposed in Section 4.4.

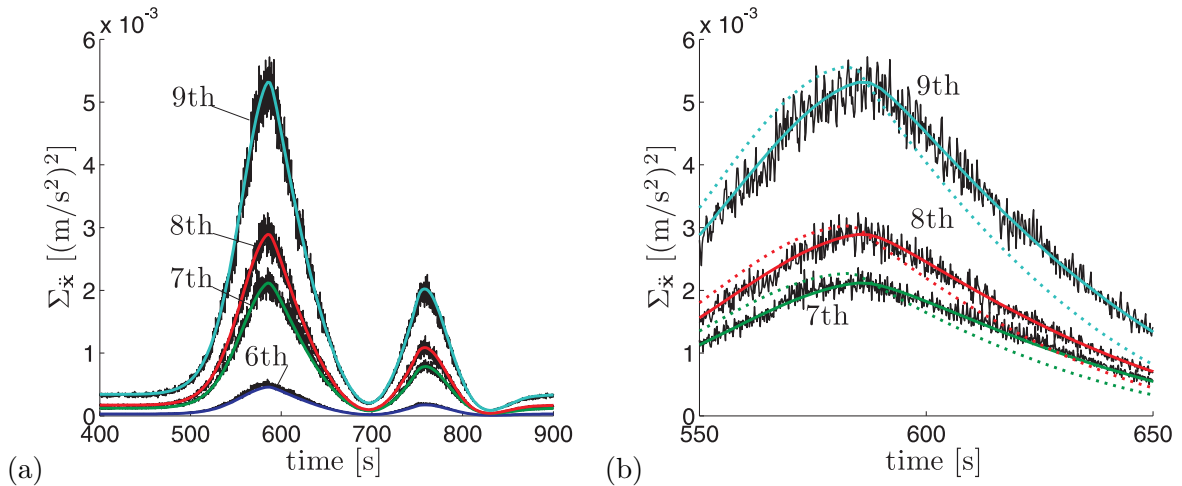


Figure 5.13: Tower subject to downburst with TLCD. Comparison of the asymptotic expansion-based method with Monte Carlo simulation (black curves) and the quasi-stationary assumption (dot curves).

Often in the previous Chapters it is referred to the *quasi-stationary hypothesis*: the variance of the equivalent stationary response of the structure is readily modulated by the square of the time window. In Figure 5.13, this method is also applied to this example in order to show that this assumption leads to satisfactory results in this case especially in terms of the maximum variance, but the time evolution is poorly analyzed with this assumption.

5.5 Update of the equivalent modal basis

In Section 3.5, a Newton-Raphson iterative method combined with the asymptotic expansion based-method is applied to the stochastic linearization of a nonlinear structure in a stationary setting. That application is noteworthy, as the intensity of the nonlinear stiffness forces requires a modification of the modal basis in order to converge to the correct equivalent statistical linearization. In this Chapter, the question of the modal basis adaptation has not been addressed yet, because it requires a specific attention and some adaptations in the previous developments. Indeed, in the analysis of equivalent LTV systems, nothing guarantees that the initial basis (e.g. the modal basis of the linear subsystem) will remain suitable throughout the solution.

The problematic of modal basis adaptation in a transient equivalent linearization is considered by Proppe [Prop 03b] as central, a reason for extending our developments. The interest in the modal basis is mainly in the size reduction of the problem, but also in the estimation of the natural frequencies, gathered in Ω_d , since the excitation psd is not constant.

The pertinence of a modal basis depends mainly on the intensity and the spatial distribution of the nonlinear stiffness forces. In this circumstance, the matrix $\tilde{\mathbf{K}}$ introduced in Section 3.3 may be changed with respect to the intensity of these forces. Naturally, a criterion to adapt this basis depends on the index $\rho_{\mathbf{J}}$. If this criterion becomes close to one, another linearized modal basis may be computed in order to ensure the convergence of the asymptotic expansion. Actually, no modal adaptation is required in the example presented in Section 5.3.4, especially because the linearized stiffness remains small compared with the linear one.

Let consider the vector of *nodal* coordinates $\mathbf{x}(t)$ and the related velocity vector $\dot{\mathbf{x}}(t)$ in the interval $[t_k, t_{k+1}]$ projected in a modal basis $\Phi_{[k+1]}$, this operation gives the state vector $\mathbf{z}_{[k+1]}(t)$. In this interval, the solution of the piecewise LTI system subject to evolutionary rand excitations is given by

$$\mathbf{z}_{[k+1]}(t_{k+1}) = \Psi_{[k+1]}(t_{k+1}, t_k) \mathbf{z}_{[k+1]}(t_k) + \int_{\mathbb{R}} \Upsilon_{[k+1]}(t_{k+1}, t_k, \omega_1) d\tilde{\mathbf{p}}_{[k+1]}(\omega_1) \quad (5.5.1)$$

with $\tilde{\mathbf{p}}(\omega, \theta) : \mathbb{R} \times \Theta \mapsto \mathbb{C}^{2m}$. The underscored index in brackets indicates the considered modal basis in the interval. At the initial time t_0 , the structure is supposed to be at rest. The state vector at t_1 , in the modal basis $\Phi_{[1]}$, reads

$$\mathbf{z}_{[1]}(t_1) = \int_{\mathbb{R}} \Upsilon_{[1]}(t_1, t_0, \omega_1) d\tilde{\mathbf{p}}_{[1]}(\omega_1). \quad (5.5.2)$$

At this stage, the delicate operation consists in calculating the covariance matrix (or the correlation function) of the vector $\mathbf{z}_{[k+1]}(t_{k+1})$, because the system must now be seen

piecewisely as an LTI system with random initial condition depending upon a past history that cannot be readily forgotten. The covariance matrix at t_{k+1} , i.e. the expectation operator of the product $\mathbf{z}_{[k+1]}\mathbf{z}_{[k+1]}^*$, is given by

$$E \left[\mathbf{z}_{[k+1]}(t_{k+1})\mathbf{z}_{[k+1]}^*(t_{k+1}) \right] = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_2^* + \mathbf{T}_3 \quad (5.5.3)$$

with

$$\mathbf{T}_1 = \Psi_{[k+1]}(t_{k+1}, t_k) E \left[\mathbf{z}_{[k+1]}(t_k)\mathbf{z}_{[k+1]}^*(t_k) \right] \Psi_{[k+1]}^*(t_{k+1}, t_k) \quad (5.5.4)$$

$$\mathbf{T}_2 = \Psi_{[k+1]}(t_{k+1}, t_k) \int_{\mathbb{R}} E \left[\mathbf{z}_{[k+1]}(t_k) d\tilde{\mathbf{p}}_{[k+1]}^*(\omega_1) \right] \Upsilon_{[k+1]}^*(t_{k+1}, t_k, \omega_1) \quad (5.5.5)$$

$$\mathbf{T}_3 = \int_{\mathbb{R}^2} \Upsilon_{[k+1]}(t_{k+1}, t_k, \omega_1) E \left[d\tilde{\mathbf{p}}_{[k+1]}(\omega_1) d\tilde{\mathbf{p}}_{[k+1]}^*(\omega_2) \right] \Upsilon_{[k+1]}^*(t_{k+1}, t_k, \omega_2). \quad (5.5.6)$$

The three terms in (5.5.3) have different meanings. The operation (5.5.3) is not easily explained, whilst it is not difficult to implement. The first term expresses the randomness at the lower bound of the interval $[t_k, t_{k+1}]$, while the last one deals with the effect of new loadings acting during this interval. However, as previously mentioned, the condition at t_k cannot be considered as independent of the past of the evolutionary process, i.e. independent of the time interval from t_0 to t_k . The second term expresses the correlation between the random state vector at t_k and the evolutionary process $\mathbf{p}(t)$. This operation presents some difficulties to be computed, but a relevant and helpful implementation is provided.

As previously said, an expensive operation in modal analysis is the projection of the psd matrix of the nodal forces into the modal basis. Adapting the modal basis requires also an adaptation of this psd. Instead of projecting the nodal forces into any modal basis, this operation may be replaced by a linear application projecting the forces from one basis to another, i.e.

$$d\tilde{\mathbf{p}}_{[k+1]}(\omega_1) = \hat{\Phi}_{[k+1]}^T \left(\hat{\Phi}_{[k]}^T \right)^\dagger d\tilde{\mathbf{p}}_{[k]}(\omega_1) = \hat{\mathbf{V}}_{[k]}^{[k+1]} d\tilde{\mathbf{p}}_{[k]}(\omega_1) \quad (5.5.7)$$

with

$$\hat{\Phi}_{[k]} = \begin{pmatrix} \Phi_{[k]} & \mathbf{0} \\ \mathbf{0} & \Phi_{[k]} \end{pmatrix} \quad (5.5.8)$$

a $2n \times 2m$ -dimensional matrix and the operator $\dagger$ defining the pseudo-inverse operation, since a modal basis is usually truncated. The same relation is applied to the state space vector, such that

$$\mathbf{z}_{[k+1]}(t_k) = \hat{\Phi}_{[k+1]}^\dagger \hat{\Phi}_{[k]} \mathbf{z}_{[k]}(t_k) = \hat{\mathbf{W}}_{[k]}^{[k+1]} \mathbf{z}_{[k]}(t_k). \quad (5.5.9)$$

The basis $\Phi_{[k]}$ results from the eigenproblem $(\mathbf{K} + \tilde{\mathbf{K}}_{[k]}) \Phi_{[k]} = \omega^2 \mathbf{M} \Phi_{[k]}$, as explained in Section 3.3.

Thanks to (5.5.9), the term $\mathbf{T}_1$ is readily computed: the covariance matrix of $\mathbf{z}_{[k]}(t_k)$ is projected into another basis. The term $\mathbf{T}_2$ requires more attention to highlight an efficient

recursive formulation. First, the value of $\mathbf{z}_{[k]}(t_k)$ is explicitly written from the time t_{k-1} to t_k in the modal basis $\Phi_{[k]}$, i.e.

$$\mathbf{z}_{[k]}(t_k) = \Psi_{[k]}(t_k, t_{k-1})\mathbf{z}_{[k]}(t_{k-1}) + \int_{\mathbb{R}} \Upsilon_{[k]}(t_k, t_{k-1}, \omega_1) d\tilde{\mathbf{p}}_{[k]}(\omega_1). \quad (5.5.10)$$

Then, by means of the relations (5.5.7) and (5.5.9), the expectation operation in the term $\mathbf{T}_2$ becomes

$$E \left[\mathbf{z}_{[k+1]}(t_k) d\tilde{\mathbf{p}}_{[k+1]}^*(\omega_1) \right] = \left(\widehat{\mathbf{W}}_{[k]}^{[k+1]} \right) E \left[\mathbf{z}_{[k]}(t_k) d\tilde{\mathbf{p}}_{[k]}^*(\omega_1) \right] \left(\widehat{\mathbf{V}}_{[k]}^{[k+1]} \right)^*, \quad (5.5.11)$$

and the expectation only operates on variables defined in the modal basis $\Phi_{[k]}$. Thanks to (5.5.10), the expectation operation in (5.5.11) reads

$$\begin{aligned} \mathbf{Q}_k d\omega = E \left[\mathbf{z}_{[k]}(t_k) d\tilde{\mathbf{p}}_{[k]}^*(\omega) \right] &= \Psi_{[k]}(t_k, t_{k-1}) E \left[\mathbf{z}_{[k]}(t_{k-1}) d\tilde{\mathbf{p}}_{[k]}^*(\omega) \right] \\ &+ \int_{\mathbb{R}} \Upsilon_{[k]}(t_k, t_{k-1}, \omega_2) E \left[d\tilde{\mathbf{p}}_{[k]}(\omega_2) d\tilde{\mathbf{p}}_{[k]}^*(\omega) \right] \end{aligned} \quad (5.5.12)$$

In summary, this last equation becomes

$$\mathbf{Q}_k = \Psi_{[k]}(t_k, t_{k-1}) \left(\widehat{\mathbf{W}}_{[k-1]}^{[k]} \right) \mathbf{Q}_{k-1} \left(\widehat{\mathbf{V}}_{[k-1]}^{[k]} \right)^* + \Upsilon_{[k]}(t_k, t_{k-1}, \omega) \mathbf{S}_{\mathbf{p},[k]}(\omega), \quad (5.5.13)$$

as a result from the orthogonality property of the increments of $\mathbf{p}(t)$, in any modal basis. As the structure starts vibrating from rest, the initial term $k = 1$ in the recurrence relation (5.5.13) is

$$\mathbf{Q}_1 d\omega = E \left[\mathbf{z}_{[1]}(t_1) d\tilde{\mathbf{p}}_{[1]}^*(\omega) \right] = \Upsilon_{[1]}(t_1, t_0, \omega) \mathbf{S}_{\mathbf{p},[1]}(\omega) d\omega. \quad (5.5.14)$$

In these developments, the modal basis is supposed to be different between the k -th and $(k+1)$ -th intervals. However, modal basis must not be necessarily modified on each interval, quite the opposite actually. In this case, the transformation matrices $\widehat{\mathbf{V}}_{[k]}^{[k+1]}$ and $\widehat{\mathbf{W}}_{[k]}^{[k+1]}$ are equal to the identity matrix $\mathbf{I}$.

Application

Inspired by the illustration in stationary equivalent linearization in which the matrix $\tilde{\mathbf{K}}$ is updated, we study two neighbor shear-type buildings subject to earthquake. In some circumstances, depending on the separation distance ℓ and the intensity of the earthquake, these buildings may collide each other. At the contact point, nonlinear forces occur depending on the penetration. After statistical linearization, these forces are replaced by an equivalent spring joining the two building tops, as illustrated in Figure 5.14.

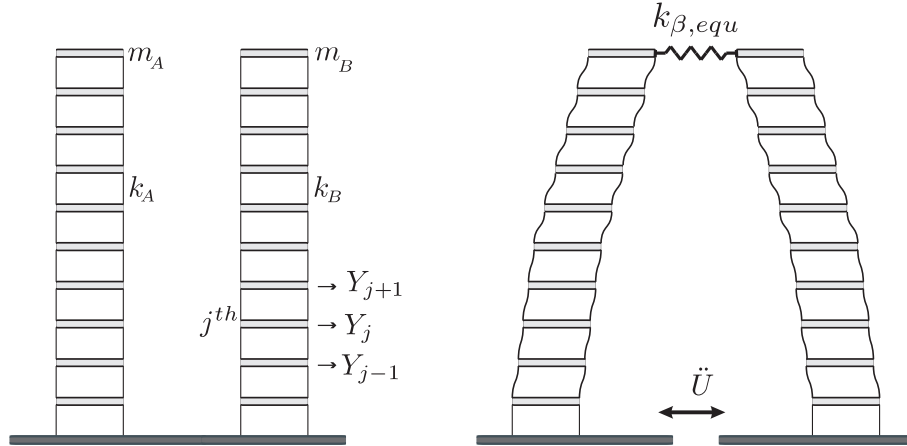


Figure 5.14: Sketch of the pounding problem between two adjacent buildings. In stochastic linearization, the nonlinear contact forces are replaced by a linear equivalent spring at the contact point.

The nonlinear contact force $F_c(t)$ is naturally equal to zero for negative penetration and may be modeled by a Kelvin law for positive penetration, such that

$$F_c(t) = k_\beta \delta^{3/2} \mathbb{H}(\delta), \quad (5.5.15)$$

with δ the penetration length, $\mathbb{H}(\cdot)$ the Heaviside function and k_β a deformation coefficient. The penetration is calculated as

$$\delta = x_A - x_B - \ell = x_{AB} - \ell, \quad (5.5.16)$$

with x_A and x_B the top displacements of buildings A and B , respectively. The nonlinear force only depends on displacements of the structures. Therefore, the equivalent statistical linearization transforms $F_c(t)$ into a linearized stiffness matrix. Applying the Kazakov formula (3.5.5), the equivalent stiffness matrix reads

$$\mathbf{K}_{eq} = \frac{3}{2} \beta E \left[\delta^{1/2} \mathbb{H}(\delta) \right] \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad (5.5.17)$$

with

$$E \left[\delta^{1/2} \mathbb{H}(\delta) \right] = \int_0^{+\infty} \sqrt{x} \psi_\delta(t, x) dx \quad (5.5.18)$$

and $\psi_\delta(t, x) = \mathcal{N}(-\ell, \sigma_{x_{AB}}(t))$ the pdf of the penetration length. Here, we suppose that x_A and x_B have negligible mean displacement. We can imagine the shape of the exact pdf of the relative displacement x_{AB} . The structures have linear behavior except as $\delta > 0$. In this situation, they are subject to huge nonlinear forces limiting drastically their displacements. Therefore, for $x_{AB} < \ell$ the behavior is linear and for $x_{AB} > \ell$ it is not, leading to a possibly highly skewed pdf¹. Nonetheless, in a Gaussian equivalent linearization, we approximate the

¹An example is treated in Section 8.2.2 - Part II in which such pdf's are depicted.

exact pdf by a Gaussian one, i.e. a symmetric distribution. This approximation may lead to poor results independently of our implementation.

In this example, we consider two 10-storey shear-type buildings A and B as described in Section 3.5.2 with the following characteristics : $m_A = 1000\text{tons}$, $m_B = 1500\text{tons}$, $k_A = 10^9\text{N/m}$, $k_B = 2 \cdot 10^9\text{N/m}$. The parameter k_β is fixed at $10^9\text{N/m}^{3/2}$. A damping ratio of 1% is chosen in the first two modes of each structure. The excitation is a Kanai-Tajimi process with $S_0 = 2 \cdot 10^{-3}\text{m}^2/\text{s}^3$ (other parameters in Section 3.5.2) modulated by a Shinozuka window $b_1 = 0.0045\pi$ and $b_2 = 0.0050\pi$.

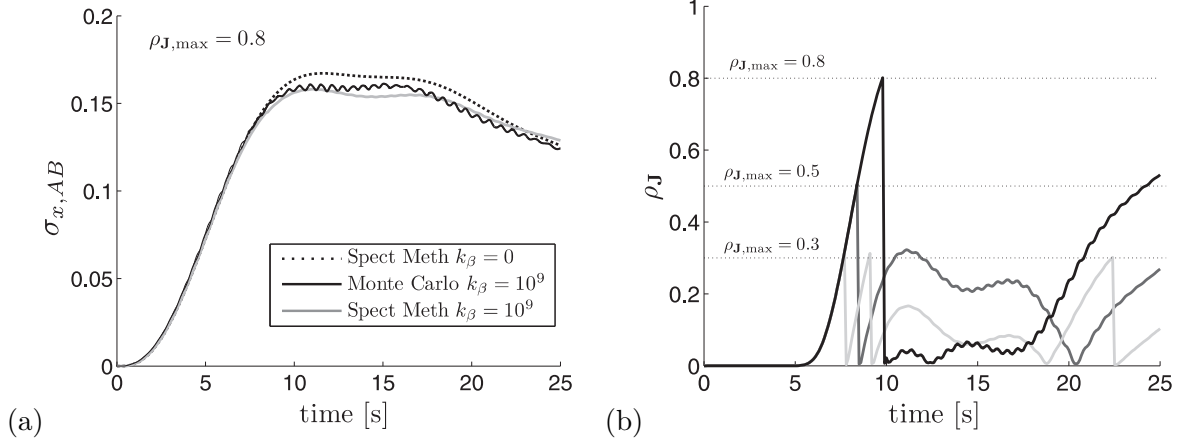


Figure 5.15: Update of the equivalent modal basis and pounding problem. (a) Time evolution of the standard deviation of the relative displacement between the two building tops. (b) Time evolution of the convergence criterion ρ_J for different values of $\rho_{J,\max}$ (value at which the basis is updated).

Figure 5.15 shows the main results of this example. Figure 5.15-a depicts the time evolution of the standard deviation of the relative displacement between the two building tops. We compare our results obtained with $N = 1$ with Monte Carlo simulation (3200 samples, $\Delta t = 2\text{ms}$) and with the response of the linear structure $k_\beta = 0$ (in order to quantify the non-linear forces). In Figure 5.15-a, we see the good agreement between our method and Monte Carlo simulation. Figure 5.15-b shows the time evolution of the index ρ_J for different values of $\rho_{J,\max}$. Once the index overcrosses the chosen maximum value, the basis is updated. We notice that without this limitation $\rho_{J,\max}$, the index ρ_J would exceed one. However, no difference has been noticed on the standard deviation between the three thresholds. This is in accordance with the results highlighted throughout this Chapter about the index of convergence.

5.6 Conclusions

Due to the difference between the fast structural and the slow statistical timescales, we are able to highlight in the equations of motion that the equivalent stiffness and damping matrices evolve according to the slow timescale. This is an important result obtained by a rigorous *multiple scales approach*. Indeed, the equivalent statistical linearization transforms an original *nonlinear time-invariant* system into a *linear time-variant* system. However, with the multiple

scales approach, we demonstrate that this linearized system may be considered to respond as a piecewise time-invariant system.

The multiple scales approach allows to bridge the gap between the Gaussian equivalent statistical linearization exposed in Chapter 3 and the evolutionary spectral analysis presented in Chapter 4.

Therefore, the recurrence relation obtained in Section 4.4 can be applied as such for the equivalent statistical linearization in a nonstationary setting. The difference between the linear and linearized problems is found in the computation of the equivalent structural matrices. Once again, the multiple scales approach justifies that the equivalent properties of the structure can be computed at the beginning of a time interval and kept constant over this interval. The length of these intervals must principally take into account the shape of the time window, while possibly including several periods of the structural response. Indeed, the envelope is the main factor influencing the statistical timescale.

The evolutionary spectral analysis in equivalent statistical linearization, is applied to two examples in wind and earthquake engineering, in presence of coherent random loadings. Indeed, for these loadings, the evolutionary spectral analysis finds a great interest. In these examples, the nonlinear forces result from nonlinear damping devices. The modal basis is thus calculated with the linear subsystem. In both cases, we highlight the good agreement between our method and Monte Carlo simulation, refuting definitely the quasi-stationary assumption. The computational time required for our method is much less than for Monte Carlo simulation, especially in the wind application.

The last problem addressed in this Chapter concerns the possibility of adaptation of a modal basis during the solution. The objective is to avoid that the convergence criterion $\rho_{\mathbf{J}}$ exceeds one. This problem is studied piecewisely, as previously, but a generalized (or modal) basis update takes place time to time. We derive a recurrence relation allowing a smooth and simple basis update, while preserving the statistical continuity and the consistency in the asymptotic expansion-based method. We can also notice the difficulty with equivalent linearization to deal with problems involving nonlinear stiffness forces. Indeed, oscillations may occur because of time modifications of the natural frequencies of the decoupled system. These oscillations may create instability in the procedure of linearization. Though this problem has been only encountered for strongly nonlinear structures, it remains a source of possible improvements, not of our proposed solutions, but rather of the statistical linearization concept.

Conclusion

To conclude this first part of the dissertation, we would like to summarize the original objectives, our methodology, our main results and some tracks for further improvements.

The first motivation of this part was to refute the common idea that a transient problem, in linear or nonlinear probabilistic analysis, can only be solved with Monte Carlo simulation. This idea is fallacious, especially for large structures subject to coherent random loads, where our approach proves really efficient. Secondly, we were concerned by the common opinion that spectral analysis is devoted to stationary problems. In this part, we have shown that these opinions can be denied by the development of an efficient evolutionary spectral analysis, for linear structures and for equivalent statistical linearization.

The main difficulties in the application of evolutionary methods are the computation of the impulse response matrix of a structure and the convolution between this matrix and a time envelope. We propose a methodology to circumvent them. First, the equation of motion is projected in a modal basis, which may take into account the intensity and the spatial distribution of the nonlinear forces. Actually, the equivalent structural matrices may be not diagonal in this basis, such that a mechanical coupling occurs between the modes. For slightly to moderately coupled modes, the out-of-diagonal elements can be seen as perturbations of a main diagonal matrix. Based on this assumption, we asymptotically expand the modal transfer matrix of the system around the one of a virtual decoupled system.

The *asymptotic expansion-based method* circumvents the two main difficulties, by returning to linear algebra. As the modal transfer matrix of a structure and its impulse response matrix form a Fourier transform pair, we can find an asymptotic expansion of this matrix in a closed form. Afterward, the aforementioned convolution is analytically computed for some well-known time envelopes.

The asymptotic expansion of the modal transfer matrix of a structure is the key point of our methodology and an original contribution of this part. It is grounded on a solid mathematical statement and on a trustworthy, although quite strict, convergence criterion.

Based on this expansion, we develop an original and relevant formulation to perform *linear* evolutionary spectral analysis. Thanks to the semi-group property characterizing linear systems, we find an efficient recurrence relation allowing to compute, at a low computational cost, the state evolutionary transfer matrix, key of the evolutionary spectral analysis. Furthermore, we take advantage of the limited number of correction terms required in the expansion of the impulse response matrix on short time intervals. This *recurrence formulation* has shown its relevance on different realistic applications, in earthquake and wind engineering.

This formulation, initially developed for linear analysis, can also be applied to nonlinear transient analysis through an equivalent statistical linearization. The gap between linear

analysis and stochastic linearization in a nonstationary setting, is bridged with a *multiple scales approach* of the problem. A nonlinear (time-invariant) system is transformed into an equivalent, linearized and time-variant system. Then, the multiple scales approach formalizes the consideration of this equivalent system as piecewise time-invariant, since the equivalent properties of the structure evolves at slow timescale with the probabilistic characteristics of the response.

The different perturbation methods applied in this work allow us to take advantage of a rigorous and efficient semi-analytical approach. The construction of blocks of elementary analytical results, incorporated in a numerical procedure, is a significant source of improvement in probabilistic analysis.

As tracks of future investigations, we may point out three axes : (i) the developments of an efficient procedure to perform linear evolutionary spectral analysis in a non-Gaussian framework, (ii) the improvement of the procedure of modal basis adaptation in transient stochastic linearization and (iii) the integration of the method to implicit nonlinear behaviors.

(i) The developments of *higher-order spectral analysis*, based on the Volterra series, is an old topic in probabilistic analysis. Its statement in a stationary setting is well-known and its development for evolutionary spectra has been timidly initiated. The main issue in this approach is the integration of spectral function in large-dimensional frequency domain. Any improvement of these integration techniques would create an emulation around non-Gaussian spectral analysis. In this context, the asymptotic expansion-based method used to compute evolutionary Volterra kernel would find a practical interest. Naturally, these reflexions also depend on the determination of non-Gaussian properties of wind pressure or ground acceleration. Although these properties are more available in wind engineering, it remains an issue in earthquake engineering.

(ii) The concept of *generalized basis updating* proposed in this work has highlighted some difficulties to deal with important nonlinear stiffness forces. Indeed, due to realistic excitation models, it may become intricate to manage the time evolution of the equivalent natural frequencies. Although we investigate this adaptation procedure and we show its relevance in some applications, its application to highly nonlinear problems remains an issue. The problems encountered are not related to the asymptotic expansion based-method, but more to the modal reduction. Mathematically speaking, other paradigms than the modal basis could be investigated, as Krylov-Bogoliubov subspaces, while keeping in mind the difficulties to deal with nonlinear random forces.

(iii) One of the drawbacks of equivalent linearization, as introduced in this work or by many authors before, is the necessity to deal with explicit nonlinear forces. Indeed, the equivalent statistical linearization is performed on explicit nonlinear laws. Those laws are not necessarily established in the initial formulation of the problem. Contrary to nonlinear viscous dampers or tuned liquid column dampers, the geometric nonlinearities in civil structures are usually calculated on an incremental basis. Furthermore, the location of material nonlinearities (e.g. plastic hinges) are not necessarily known at the beginning of a computation. These observations are beyond the humble scope of this work, but the author thinks that this limitation should be addressed in order to make the equivalent statistical linearization, and possibly the developments of this work, more relevant for practitioners.

Part II

A Lagrangian Method for Fokker-Planck-Kolmogorov Equation

Chapter 6

Fokker-Planck Equation : Issues and Challenges

*Le "déterminisme" est la seule manière de se représenter le monde.
Et l'indéterminisme, la seule manière d'y exister.*

Paul Valéry

-
- 6.1 The Fokker-Planck-Kolmogorov equation
 - 6.2 Resolution methods : an overview
 - 6.3 Purpose of this part
-

6.1 The Fokker-Planck-Kolmogorov equation

The Fokker-Planck-Kolmogorov (FPK) equation, in its transient form, models the time evolution of the probability density function (pdf) of a random state vector of a random system. In Section 2.3, we show how to solve some classes of stochastic differential equations (sde) with this equation. This part of the dissertation focuses on this approach exclusively. According to Itô's calculus, this equation establishes a relation between a purely stochastic paradigm using Brownian motions (or other martingales) and a deterministic one. Indeed, the FPK equation allows to directly compute the pdf, which is a central *deterministic* information about a random process. Actually, the solution of random problems with deterministic representation is also applied in the spectral analysis, because a psd remains a deterministic model of the energy-frequency distribution of a random process. For sake of fluidity of this manuscript, some notions about FPK equation are recalled hereinafter.

Considering an ordinary differential equation with *noises* as external loading terms is a meaningless notion, because the Brownian motion is never and nowhere differentiable. However, a stochastic integral or a stochastic differential equation can be written from the initial ordinary equation provided the original drift coefficient is corrected by the Wonk-Zakai correction term [Wong 65]. For deterministic systems driven by random inputs, the time evolution of the random state vector $\mathbf{X}(t) \in \mathbb{R}^n$ is thus mathematically described by a stochastic differential equation in Itô's formalism [Okse 03]

$$d\mathbf{X}(t) = \mathbf{a}(t, \mathbf{X})dt + \mathbf{b}(t, \mathbf{X}) d\mathbf{B}(t), \quad (6.1.1)$$

where $\mathbf{a}(t, \mathbf{X}(t)) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear drift vector function, $\mathbf{b}(t, \mathbf{X}(t)) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ a nonlinear matrix function and $\mathbf{B}$ an m -dimensional vector of independent Brownian motions. The FPK equation expresses the diffusion of probability density of $\mathbf{X}(t)$ and the conservation of the total probability [Risk 96]. It takes the form of a second-order partial differential equation

$$\frac{\partial \psi}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial x_i} (a_i \psi) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (D_{ij} \psi) \quad (6.1.2)$$

with $\mathbf{D}(t, \mathbf{x})$ the *diffusion matrix* defined as $2D_{ij} = (\mathbf{b}\mathbf{b}^T)_{ij}$, a_i the i -th element of $\mathbf{a}$, and where the conditional probability density function (pdf) $\psi(t, \mathbf{x} | t_0, \mathbf{x}_0)$, abbreviated in $\psi(t, \mathbf{x}) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^+$ is the joint-pdf of components of $\mathbf{X}(t)$, with *initial state distribution* $\psi_0(\mathbf{x}) (= \psi(t_0, \mathbf{x}))$, respecting the first two axioms of Kolmogorov

$$\int_{\mathbb{R}^n} \psi d\mathbf{x} = 1 \quad \text{and} \quad \psi(t, \mathbf{x}) > 0, \forall \mathbf{x} \in \mathbb{R}^n. \quad (6.1.3)$$

Obviously, the time evolution of $\psi(t, \mathbf{x})$ is conditioned by the initial condition $\psi_0(\mathbf{x})$. For a dynamical system, the FPK equation models the transport of the probability density of the response across the state space of the system. A well-posed problem is obtained by complementing (6.1.2) with the *far-field conditions* given in (2.3.3), expressing at least that $\psi(t, \mathbf{x})$ tends to zero as $\|\mathbf{x}\| \rightarrow \infty$. However, other boundary conditions may be encountered, depending on the mechanical system modeled by (6.1.1).

The FPK equation (6.1.1) is not only used in civil or mechanical engineering, quite the opposite actually. Originally, this equation has been derived by physicians studying the radiation of electric dipoles [Fokk 14], to describe the Brownian motion of particles. A broad range of scientific disciplines have used hitherto this equation to measure the propagation of random perturbations, random loadings or random parameters on *systems* (in a broad sense). In biological systems, neurosciences or ecosystem modeling, the FPK equation provides an elegant method to characterize random processes or system driven by random inputs.

In those fields, most of the applications are stationary. Hence, the time, seen as the $(n + 1)$ -th dimension of the problem, is not considered at all. As for many convection-diffusion equations encountered in fluid dynamics, heat transfer or solid mechanics, closed-form solutions are seldom available. For transient problems, closed-form solutions are even rarer, almost exceptional. In [Preu 94], an explicit solution of (6.1.2) is given for an SDOF linear system starting from a deterministic initial condition. In the context of approximate analytical methods, the size of the system may be interestingly reduced by an averaging procedure [Grig 02] before writing the FPK equation in term of a random amplitude [Cai 04]. Nonetheless, this method is not convenient for general purposes.

It is thus clear that an efficient application of the method should be based on a numerical solution of the convection-diffusion equation. However, some issues we need to cope with are encountered as well.

From a mathematical viewpoint, the pdf $\psi(t, \mathbf{x})$ can be interpreted as a scalar field defined in a n -dimensional state space, possibly unbounded. This first interpretation highlights two difficulties in the numerical solution of this equation, yet. First of all, the computational method must be as readily as possible adapted to cope with n dimensional spaces. Then, the unboundedness of the resolution domain leads to manage the intricate far-field boundary condition with a limited computational power. Indeed, these two questions raise the issue of dealing with a sufficiently large domain. For instance, the initial pdf ψ_0 may have dispersion parameters (roughly speaking, the standard deviation) much smaller than the stationary one. Therefore, a suitable method must be able to deal with a large range of transient distributions taken by the pdf.

Secondly, the first two Kolmogorov axioms must be fulfilled. Though the conservation of the total probability may be adapted by normalization, the positivity of the pdf is hard to be imposed with existing methods, especially as the probability is extremely small. However, the distribution in the tails is of major importance, in reliability problems notably. Thence, the accuracy of a method must not only be judged by a norm of the error with respect to a reference solution, because a negative pdf is inconceivable from the point of view of mathematical rigor, even if it is small.

6.2 Resolution methods : an overview

The exact explicit solution for stationary FPK equation may be obtained in few cases only. One of them is ground on the assumption of isotropy in the *diffusion matrix*, i.e. the diffusion matrix $\mathbf{D}(\mathbf{x})$ is diagonal such that $(\mathbf{D}(\mathbf{x}))_{ij} = D(\mathbf{x})\delta_{ij}$. In this case, the stationary solution

of (6.1.2) reads

$$\psi(\mathbf{x}) = \frac{C}{D(\mathbf{x})} \exp \left(\sum_i \int \frac{2a_i(\mathbf{x})}{D(\mathbf{x})} dx_i \right) \quad (6.2.1)$$

under some additional integrability assumptions, with C a normalization constant. Concerning the transient FPK equation, exact explicit solutions may be obtained for some nonlinear Hamiltonian systems [Risk 96]. The technique of *detailed balance* allows to deal with a family of random processes having the property that each transition from one state to another is balanced statistically by a reversed transition as explained in [Risk 96, Lin 04b]. Some nonlinear systems can thus be solved with this method, but it is quite cumbersome in realistic applications. Hence, for most of the linear or nonlinear systems driven by colored noises, such exact solutions cannot be calculated explicitly and the use of numerical methods is inevitable [Soiz 88].

The first numerical technique used to solve the FPK equation was the finite difference method [Chan 70, Pich 11]. In further attempts, the finite element (FE) method has also been used [Spen 93, Lang 85]; it has been extended up to a four-dimensional state space [Wojt 00]. This method allows to solve the unsteady equation through a step-by-step analysis or, immediately, the stationary equation through an eigenvalue problem. Because it has been implemented by the present author, the method is exposed in Appendix C.1 in a two-dimensional setting. Some comments are also formulated with respect to [Spen 93], especially in the formulation of the weak form.

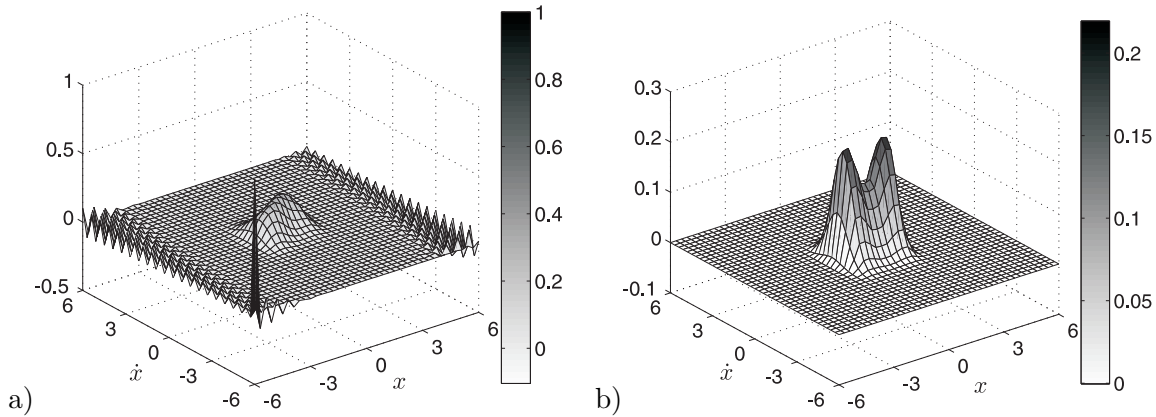


Figure 6.1: Illustration of instability inspired from [Spen 93] on a Duffing oscillator. (a) Instability in the far-field in a transient resolution with FE. (b) Stationary pdf obtained by an eigenvalue problem with FE. Details of implementation in Appendix C.1.

Nevertheless, the finite element method encounters difficulties in modeling the far-field boundary conditions. To overcome this problem, Spencer [Spen 93] proposes to mesh a sufficiently large domain to eliminate the diffusion of probability. Langtangen [Lang 91] proposes to impose a zero flux condition at the boundaries of the integration domain. In any case, the positivity of the pdf is not properly ensured: if elements are too large or the domain too small some spurious waves can propagate through the state space and spoil the quality of the solution. This FE method may thus fail in creating artificially and numerically the vanishing condition $\psi(t, \mathbf{x}) \rightarrow 0$ for $\|\mathbf{x}\| \rightarrow +\infty$ at any time t . Figure 6.1-a shows an example

of instability at the boundary of the truncated domain. In this example, the FPK equation is related to a classic Duffing oscillator (see Appendix A.1) subject to a stationary white noise. Figure 6.1-b shows the stationary response of the Duffing oscillator computed with the FE method, by an eigenvalue problem. Some values of the pdf are negative, although very small.

Furthermore, the finite element method is laboriously extended to high multidimensional systems, because of some difficulties inherent to discretization of high-dimensional state spaces. Another argument out of favor with mesh-based methods is the need to mesh useless subspaces: at a certain time step, probability density in some subspaces is so low that its representation is useless. For a given computing capacity, it is more efficient to refine appropriate zones of the state space. Adaptive meshing and remeshing methods exist, but they have not been applied to our problem, yet.

To circumvent the issues encountered with mesh-based methods, recent developments propose to apply meshless methods to the resolution of FPK equation. Kumar *et al.* [Kuma 06] propose the use of a multi-resolution meshless method based on Meshless Petrov-Galerkin method. Unfortunately, this method requires an important time of execution and the convergence is an issue, as pointed in [Kuma 09]. Then, Kumar *et al.* propose the use of PUFEM (partition of unity finite element method) and apply it to the stationary [Kuma 09] and the transient [Kuma 10] FPK equation up to a four-dimensional state space. The examples illustrating the method are classic nonlinear mechanical systems as Duffing oscillator or Lorenz attractor.

In the field of kinetic theory, the FPK equation presents a great interest for modeling complex fluid dynamics. Chauvière and Lozinski [Lozi 03, Chau 04] present a spectral discretization approach to solve FPK equation, another mesh-based method. In this approach, some drawbacks similar to the FE method may be encountered as the dependence of the positivity with the refinement of the mesh and instability for high gradients in the pdf [Chau 04]. With the objective of solving multidimensional partial differential equations, Ammar *et al.* [Chin 03, Amma 06, Amma 07] develop a method based on a separate representation of variables, which allows to decouple the numerical integration in each dimension of the system. This method shows encouraging results for high-dimensional time-invariant systems on a bounded domain. In complex fluid theory, the definition of FPK equation differs from (6.1.1) in the domain of the state vector $\mathbf{X}(t)$ which is bounded [Chup 10]. However, the occurrence of rare events is not a major issue in this field as it is the case in mechanical or civil engineering [Lin 04b] and so the question of the tail distribution is not addressed. Furthermore, in a bounded domain, the stability and the efficiency of the method is not the same issue as it is in an unbounded state space.

The time dependence of systems modeled with (6.1.1) is explicitly considered in the time dependence of drift and diffusion coefficients. Some of these methods may also reveal some numerical drawbacks in the resolution of FPK equation related to time-variant systems. In the classic FE method or in the PUFEM approach, the matrices of the system must be computed at each time step and the modal reduction for time-variant problem is not possible. Yet, the management of the explicit time dependence of drift and diffusion coefficients is an issue in transient dynamics. For instance, in earthquake engineering [Kiur 96], the seismic excitation is usually modeled as a diffusive process modulated by a time window, as amply exposed in the previous Part. Therefore, the FPK equation has explicit time-dependent coefficients.

6.3 Purpose of this part

In this part of the work, a particle strategy is explored to deal with the FPK equation related to dynamical diffusive systems. The FPK equation is viewed as a *convection-diffusion equation* in a *Lagrangian paradigm*. The **Smoothed Particle Hydrodynamics** (SPH) method, a *meshfree particle method*, is applied to the resolution of FPK equation. Indeed, we think that this method may be efficient for our purpose.

An efficient method for solving the transient FPK equation in an unbounded domain must be able

- i . to adaptively cover all the state space from initial condition to steady-state,
- ii . to ensure the vanishing condition in the far field,
- iii . to precisely capture important distortions of the pdf,
- iv . to ensure the positivity of pdf,
- v . to ensure the conservation of the total probability,
- vi . to deal with n -dimensional problem (at least in the formalism).

The first five points have been commented in Section 6.1. Concerning the sixth one, we insist about the formalism and not about the straight applicability for two reasons. The *stricto sensu* applicability in n -dimensional state space is dependent on the technological and computational capacities we dispose. The strategy of implementation (e.g. using rather *cpu* than *gpu*) influences also the final computational burden. Nonetheless, the flexibility of a formalism can be considered independently of its present applicability.

Furthermore, as explained in the Introduction of this dissertation, the deterministic methodology does not aim at solving large scale problems, but is more focused on small size systems with simple random excitations. Hence, the method developed in this Part will be applied to different families of problems than those of Part I.

The SPH method, with the implementation details given next, fulfills most of these requirements. The SPH method is a meshless method based on a Lagrangian formulation of the equations governing a phenomenon and on the *integral representation* of fields. This method as other particle methods, has been amply used for the resolution of diffusion equations.

This method was developed by Lucy in 1977 [Lucy 77] (and simultaneously by Gringold and Monaghan [Ging 77]) and first applied to astrophysical problems. Thus far, the method has been applied to many problems of continuous and discontinuous dynamics as fluid flow problems [Feld 07, Morr 00], damage and fracture [Rabc 03, Rabc 07], impact computation [John 96] or heat conduction [Clea 99, Jeon 03]. The main principles of the SPH method are amply developed in [Rand 96, Mona 05, Liu 03a, Liu 10]. In the context of kinetic theory, Chaubal *et al.* [Chau 97, Chau 99] apply first SPH technique to the dynamics of liquid crystalline polymers on a closed domain. Particle strategies for solving the Fokker-Planck equation are commonly used in this theory, as illustrated in [Chin 03, Amma 05, Prul 07]. Inspired by these previous works, the present part of the dissertation proposes an application

of the SPH formalism to solve the transient FPK equation related to diffusive random systems independently of their size. In this context, a new formulation of the conservation equation using the Lagrangian formalism is proposed and some considerations to deal with the vanishing condition are exposed.

In the following Chapters, the SPH method is briefly exposed, then the FPK equation is formulated with the SPH formalism. Finally, the developments are applied to dynamical systems usually encountered in the literature, but also to more original problems. The main results are extensively discussed and commented.

The content of this Chapter was concisely presented in [Cano 11, Cano 13].

Chapter 7

SPH Method and Fokker-Planck Equation

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- 7.1 Smoothed particle hydrodynamics
 - 7.2 SPH formulation of Fokker-Planck-Kolmogorov equation
 - 7.3 Computational aspects
 - 7.4 Description of the algorithm
-

Purpose of the chapter

In Section 7.1, the statement of SPH method is presented. The concept of integral representation of a field and kernel approximation are briefly exposed. Then, we emphasize how to transform the FPK equation into a Lagrangian formalism in Section 7.2. We show that the solution of FPK equation with SPH method does not require to solve any algebraic equation, since the particles move by themselves according to a conservation and a transport equations. A method to compute statistical moments and probability of exceedance are also provided. Finally, Section 7.3 highlights some computational aspects, as the initialization of particle masses, the computation of their interaction and the treatment of boundary conditions.

7.1 Smoothed particle hydrodynamics

The philosophy of the SPH method is to transform a set of partial differential equations into integral equations by using an interpolation function that gives an estimation of a field (density, velocity, energy) at a given point. Vector or scalar fields are only known in a number of discrete points, so-called *particles*, free to move in the domain of resolution. Thus, the evaluation of an integral is transformed into a sum a set of some neighboring particles. In this method, a grid is unnecessary. Indeed, interactions between neighboring particles are modeled with interpolation functions, so-called *kernel functions*, which depend on the distance between particles.

7.1.1 Integral representation of a field

The concept of *integral representation* of a continuous field $f(\mathbf{x})$ in SPH method uses the properties of the Dirac-delta function $\delta(\mathbf{x})$,

$$\int_{\mathbb{R}^n} f(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') d\mathbf{x}' = f(\mathbf{x}) \quad (7.1.1)$$

with $\delta(\mathbf{x} - \mathbf{x}') = 0$ if $\mathbf{x} \neq \mathbf{x}'$.

Though Equation (7.1.1) is theoretically rigorous and applicable to any continuous field $f(\mathbf{x})$, this operation cannot be *stricto sensu* numerically implemented. The philosophy of the SPH method is here to estimate (7.1.1) as accurately as possible by some numerical artifacts. First, the Dirac-delta function is approximated by a suitable function, fulfilling some analytical properties. Then, the integral operation is transformed in order to be efficiently approximated numerically. Finally, the field $f(\mathbf{x})$ is estimated (or “particularized”) for each integration point, i.e. for each particle. We now introduce formally these three steps.

Numerically, the Dirac-delta function is thus replaced by the *kernel function* $W(|\mathbf{x} - \mathbf{x}'|, h)$, such that

$$\lim_{h \rightarrow 0} W(|\mathbf{x} - \mathbf{x}'|, h) = \delta(\mathbf{x} - \mathbf{x}') \quad (7.1.2)$$

with h the *smoothing length*. The *kernel approximation* of $f(\mathbf{x})$ with the kernel function, noted $\langle f(\mathbf{x}) \rangle$, is

$$\langle f(\mathbf{x}) \rangle = \int_{\mathbb{R}^n} f(\mathbf{x}') W(|\mathbf{x} - \mathbf{x}'|, h) d\mathbf{x}'. \quad (7.1.3)$$

According to the *Dirac-delta function property* (7.1.2), $\langle f(\mathbf{x}) \rangle$ tends to $f(\mathbf{x})$ when h tends to zero. This latter expression is a smoothed approximation of $f(\mathbf{x})$ and the integral is discretized, for a number N_p of particles, as

$$\langle f(\mathbf{x}) \rangle \approx \sum_{j=1}^{N_p} f(\mathbf{x}_j) W(|\mathbf{x} - \mathbf{x}_j|, h) \Delta V_j = \sum_{j=1}^{N_p} \frac{m_j}{\rho_j} f(\mathbf{x}_j) W(|\mathbf{x} - \mathbf{x}_j|, h), \quad (7.1.4)$$

where m_j , ρ_j and ΔV_j are the mass, the density and the volume of particle j , respectively. Equation (7.1.4) is the *particle approximation* of the field $f(\mathbf{x})$. This approximation can be particularized at a particle i ,

$$\langle f(\mathbf{x}_i) \rangle = \sum_{j=1}^{N_p} f(\mathbf{x}_j) W_{ij} \Delta V_j \quad (7.1.5)$$

with $W_{ij} = W(r_{ij}, h_i)$ and r_{ij} the distance between particles i and j . Equation (7.1.5) highlights the difference between the particle value $f(\mathbf{x}_i)$ and the kernel approximation $\langle f(\mathbf{x}_i) \rangle$. These values must not be confused [Feld 07], but they can be sufficiently close if h is small enough according to (7.1.2) and (7.1.3).

Figure 7.1 shows the concept of discretized integral representation of a field $f(\mathbf{x})$ with SPH and particles.

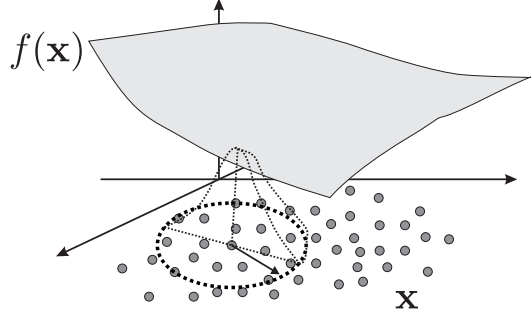


Figure 7.1: Smoothed particle hydrodynamics. Integral representation of a field.

7.1.2 Kernel function

To be used in SPH method, a kernel function must fulfill several conditions [Liu 03b]:

- *Unity*: the kernel function must be normalized

$$\int_{\mathbb{R}^n} W(|\mathbf{x} - \mathbf{x}'|, h) d\mathbf{x}' = 1 \quad \text{or} \quad \sum_{j=1}^N W(|\mathbf{x} - \mathbf{x}_j|, h) \Delta V_j = 1, \quad (7.1.6)$$

- *Delta function property* as formulated in (7.1.2),
- *Compact support*: $W(r, h) = 0$ for $r \geq \kappa h$, where κ is a constant depending on the kernel function,
- *Decay*: the smoothing function should decrease monotonically with the distance between a particle and the neighboring ones,
- *Symmetry, positivity and smoothness*.

The *B-spline function*, introduced by Monaghan in 1985, is a broadly used smoothing function. Here, the compact support is divided in three subspaces,

$$W(R, h) = \alpha_d \begin{cases} \frac{2}{3} - R^2 + \frac{1}{2}R^3, & \text{if } 0 \leq R < \frac{\kappa}{2} \\ \frac{1}{6} (2 - R)^3, & \text{if } \frac{\kappa}{2} \leq R < \kappa \\ 0, & \text{if } R > \kappa \end{cases} \quad (7.1.7)$$

with κ equal to 2 and $R = r/h$. The parameter α_d depends on the dimension n of the state space. From (7.1.6), the parameter α_d equals $5/7\pi h^2$ and $3/2\pi h^3$ in two- and three-dimensional space, respectively. This function and its first derivative are both continuous. Nevertheless, the second derivative is piecewise linear. Because of this discontinuity, the stability properties of this function are inferior to those of smoother functions.

In this work, the original *Lucy kernel function* is used. Introduced by Lucy in 1977 [Lucy 77], this kernel is a polynomial function continuously differentiable over its compact support ($\kappa = 1$),

$$W(R, h) = \alpha_d \begin{cases} (1 + 3R)(1 - R)^3, & \text{if } R \leq \kappa \\ 0, & \text{if } R > \kappa \end{cases}, \quad (7.1.8)$$

where α_d equals $5/\pi h^2$, $105/16\pi h^3$ and $28/\pi^2 h^4$ in a two-, three- or four-dimensional state space, respectively.

Those two kernel functions are simple and do not present any particularities. For a sake of completeness, it should be pointed out that it exists different kernel functions and their use depends on the problem. Indeed, the properties imposed to the kernel function may be too restrictive in some problems, leading to some instabilities. For instance, the boundedness of the support is contravened in the Gaussian kernel function, while the monotonic decay is broken in the Johnson kernel function.

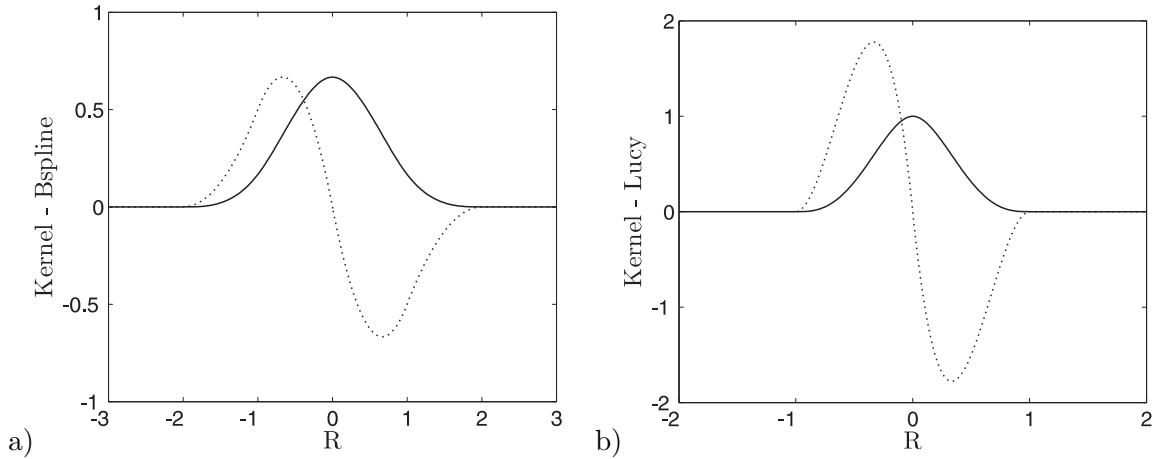


Figure 7.2: Kernel functions (-) and their derivatives ($\cdots$) : B-spline function (a) and Lucy function (b) - the functions are normalized by their respective α_1 .

7.2 SPH formulation of Fokker-Planck-Kolmogorov equation

Since we have defined the basics of SPH method, we are now able to apply this method to the FPK equation. Indeed, the field defined in (7.1.1) can be seen as the probability density function ψ related to the random vector defined in Section 2.1.2. However, some adaptations must be first operated on the convection-diffusion equation (6.1.2). Indeed, written in this way, Equation (6.1.2) is expressed in a Eulerian formalism, while the SPH method requires to

be transformed into a Lagrangian formalism. Then, the conservation and transport equations will be explicitly written in the formalism of the method.

7.2.1 Lagrangian formalism

In the context of fluid dynamics, an *Eulerian description* of motion can be imagined as the one given by a fixed observer looking at the evolution of fluid properties in a finite, fixed and non-deformable volume. Contrarily, a *Lagrangian description* of motion can be viewed as the motion described by observers sitting on moving particles. These views modify the formulation and the interpretation of the equations of motion: in a Lagrangian formalism, the integration points are moving according to trajectories depending on the modeled system.

The Eulerian formulation of a convection-diffusion phenomenon of a scalar field ϕ is transformed into a Lagrangian formulation by introducing the concept of *material derivative* D/Dt ,

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \mathbf{v} \cdot (\nabla\phi), \quad (7.2.1)$$

where $\mathbf{v}$ is the medium velocity.

The FPK equation (6.1.2), as a convection-diffusion equation, can be recast into such a Lagrangian formalism. Indeed, Equation (6.1.2) can be written as

$$\frac{\partial\psi}{\partial t} = - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\psi \left(\tilde{a}_i - \sum_{j=1}^n \frac{\partial D_{ij}}{\partial x_j} - \frac{1}{\psi} \sum_{j=1}^n D_{ij} \frac{\partial\psi}{\partial x_j} \right) \right) = - \sum_{i=1}^n \frac{\partial}{\partial x_i} (v_i \psi), \quad (7.2.2)$$

where it is readily seen that the i -th component of $\mathbf{v}(t, \mathbf{X})$, defined as

$$v_i = \tilde{a}_i - \sum_{j=1}^n \frac{\partial D_{ij}}{\partial x_j} - \frac{1}{\psi} \sum_{j=1}^n D_{ij} \frac{\partial\psi}{\partial x_j}, \quad (7.2.3)$$

provides a form similar to (7.2.1). The *diffusion velocity* $\mathbf{v}$ [Laco 99, Comb 86] has a term divided by ψ , but there is no hidden difficulties. Indeed, the field ψ represents a density of probability, i.e. a quantity of probability divided by a given volume. The factor $1/\psi$ remains finite as particles can neither have zero mass (as shown in section 7.3.1), nor occupy an infinite volume.

The conservation equation (7.2.2) is further developed to introduce the total derivative

$$\frac{\partial\psi}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial x_i} (v_i \psi) = \frac{D\psi}{Dt} + \sum_{i=1}^n \psi \frac{\partial v_i}{\partial x_i} = 0. \quad (7.2.4)$$

and the vectorial equation of transport across the state space simply reads

$$\frac{d\mathbf{X}}{dt} = \mathbf{v} \quad (7.2.5)$$

with $\mathbf{v}$ the velocity field defined in (7.2.3).

In this method, the steady-state regime must be interpreted from a Lagrangian perspective: the particle velocity is not null, but the particles keep on moving along $(n - 1)$ -dimensional

isoprobability manifolds. This is shown as follows. In steady-state regime, the material derivative of ψ and the partial derivative with respect to time of ψ are equal to zero. Therefore, Equation (7.2.1) leads to $\mathbf{v} \cdot (\nabla \psi) = 0$, which shows that the steady-state velocity field is always normal to the gradient of the pdf, like the tangent vector to isoprobability manifolds.

7.2.2 Conservation equation

The *quantity of probability* μ_j (equivalent to mass in hydrodynamics) carried by a particle j is defined as the product of the particle volume ΔV_j and the particle density ψ_j . By analogy with fluid dynamics, this quantity of probability is referred to as *mass* in the following. Taking into account (3.3.3), the particle approximation of the density of probability ψ at a particle i is

$$\langle \psi(\mathbf{x}_i) \rangle = \sum_{j=1}^{N_p} \psi_j W_{ij} \Delta V_j = \sum_{j=1}^{N_p} \mu_j W_{ij}, \quad (7.2.6)$$

where $\mu_j = \psi_j \Delta V_j$. This particle approximation of the probability density function is equivalent to what is usually done in hydrodynamics [Rand 96]. In dedicated literature, Equation (7.2.6) is often mentioned as the *summation density* approach. In the sequel, as the mass of each particle is invariant, the total unit probability is conserved. From a computational point of view, there is no need to solve the conservation equation as a differential equation. Furthermore, the positivity of the pdf is ensured, because each particle approximation is a sum of positive terms ($\mu_j, W_{ij} > 0$).

7.2.3 Estimation of statistical moments

The SPH method and the integral representation of pdf also allow an approximation of statistical moments of the random state vector $\mathbf{X} = [X_1, \dots, X_n]^T$. Indeed, the particle approximation of statistical moments reads

$$\langle E[X_1^{\alpha_1} \dots X_n^{\alpha_n}] \rangle = \int_{\mathbb{R}^n} x_1^{\alpha_1} \dots x_n^{\alpha_n} \langle \psi(\mathbf{x}) \rangle d\mathbf{x}, \quad (7.2.7)$$

where $E[\cdot]$ is the expectation operator and $\alpha_i \in \mathbb{N}$ for $i = 1, \dots, n$. Therefore, if the masses are reasonably well calculated by the previous relations, Equation (7.2.7) in an SPH formalism yields

$$\langle E[X_1^{\alpha_1} \dots X_n^{\alpha_n}] \rangle \approx \sum_{i=1}^{N_p} \left(\prod_{k=1}^n (\mathbf{X}_i)_k^{\alpha_k} \right) \langle \psi_i \rangle \Delta V_i. \quad (7.2.8)$$

In this approximation technique, particles are seen as *integration points*. The volume ΔV_i occupied by the i -th particle is also considered as the integration volume. This approximation means that the kernel approximation $\langle \psi_i \rangle$ of the pdf is constant in the volume of the particle. This integration technique is similar to the rectangle integration method. The computation of statistical moments is subsequent to the solution of the FPK equation.

The choice of ΔV_i as integration volumes is justified, if the reciprocity in the interaction between particles is imposed. The reciprocity exists, if the matrix $\mathbf{W}$ gathering the elements

W_{ij} , is symmetric. Actually, the unity principle is expressed as

$$\Delta V_j = \sum_{k=1}^{N_p} (\mathbf{W}^{-1})_{jk}. \quad (7.2.9)$$

So, if one assumes that the volumes ΔV_j are integration volumes for the kernel approximation $\langle \psi(\mathbf{x}_j) \rangle$, the conservation of total probability leads to

$$\sum_{j=1}^{N_p} \langle \psi(\mathbf{x}_j) \rangle \Delta V_j = \sum_{i=1}^{N_p} \mu_i \left(\sum_{j=1}^{N_p} \sum_{k=1}^{N_p} (\mathbf{W})_{ji} (\mathbf{W}^{-1})_{jk} \right). \quad (7.2.10)$$

The summation (7.2.10) is constant if and only if $\mathbf{W}$ is symmetric, which is not *ipso facto* the case. Among available ways to ensure that $\mathbf{W}$ is built symmetric, the method adopted here consists in averaging the reciprocal kernel functions between two particles as

$$W_{ij} = W_{ji} = \frac{1}{2} (W(r_{ij}, h_i) + W(r_{ij}, h_j)). \quad (7.2.11)$$

An important remark must be formulated at this stage. Other expressions have been proposed in order to compute (7.2.7) as in [Chau 97]. Indeed, the explicit analytical expression of the kernel function could be used to perform (7.2.7), avoiding the numerical approximation of the integration volumes (7.2.8). Nonetheless, disadvantages associated with this approach must be pointed out, especially in the density field. The SPH method is used in solid and fluid dynamics. In the last one, even if a fluid is compressible, the spatial gradient of the density field remains tractable. In the FPK problem, there can be huge differences in the probability density field between two zones of the domain. Therefore, using the analytical formulation of the kernel function, while all the quantities in the problem are discretized and particularized, is intricate and leads to poor results, as experienced by the author. We will keep on discussing this assumption in the following Sections; additional arguments will be given next.

7.2.4 Probability of exceedance

The proposed SPH method also allows the computation of probability function depending on the pdf of the state vector $\mathbf{X}(t)$. The mathematical formulation of a structural reliability problem [Kiur 96], as illustrated in Figure 7.3, consists in calculating the *probability of exceedance* $\mathbb{P}_f$ defined as the probability integral

$$\mathbb{P}_f(t) = \int_{\Omega_f} \psi(t, \mathbf{x}) d\mathbf{x}, \quad (7.2.12)$$

where Ω_f denotes the subspace where exceedance (or failure) occurs, obviously conditioned by an initial distribution. In the presented SPH method, the calculation of such a probability may be estimated directly : at a given time, only the mass of particles contained in the domain Ω_f contributes to the probability of failure. Therefore, an estimation of the probability of failure $\langle \mathbb{P}_f(t) \rangle$ at a given time t is

$$\langle \mathbb{P}_f(t) \rangle = \sum_{j=1}^{N_p} \mu_j \mathbb{I}(\mathbf{X}_j(t) \in \Omega_f), \quad (7.2.13)$$

where $\mathbb{I}(\mathcal{A})$ is a set function equal to one if the condition $\mathcal{A}$ is fulfilled, and zero otherwise. Obviously, the discussion initiated in the previous Section applies also in (7.2.12).

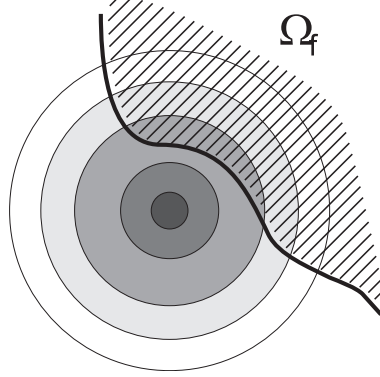


Figure 7.3: Reliability problems. Illustration of the probability of exceedance.

7.2.5 Transport equation

According to (7.2.6), the probability density of a particle i depends on W_{ij} and therefore on the relative positions of particles at time t . To calculate the position of a particle i , Equation (7.2.5) is integrated with a first-order forward *Euler integration scheme*

$$\mathbf{X}_i^{t+\Delta t} = \mathbf{X}_i^t + \mathbf{v}_i^t \Delta t. \quad (7.2.14)$$

An adaptive time step is implemented to ensure the stability and the accuracy of the method as presented in [Cano 11],

$$\Delta t = C_r \min_{i=1, \dots, N_p} \left(\frac{\min_{j=1, \dots, N_p, j \neq i} (r_{ij})}{\|\mathbf{v}_i\|} \right) \quad (7.2.15)$$

with $C_r (< 1)$ the Courant number. An adaptive time step is necessary in the resolution of FPK equation, at least when particles are initially slightly dispersed, since particles are very close and an inappropriate time step can spoil the distribution of particles or cause collisions between them. This proposed *Courant condition* may be severe compared with some usual Courant conditions proposed in the literature usually based on the smoothing length and not on the distance between particles [Liu 03a]. Nevertheless, the Courant criterion as formulated in (7.2.15) conserves the particle meaning of this condition. Hence, according to this criterion, a particle in a time step cannot cross a distance greater than a fraction of its distance to the closest neighbor particle. Furthermore, according to (7.2.15), the time step depends only on geometrical considerations and not on the probability density (as it would be if the smoothing length was considered). From the author's experience, this option increases the stability performances of the algorithm.

In the numerical implementation, the velocity field is established, according to the concept of integral approximation of a vector field and Equation (7.2.3) becomes

$$\mathbf{v}(\mathbf{X}) = \mathbf{f}(\mathbf{X}) - \frac{1}{\psi(\mathbf{X})} \mathbf{D} \nabla \psi(\mathbf{X}), \quad (7.2.16)$$

so that the particle approximation at time t of (7.2.16) is

$$\mathbf{v}_i^t = \mathbf{f}(\mathbf{X}_i^t) - \frac{1}{\langle \psi(\mathbf{X}_i^t) \rangle} \mathbf{D} \langle \nabla \psi(\mathbf{X}_i^t) \rangle, \quad (7.2.17)$$

where $\langle \psi(\mathbf{X}_i^t) \rangle$ is calculated with (7.2.6) and $\langle \nabla \psi(\mathbf{X}_i^t) \rangle$ calculated as

$$\langle \nabla \psi(\mathbf{X}_i^t) \rangle = \sum_{j=1}^{N_p} \mu_j \nabla_i W_{ij}. \quad (7.2.18)$$

The Lagrangian formalism makes the method relatively robust with respect to the dissemblance between initial and steady-state distributions. With this particle approximation method, particles move from the initial to the stationary distribution by themselves. This is a noticeable difference with mesh-based methods where the mesh must cover a sufficiently large space in order to represent every step of the transient solution.

In SPH method, *leapfrog integration scheme* is usually preferred, because it is simply implemented and second-order accurate in both time and space [Pres 07]. With the leapfrog algorithm, Equation (7.2.5) is integrated as [Liu 03a]

$$\mathbf{X}_i^{t+\Delta t} = \mathbf{X}_i^t + \mathbf{v}_i^{t+\frac{\Delta t}{2}} \Delta t \quad (7.2.19)$$

with the midpoint velocity field estimated as

$$\mathbf{v}_i^{t+\frac{\Delta t}{2}} = \mathbf{f}(\mathbf{X}_i^{t+\frac{\Delta t}{2}}) - \frac{1}{\langle \psi(\mathbf{X}_i^{t+\frac{\Delta t}{2}}) \rangle} \mathbf{D} \langle \nabla \psi(\mathbf{X}_i^{t+\frac{\Delta t}{2}}) \rangle \quad (7.2.20)$$

and

$$\mathbf{X}_i^{t+\frac{\Delta t}{2}} = \mathbf{X}_i^t + \mathbf{v}_i^t \frac{\Delta t}{2}. \quad (7.2.21)$$

The choice between an Euler explicit scheme and a leapfrog scheme is typically driven by accuracy and computational efficiency. On the one hand, if accuracy is leading, the leapfrog algorithm is more efficient for the same adaptive time step procedure. On the other hand, if the choice is motivated by efficiency, Euler's scheme is more advantageous, because the estimation of the velocity field, i.e. the more consuming operation, is performed just once per time step. In any case, the selection of a small time step reduces the error due to the time discretization but barely influences the residual error due to particle interpolation, as shown in the following illustrations.

7.3 Computational aspects

7.3.1 Initialization of particle masses

The initialization of masses is a key issue in the method. From the arbitrary initial distribution of particles in the state space, the reconstructed field is calculated where particles are initially located $\langle \psi(\mathbf{x}_j) \rangle$. Then, the solution of (7.2.6) should in principle give the particle masses. Nevertheless, for computational efficiency reasons and also because the system (7.2.6) may be ill-conditioned, an alternative approach is followed by the author. The masses μ_j and the initial particle values $\psi_{0,j}$ are related by $\mu_j = \psi_{0,j} \Delta V_{0,j}$, where $\Delta V_{0,j}$ is the initial volume

associated with the j^{th} particle, which corresponds to a hypercubic subspace if the initial arrangement of particles is regular, as illustrated in Figure 7.4.

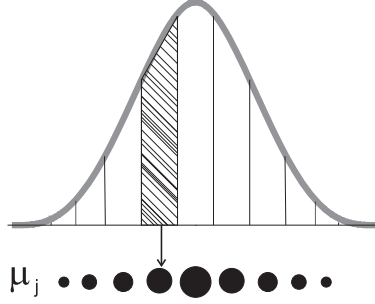


Figure 7.4: Initialization of the particle masses.

Although conceptually different, the initial condition ψ_0 may be understood as the kernel approximation of the probability density function. Indeed, a reasonable approximation for $\psi_{0,j}$ is the initial reconstructed field $\langle \psi_0(\mathbf{x}_j) \rangle$ provided the smoothing length is small enough according to (7.1.3). With this approach, the sum of particle masses is not strictly equal to unity, because of the limited integration domain and the approximation on the integration volumes. However, this error may be reduced by increasing the number of particles and so reducing the smoothing length as presented hereafter. After initialization, the particle masses are kept constant in time. Consequently, with this initialization technique, there is no equation to solve, neither for the initialization of masses, nor during the transient resolution.

7.3.2 Particle interaction and smoothing length

The determination of the smoothing length h is an important issue in the SPH method. If the smoothing length is too small, a particle does not interact with its neighboring particles. On the other hand, if the smoothing length is too large, local properties of the studied field could be excessively smoothed. It is commonly assumed that the accuracy depends on h^2 [Liu 03a].

The smoothing length also influences the computation time. For these reasons, the smoothing length associated to each particle is adapted at each time step. Liu [Liu 03a] suggests to maintain an appropriate number of particles in the neighborhood of each particle: 5, 21 and 57 in one-, two- and three-dimensions, respectively.

To adapt the smoothing length with particle positions, different authors suggest relations to maintain the number of neighboring particles constant [Liu 03a]. The simplest relation consists in keeping the initial product $(h_0 \sqrt[n]{\psi_0})_i$ constant in time for each particle, where n is the dimension of the considered space. The smoothing length is updated with the evolution of ψ_i , as

$$h_i = h_{0,i} \left(\sqrt[n]{\frac{\psi_0}{\psi}} \right)_i. \quad (7.3.1)$$

Equation (7.3.1) expresses also the limited risk that a particle gets isolated. Indeed, if particles are too close, their densities increase and their smoothing lengths decrease. Conversely, if a particle is too distant from the others, its density decreases and its smoothing length increases. Therefore, a particle cannot be isolated without any interaction with its neighbors.

Smoothing length and particle interaction are closely related in SPH method. Indeed, a particle i interacts only with particles contained in its compact support. Therefore, it is useless to check possible interaction if two particles are too far from each other. Specific literature amply covers this topic [Mona 05, Liu 03a].

The discussion about particle interaction and smoothing length is relevant only in the standard version of the method based on Eulerian kernels. In this approach, the compact support has a constant shape (though the size is updated) in time. Only the particles present in this support change. Besides, the particle interaction may also be interpreted in a Lagrangian paradigm. A major difficulty in SPH method with Eulerian kernel concerns the tensile instability [Sweg 95] which occurs in elasticity problems when particles tend to clump in pairs under tension. This instability is mathematically due to the calculation of the second derivative of the kernel [Bely 00]. A fully Lagrangian kernel [Rabc 04] or an updated Lagrangian kernel [Vida 07] can be used to discard occurrence of this instability. The Lagrangian kernel is a robust and stable method to deal with solid mechanics. Nevertheless, this method is not suitable for fluid flow problems [Rabc 04], because the deformation can be much larger in fluids than in solids. In our particular case, the transient FPK equation, as a convection-diffusion equation, is closer to the modeling of an artificial fluid without viscosity or constitutive law than an elastic solid, a reason why we have opted for Eulerian kernels. In spite of some drawbacks, Eulerian kernels also offer a simple management of the mixing of particles or the large distortions as illustrated hereinafter.

7.3.3 Boundary conditions

In civil engineering, the encountered problems are usually defined on an unbounded state space. Nonetheless, for a sake of completeness of this manuscript, we want to consider other classes of boundary conditions sometimes encountered in stochastic dynamics in other fields of science.

Unbounded domain

In the standard SPH method, the treatment of boundary conditions is sometimes neglected, as in astrophysical problems [Lucy 77] which deal with theoretically infinite domains. In the treatment of free surface boundary conditions, the standard SPH method behaves quite well, because the deficiency of particles near the boundary acts as a zero pressure condition. Nevertheless, the term “free surface” is not suitable for this application, because there is no clear separation between two media as in fluid or solid and the probability density function is defined everywhere in the state space, with a continuously decreasing pdf in the far field.

In the normalized SPH method, Randles and Libersky [Rand 96] suggest to define the density of a boundary particle with the kernel summation corrected with regard to the boundary deficiency. In this method, the definition of boundary particles remains an issue, especially for large modification of particle distribution compared with the initial one. Furthermore, the estimation of the gradient is not relevant.

In our context, with only far-field conditions, it is proposed to define an initial continuous probability distribution on a sufficiently large domain to have surrounding particles with a very low mass. These peripheral particles introduce implicitly a condition similar to $\psi(t, \mathbf{x}) \rightarrow 0$ when $\|\mathbf{x}\| \rightarrow \infty$. Their interactions with inner particles of higher masses are limited. So, the

absence of boundary treatment in the far field may be justified in this manner.

In Section 7.2.3, a discussion has been initiated about the pertinence of dealing with analytical form of kernel functions to compute statistical moments or exceedance probability. From (7.3.1), we can understand how fast the smoothing length related to the particle i evolves in time. Indeed, the ratio between the density of a particle at two time instants might change by one or several orders of magnitude. The smoothing length is therefore not necessarily a smoothed field, because huge gradients are encountered, especially in the tails. Actually, the lack of particles in the tails, playing the role of a *pseudo*-free surface, is a source of inaccuracy. Thence, the unity principle is consciously contravened, but it is not involved in the whole equilibrium of the particles [Lucy 77]. We cannot put up with this approximation, than coming back to the analytical formulation.

Periodic boundary conditions

In [Lin 04b], the problem of periodic boundary condition is at once introduced with the case of a one-dimensional diffusion process defined on a circle. In this case, both the probability density and the flow of probability must be periodic. More generally, the periodic boundary conditions are often used to simulate large systems (even infinite) by modeling a small piece of them.

In the context of Fokker-Planck equation, if the periodicity is assumed on the j -th dimension of the state space, the periodic conditions on the probability field in this case are such that

$$\psi(t, \bar{\mathbf{x}} | t_0, \mathbf{x}_0)|_{\bar{x}_j} = \psi(t, \bar{\mathbf{x}} | t_0, \mathbf{x}_0)|_{\bar{x}_j+L}, \quad (7.3.2)$$

$$\left. \frac{\partial}{\partial x_k} \psi(t, \bar{\mathbf{x}} | t_0, \mathbf{x}_0) \right|_{\bar{x}_j} = \left. \frac{\partial}{\partial x_k} \psi(t, \bar{\mathbf{x}} | t_0, \mathbf{x}_0) \right|_{\bar{x}_j+L} \quad (7.3.3)$$

with L the period of the domain, provided $\bar{\mathbf{x}}$ belongs to the boundary and $\bar{x}_j$ its j -th component. Equations (7.3.2) and (7.3.3) express that the probability density field is equal in $\bar{x}_j$ and $\bar{x}_j + L$ and that the flow of probability penetrating the domain at $\bar{x}_j$ is equal to the flow leaving it at the same time.

These conditions are readily transposed into the Lagrangian formalism. Indeed, referring to the SPH method, it consists in transferring a particle leaving the domain from the border $\bar{x}_j + L$ to $\bar{x}_j$, and *vice versa*. Therefore, the mass is preserved, as well as the probability flow, and the probability at the boundary is unchanged. The periodic boundary condition is introduced thus by connecting physically those boundaries and by maintaining the continuity of the particle trajectories. However, in the SPH formalism, the interaction between the particles must also take the periodicity of the domain into account: the distance between two particles is thus computed knowing the fact that they may interact across boundaries.

Absorbing boundary conditions

The concept of absorbing boundary condition is naturally understood. If a sample function is bounded by an absorbing condition, a realization of this process is stopped once it reaches this boundary, also called *exit condition*. According to [Lin 04b], this realization is “removed” from

the population of samples, but this interpretation may be fallacious, because this realization has a role in the population, the role of having reached the boundary or of having left the resolution domain forever.

Therefore, the probability density on an absorbing boundary is always equal to zero,

$$\psi(t, \bar{\mathbf{x}} | t_0, \mathbf{x}_0) = 0 \quad (7.3.4)$$

with $\bar{\mathbf{x}} \in \mathbb{R}^n \setminus \Omega$ and Ω the domain of resolution.

When a particle leaves the domain, its carrying mass is lost, but the interaction with the other particle is also perturbed. In this case, the use of *ghost particles* is a way to smooth the field ψ at the boundary and to maintain the unity principle, as far as possible. We propose to use the so-called ghost particles of Type II [Liu 03a] constructed with the method illustrated in Figure 7.5 : for a particle i located within the distance of κh_i from the boundary, a ghost particle is placed symmetrically out of the domain. These ghost particles have the same density as the corresponding effective particles. The parameters of these particles do not change in time, because they are just reproduced symmetrically to the boundary at each time step. More details about the treatment of boundary conditions can be found in [Rand 96, Mona 05].

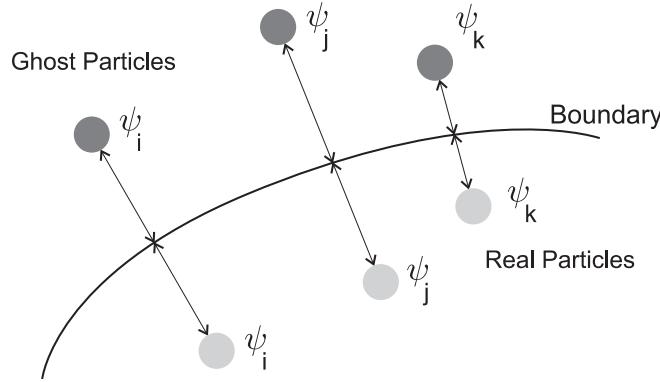


Figure 7.5: Ghost particles constructed symmetrically with the boundary.

7.4 Description of the algorithm

Initially, the particles are regularly spread out and the pdf is calculated at the position of each particle. A regular distribution allows to choose a smoothing length h for each particle that ensures an adequate number of neighboring particles in the compact support. Masses μ_j are initialized as in Section 7.3.1 and are kept constant throughout the resolution.

In a time loop, the computation of the conservation equation (7.2.6) and the gradient of the pdf (3.3.27) requires to identify (as fast as possible) the particles contained in the compact support of a given particle. This operation becomes time consuming if the number of particles is too important. The implemented algorithm, presented hereafter, is based on a *cell mapping* [Liu 03a], which is commonly used in SPH applications. Our aim has not been to improve or optimize existing method, but to use a readily implemented method readily extendable to higher-dimensional space.

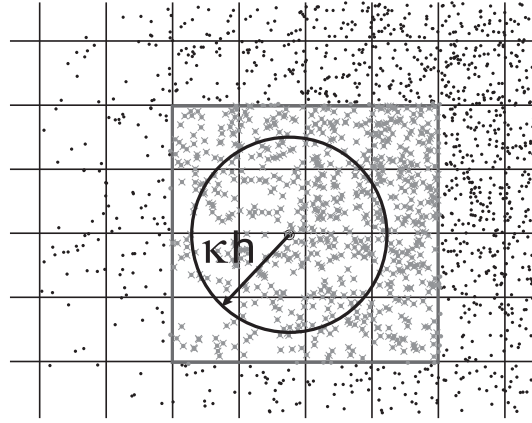


Figure 7.6: Cell mapping in a 2-dimensional state space.

To compute interactions, particles are first sorted in cells, as illustrated in Figure 7.6. This mapping consists in dividing the state space in a number of cells or n -dimensional hypercubes. In some applications, the edge-size of hypercubes is chosen as the maximum smoothing length h over all particles. This choice is adequate only if the density in the system is sensibly homogeneous. In the case of probability density evolution, there are large differences between zones of the state space. Therefore, the mapping is constructed with a given number of cells and the size of edges, identical for each cell, are modified at each time step. The method consists in exploiting the regularity of the cell map to find the smaller set of cells including completely the hypercube centered on the particle and with an edge equal to the diameter of the compact support.

Finally, when the kernel approximations of the fields $\psi(\mathbf{x})$ and $\nabla\psi(\mathbf{x})$ are calculated at each particle position, the velocity field (3.3.26) is evaluated for each particle and the particle positions are immediately modified according to (3.3.13). At this step, the smoothing length and the time step are updated according to Equations (3.3.28) and (3.3.22), respectively.

Chapter 8

Applications

-
- 8.1 Verification and validation
 - 8.2 Nonlinear single DOF systems
 - 8.3 Probability of exceedance of a hysteretic oscillator
 - 8.4 Examples from other scientific disciplines
-

Purpose of the chapter

In order to illustrate extensively the applicability of our method, we start by a validation of our algorithm on a 2-DOF linear system. Then, we deal with three examples more related to civil engineering problems: a nonlinear viscous damper set up on a structure, the pounding between two adjacent buildings and an elasto-plastic oscillator. Finally, to give other perspectives to this work, we deal with three other problems related to other scientific disciplines: a randomly excited pendulum, the Lorenz attractor and a random ecosystem model.

8.1 Verification and validation

First, to illustrate and validate the method, a dimensionless SDOF linear oscillator is considered,

$$\ddot{x} + 2\xi\dot{x} + x = W \quad (8.1.1)$$

where $W(t)$ is a normalized Gaussian δ -correlated process with zero mean and $E[W(t)W(t+\tau)] = \delta(\tau)$ and ξ is the damping ratio equal to 0.1 in this example. The exact solution of this problem, aiming at determining the time evolution of first and second-order moments is obtained for instance with Gaussian closure method [Lin 04b]. The initial uncorrelated Gaussian distribution has mean position and mean velocity equal to 1 and standard deviations of position and velocity equal to 0.1, in order to demonstrate that the developed method is able to cope with initial conditions with few dispersion.

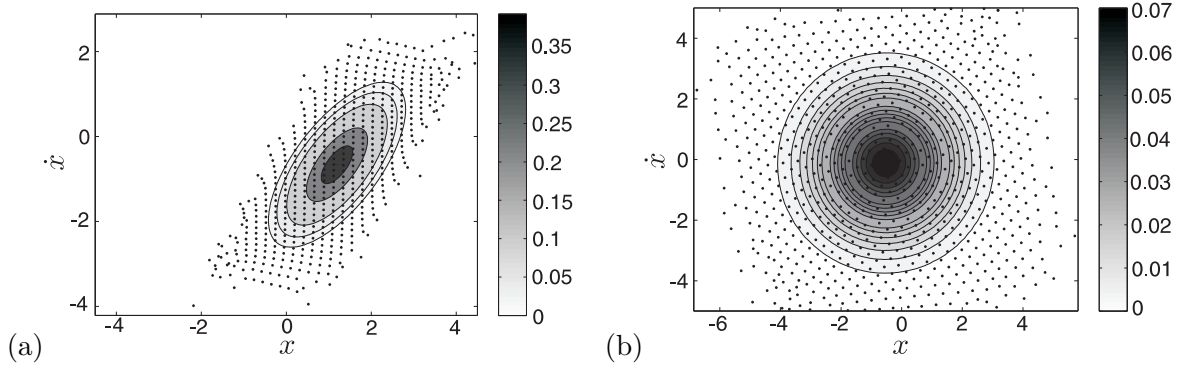


Figure 8.1: Reconstructed probability density function ψ and position of particles at (a) $t = 1.3$ and (b) $t = 10$ for $N_p = 1681$.

Figure 8.1 presents two snapshots at different times of the pdf reconstructed from the particle positions. The particles automatically move from the initially meshed zone to the subspace of interest at any time step and they finally reach the steady-state distribution.

In order to study the accuracy of the method, the FPK equation is solved for different numbers of particles N_p and a Courant number C_r fixed to 0.2. The relative error on the pdf is calculated for each time step as

$$\mathcal{E}_\psi = \frac{\|<\psi> - \psi_{ana}\|_2}{\|\psi_{ana}\|_2} = \frac{\sqrt{\sum_{j=1}^{N_p} |<\psi> - \psi_{ana}|_j^2}}{\sqrt{\sum_{j=1}^{N_p} |\psi_{ana}|_j^2}} \quad (8.1.2)$$

with $\psi_{ana}(t, \mathbf{x}(t)) : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}_0^+$ the analytical solution of the FPK equation related to (8.1.1). In Figure 8.2-a the time evolution of the error $\mathcal{E}_\psi$ for three numbers of particles N_p is shown. For a small number of particles, the error $\mathcal{E}_\psi$ is rather significant in the early transient phase, but it decreases when N_p increases, because the smoothing length decreases.

The second-order moment of x calculated with (3.3.16) is illustrated in Figure 8.2-b. In the early transient phase, it is well estimated, even with a small number of particles. In the steady-state distribution, there are less particles in the bulk of the pdf and a larger amount of particles is necessary for an accurate estimation of the second moment. Nevertheless,

instead of using (3.3.16), translating the rectangle approximation, a better approach consists in interpolating pdf values between the values calculated at particle positions and then to calculate moments with (3.3.14). This operation reduces the difference between the exact moment and the approximation (calculated with N_p equal to 1681), as illustrated in Figure 8.2-b.

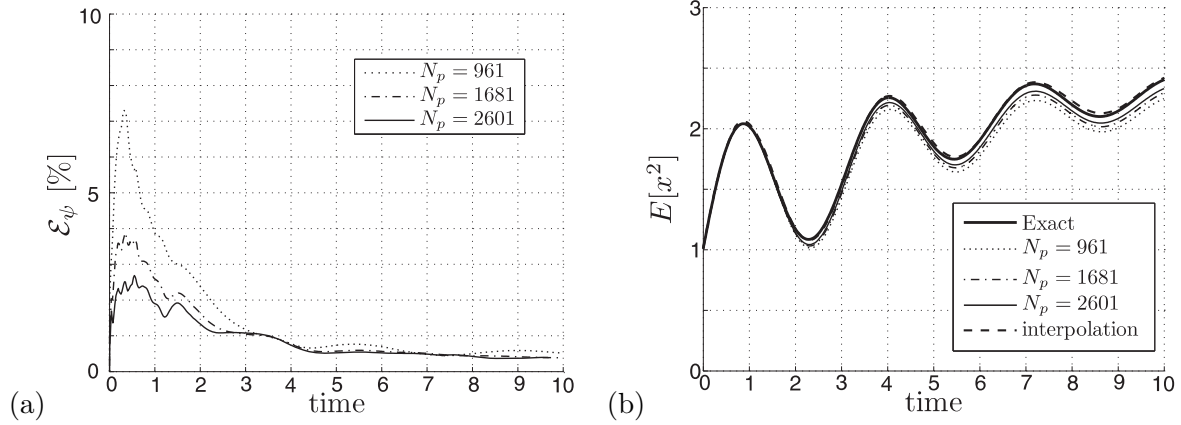


Figure 8.2: Two-dimensional linear system: (a) relative error on the pdf $\mathcal{E}_\psi$ with respect to time as defined in (8.1.2), (b) the time evolution of the second-order moment of x .

The reduction of the errors while increasing N_p also implies an increase of the computation time for two different reasons: (i) the evaluation of the particle interaction increases and (ii) the particles are closer so the time step is reduced with regard to (3.3.22).

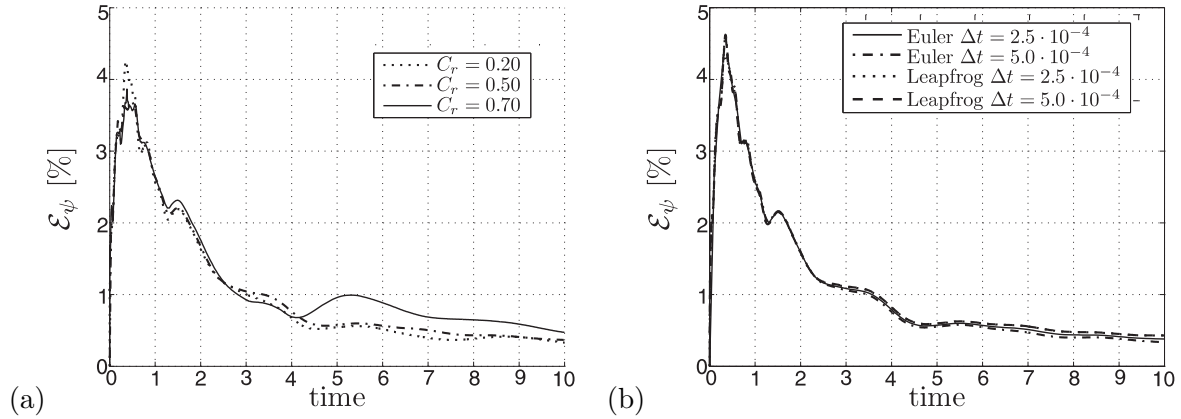


Figure 8.3: Two-dimensional linear system ($N_p = 1641$): (a) influence of Courant number C_r on the relative error $\mathcal{E}_\psi$ with respect to time and a forward Euler scheme, (b) the time evolution of the relative error $\mathcal{E}_\psi$ for two different integration schemes and two different time steps.

Figure 8.3 shows the influence of time integration parameters on the error $\mathcal{E}_\psi$: the influence of the Courant number and the integration scheme are illustrated in this example. Figure 8.3-a compares the error $\mathcal{E}_\psi$ committed with an Euler explicit scheme and different Courant numbers. The error is not significantly influenced by this parameter, even if the

error is less for Courant numbers lower than 0.5. Figure 8.3-b compares the two integration schemes previously presented. For the comparison, the time step is chosen constant and small enough to keep the Courant number lower than 1 for the whole computation. As announced previously, the reduction of the error between an Euler explicit integration scheme and a leapfrog algorithm is not significant. The residual error on ψ preponderantly depends on the particle discretization and not on the time integration.

The accuracy and the convergence of the method have been also illustrated with two-dimensional nonlinear systems, as a Duffing-Van der Pol oscillator in [Cano 11].

8.2 Nonlinear single DOF systems

8.2.1 Probability of exceedance of a nonlinear SDOF oscillator

The transient FPK equation finds a potential interest in seismic design. In this topic, reliability and probability of failure are usual questions. The estimation of a probability of exceedance is therefore required. The resolution of FPK equation with SPH method is applied to a structure damped by nonlinear fluid viscous devices and subjected to seismic loading. Fluid dampers installed on civil structures are commonly used as energy sinks for seismic protection. The nonlinear behavior of the damping device deeply influences the response of the structure [DiPa 07].

The structure is supposed to mainly respond in its first mode and a nonlinear viscous damper is considered to mitigate the vibrations due to an earthquake. The equation of motion for the modal coordinate q is

$$\ddot{q} + 2\xi\omega_0\dot{q} + \eta\text{sign}(\dot{q})|\dot{q}|^\alpha + \omega_0^2q = aW \quad (8.2.1)$$

where ω_0 and ξ are the natural circular frequency and the damping ratio of the system, respectively. The nonlinear viscous force is characterized by the damper exponent α typically ranging from 0.2 to 1 [Pekc 99] and η the damping coefficient (related to the generalized mass of the system). The excitation is modeled by a white noise $W(t)$ modulated by a time window $a(t)$ defined as

$$a(t) = \begin{cases} (t/t_1)^{n_1}, & t \leq t_1 \\ 1, & t_1 < t < t_2 \\ \exp(n_2(t_2 - t)) & t_2 \leq t \end{cases} \quad (8.2.2)$$

Thus, the only non-zero element of the diffusion matrix is $D_{22}(t) = S_0a^2$ with $E[W(t)W(t + \tau)] = 2S_0\delta(\tau)$. The approximation of the probability of exceedance $\mathbb{P}_f$ developed in Section 7.2.4 is illustrated in this example with an outcome space defined as $\Omega_{\mathcal{T}} = \{q \in \mathbb{R} : |q| > \mathcal{T} > 0\}$ with $\mathbf{x} = [q \ \dot{q}]^T$ the state vector of the system with $\mathcal{T}$ a threshold related to the ductility of the system.

The parameters have the numerical values: $\omega_0 = \pi$, $\xi = 1\%$, $\eta = 0.25$, $\alpha = 0.35$ and $S_0 = 0.2$. The time window is characterized by $n_1 = 1$, $t_1 = 2$, $n_2 = 2$ and $t_2 = 6$. The initial uncorrelated Gaussian distribution has standard deviations equal to 0.15 for q and 0.10 for $\dot{q}$ and mean values equal to zero. The number of particles is 1650 and the Courant number is 0.8. Initially, particles are regularly arranged in a rectangle with edge sizes of 1.2 and 0.8 in q and $\dot{q}$ directions, respectively.

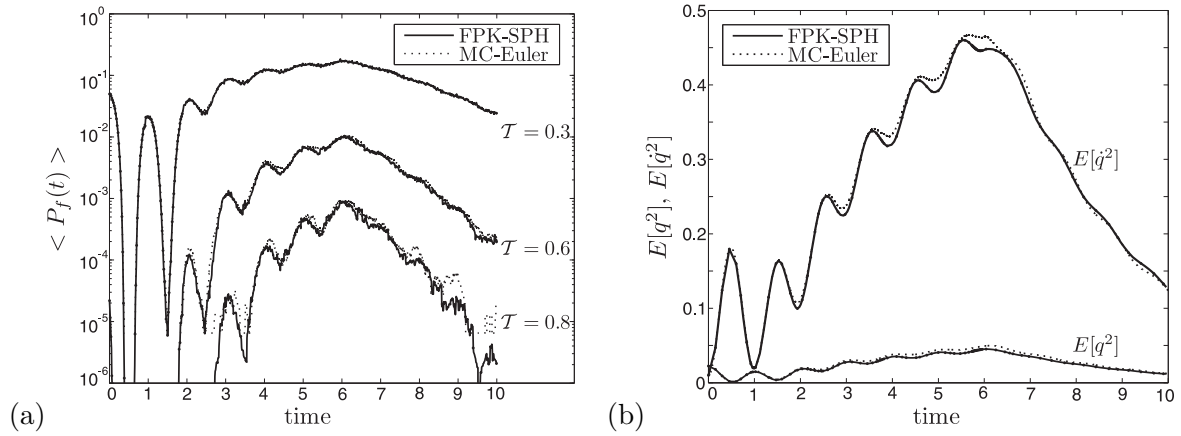


Figure 8.4: SDOF system with nonlinear viscous damper ($N_p = 1650$). The probability of exceedance $\mathbb{P}_f$ (a) for the system (8.2.1) is estimated for different thresholds $\mathcal{T}$. The results obtained by solving FPK with SPH are compared with Monte Carlo (MC) simulations (10^5 samples). The estimated second-order moments of q and $\dot{q}$ are compared with simulations (b).

Figure 8.4-a shows $\langle \mathbb{P}_f \rangle$ for different thresholds $\mathcal{T}$ calculated with FPK equation and Equation (7.2.13). These results are compared with Monte Carlo simulations (10^5 samples and 2^{16} time steps chosen after a convergence study). For the Monte Carlo method, Equation (8.2.1) is integrated by means of a forward Euler scheme [Kroe 11]. This figure emphasizes the good agreement between results obtained with both methods, especially for high probabilities of exceedance. As expected, the convergence of stochastic simulations for low probabilities (in the order of 10^{-4}) is not achieved. Because the response is not ergodic, a larger number of simulations is required to obtain accurate results. However, the computation of FPK equation has required about 11,000 adaptive time steps and a single simulation has covered six orders of magnitude. Figure 8.4-b shows the estimation of the second-order moments of q and $\dot{q}$ estimated with (3.3.14) and highlights the ability of the method to capture the structural and statistical timescales, besides the oscillations of the moments. Some small differences with simulation results are observed, but the second-order moments of q and $\dot{q}$ are properly estimated, despite an order of magnitude of difference between them, because the pdf is more stretched in the $\dot{q}$ direction than in the q direction.

8.2.2 Reduced pounding problem

As a second SDOF nonlinear system, we propose to illustrate the method with an example of pounding risk assessment. Pounding refers to the contact taking place between close buildings when they are shaken by an intense earthquake. The pounding risk assessment consists in computing the probability of contact assuming a certain scenario [Barb 10, Barb 13]. In the literature, this probability measure is limited to transient dynamics of linear systems subject to a fully coherent excitation and the contact is not taken into account.

Contact can be integrated in the model, but the problem becomes therefore nonlinear. Different models may be used to integrate the contact between the structures. According to [Jank 05], the pounding forces may be linear viscoelastic, nonlinear elastic or nonlinear viscoelastic. In this application, we will consider only nonlinear elastic forces based on a

Kelvin contact model evolving with a power $3/2$ of the penetration. Secondly, we assume that one of the two buildings is much stiffer than the other one, such that it may be considered as not deformable. The dimensionless equation of motion, for relative displacement with respect to the ground, is given by

$$\ddot{q} + 2\xi\dot{q} + q + \beta(q-1)^{3/2}\mathbb{H}(q-1) = a(t)W(t) \quad (8.2.3)$$

with $\mathbb{H}(\cdot)$ the Heaviside function in order to take into account the unilateral behavior in the Kelvin nonlinear forces, though the problem remains continuous. The dimensionless formulation is readily obtained by scaling the time with the natural pulsation of the linear oscillator ($\beta = 0$) and the displacements and positions are scaled with respect to the initial gap. The coefficient β is a dimensionless parameter quantifying the intensity of the nonlinear stiffness. Obviously, this coefficient is much greater than one.

We propose to study this example inspired from [Jank 05] in order to illustrate a problem solved with a penalty method. Actually, a particle entering in the domain $q > 1$ is subject to large nonlinear forces, inducing an important deceleration and preventing a deep penetration in the domain. The nonlinear domain can thus be seen as a penalty layer. The damping ratio ξ is fixed at 0.05 and the white noise intensity at 0.25. The coefficient β is equal to 200 and 1000 in order to illustrate the influence of the nonlinear stiffness on the penetration domain. We consider the time window (8.2.2) with $n_1 = 1$, $t_1 = 10$, $n_2 = 2$ and $t_2 = 25$.

We are also interested in computing the probability of contact and penetration and its time evolution. The approximation of the probability of exceedance $\mathbb{P}_f$ developed in Section 7.2.4 is illustrated in this example with an outcome space defined as $\Omega_{\mathcal{T}} = \{q \in \mathbb{R}, \mathcal{T} \geq 1 : q > \mathcal{T}\}$ with $\mathcal{T}$ a threshold related to the penetration distance.

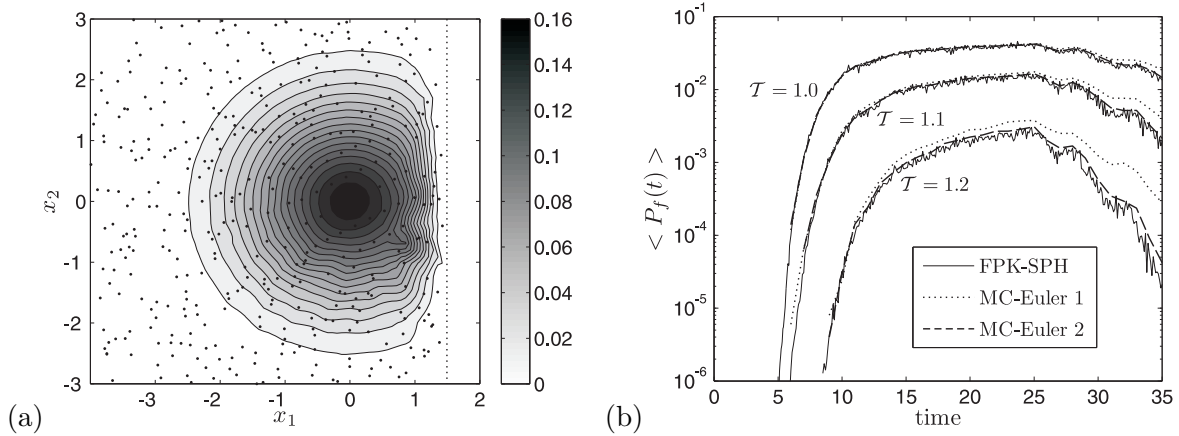


Figure 8.5: Pounding problem with Kelvin contact with $\beta = 200$ and $N_p = 2600$ ($x_1 = q$, $x_2 = \dot{q}$). (a) Sketch of the probability density function of q and $\dot{q}$ at $t = 25$, final time of the plateau of the envelope. (b) The probability of exceedance $\mathbb{P}_f$ for the system (8.2.3) is estimated for different thresholds $\mathcal{T}$. The results obtained with SPH are compared with Monte Carlo (MC) simulations (10^6 samples) with two different time steps $\Delta t = 4 \cdot 10^{-3}$ (MC-1) and $\Delta t = 10^{-3}$ (MC-2).

The mains results are depicted in Figures 8.5 and 8.6. In Figures 8.5-a and 8.6-a, we show clearly that some particles penetrate into the domain $\{q > 1\}$, but they are stopped by the

intensity of the nonlinear forces. The Kelvin force and the high values of β explain the shape of the pdf. Actually, the coefficient β could be chosen significantly greater, but in that case, there would be almost no chance to overcome the threshold $\mathcal{T}$.

More interesting are the Figures 8.5-b and 8.6-b in which we compute the probability of exceedance $\mathbb{P}_f(t)$ defined in (7.2.12). They are compared with Monte Carlo simulations. We see the good agreement between the two approaches, but we emphasize also here the problem of the time step in Monte Carlo simulation and the importance to check the convergence, especially for contact problems and also because the excitation is a white noise. The SPH method with 2500 particles is able to cover at least four orders of magnitude in terms of the total probability which is remarkable. In Figure 8.6-b, the curves obtained with our method are more erratic; this is explained by the values of the probability mass μ_j and the logarithmic axis. Actually, with (7.2.13) we cannot compute a probability lower than the particles masses. The curves are thus erratic because particles are entering and leaving the domain $\Omega_{\mathcal{T}}$, modifying the probability $\mathbb{P}_f$ by their amount of probability. The calculation of $\mathbb{P}_f$ only based on the particle probability masses can be seen as a limitation in the computation of very low probability, as we cannot compute probability lower than the lowest mass.

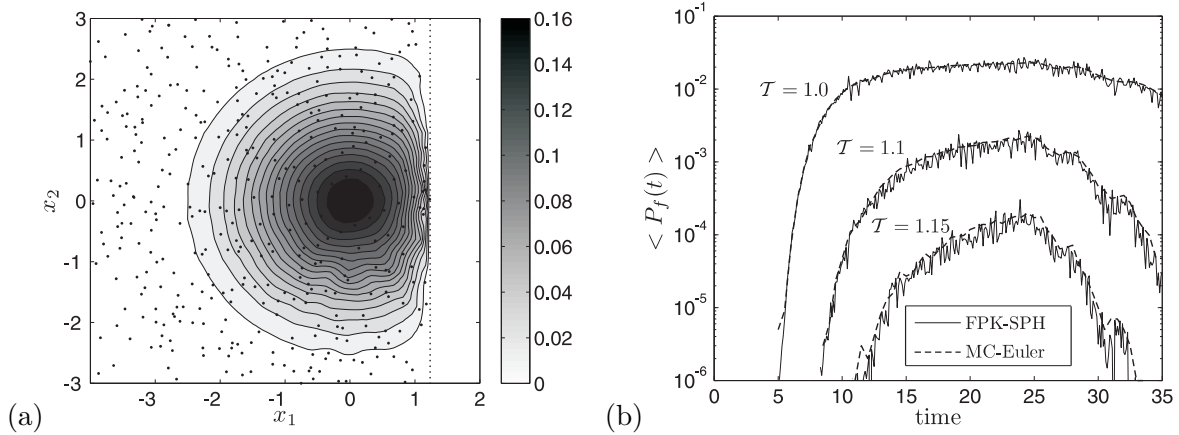


Figure 8.6: Pounding problem with Kelvin contact with $\beta = 1000$ and $N_p = 2600$ ($x_1 = q$, $x_2 = \dot{q}$). (a) Sketch of the probability density function of q and $\dot{q}$ at $t = 25$, final time of plateau of $a(t)$. (b) The probability of exceedance $\mathbb{P}_f$ for the system (8.2.3) is estimated for different thresholds $\mathcal{T}$. The results obtained by solving FPK with SPH are compared with Monte Carlo (MC) simulations (10^6 samples) with $\Delta t = 5 \cdot 10^{-4}$.

8.3 Probability of exceedance of a hysteretic oscillator

As another application in earthquake engineering, the resolution of FPK equation with SPH method is applied to an SDOF system presenting a nonlinear elastoplastic behavior and subjected to a transient seismic loading. This example aims at showing that the proposed method gives accurate information about the tail distribution through a reliability problem, even in a three dimensional state space.

The equations of motion for a second-order structural system with hysteretic behavior [Isma 09] are

$$\begin{cases} \ddot{q} + 2\xi\omega_0\dot{q} + \Phi = aW \\ \Phi = \omega_0^2 (\alpha q + (1 - \alpha)z) \\ \dot{z} = \dot{q} (A - (\beta \text{sign}(\dot{q}z) + \gamma) |z|^n) \end{cases} \quad (8.3.1)$$

with q the dimensionless coordinate of the SDOF system, ξ the structural damping ratio and ω_0 the natural circular frequency. The excitation is a white noise $W(t)$ ($E[W(t)W(t + \tau)] = 2S_0\delta(\tau)$) modulated by a time window $a(t)$ and $\Phi(q, z)$ is the elastoplastic restoring force. This force is a convex combination of a linear stiffness component and a hysteretic component depending on the hardening parameter α . The hysteretic behavior is controlled by the Bouc-Wen model [Wen 76] characterized by the dimensionless variable z gathering the loading history and by the four parameters A , γ , β and n [Isma 09]. The outcome space is defined as $\Omega_{\mathcal{T}} = \{q \in \mathbb{R} : |q| > \mathcal{T} > 0\}$ for different values for the threshold $\mathcal{T}$.

In this example inspired by [Lin 04b, Ikho 05], the parameters of the system are chosen as $\omega_0 = \pi$, $\xi = 5\%$, $\alpha = 0.5$ and $S_0 = 1.0$ and those of the Bouc-Wen model are $A = 1$, $\gamma = 0.5$, $\beta = 0.5$, $n = 1$. This set of data provides a significantly nonlinear response. The time window is chosen as

$$a(t) = f_{max} \frac{t}{t_{max}} \exp\left(1 - \frac{t}{t_{max}}\right) \quad (8.3.2)$$

with $t_{max} = 2$ and $f_{max} = 1$. The initial uncorrelated Gaussian distribution is slightly dispersed with standard deviations equal to 0.05 and means equal to zero for q , $\dot{q}$ and z . The number of particles is 9261 ($=21^3$) and the Courant number is 0.8. Initially, particles are regularly arranged in a cube with an edge size of 0.4, and thus with 21 particles along each dimension.

This example shows in a three-dimensional state space that information about the tail distribution is accurately obtained with a reasonable number of particles during the whole duration of the earthquake, with a number of time steps limited to 35000. With only one run, the proposed method covers six orders of magnitude in the probability of exceedance, see Figure 8.7-d, whilst the Monte Carlo simulation, presented here with $5 \cdot 10^5$ runs, poorly estimates probabilities lower than 10^{-5} . It would require approximately 10^8 runs to capture probabilities less than 10^{-6} .

In the SPH simulation, the number of particles can be increased to smooth the mass distribution and consequently further ameliorate the estimation of low probabilities of exceedance. Also these results must be analyzed under the light of the significant dispersion from the slightly spread initial Gaussian distribution and the probability function illustrated in Figure 8.7; as an indication the second-order moments of q and $\dot{q}$ are multiplied by 80 and 500, respectively, for original problems. The main results are extensively discussed and commented.

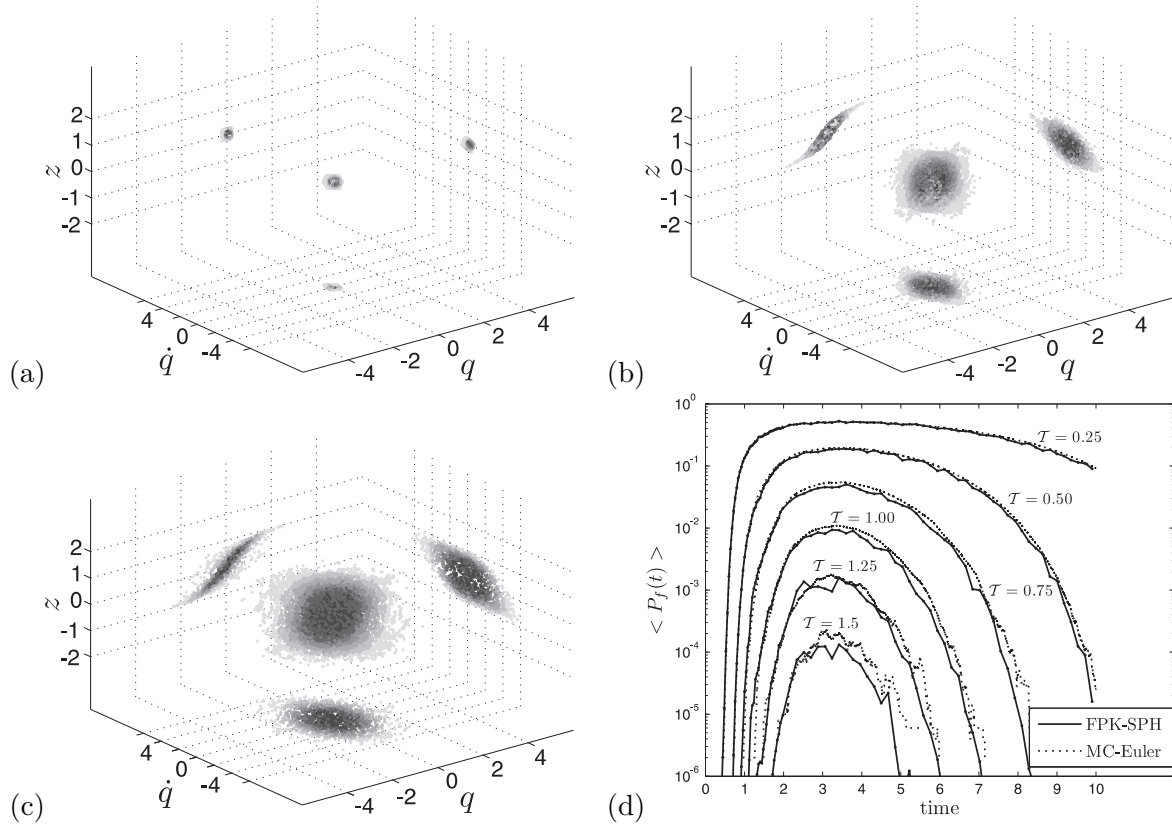


Figure 8.7: Probability density function ψ of the elastoplastic oscillator calculated with 9261 particles at time (a) $t = 0.2$ (b) $t = 1.2$ (c) $t = 3.6$ corresponding to the maximum variance. The darker the larger pdf, the lighter the lower pdf. Side and bottom views are projections on lateral faces; they illustrate the saturation effect on $z \in [-1; 1]$ although the solution space is $\mathbb{R}^3$. (d) Probability of exceedance P_f for system (8.3.1) estimated for different thresholds $\mathcal{T}$. Results obtained with SPH are compared with Monte Carlo (MC) simulations ($5 \cdot 10^5$ samples).

Figures 8.7-a, -b and -c show the time evolution and the dispersion of the three-dimensional pdf, from initial condition to their maximum dispersion ($t=3.6$ s). The more the probability is stretched compared with the initial distribution, the greater the number of particles is required. However, as illustrated in Figure 8.7-d, very good accuracy on the probability of exceedance for $\Omega_{\mathcal{T}}$ is achieved with few particles.

Besides this apparent robustness regarding the adaptation of the particle field from initial to transient states, the SPH algorithm proves to be efficient in containing the particles in the authorized subspace of the response domain. Indeed, the history variable z in this model is known to be limited to $z \in [-A; A]$. Although no specific boundary condition is imposed, i.e. particles are thus free to move in the state space according to the governing equations, the SPH algorithm contains particles in the subspace limited by planes $z = \pm A$, as shown by the S-shaped projection of the pdf in the (q, z) plane.

These examples illustrating well-known mechanical and civil engineering applications show that the resolution of the FPK equation with SPH may be seen as an interesting alternative to simulation methods.

8.4 Examples from other scientific disciplines

In this Section, some complementary examples are given in order to demonstrate the applicability of the method in other fields of science. We propose to deal with three examples. A nonlinear *pendulum randomly excited* by a white noise is first studied in three different configurations. The second example is the random *Lorenz oscillator*, chosen to show the adaptability of the method to effectively cope with initial three-dimensional pdf degenerating into a surface pdf. Finally, the *Lotka-Volterra prey-predator* system is used to highlight the crucial interpretation of the stochastic integral before solving the FPK equation.

8.4.1 Randomly excited pendulum

In this Section, a *pendulum randomly excited* is studied in three different configurations in order to illustrate problems with periodic and reflective boundaries. The equation of motion of a nonlinear random pendulum subject to *additive* excitation (by opposition to multiplicative) is given by

$$\ddot{\theta} + 2\xi\omega_0\dot{\theta} + \omega_0^2 \sin \theta = W(t) \quad (8.4.1)$$

where $\theta(t)$ is the angle between the pendulum and the gravity reference (vertical line $\theta = 0$) as illustrated in Figures 8.8-a and -b, ω_0 denotes here the natural pulsation of the oscillator and ξ is a damping ratio. Figure 8.8-a shows the pendulum excited at its support, free to move from $-\pi$ to π , while Figure 8.8-b shows the same pendulum but the rotation are limited to vary in $[-\theta_c, \theta_c]$ by two *walls*. The role of these walls is adapted in the following in order to illustrate the way to manage different boundary conditions. First, they are considered as absorbing boundaries, i.e. a trajectory reaching a wall is directly stopped. Secondly, inelastic contact is considered together with a restitution coefficient e .

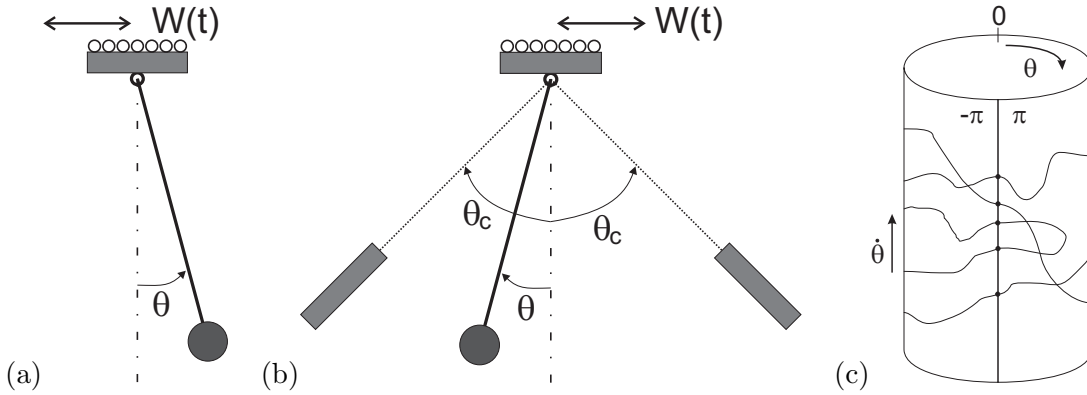


Figure 8.8: (a) Pendulum under additive random excitation. (b) Pendulum with walls limiting the rotations. (c) Concept of periodic boundary condition applied to the pendulum and continuity of the trajectories.

Periodic boundary condition

First, we focus on the unbounded periodic pendulum depicted in Figure 8.8-a. With regard to FPK equation, the difficulty to deal with this system starts when complete loops occur,

meaning that the pendulum oscillates out of the *homoclinic* orbit of nonlinear deterministic dynamics. In this case, the angle $\theta(t) : \mathbb{R}^+ \mapsto]-\pi, \pi]$ crosses the boundary between $-\pi$ and π (in one way or another) introducing a discontinuity in the mathematical representation of the response. This discontinuity is not physical and pertains only to an inappropriate representation of the rotation angle in $]-\pi, \pi]$. In fact, a periodic boundary condition must be imposed between $\theta = \pi$ and $\theta = -\pi$, to ensure that the pdf is equal for these two angles at any time. This periodicity condition may be easily explained by use of the geometrical considerations shown in Figure 8.8-c : a trajectory of this oscillator is described by a “point” moving on the surface of a cylinder. From this view, there is actually no discontinuity, because the particle does not “feel” the crossing of the boundary.

With respect to SPH method, the FPK equation obviously cannot be solved in an infinite domain like in the previous examples. Indeed, a probability associated with an angle θ is actually related to $\text{mod}(\theta, 2\pi)$. However, the Lagrangian formalism allows to avoid the discontinuity in π , while other artifact are employed in a Eulerian formalism (like a penalisation). Hence, a particle crossing “numerically” the boundary at a time t^* from $\theta(t^*) = -\pi + \varepsilon_1$ to $\theta(t^* + \Delta t) = -\pi - \varepsilon_2$ ($\varepsilon_1, \varepsilon_2 > 0$) is readily transposed from $\theta(t^*)$ to $\theta(t^* + \Delta t) = \pi - \varepsilon_2$, with the same velocity. The same explanation applies around π . The interaction between the particles must take the following fact into account: the distance between two particles is thus computed with the smaller arch of circle over the cylinder. Furthermore, a particle i at a position θ_i may have an influence domain crossing the periodic boundary, as illustrated in Figure 8.9. With this short explanation, we understand the advantage of a Lagrangian description of the FPK equation. The periodic boundary condition is readily introduced by connecting physically those boundaries and by maintaining the continuity of the particle trajectories.

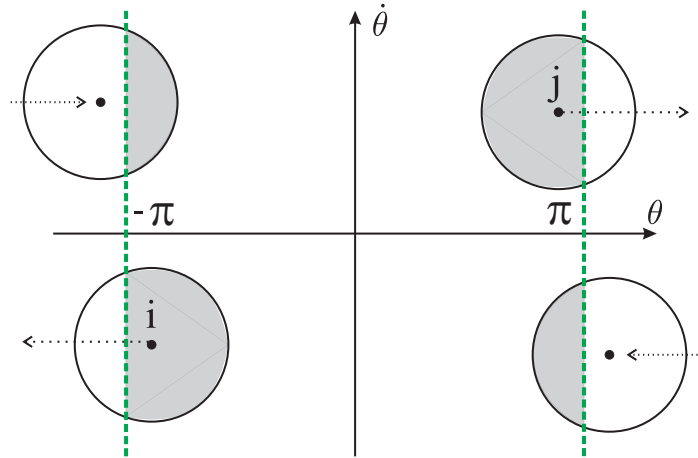


Figure 8.9: Sketch of the particle interaction in a periodic domain. The compact support is continuous through the boundary.

The main results are shown in Figure 8.10. The pdf of $(\theta, \dot{\theta})$, obtained with the SPH method ($N_p = 1876$ particles, $C_r = 0.8$, 20000 adaptive time steps) and Monte Carlo simulation (10^6 samples, $\Delta t = 10^{-3}$ s), at three different times are depicted. The natural pulsation ω_0 is equal to 1rad/s, while the damping ratio ξ is set equal to 0.1. The initial uncorrelated Gaussian distribution is centered on the vertical position and the standard deviations of angle

and angular velocity equal to 0.1 rad and 0.1 rad/s, respectively. The intensity of the white noise D_W is fixed to $5.0\text{m}^2\text{s}^{-2}$.

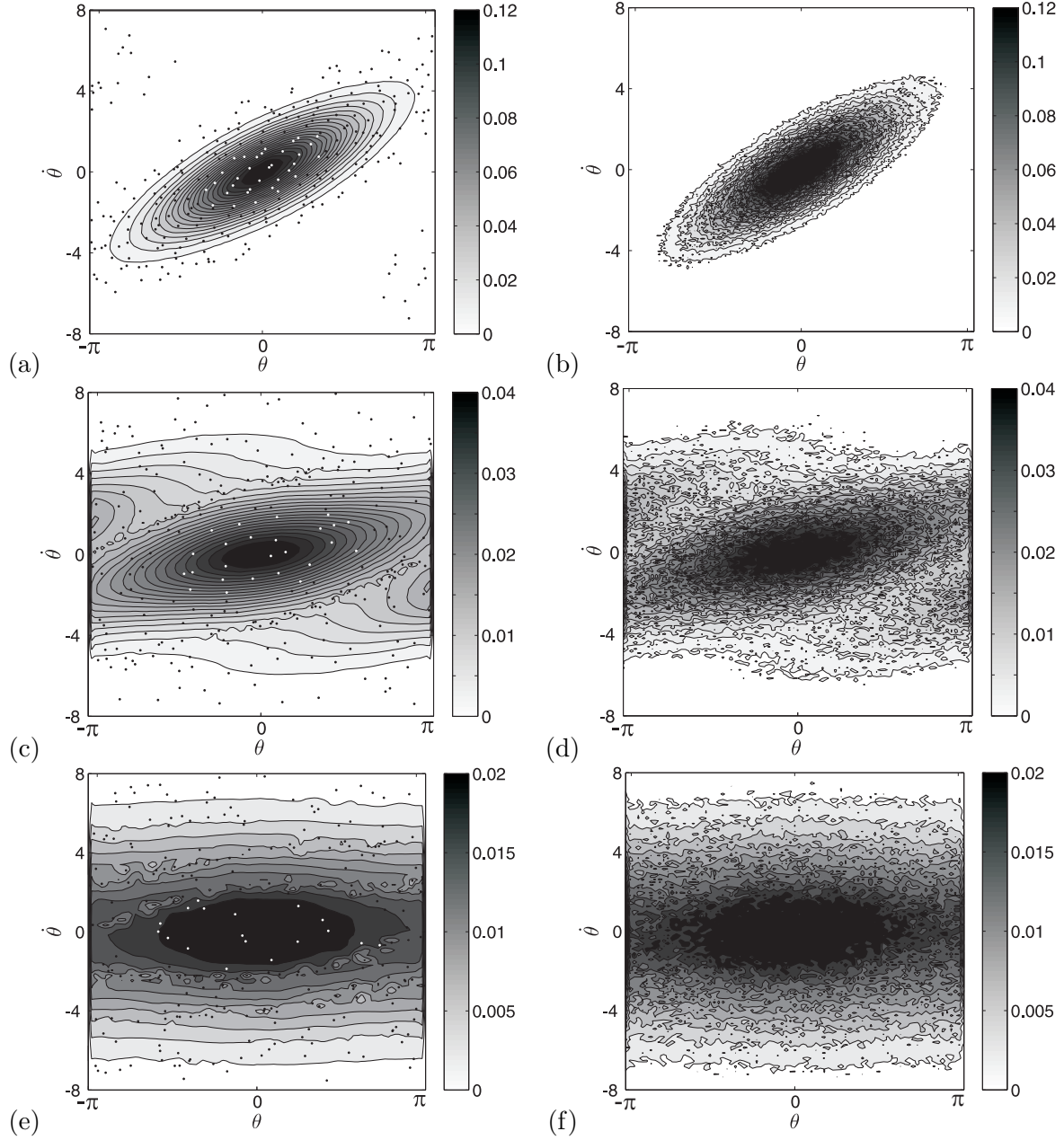


Figure 8.10: Pendulum with large amplitudes. SPH resolution (left column) with $N_p = 1876$. Monte Carlo simulation with 10^6 samples and $\Delta t = 10^{-3}\text{s}$ (right column). Probability density function $\psi(\theta, \dot{\theta})$ at the time $t = 1\text{s}$ (a,b), $t = 2\text{s}$ (c,d) and $t = 4\text{s}$ (e,f).

In Figure 8.10, the SPH results agree the trend obtained with Monte Carlo simulations, though 10^6 samples are not sufficient to converge to the exact pdf. Looking more attentively in Figures 8.10-c and -d, we notice the continuity of isoprobability lines through the limits of

the domain. Noticing the maximum of the initial pdf is about 2.0, the method is thus able to continuously cope with two orders of magnitude for the pdf.

Absorbing boundary condition

The pendulum is now modified according Figure 8.8-b and the rotation is now bounded, i.e. a trajectory reaching a wall is stopped. In this configuration, the problem is constrained by two walls for $|\theta| = \theta_c$ such that $\theta(t) : \mathbb{R}^+ \mapsto [-\theta_c, \theta_c]$. The walls play the role of absorbing boundaries. Hence, this example aims to illustrate our method in presence of such boundary conditions. The characteristics of the pendulum are similar to those of the previous application. However, the number of particles is increased to $N_p = 2436$ in order to reduce the average mass carried by each particle. For the comparison with step-by-step simulations, the number of samples is equal to 10^6 . The intensity of the white noise D_W is fixed to 0.05 and $0.1\text{m}^2\text{s}^{-2}$, in order to show two cases with different probability to hit a wall. The white noise is also modulated by the time window (8.2.2) with the parameters $n_1 = 2$, $n_1 = 1/2$ and $t_1 = 2$ and $t_2 = 8$. The modulation of the excitation is motivated by a probabilistic reason: in a stationary setting, after a time depending on the parameters of the system and of the excitation, the probability to have hit the wall is equal to one. In a SPH method, this means that there is no more particle in the domain.

Figure 8.11 emphasizes results allowing to validate our approach for this problem. Indeed, this figure shows the time evolution of the total probability for the pendulum to oscillate between the walls. In other words, this is the probability not to have impacted a wall at this time. In the SPH formalism, this quantity is calculated by readily summing the quantity of probability (μ_j) of all the remaining particles, similarly to (7.2.13). Indubitably, the agreement between Monte Carlo simulations and the SPH approach is patent for both white noise intensities, for small or more important loss of probability. We shall also notice in Figure 8.11-a that the results obtained by the FPK approach does not exactly start at one, which is due to the discretization of the initial pdf as explained in Section 7.3.1.

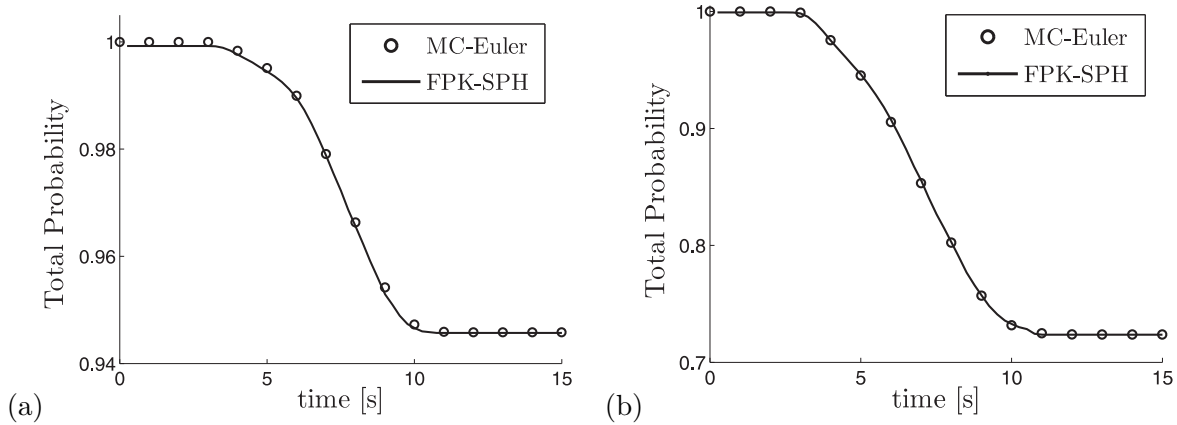


Figure 8.11: Time evolution of the total probability for the pendulum to oscillate between the walls for $D_W = 0.05\text{m}^2\text{s}^{-2}$ (a) and $D_W = 0.10\text{m}^2\text{s}^{-2}$ (b). Comparison between the proposed SPH method ($N_p = 2436$) and Monte Carlo simulations (10^6 samples).

The computation of this probability is based on the particle positions and the quantities

of probability and not on the density. The density field influences the diffusion velocity but not directly the calculated probability. In other words, the computation of the correct amount of probability still in the domain does not imply that this amount is correctly distributed or spread out in the domain. In Figure 8.12, we can see the pdf at two different times with the two approaches. We notice that there is no particle in the low density zone, the reconstructed pdf is thus equal to zero. In Figure 8.12-c, a sharp discontinuity can be identified, which is not the case with Monte Carlo simulations. We must point out this drawback of our method in its current implementation. Some ideas to circumvent this limitation would be to split particles in order to generate new trajectories.

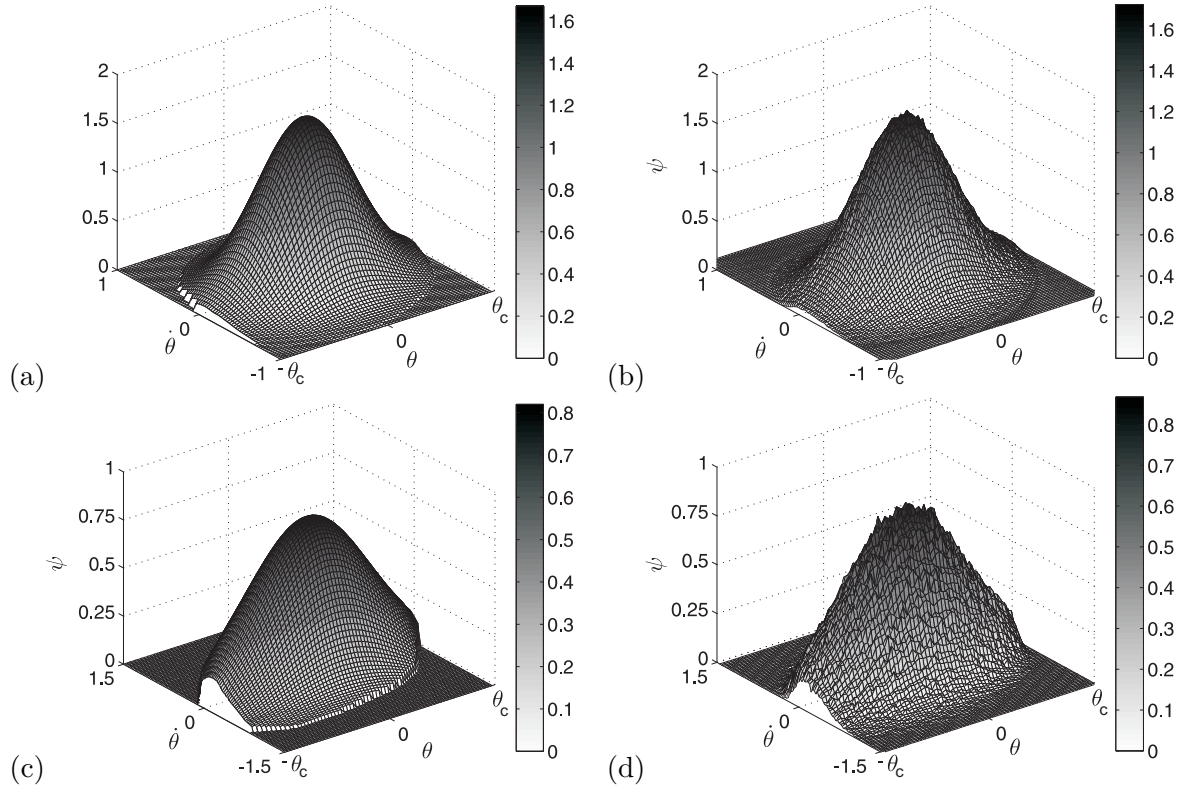


Figure 8.12: Pendulum with absorbing boundary with $\theta_c = \pi/4$ and $D_W = 0.05\text{m}^2\text{s}^{-2}$. SPH resolution (left column) with $N_p = 2436$. Monte Carlo simulation with 10^6 samples and $\Delta t = 10^{-3}\text{s}$ (right column). Probability density function $\psi(\theta, \dot{\theta})$ at the time $t = 4\text{s}$ (a-b) and $t = 8\text{s}$ (c-d).

Reflective pendulum

In this third configuration, the pendulum is still constrained as previously explained, but it is now reflected. When the pendulum impacts one of the walls at a time t^* , the angle is unchanged (the wall is not deformed), but the velocity becomes $\dot{\theta}(t^* + \varepsilon) = -e\dot{\theta}(t^*)$ ($\varepsilon > 0$) with e a *restitution coefficient*. This mechanical boundary condition expresses that a pendulum trajectory reaching the border is reflected from it, back into the simulation domain. In the SPH formalism, the reflection condition is implemented with the same philosophy than in the periodic domain. Indeed, a particle i impacting the wall θ_c is transported in the state space from the position $(\theta_c, \dot{\theta}(t^*))_i$ to $(\theta_c, -e\dot{\theta}(t^*))_i$, thanks to the Lagrangian formalism.

Other adaptations must be now formulated, because the problem is here discontinuous. The Fokker-Planck equation can be written as usual, even if the drift vector or the diffusion matrix are discontinuous (discontinuity of the first kind) [Stra 65]. In this case, another adaptation is required : the domain of interaction between particles is also discontinuous. A particle impacting a wall and transported in the state space still interacts with some of its previous neighbors, because the compact support also becomes discontinuous. The implementation of this discontinuity in the interaction is illustrated in Figure 8.13.

This is our interpretation of the conservation of the probability flow given in [Stra 65]. Nonetheless, some drawbacks have been encountered as shown in Figure 8.14. The comparison with Monte Carlo simulation highlights the lack of accuracy at the limits of the domain. This is surely a source of further improvements and developments. In our opinion, this case is really particular and the difficulties encountered here do not ruin the proposed method. Contrary, it is a source of future investigations.

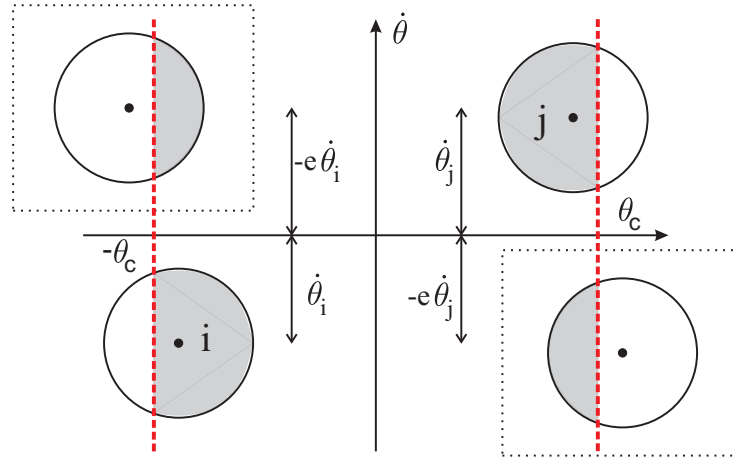


Figure 8.13: Sketch of the particle interaction in discontinuous domain. The compact support is discontinuous through the reflection line with the restitution coefficient e .

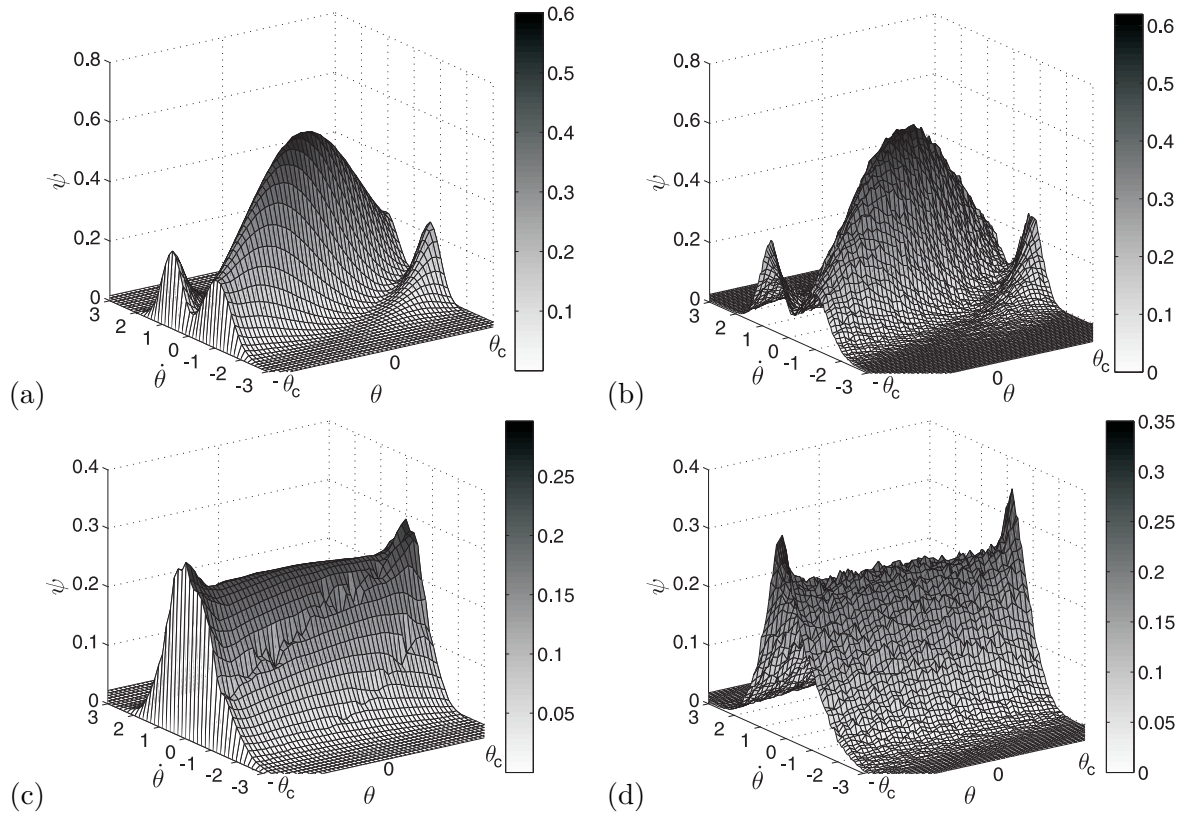


Figure 8.14: Reflective pendulum with $\theta_c = \pi/4$ and $D_W = 0.05\text{m}^2\text{s}^{-2}$. SPH resolution (left column) with $N_p = 2436$. Monte Carlo simulation with 10^6 samples and $\Delta t = 10^{-3}\text{s}$ (right column). Probability density function $\psi(\theta, \dot{\theta})$ at the time $t = 1\text{s}$ (a-b) and $t = 2\text{s}$ (c-d).

8.4.2 Lorenz attractor

In this second example, the Lorenz attractor [Khal 02], a three-dimensional nonlinear oscillator, is studied. Governing equations are

$$\begin{aligned}\dot{x} &= \sigma(y - x) + W_1 \\ \dot{y} &= \rho x - y - xz + W_2 \\ \dot{z} &= xy - \beta z + W_3\end{aligned}\tag{8.4.2}$$

with $\sigma = 10$, $\rho = 28$, $\beta = 8/3$ and W_1, W_2, W_3 three independent unit Gaussian δ -correlated noises with $E[W_i(t)W_j(t + \tau)] = \delta_{ij}\delta(\tau)$. This example allows to present the method on a more complex system. Figure 8.15 illustrates the positions of the 9261 particles in the three-dimensional state space at four time steps. The use of different colors is a didactic way to represent the density associated with each particle. This example shows how the particles initially regularly spread out can produce complex distributions. Starting from a Gaussian initial distribution, particles move in the three-dimensional space to finally produce the steady-state distribution, i.e. the butterfly wings of the Lorenz attractor.

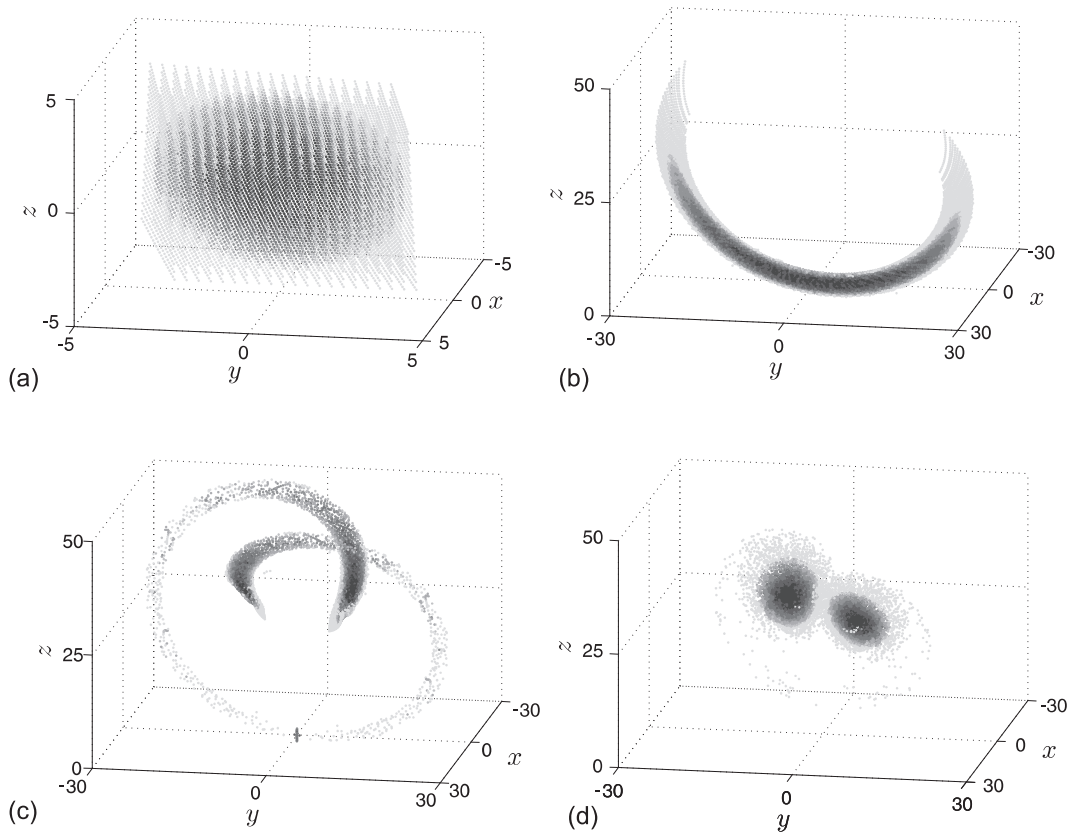


Figure 8.15: Probability density function ψ of Lorenz oscillator calculated with 9261 particles at four different time steps: (a) $t = 0.01$, (b) $t = 0.2$, (c) $t = 0.5$ and (d) $t = 3.0$. The darker the larger pdf, the lighter the lower pdf.

This example shows the adaptability of the method to follow a pdf with large distortions compared with the initial distribution and to manage the interaction between particles when their relative distance is reduced. The Eulerian kernel also shows its interest in the present application. The particles are initially distributed in a cube, then they are chaotically divided into two groups. This fact highlights the capability of the method to deal with very low density zones and with distribution becoming discontinuous. In the steady state, the particles are moving on a surface and the interaction pattern has been significantly changed compared to the regular initial state.

It is now clear that moving particles provide an interesting way to represent the zones of the state space with higher density using a limited number of degrees of freedom whereas mesh-based methods would require to mesh a substantially larger volume. About the computation time, the CPU time for one iteration with 9261 particles is about one second (single processor 2.3GHz Intel i5); about 30000 steps have been performed in total for this simulation.

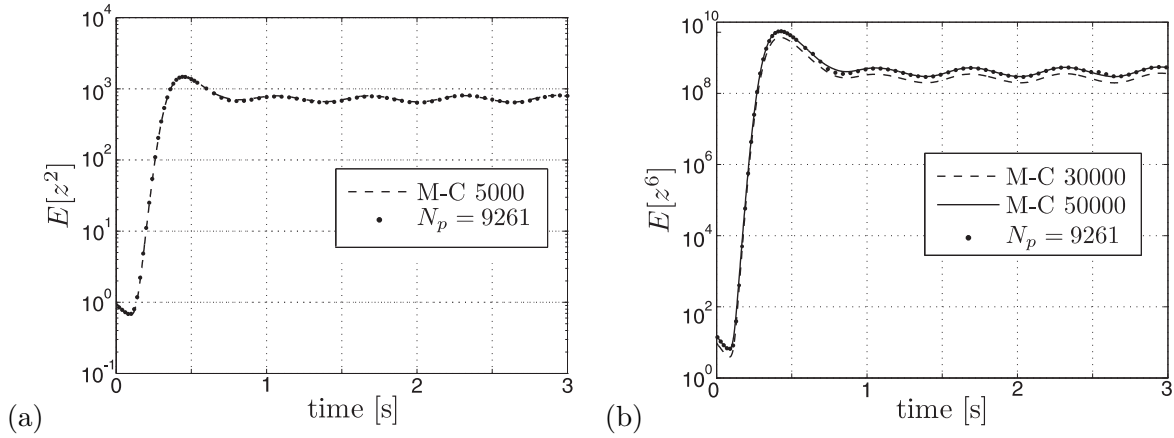


Figure 8.16: Time evolutions of the (a) second-order and (b) sixth-order moments of z with SPH method. Comparison with Monte Carlo simulation : (a) 5000 samples, (b) 30000 and 50000 samples.

Figure 8.16 shows the time evolution of the second-order and sixth-order moments of the state variable z calculated with (3.3.16). The comparison with Monte Carlo simulations highlights the validity of the rectangle approximation of moments for a reasonable number of particles. Of course, the calculation of the whole pdf provides a wider panel of information about the response of the random system, in particular about cross-moments.

The accurate representation of a high-order moment suggests an acceptable representation of the tails of the distribution with a limited number of particles. Also 30,000 samples are not sufficient for the Monte Carlo simulation to converge in the estimation of the sixth-order moment.

8.4.3 Stochastic Lotka-Volterra system

This last example comes back to the discussion initiated in Chapter 2 as it is concerned with the difference between the Stratonovitch and Itô formalisms to derive the FPK equation related to a dynamical random system. This is a reason for studying carefully the Lotka-Volterra system. Moreover, the definition of the diffusion velocity (7.2.3) requires here all the terms, because the diffusion matrix is not constant in this application.

This dynamical system is famous in biology and ecosystems, because it models the intrinsic interaction between preys and predators evolving in a shared environment. A self-competition term in the prey equilibrium is usually introduced to avoid an unbounded growth of those species. Then, a source of randomness within the *milieu* may be integrated in the model by considering stochastic variations in the birth rate of the prey and the death rate of the predators. All the scientific, historical and mathematical details about this model may be found in [Cai 04, Cai 07].

The governing equation of the model can be written as

$$\begin{cases} \dot{x}_1 = x_1(a - sx_1 - bx_2) + "x_1W_1" \\ \dot{x}_2 = x_2(-c + fx_1) + "x_2W_2" \end{cases} \quad (8.4.3)$$

with x_1 and x_2 the number of preys and predators in the environment, $W_1(t)$ and $W_2(t)$ two white noises, such that $E[W_i(t)W_j(t+\tau)] = D_i\delta_{ij}\delta(\tau)$ with $i = 1, 2$ and with a, b, c, f, s some parameters. According [Okse 03] and the comments in Section 2.1.2, Equation (8.4.3) is meaningless because a white noise (seen as continuous δ -correlated) is not differentiable anywhere and the formalism of ordinary differential equation does not apply anymore. The stochastic integral related to (8.4.3) should be written.

Due to the parametric terms " x_1W_1 " and " x_2W_2 " in (8.4.3), the equivalence between Itô's and Stratonovitch's forms is broken. Applying the Itô differential rule, or the Stratonovitch definition of the stochastic integral modified by the Wonk-Zakai correction term, the stochastic differential equation (sde) related to (8.4.3) reads

$$d\mathbf{X} = \begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{pmatrix} X_1(a - sX_1 - bX_2) + \frac{1}{2}D_1X_1 \\ X_2(-c + fX_1 + \frac{1}{2}D_2) + \frac{1}{2}D_2X_2 \end{pmatrix} dt + \begin{pmatrix} \sqrt{D_1}X_1 & 0 \\ 0 & \sqrt{D_2}X_2 \end{pmatrix} d\mathbf{B} \quad (8.4.4)$$

with $\mathbf{X}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}_0^+ \times \mathbb{R}_0^+$ and $\mathbf{B}(t)$ gathering the Brownian motions related to two δ -correlated processes with unit variance. The capital letters indicate symbolically here that $\mathbf{X}(t)$ is a random vector coming from a rigorous definition of sde. In the previous applications, this distinction was not required.

In Section 7.2, the Lagrangian formalism for the FPK equation is derived from an sde similar to (8.4.4). The velocity field can thus be written according to (7.2.3), taking into account that the diffusion matrix depends on $\mathbf{X}$ (so the term $\partial_{x_i}D_{ij} \neq 0$) and that the Wonk-Zakai correction terms are added in the drift vector. The velocity field reads

$$\mathbf{v}(X_1, X_2) = \begin{pmatrix} X_1(a - sX_1 - bX_2) + \frac{1}{2}D_1X_1 \\ X_2(-c + fX_1 + \frac{1}{2}D_2) + \frac{1}{2}D_2X_2 \end{pmatrix} - \begin{pmatrix} D_1X_1 & 0 \\ 0 & D_2X_2 \end{pmatrix} - \begin{pmatrix} D_1X_1^2 & 0 \\ 0 & D_2X_2^2 \end{pmatrix} \frac{\nabla\psi}{2\psi}. \quad (8.4.5)$$

Figure 8.17 shows the pdf obtained with the proposed SPH method and Monte Carlo simulations. The parameters chosen for the computation are $s = 0.1$, $a = b = 1$, $c = f = 0.5$ and $D_1 = D_2 = 0.1$. The initial condition of $\mathbf{X}$ is an uncorrelated Gaussian distribution with the mean position vector $[0.5, 0.5]$ and the standard deviations $[0.1, 0.1]$. For the SPH computation, the number N_p is equal to 1500 particles and the Courant number C_r is equal to 0.75. For the digital simulations, 10^6 samples are considered and a time step of 10^{-3} . In Figure 8.17, the agreement between the pdf obtained with the two methods is noteworthy. No essential boundary condition is imposed, because the trajectory must remain in the domain $\{X_1 > 0, X_2 > 0\}$ according to (8.4.3). Therefore, the random internal perturbation should not be too important at the risk of having negative amount of species. This example highlights one more time the capability of the method to cope with highly dispersed pdf and to cover a large zone of the domain of resolution, compared with the initial distribution and the initial particle location.

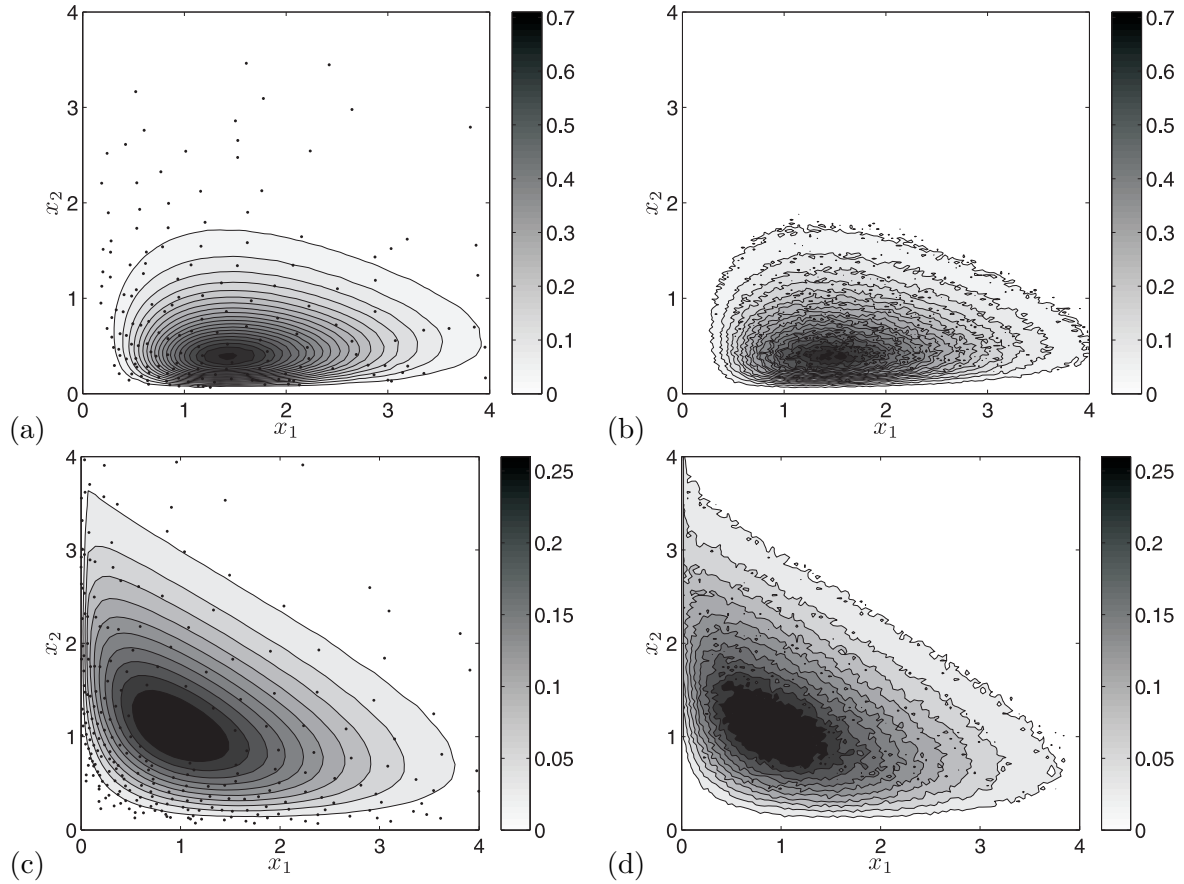


Figure 8.17: Lotka-Volterra system with the aforementioned parameters. SPH resolution (left column) with $N_p = 1500$. Monte Carlo simulation with $2 \cdot 10^6$ samples and $\Delta t = 10^{-3}$ (right column). Probability density function $\psi(x_1, x_2)$ at the time $t = 3$ (a-b) and $t = 5$ (c-d).

Figure 8.18 shows the time evolutions of the second and fourth statistical moments of both variables x_1 and x_2 . Comparing the moments is also a manner to compare numerically the results given by both methods. For the second-order moment, the agreement with the

simulations and the SPH approach with the approximation (7.2.8) is noteworthy. However, the accuracy is lesser on the fourth-order moment, in spite of a good approximation. This can be explained by the low concentration of particles in the bulk of the pdf. Furthermore, the volume ΔV_i attributed to a particle i is assumed to be the ratio between the mass (μ_i , invariant in time) and the calculated density ($\langle \psi_i \rangle$). This is obviously a rough approximation that could be refined. As rough as it is, this approximation gives good results, even for highly stretched distributions.

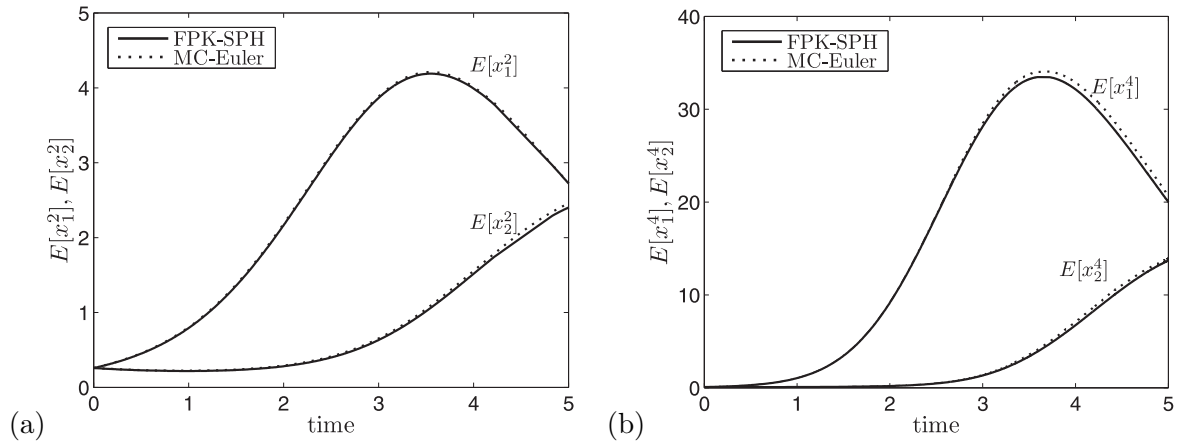


Figure 8.18: Lotka-Volterra system with the aforementioned parameters. Time evolutions of the (a) second- and (b) fourth-order moments.

Conclusion

In this part of the dissertation, we explore the solution of the FPK equation with the SPH method. Some results in two and three dimensions are shown to illustrate the relevance of the proposed developments and assumptions. To conclude, some advantages, limitations and possible improvements are summarized.

The Lagrangian formalism makes the method robust to deal with a large range of initial conditions even very different from the steady-state distribution. With the SPH method, particles move from the initial to the stationary distribution by themselves. Therefore, only an accurate representation of the initial condition (even slightly dispersed) must be worried about. Furthermore, the implemented SPH method ensures the positivity of the pdf. No equation must be solved in the implementation: the particles are propagated according the dynamics of the physical system and the FPK formalism.

From the point of view of reliability problems, the method is able to cope with very low probabilities of exceedance as illustrated in different applications. The estimation of the moments is also possible; illustrations have shown that high-order moments could be reasonably well estimated. Moreover, thanks to the Lagrangian formalism, far-field conditions, periodic conditions or absorbing boundaries are easily implemented. As the interaction among the particles is not governed by interaction forces, the implementation of ghost particles is simplified.

Nevertheless, the method has also some limitations. For instance, the stationary distribution cannot be directly computed (contrarily to finite element methods). For problems with variable total probability, the SPH method may have some drawbacks, since particles are not necessary located in low probability zones, even if the total probability is preserved or well estimated. This observation pertains essentially to the distribution of the total probability is between the high- and low-probability zones.

For a large amount of particles, the computation of the interaction can turn out to be time consuming and the recourse to advanced cell mapping or sorting algorithm could be mandatory. Finally, we see that some improvements are possible in order to deal with discontinuous systems and discontinuous interaction domain. In the future, examples in higher-dimensional state space should be studied, as well. In the field of reliability theory, the SPH implementation of the backward-Kolmogorov equation, central in first passage problems, is a challenge, since the Lagrangian form is not readily derived.

Although being mainly exploratory, this part throws light on a new possible use of the SPH method. As presented throughout this part, the method offers already the possibility to rapidly and easily extend the formalism to multidimensional spaces and to have an accurate representation of the transient regime of the FPK equation.

Finally, to our knowledge, application of the SPH technique to the FPK equation is the first opportunity to develop this method in a n -dimensional space, with no limitation on n other than the capacity of the processors.

Part III

Closure

Throughout this work, we have been concerned about the developments and the improvements of both analytical and numerical methods to perform nonstationary probabilistic analysis. We have successively explored two different perspectives, at two different scales, to compete against Monte Carlo simulation.

First, we have focused on the evolutionary spectral analysis, together with Gaussian equivalent linearization, of large dimensional structures subject to coherent random loads. Then, we have applied the smoothed particle hydrodynamics method, a numerical Lagrangian method, to solve the famous, but no less dreaded, Fokker-Planck equation. Since these two subjects are clearly separated in the field of probabilistic analysis, we have endeavored in this work to explore them independently of each other in order to improve them in accordance with their own limitations and own fields of applications.

The first part of the work is more related to usual structural dynamics problems encountered in civil engineering. We have aimed at developing a method that could be applied by practitioners and concerned engineers to efficiently perform nonstationary probabilistic analysis. At different steps of our developments, we have used relevant results obtained by means of perturbation methods, like asymptotic expansion or multiple scales approach. We have emphasized that these methods can be used first to justify some assumptions concerning structural dynamics, and then to build up a semi-analytical procedure that now makes evolutionary spectral analysis affordable for large civil engineering structures.

The second part of the work, more exploratory, is focused on small size systems. We have been concerned not to limit our investigations at the computation of the probability function, but also to apply our method to academic reliability problems. We have highlighted that the Lagrangian formalism presents many advantages to deal with vanishing far-field boundary conditions, to maintain the positivity of the probability density field and to ensure the conservation of the total probability. As shown in this work, the SPH method also offers a formalism readily applied to high-dimensional problems.

While having no claim of having exhaustively covered the topic of nonstationary random analysis, this work is an exploratory examination of some well-known concepts with the final objective to enhance their applicability in engineering and science.

Appendices - Bibliography

Appendix A

Addendum to background in probabilistic analysis

A.1 Approximate Methods - Closure methods

The equivalent linearization is parametric method, since a pdf shape must be defined before calculating the equivalent terms. The closure methods are non-parametric approximate methods allowing to estimate higher-order moments of a random process $\mathbf{X}(t)$. According to (3.3.2), recalled here

$$\frac{\partial}{\partial t} E[g(\mathbf{X})] = \sum_{i=1}^n E \left[a_i(t, \mathbf{X}) \frac{\partial g(\mathbf{X})}{\partial x_i} \right] + \sum_{i,j=1}^n E \left[D_{ij}(t, \mathbf{X}) \frac{\partial^2 g(\mathbf{X})}{\partial x_i \partial x_j} \right], \quad (\text{A.1.1})$$

the function $g(\mathbf{x})$ is a twice-differentiable deterministic function. Here, we consider it as a monomial of degree α , such that $g(\mathbf{x}) = \prod_{i=1}^n \mathbf{x}_i^{\lambda_i}$ with $\lambda_i \in \mathbb{N}$ for $i = 1, \dots, n$ and $\sum_{i=1}^n \lambda_i = \alpha$. The expectation operator allows thus to transform this partial differential equation into a linear ordinary differential equation in terms of the statistical moments of the process $\mathbf{X}$. An issue in this method is that the statistical moments of order α may depend on higher-order moments (than α), especially for nonlinear systems. In this case, the set of equations constitutes an *infinite hierarchy* of linear ordinary differential equations. A mathematical artifact is therefore required to solve these equations. Nevertheless, for linear systems, moments of order α only depend on lower order moments. The main discussion concerns thus the method to close this infinite hierarchy.

Moment equations and closure techniques are well-known methods in probabilistic analysis (see [Mano 95, DiPa 08]). The first studies on closure techniques are due to Bellman and Richardson [Bell 68]. They were also the first to state the limitations of closure methods. Indeed, the effective preservation of moments properties in closure techniques as positivity of even marginal moments or the verification of the Schwartz inequality in terms of even cumulants, is not ensured

A famous closure procedure was independently developed and formalized by Wu and Lin [Wu 84] and by Ibrahim *et al.* [Ibra 85]. This procedure, so-called **cumulant neglect-closure**, consists in assuming that cumulants beyond a given order α (so-called the **closure level**) are negligible. Thus, the α -closure gives approximations on cumulants of order less or equal than α . The relation between cumulants $\kappa[\mathbf{X}]$ and moments $E[\mathbf{X}]$ for a $\mathbb{R}^n$ -valued

random vector $\mathbf{X}(t)$ is obtained by expressing them in terms of the *characteristic function* of $\mathbf{X}$, such that $\varphi_{\mathbf{X}}(\mathbf{u}) : \mathbb{R}^n \mapsto \mathbb{C}$,

$$\kappa[X_1, X_2, \dots, X_n] = \frac{1}{\iota^n} \left[\frac{\partial^n}{\partial u_1 \partial u_2 \dots \partial u_n} \ln(\varphi_{\mathbf{X}}(\mathbf{u})) \right]_{\mathbf{u}=\mathbf{0}} \quad (\text{A.1.2})$$

and

$$E[X_1, X_2, \dots, X_n] = \frac{1}{\iota^n} \left[\frac{\partial^n}{\partial u_1 \partial u_2 \dots \partial u_n} \varphi_{\mathbf{X}}(\mathbf{u}) \right]_{\mathbf{u}=\mathbf{0}} \quad (\text{A.1.3})$$

with $\iota = \sqrt{-1}$. Furthermore, the characteristic function of $\mathbf{X}$ is defined as the Fourier transform of $\psi_{\mathbf{X}}$, the probability density function of $\mathbf{X}$, such that

$$\varphi_{\mathbf{X}}(\mathbf{u}) = \int_{\mathbb{R}^n} \psi_{\mathbf{X}}(\mathbf{x}) \exp(\iota \mathbf{u}^T \mathbf{x}) d\mathbf{x}. \quad (\text{A.1.4})$$

The characteristic function $\varphi_{\mathbf{X}}(\mathbf{u})$ is therefore expressed in terms of moments and cumulants, such that

$$\varphi_{\mathbf{X}}(\mathbf{u}) = \exp \left(\sum_{i_1=1}^n \iota u_{i_1} \kappa[X_{i_1}] + \frac{(\iota)^2}{2!} \sum_{i_1, i_2=1}^n u_{i_1} u_{i_2} \kappa[X_{i_1} X_{i_2}] + \dots \right) \quad (\text{A.1.5})$$

and

$$\varphi_{\mathbf{X}}(\mathbf{u}) = 1 + \sum_{i_1=1}^n \iota u_{i_1} E[X_{i_1}] + \frac{(\iota)^2}{2!} \sum_{i_1, i_2=1}^n u_{i_1} u_{i_2} E[X_{i_1} X_{i_2}] + \dots \quad (\text{A.1.6})$$

Although the cumulant neglect closure remains the most famous, its mathematical background is not really established. This method may origin in the intuitive extension of the Gaussian properties. Indeed, all the cumulants of order higher than two of a Gaussian variable are equal to zero. However, this method has a major drawback, since it transforms an infinite set of linear differential equations into a finite set of nonlinear equations. This mathematical artifact therefore introduces multiple solutions, sensitivity to initial conditions and possible instabilities. This could be illustrated on a simple Duffing oscillator subject to a white noise excitation.

Application

To illustrate the cumulant-neglect closure, a SDOF nonlinear oscillator with cubic stiffness is considered, such that

$$\ddot{x} + 2\xi\omega_0\dot{x} + \omega_0^2(\varepsilon_1 x + \varepsilon_2 x^3) = W(t), \quad (\text{A.1.7})$$

where $\mathbf{X}(t) = [x_1 \ x_2]^T$ the state space vector and $W(t)$ a δ -correlated process such that $E[w(t)w(t+\tau)] = 2D_w\delta(\tau)$. The moment equation yields

$$\begin{aligned} \frac{d}{dt} E[x_1^{\lambda_1} x_2^{\lambda_2}] &= E \left[\lambda_1 x_1^{\lambda_1-1} x_2^{\lambda_2+1} - \omega_0^2 \lambda_2 \left(\varepsilon_1 x_1^{\lambda_1+1} + \varepsilon_2 x_1^{\lambda_1+3} \right) x_2^{\lambda_2-1} \right. \\ &\quad \left. - 2\xi\omega_0 \lambda_2 x_1^{\lambda_1} x_2^{\lambda_2} + D_w \lambda_2 (\lambda_2 - 1) x_1^{\lambda_1} x_2^{\lambda_2-2} \right] \quad (\text{A.1.8}) \end{aligned}$$

Due to the term $x_1^{\lambda_1+3}$ in (A.1.8), the set of moment equations constitutes an infinite hierarchy. In this illustration, both the intensity of the white noise D_w (equal to 0.1) and the natural pulsation ω_0 are set equal to 1 and the damping ratio ξ is equal to 0.1. Figure A.1 shows the stationary second-order moment of x , for different closure levels. We notice that the methods are more accurate for $\varepsilon_1 > 0$. Actually, for $\varepsilon_1 < 0$, the exact pdf is the classical double-well pdf, as shown in Figure 6.1-b, which explains this huge difference.

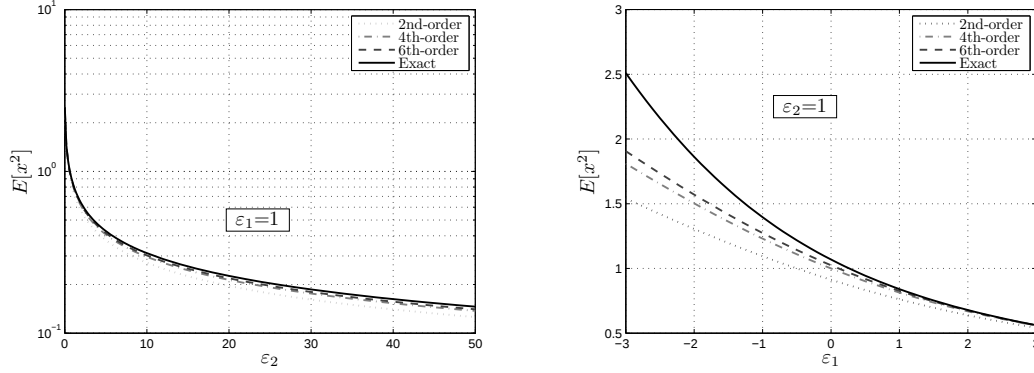


Figure A.1: Cumulant neglect-closure and Duffing oscillator.

The cumulant-neglect closure presents also some drawbacks. The computation of the cumulant is cumbersome for large-dimensional systems. Furthermore, the resolution of the nonlinear set of algebraic equations (in a stationary setting) is difficult due to the existence of several solutions [Wojt 96]. The algebraic solution may be avoided by solving the set of differential equations (A.1.8). Nevertheless, the convergence to a stable solution is not ensured, as shown in Figure A.2. Slightly modifying the initial conditions may induce instability for higher-order closure levels (e.g. 6th-order).

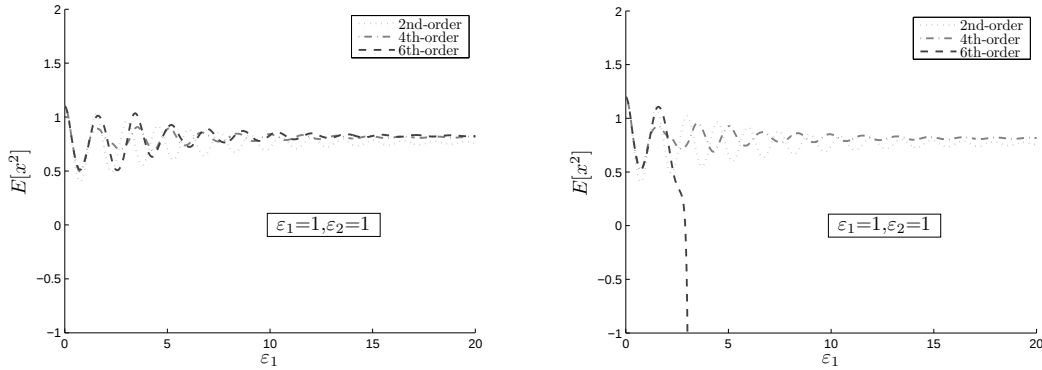


Figure A.2: Stability vs instability in cumulant neglect-closure for three different closure levels and two different uncorrelated and centered Gaussian initial conditions (a) $(\sigma_x, \sigma_{\dot{x}}) = (1.1, 1.1)$ and (b) $(\sigma_x, \sigma_{\dot{x}}) = (1.2, 1.2)$.

Appendix B

Addendum to Part I

B.1 Extension of the background-resonant approximation

First, the integral $\mathcal{I}_{iii,0}$ is considered. The background component is readily introduced by adding and subtracting simultaneously a term $|(\mathbf{H}_d(0))_i|^3$, such that

$$\begin{aligned} \int_{\mathbb{R}} (\mathbf{H}_d)_i (\mathbf{H}_d)_i (\mathbf{S}_{\mathbf{p}})_{ii} (\mathbf{H}_d^*)_i d\omega &= \frac{1}{\omega_i^6} \int_{\mathbb{R}} (\mathbf{S}_{\mathbf{p}})_{ii} d\omega + \int_{\mathbb{R}} \left((\mathbf{H}_d)_i (\mathbf{H}_d)_i (\mathbf{H}_d^*)_i - \frac{1}{\omega_i^6} \right) (\mathbf{S}_{\mathbf{p}})_{ii} d\omega \\ &= \frac{1}{\omega_i^6} (\mathbf{S}_{\mathbf{p}})_{ii} + \int_{\mathbb{R}} \phi(\omega) (\mathbf{S}_{\mathbf{p}})_{ii} d\omega, \end{aligned} \quad (\text{B.1.1})$$

where $\mathbf{S}_{\mathbf{p}}$ is the covariance matrix of the generalized forces. The first term is the background contribution, while the second integral is the resonant one. Two identical and symmetric contributions to the resonant component are concentrated around the natural frequency $\pm\omega_i$, so just one of those is computed for $\omega > 0$. A coordinate η is used to stretch the frequency content around ω_i , such as $\omega = \omega_i(1 + \xi\eta)$. The integral (B.1.1) now reads

$$\int_{\mathbb{R}} \phi(\omega) (\mathbf{S}_{\mathbf{p}}(\omega))_{ii} d\omega \approx 2\xi_i\omega_i \int_{\mathbb{R}^+} \phi(\eta) (\mathbf{S}_{\mathbf{p}}(\omega))_{ii} d\eta. \quad (\text{B.1.2})$$

However, the function $\phi(\eta)$ remains non-integrable on $+\infty$. Since this integral is real-valued, so the real part of $\phi(\eta)$ is replaced by a local Padé approximation $\tilde{\phi}(\eta)$. Everywhere else, the Padé approximation $\phi(\eta)$ is continuous and tends to zero for η tending to 0, $+\infty$ and $-\infty$.

The resonant contribution is now approximated by

$$\int_{\mathbb{R}} \phi(\omega) (\mathbf{S}_{\mathbf{p}}(\omega))_{ii} d\omega \approx 2 \int_{\mathbb{R}} \tilde{\phi}(\eta) (\mathbf{S}_{\mathbf{p}}(\omega))_{ii} \xi_i \omega_i d\eta. \quad (\text{B.1.3})$$

Locally, the spectrum $\mathbf{S}_{\mathbf{p},ii}$ is approximated by its first-order Taylor expansion and the integral (B.1.3) becomes

$$\int_{\mathbb{R}} \phi(\omega) (\mathbf{S}_{\mathbf{p}}(\omega))_{ii} d\omega \approx 2 \int_{\mathbb{R}} \tilde{\phi}(\eta) \left((\mathbf{S}_{\mathbf{p}}(\omega_i))_{ii} + \eta \xi_i \omega_i (\mathbf{S}'_{\mathbf{p}}(\omega_i))_{ii} \right) \xi_i \omega_i d\eta. \quad (\text{B.1.4})$$

The remaining integrals can be performed with the residue theorem. The integral $\mathcal{I}_{iii,\alpha}$ with $\alpha > 0$ has no background contribution. The same methodology is also applied in order to obtain simplified expression. Some pertinent results are

$$\mathcal{I}_{iii,1} = \frac{\pi}{8\xi_i^2\omega_i^4} (\mathbf{S}_p(\omega_i))_{ii}, \quad \mathcal{I}_{iii,2} = \frac{\pi}{8\xi_i\omega_i^2} (\mathbf{S}'_p(\omega_i))_{ii}, \quad \mathcal{I}_{iii,3} = \frac{-\pi}{8\xi_i^2\omega_i^2} (\mathbf{S}_p(\omega_i))_{ii}. \quad (\text{B.1.5})$$

B.2 Linear evolutionary spectral analysis - Usefull constants

Constants related to the second correction term of the asymptotic expansion of the impulse response matrix of a linear structure presented in Section 4.3. The constants for $i \neq j \neq k$ are given by

$$\begin{aligned} (\mathbf{V}'_{1,1})_{ijk} &= \frac{-(\mathbf{V}_{1,1})_{ij}}{(\Omega_i^+ - \Omega_k^-)(\Omega_i^+ - \Omega_k^+)}, & (\mathbf{V}'_{2,1})_{ijk} &= -(\mathbf{V}'_{1,1})_{ijk} \Omega_i^+, & (\mathbf{V}'_{3,1})_{ijk} &= -(\mathbf{V}'_{2,1})_{ijk} \Omega_i^+ \\ (\mathbf{V}'_{1,2})_{ijk} &= \frac{-(\mathbf{V}_{1,2})_{ij}}{(\Omega_i^- - \Omega_k^-)(\Omega_i^- - \Omega_k^+)}, & (\mathbf{V}'_{2,2})_{ijk} &= -(\mathbf{V}'_{1,2})_{ijk} \Omega_i^-, & (\mathbf{V}'_{3,2})_{ijk} &= -(\mathbf{V}'_{2,2})_{ijk} \Omega_i^- \\ (\mathbf{V}'_{1,3})_{ijk} &= \frac{-(\mathbf{V}_{1,1})_{ji}}{(\Omega_j^+ - \Omega_k^-)(\Omega_j^+ - \Omega_k^+)}, & (\mathbf{V}'_{2,3})_{ijk} &= -(\mathbf{V}'_{1,3})_{ijk} \Omega_j^+, & (\mathbf{V}'_{3,3})_{ijk} &= -(\mathbf{V}'_{2,3})_{ijk} \Omega_j^+ \\ (\mathbf{V}'_{1,4})_{ijk} &= \frac{-(\mathbf{V}_{1,2})_{ji}}{(\Omega_j^- - \Omega_k^-)(\Omega_j^- - \Omega_k^+)}, & (\mathbf{V}'_{2,4})_{ijk} &= -(\mathbf{V}'_{1,4})_{ijk} \Omega_j^-, & (\mathbf{V}'_{3,4})_{ijk} &= -(\mathbf{V}'_{2,4})_{ijk} \Omega_j^- \\ (\mathbf{V}'_{1,5})_{ijk} &= \frac{-(\mathbf{V}_{1,1})_{ki}}{(\Omega_k^+ - \Omega_j^+)(\Omega_k^+ - \Omega_j^-)}, & (\mathbf{V}'_{2,5})_{ijk} &= -(\mathbf{V}'_{1,5})_{ijk} \Omega_k^+, & (\mathbf{V}'_{3,5})_{ijk} &= -(\mathbf{V}'_{2,5})_{ijk} \Omega_k^+ \\ (\mathbf{V}'_{1,6})_{ijk} &= \frac{-(\mathbf{V}_{1,2})_{ki}}{(\Omega_k^- - \Omega_j^+)(\Omega_k^- - \Omega_j^-)}, & (\mathbf{V}'_{2,6})_{ijk} &= -(\mathbf{V}'_{1,6})_{ijk} \Omega_k^-, & (\mathbf{V}'_{3,6})_{ijk} &= -(\mathbf{V}'_{2,6})_{ijk} \Omega_k^- \end{aligned}$$

B.3 Illustration of the convergence of $\mathbf{h}_N$

In Section 4.3.2, we highlight that the convergence of the series $\mathbf{h}_N(t)$ is time dependent. The criterion $\rho_{\mathbf{J}} < 1$ ensures that the series $\mathbf{H}_N(\omega)$ converges. Therefore, the series $\mathbf{h}_N(t)$ is also convergent. We also notice that for short time intervals $[0, t]$, the number of correction terms N required for a certain accuracy is thus smaller, than for the stationary solution. We provide now a simple example to illustrate this property.

We study the system described in Section 4.7 with the same characteristics. The parameter ζ is taken equal to 0.05 and 0.10 in order to create a moderate and strong modal coupling, with $\rho_{\mathbf{J}}$ equal to 0.60 and 0.80, respectively.

Figure B.1 shows the convergence of the elements of $\mathbf{H}_N(\omega)$ for different values of N . For $\rho_{\mathbf{J}}$ equal to 0.6 ($\zeta = 0.05$), two correction terms are sufficient to correctly represent the target $\mathbf{H}_\infty(\omega)$. In the case of $\zeta = 0.10$, more terms are required. These remarks are relevant to the diagonal $(\mathbf{H}_N)_{11}$ and the out-of-diagonal term $(\mathbf{H}_N)_{12}$ as well. In Figures B.1-c and -d, just one term is required, since $N = 0$ gives the decoupled approximation and $N = 2$ is identical to $N = 1$.

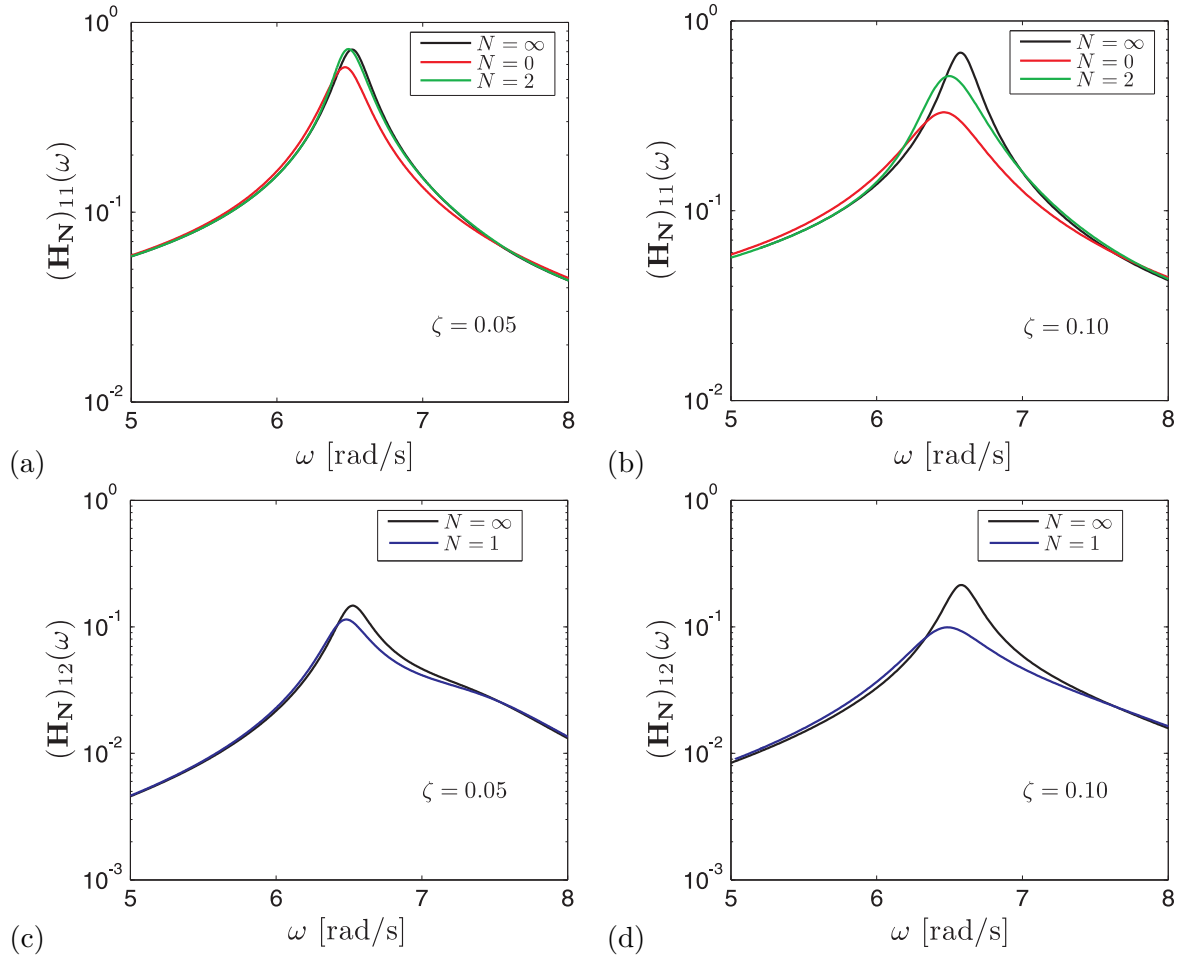
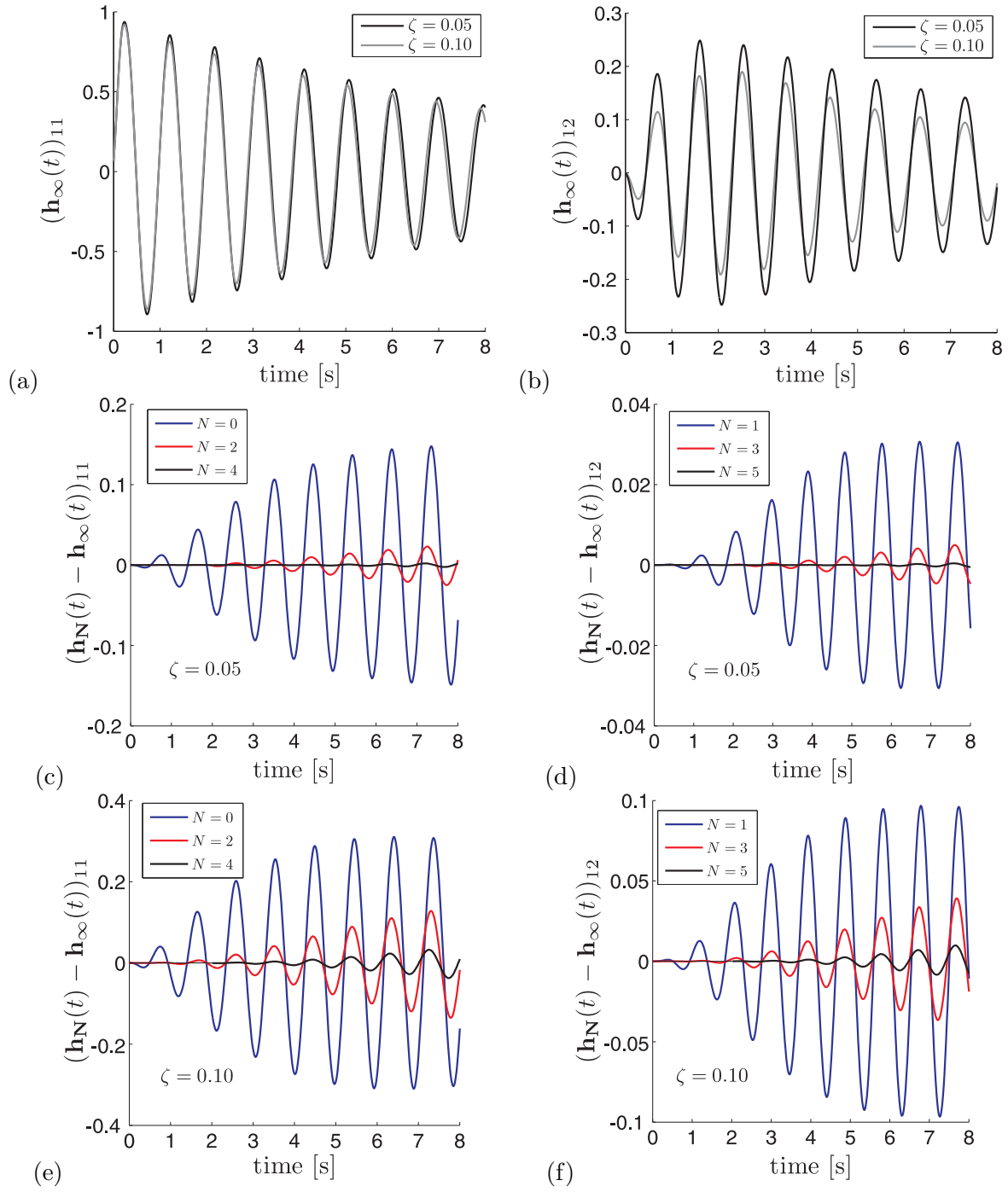


Figure B.1: Convergence of $\mathbf{H}_N(t)$ for different values of N compared with $\mathbf{H}_\infty(\omega)$

We now investigate how these approximations of $\mathbf{H}_N(\omega)$ influence the impulse response matrix $\mathbf{h}_N(t)$. The elements of the matrix $\mathbf{h}_\infty(t)$, supposed to be our reference, are depicted in Figures B.2-a and -b. Then, the differences between these references and the different approximations are shown in Figures B.2-c to -d, for the two same values of ζ .

Our purpose is here perfectly highlighted. We clearly see that the differences between the approximations and the references are smaller at the beginning than at the end. Besides, the differences increase as time goes by. Therefore, we emphasize the rate of convergence is not mainly influenced by N on a short time interval.

Figure B.2: Convergence of $\mathbf{h}_N(t)$ for different values of N compared with $\mathbf{h}_\infty(\omega)$

Appendix C

Addendum to Part II

C.1 Finite-Element method for FPK equation

In this Section, we highlight the basic notions required to solve FPK equation with FEM (Finite Element Method). This summary is widely inspired from [Spen 93].

Weak Form of the FPK equation

First, we recall the usual Eulerian form of the FPK equation,

$$\frac{\partial \psi}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial x_i} (a_i(\mathbf{x})\psi) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial}{\partial x_i \partial x_j} (D_{ij}(\mathbf{x})\psi), \quad (\text{C.1.1})$$

We multiply this equation by a virtual pdf $\delta\psi$ having the same properties of ψ , i.e.

$$\int_{\Omega} \delta\psi d\mathbf{x} = 1, \quad \text{and} \quad \delta\psi > 0 \quad (\text{C.1.2})$$

and

$$\delta\psi \rightarrow 0 \text{ as } |\mathbf{x}| \rightarrow +\infty. \quad (\text{C.1.3})$$

Then, left- and right- terms of (C.1.1) are integrated over the entire state space $\Omega = \mathbb{R}^n$, such that

$$\int_{\Omega} \frac{\partial \psi}{\partial t} \delta\psi d\omega + \int_{\Omega} \sum_{i=1}^N \frac{\partial}{\partial x_i} (a_i(\mathbf{x})\psi) \delta\psi d\omega = \int_{\Omega} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} (D_i\psi) \delta\psi d\omega. \quad (\text{C.1.4})$$

After an integration by parts, we have

$$\int_{\Omega} \frac{\partial \psi}{\partial t} \delta\psi d\omega + \left[\sum_{i=1}^N a_i(\mathbf{x})\psi \delta\psi \right]_{\delta\Omega} - \int_{\Omega} \sum_{i=1}^N a_i(\mathbf{x})\psi \frac{\partial \delta\psi}{\partial x_i} d\omega = \left[\sum_{i=1}^N D_i \frac{\partial \psi}{\partial x_i} \delta\psi \right]_{\delta\Omega} - \int_{\Omega} \sum_{i=1}^N D_i \frac{\partial \psi}{\partial x_i} \frac{\partial \delta\psi}{\partial x_i} d\omega. \quad (\text{C.1.5})$$

The boundary $\delta\Omega$ has no physical meaning in this context, as we assume an infinite state space. Therefore, $\delta\Omega$ might represent the infinity in each direction of the state space, i.e. $|\mathbf{x}| \rightarrow +\infty$. The aforementioned conditions on ψ and $\delta\psi$ allow to cancel some terms in (C.1.5), such that,

$$\int_{\Omega} \frac{\partial \psi}{\partial t} \delta \psi d\omega + \int_{\Omega} \sum_{i=1}^N D_i \frac{\partial \psi}{\partial x_i} \frac{\partial \delta \psi}{\partial x_i} d\omega - \int_{\Omega} \sum_{i=1}^N a_i(\mathbf{x}) \psi \frac{\partial \delta \psi}{\partial x_i} d\omega = 0. \quad (\text{C.1.6})$$

Discretization of FPK equation

The residue equation (C.1.6) is discretized over each finite element using the shape function $N_r(\mathbf{x})$, such as

$$\psi = \sum_{r=1}^{N_e} \psi_r^e(t) N_r(\mathbf{x}) \quad \text{and} \quad \delta \psi = \sum_{s=1}^{N_e} \delta \psi_s^e(t) N_s(\mathbf{x}).$$

Therefore, the weak form of FPK equation becomes

$$\sum_{r,s=1}^{N_e} \left\{ R_{rs} \frac{\partial \psi_r^e}{\partial t} \delta \psi_s^e + G_{rs} \psi_r^e \delta \psi_s^e \right\} = 0$$

with the two elementary matrices calculated as

$$R_{rs} = \left(\int_{\Omega_{el}} N_r(\mathbf{x})(t) N_s(\mathbf{x}) d\omega_{\mathbf{x}} \right)$$

$$G_{rs} = \int_{\Omega_{el}} \left(\sum_{i=1}^N D_i \frac{\partial N_r(\mathbf{x})}{\partial x_i} \frac{\partial N_s(\mathbf{x})}{\partial x_i} - \sum_{i=1}^N a_i(\mathbf{x}) N_r(\mathbf{x}) \frac{\partial N_s(\mathbf{x})}{\partial x_i} \right) d\omega_{\mathbf{x}}.$$

For a Q-4 finite element, the shape functions N_r are

$$N_r = \frac{1}{4L^2} (L + (\boldsymbol{\xi}_1)_r (x - a)) (L + (\boldsymbol{\xi}_2)_r (y - b)),$$

where a and b are the center of a considered finite element, L the length of the square (we suppose square elements) and the vectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are

$$\boldsymbol{\xi}_1 = [-1 \quad -1 \quad +1 \quad +1]$$

$$\boldsymbol{\xi}_2 = [-1 \quad +1 \quad +1 \quad -1].$$

The global system can be rewritten in a matrix formalism, as

$$\mathbf{R} \dot{\boldsymbol{\Psi}} + \mathbf{G} \boldsymbol{\Psi} = \mathbf{0}$$

with $\boldsymbol{\Psi}$ the vector gathering the unknown values of the pdf at the different nodes of the mesh.

Appendix D

Curriculum vitae

19/09/2014

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Born on 18/04/1987
 Belgian, Cohabiting
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Education

- 2014 **Ph.D in Applied Sciences and Civil Engineering - ULg**
 Thesis entitled "*New perspectives on probabilistic methods for nonlinear transient dynamics in civil engineering*"; Defence 06/05
- 2010 **Master in Civil Engineering - ULg - *summa cum laude***
 Thesis entitled "*Amortissement non-linéaire des structures par des câbles*"
- 2008 **Bachelor in Applied Sciences - ULg - *summa cum laude***
 Major option in civil engineering, Minor option in mechanical engineering

Work Experience

- 2010 **Research Fellow** - Grant from **F.R.S.-FNRS**
- 2014 **University of Liege** - Structural and Stochastic Dynamics
 Advised by Pr. Vincent Denoël
- Activities** : probabilistic structural dynamics, earthquake engineering, wind engineering, expertise in footbridge and building vibrations
- 2013 **Visiting researcher** at the **Northeastern University** in Boston (MA, USA)
 (February-April & November-December) advised by Pr. Luca Caracoglia
- 2009 **Internship - Besix Group** - Brussels - Design Department.
Project : Design of a jetty in Ruwais (Abu Dhabi)

Training

- 2013 *Financial Mathematics* by P. Devolder at UCL (Belgium)
- 2013 *Internship in a wind tunnel* at CSTB (Nantes, France) with O. Flamand
- 2012 *Verification and Validation of Struct. Dynamics* by F. Hemez at KUL (Belgium)
- 2011 *Non-deterministic Mechanics*, class organized by CISM (Udine-Italy)
- 2010 *Exploiting Nonlinear Dynamics*, class organized by CISM (Udine-Italy)

Prizes and Honours

- 2010 Laureate of the AILg-Greisch Award for the best Master thesis
 Nominate for the Infosteel-Student Prize for the best Master thesis
- 2005 Entrance examination in Applied Sciences ULg - *Second place*
 Laureate of the Fernand Pisart Award - Study Grant - ULg
 First laureate of the Belgian Chemistry Olympiads
 Member of the National Team of Chemistry
 Participation in *37th International Chemistry Olympiad* (Taiwan)

Scientific Contributions

Author or co-author of 5 journal papers and of 12 full-length conference papers
 Publications available on <http://orbi.ulg.ac.be/simple-search?query=canor+thomas>

Articles in International Journals (with peer review process)

- Canor T., Caracoglia L. and Denoël V., *Perturbation methods in evolutionary spectral analysis for linear dynamics and equivalent statistical linearization*, 2015, Nonl. Dyn.,
- Canor T., Blaise N. and Denoël V., *An asymptotic expansion-based method for a spectral approach in equivalent statistical linearization*, 2014, Prob. Eng. Mech., 38(2014):1-12
- Canor T. and Denoël V., *Transient FPK equation solved with Smoothed Particle Hydrodynamics method*, 2013, Int. J. Num. Method. Eng., 94(6): 535-553
- Denoël V. and Canor T., *Patching Asymptotics Solution of a Cable with a small bending stiffness*, 2013, J. Struct. Eng.-ASCE, 139: 180-187
- Canor T., Blaise N. and Denoël V., *Efficient uncoupled stochastic analysis with non-proportional damping*, 2012, J. Sound Vib., 331(2012): 5283-5291

International Conferences

- Canor T. and Denoël V., SEMC 2013, 2-4 September, Cape Town, South-Africa
Vibrations of cables with bending stiffness by an asymptotic approach
- Canor T., Blaise N. and Denoël V., EACWE 2013, 2-4 July, Cambridge, Great-Britain
Uncoupled Spectral Analysis with non-proportional damping
- Canor T. and Denoël V., COMPDYN 2013, 12-14 June, Kos Island, Greece
Transient FPK equation with SPH method - Seismic Applications
- Canor T. and Denoël V., ACOMEN 2011, 14-17 November, Liege, Belgium
Transient FPK Equation with Smoothed Particle Hydrodynamics
- Canor T. and Denoël V., ISCD 2011, 18-20 October, Shanghai, China
On the influence of background component on cable resonance
- Canor T., Kerschen G. and Denoël V., Footbridge 2011, 6-8 July, Wroclaw, Poland
Vandalism prevention of a footbridge with cable vibrations

Internal Reports

- Structural Analysis under Differential Seismic Excitation*, Canor T., Denoël V.
- Modal Identification of Santo Agostino Church in l'Aquila*, Canor T., Denoël V.
- Mesures des vibrations de la passerelle de Dohlain-Limbours*, Canor T., Denoël V.
- Mécanique des Matériaux II - Notes de cours Pr. Cescotto*, Belligoï T., Canor T.

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