Muti-scale methods with strain-softening: damage-enhanced MFH for composite materials and computational homogenization for cellular materials with micro-buckling

L. Noels, G. Becker, V.-D. Nguyen, L. Wu, L. Adam (x-Stream), I. Doghri (UCL)

Non-local damage mean-field-homogenization

FE2 homogenization of cellular structures
Multi-scale modelling: Why?

- **Materials in aeronautics**
  - More and more engineered
  - Multi-scale in nature

A350 wing lower cover
Multi-scale modelling: Why?

- Limitations of one-scale models
  - Physics at the micro-scale is too complex to be modelled by a simple material law at the macro-scale
    - Engineered materials
    - Multi-physics/scale problems
    - ...
    - See next slides
  - Lack of information of the micro-scale state during macro-scale deformations
    - Required to predict failure
    - ...
  - Effect of the micro-structure on the macro-structure response
    - Fibres distribution ...
    - ...
- Solution: multi-scale models
Content

• **Introduction**
  – Multi-scale modelling: How?
  – Strain softening issues

• **Non-local damage-enhanced mean-field-homogenization**

• **Computational homogenization for cellular materials**

• **Other researches**
  – DG-based fracture mechanics: blast, fragmentation, ...

• **Conclusions**
Multi-scale modelling: How?

- **Principle**
  - 2 problems are solved concurrently
    - The macro-scale problem
    - The micro-scale problem (Representative Volume Element)
  - Scale transitions coupling the two scales
    - Downscaling: transfer of macro-scale quantities (e.g. strain) to the micro-scale to determine the equilibrium state of the Boundary Value Problem
    - Upscaling: constitutive law (e.g. stress) for the macro-scale problem is determined from the micro-scale problem resolution

**Assumptions:**
\[ L_{\text{macro}} \gg L_{\text{RVE}} \gg L_{\text{micro}} \]
Multi-scale modelling: How?

- Computational technique: FE\(^2\)
  - Macro-scale
    - FE model
    - At one integration point $\bar{\varepsilon}$ is known, $\bar{\sigma}$ is sought
Multi-scale modelling: How?

- **Computational technique: FE$^2$**
  - Macro-scale
    - FE model
    - At one integration point $\bar{\varepsilon}$ is known, $\bar{\sigma}$ is sought

- Micro-scale
  - Usual 3D finite elements
  - Periodic boundary conditions
Multi-scale modelling: How?

• **Computational technique: FE**
  
  – **Macro-scale**
    
    • FE model
    
    • At one integration point $\bar{\varepsilon}$ is known, $\bar{\sigma}$ is sought

  – **Transition**
    
    • Downscaling: $\bar{\varepsilon}$ is used to define the BCs
    
    • Upscaling: $\bar{\sigma}$ is known from the reaction forces

  – **Micro-scale**
    
    • Usual 3D finite elements
    
    • Periodic boundary conditions
Multi-scale modelling: How?

• **Computational technique: FE²**
  
  – **Macro-scale**
    
    • FE model
    
    • At one integration point $\bar{\varepsilon}$ is known, $\bar{\sigma}$ is sought
  
  – **Transition**
    
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    • Upscaling: $\bar{\sigma}$ is known from the reaction forces
  
  – **Micro-scale**
    
    • Usual 3D finite elements
    
    • Periodic boundary conditions
  
  – **Advantages**
    
    • Accuracy
    
    • Generality
  
  – **Drawback**
    
    • Computational time

Ghosh S et al. 95, Kouznetsova et al. 2002, Geers et al. 2010, …
Multi-scale modelling: How?

- **Mean-Field Homogenization**
  - Macro-scale
    - FE model
    - At one integration point $\bar{\varepsilon}$ is known, $\bar{\sigma}$ is sought
Multi-scale modelling: How?

- **Mean-Field Homogenization**
  - **Macro-scale**
    - FE model
    - At one integration point $\bar{\epsilon}$ is known, $\bar{\sigma}$ is sought
  - **Micro-scale**
    - Semi-analytical model
    - Predict composite meso-scale response
    - From components material models
Multi-scale modelling: How?

- **Mean-Field Homogenization**
  - Macro-scale
    - FE model
    - At one integration point $\bar{\varepsilon}$ is know, $\bar{\sigma}$ is sought
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    - Downscaling: $\bar{\varepsilon}$ is used as input of the MFH model
    - Upscaling: $\bar{\sigma}$ is the output of the MFH model
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    - Predict composite meso-scale response
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Mori and Tanaka 73, Hill 65, Ponte Castañeda 91, Suquet 95, Doghri et al 03, Lahellec et al. 11, Brassart et al. 12, …
Multi-scale modelling: How?

• **Mean-Field Homogenization**
  - Macro-scale
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    • Downscaling: $\bar{\varepsilon}$ is used as input of the MFH model
    • Upscaling: $\bar{\sigma}$ is the output of the MFH model
  - Micro-scale
    • Semi-analytical model
    • Predict composite meso-scale response
    • From components material models
  - Advantages
    • Computationally efficient
    • Easy to integrate in a FE code (material model)
  - Drawbacks
    • Difficult to formulate in an accurate way
      - Geometry complexity
      - Material behaviours complexity

Mori and Tanaka 73, Hill 65, Ponte Castañeda 91, Suquet 95, Doghri et al 03, Lahellec et al. 11, Brassart et al. 12, …
Strain softening of the microscopic response

- Finite element solutions for strain softening problems suffer from:
  - The loss of uniqueness and strain localization
  - Mesh dependence

- Requires a non-local formulation of the macro-scale problem

The numerical results change with the size of mesh and direction of mesh

The numerical results change without convergence
Multi-scale simulations with strain softening

- Two cases considered
  - Composite materials
    - Mean-field homogenization
    - Non-local damage formulation
  - Honeycomb structures
    - Computational homogenization
    - Second-order FE2
    - Micro-buckling
Non-local damage-enhanced mean-field-homogenization

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Non-local damage-enhanced mean-field-homogenization

- Semi analytical Mean-Field Homogenization
  - Based on the averaging of the fields
    \[ \langle a \rangle = \frac{1}{V} \int_V a(X) dV \]
  - Meso-response
    - From the volume ratios \( v_0 + v_1 = 1 \)
      \[ \bar{\sigma} = \langle \sigma \rangle = v_0 \langle \sigma \rangle_{\omega_0} + v_1 \langle \sigma \rangle_{\omega_1} = v_0 \sigma_0 + v_1 \sigma_1 \]
      \[ \bar{\varepsilon} = \langle \varepsilon \rangle = v_0 \langle \varepsilon \rangle_{\omega_0} + v_1 \langle \varepsilon \rangle_{\omega_1} = v_0 \varepsilon_0 + v_1 \varepsilon_1 \]
    - One more equation required
      \[ \varepsilon_1 = B^\varepsilon : \varepsilon_0 \]
  - Difficulty: find the adequate relations
    \[ \begin{align*}
    \sigma_1 &= f(\varepsilon_1) \\
    \sigma_0 &= f(\varepsilon_0) \\
    \varepsilon_1 &= B_0^\varepsilon : \varepsilon_0
    \end{align*} \]
Non-local damage-enhanced mean-field-homogenization

- Mean-Field Homogenization for different materials
  - Linear materials
    - Materials behaviours
      \[
      \begin{align*}
      \sigma_1 &= \overline{C}_1 : \epsilon_1 \\
      \sigma_0 &= \overline{C}_0 : \epsilon_0
      \end{align*}
      \]
    - Mori-Tanaka assumption \( \epsilon^\infty = \epsilon_0 \)
    - Use Eshelby tensor
      \[
      \epsilon_1 = B^\epsilon \left( \mathbf{I}, \overline{C}_0, \overline{C}_1 \right) : \epsilon_0
      \]
      with \( B^\epsilon = \left[ \mathbf{I} + S : \overline{C}_0^{-1} : (\overline{C}_1 - \overline{C}_0) \right]^{-1} \)
  - Non-linear materials
    - Define a Linear Comparison Composite
    - Common approach: incremental tangent
      \[
      \Delta \epsilon_1 = B^\epsilon \left( \mathbf{I}, \overline{C}_0^{\text{alg}}, \overline{C}_1^{\text{alg}} \right) : \Delta \epsilon_0
      \]
Non-local damage-enhanced mean-field-homogenization

- **Material models**
  - Elasto-plastic material
    - Stress tensor: \( \sigma = C^{el} : (\varepsilon - \varepsilon^{pl}) \)
    - Yield surface: \( f(\sigma, p) = \sigma^{eq} - \sigma^Y - R(p) \leq 0 \)
    - Plastic flow: \( \Delta\varepsilon^{pl} = \Delta p N \) \& \( N = \frac{\partial f}{\partial \sigma} \)
    - Linearization: \( \delta\sigma = C^{alg} : \delta\varepsilon \)
Non-local damage-enhanced mean-field-homogenization

- Material models
  - Elasto-plastic material
    - Stress tensor \( \sigma = C^{el} : (\varepsilon - \varepsilon^{pl}) \)
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    - Plastic flow \( \Delta \varepsilon^{pl} = \Delta p N \quad \& \quad N = \frac{\partial f}{\partial \sigma} \)
    - Linearization \( \delta \sigma = C^{alg} : \delta \varepsilon \)
  - Local damage model
    - Apparent-effective stress tensors \( \sigma = (1 - D)\hat{\sigma} \)
    - Plastic flow in the effective stress space
    - Damage evolution \( \Delta D = F_D(\varepsilon, \Delta p) \)
Non-local damage-enhanced mean-field-homogenization

• Finite element solutions for strain softening problems suffer from:
  – The loss of uniqueness and strain localization
  – Mesh dependence

  Homogenous unique solution
  Lose of uniqueness
  Strain localized

The numerical results change with the size of mesh and direction of mesh

The numerical results change without convergence

• Implicit non-local approach [Peerlings et al 96, Geers et al 97, …]
  – A state variable is replaced by a non-local value reflecting the interaction between neighboring material points
  \[
  \tilde{a}(x) = \frac{1}{V_c} \int_{V_c} a(y)w(y; x)\,dV
  \]
  – Use Green functions as weight \( w(y; x) \)
  \[
  \tilde{a} - c\nabla^2 \tilde{a} = a
  \]
  New degrees of freedom
Non-local damage-enhanced mean-field-homogenization

- **Material models**
  - Elasto-plastic material
    - Stress tensor \( \sigma = C^{el} : (\varepsilon - \varepsilon^{pl}) \)
    - Yield surface \( f(\sigma, p) = \sigma^{eq} - \sigma^Y - R(p) \leq 0 \)
    - Plastic flow \( \Delta\varepsilon^{pl} = \Delta p N \quad & \quad N = \frac{\partial f}{\partial \sigma} \)
    - Linearization \( \delta\sigma = C^{alg} : \delta\varepsilon \)
  - Local damage model
    - Apparent-effective stress tensors \( \sigma = (1 - D)\hat{\sigma} \)
    - Plastic flow in the effective stress space
    - Damage evolution \( \Delta D = F_D(\varepsilon, \Delta p) \)
  - Non-Local damage model
    - Damage evolution \( \Delta D = F_D(\varepsilon, \Delta \tilde{p}) \)
    - Anisotropic governing equation \( \tilde{p} - \nabla \cdot (c_g \cdot \nabla \tilde{p}) = p \)
    - Linearization
      \[
      \delta\sigma = \left[ (1 - D)C^{alg} - \hat{\sigma} \otimes \frac{\partial F_D}{\partial \varepsilon} \right] : \delta\varepsilon - \hat{\sigma} \frac{\partial F_D}{\partial \tilde{p}} \delta \tilde{p}
      \]
Non-local damage-enhanced mean-field-homogenization

- Limitation of the incremental tangent method
  - Fictitious composite
    - 50%-UD fibres
    - Elasto-plastic matrix with damage
  - Due to the incremental formalism, stress in fibres cannot decrease during loading

No fibres unloading
Problem

- We want the fibres to get unloaded during the matrix damaging process
  
  - For the incremental-tangent approach
    \[
    \Delta \epsilon_1 = B^e(I,(1-D)\overline{C}_0^{\text{alg}},\overline{C}_1^{\text{alg}}):\Delta \epsilon_0
    \]
    
  - To unload the fibres (\( \epsilon_1 < 0 \)) with such approach would require \( \overline{C}_1^{\text{alg}} < 0 \)
  
  - We cannot use the incremental tangent MFH

- We need to define the LCC from another stress state

Non-local damage-enhanced mean-field-homogenization
Non-local damage-enhanced mean-field-homogenization

- **Idea**
  - New incremental-secant approach
    - Perform a virtual elastic unloading from previous solution
      - Composite material unloaded to reach the stress-free state
      - Residual stress in components

\[
\sigma \quad \varepsilon
\]

\[
\Delta \varepsilon_1^{\text{unload}} \quad \Delta \varepsilon \quad \Delta \varepsilon_0^{\text{unload}}
\]

matrix: \( \hat{\sigma}_0 \)

matrix: \( \sigma_0 \)

inclusions

\( \Delta \varepsilon_1 \quad \Delta \varepsilon \quad \Delta \varepsilon_0 \)
Non-local damage-enhanced mean-field-homogenization

• Idea
  – New incremental-secant approach
    • Perform a virtual elastic unloading from previous solution
      – Composite material unloaded to reach the stress-free state
      – Residual stress in components
    • Apply MFH from unloaded state
      – New strain increments (>0)
      – Use of secant operators
  \[ \Delta \varepsilon_{I/0} = \Delta \varepsilon_{I/0} + \Delta \varepsilon_{I/0}^{\text{unload}} \]
  – Possibility of have unloading
  \[
  \begin{cases}
    \Delta \varepsilon_{I} > 0 \\
    \Delta \varepsilon_{I} < 0
  \end{cases}
  \]
Non-local damage-enhanced mean-field-homogenization

- **Zero-incremental-secant method**
  - Continuous fibres
    - 55 % volume fraction
    - Elastic
  - **Elasto-plastic matrix (no damage)**
    - For inclusions with high hardening (elastic)
      - Model is too stiff

Longitudinal tension

Transverse loading
Non-local damage-enhanced mean-field-homogenization

- Zero-incremental-secant method (2)
  - Continuous fibres
    - 55% volume fraction
    - Elastic
  - Elasto-plastic matrix (no damage)
  - Secant model in the matrix
    - Modified if stiffer inclusions (negative residual stress)

Longitudinal tension

Transverse loading

\[
\frac{\sigma}{\sigma_0} = \frac{\bar{\varepsilon}}{\varepsilon_0}
\]

\[
\frac{\sigma}{\sigma_0} = \frac{\bar{\varepsilon}}{\varepsilon_0}
\]
Non-local damage-enhanced mean-field-homogenization

- Verification of the method
  - Spherical inclusions
    - 17 % volume fraction
    - Elastic
  - Elastic-perfectly-plastic matrix (no damage)
  - Non-radial loading

\[ \epsilon_{13} = \epsilon_{23} \]
\[ \epsilon_{33} = 2 \epsilon_{11} = 2 \epsilon_{22} \]

\[ \sigma_{13} \text{ [MPa]} \]

\[ \sigma_{33} \text{ [MPa]} \]
Non-local damage-enhanced mean-field-homogenization

- New results for damage
  - Fictitious composite
    - 50%-UD fibres
  - Elasto-plastic matrix with damage
Non-local damage-enhanced mean-field-homogenization

- **Weak formulation**
  - **Strong form**
    \[ \nabla \cdot \bar{\sigma}^T + f = 0 \quad \text{for the homogenized composite material} \]
    \[ \tilde{p} - \nabla \cdot (c_g \cdot \nabla \tilde{p}) = p \quad \text{for the matrix phase} \]
  - **Boundary conditions**
    \[ \sigma \cdot n = T \]
    \[ n \cdot (c_g \cdot \nabla \tilde{p}) = 0 \]
  - **Finite-element discretization**
    \[ \tilde{p} = N_{\tilde{p}}^a \tilde{P}^a \]
    \[ u = N_u^a u^a \]
    \[ \begin{bmatrix} K_{uu} & K_{u\tilde{p}} \\ K_{\tilde{p}u} & K_{\tilde{p}\tilde{p}} \end{bmatrix} \begin{bmatrix} du \\ d\tilde{p} \end{bmatrix} = \begin{bmatrix} F_{\text{ext}} - F_{\text{int}} \\ F_p - F_{\tilde{p}} \end{bmatrix} \]
Non-local damage-enhanced mean-field-homogenization

- **Mesh-size effect**
  - Fictitious composite
    - 30%-UD fibres
    - Elasto-plastic matrix with damage
  - Notched ply
Non-local damage-enhanced mean-field-homogenization

- Laminate plate with hole
  - Carbon-fibres reinforced epoxy
    - 60%-UD fibres
    - Elasto-plastic matrix with damage
  - $[-45_2/45_2]_S$ stacking sequence

\[
\begin{array}{|c|c|}
\hline
& 4.68 \pm 0.05 \\
\hline
39.60 \pm 0.35 & \varnothing 13 \\
\hline
40 & 220 \\
\hline
220 & 300 \\
\hline
\end{array}
\]
Non-local damage-enhanced mean-field-homogenization

• Laminate plate with hole (2)
  – Carbon-fibres reinforced epoxy
    • 60%-UD fibres
    • Elasto-plastic matrix with damage
  – \([-45_2/45_2]_S\) staking sequence
Computational homogenization for cellular materials

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Computational homogenization for foamed materials

- Challenges
  - Micro-structure
    - Not perfect with non periodic mesh

How to constrain the periodic boundary conditions?
Computational homogenization for foamed materials

• Challenges
  – Micro-structure
    • Not perfect with non periodic mesh

  How to constrain the periodic boundary conditions?

  • Thin components
  • Experiences micro-buckling

  How to capture the instability?
Challenges

- Micro-structure
  - Not perfect with non periodic mesh
    How to constrain the periodic boundary conditions?
  - Thin components
  - Experiences micro-buckling
    How to capture the instability?

- Transition
  - Homogenized tangent not always elliptic
  - Localization bands
    How can we recover the solution unicity at the macro-scale?
Computational homogenization for foamed materials

• Challenges
  – Micro-structure
    • Not perfect with non periodic mesh
      - How to constrain the periodic boundary conditions?
    • Thin components
    • Experiences micro-buckling
      - How to capture the instability?
  – Transition
    • Homogenized tangent not always elliptic
    • Localization bands
      - How can we recover the solution unicity at the macro-scale?
  – Macro-scale
    • Localization bands
      - How to remain computationally efficient
      - How to capture the instability?
Computational homogenization for foamed materials

- **Recover solution unicity: second-order FE**
  - **Macro-scale**
    - **High-order Strain-Gradient formulation**
      \[ \overline{P}(\overline{X}) \cdot \nabla_0 - \overline{Q}(\overline{X}) : (\nabla_0 \otimes \nabla_0) = 0 \]
    - **Partitioned mesh (//)**
  - **Transition**
    - **Gauss points on different processors**
    - **Each Gauss point is associated to one mesh and one solver**
  - **Micro-scale**
    - **Usual continuum**
      \[ P(X) \cdot \nabla_0 = 0 \]
Micro-scale periodic boundary conditions

- Convergence in terms of RVE size

- Periodic boundary conditions is the optimum choice for periodic structures

- Periodic boundary conditions remain the optimum choice for non-periodic structures
Computational homogenization for foamed materials

- **Micro-scale periodic boundary conditions (2)**
  - Defined from the fluctuation field
    \[
    w = u - (\bar{F} - I) \cdot X + \frac{1}{2} (\bar{F} \otimes \nabla_0) : (X \otimes X)
    \]
  - Stated on opposite RVE sizes
    \[
    \begin{align*}
    w(X^+) &= w(X^-) \\
    \int_{\partial V^-} w(X) d\partial V &= 0
    \end{align*}
    \]
  - Can be achieved by constraining opposite nodes

- **Foamed materials**
  - Usually random meshes
  - Important voids on the boundaries

- **Honeycomb structures**
  - Not periodic due to the imperfections
Computational homogenization for foamed materials

- Micro-scale periodic boundary conditions (2)
  - New interpolant method
    
    \[
    w(X^-) = \sum_k N(X)w^k \\
    w(X^+) = \sum_k N(X)w^k \\
    \int_{\partial V^-} \left( \sum_k N(X)w^k \right) d\partial V = 0
    \]

  - Use of Lagrange, cubic spline .. interpolations

  - Fits for
    - Arbitrary meshes
    - Important voids on the RVE sides

  - Results in new constraints in terms of the boundary and control nodes displacements

    \[
    \widetilde{C} \tilde{u}_b - g(\bar{F}, \bar{F} \otimes \nabla_0) = 0
    \]
Computational homogenization for foamed materials

- Discontinuous Galerkin (DG) implementation of the second order continuum
  - Finite-element discretization
  - Same **discontinuous** polynomial approximations for the
    - Test functions $\varphi_h$ and
    - Trial functions $\delta \varphi$

- Definition of operators on the interface trace:
  - **Jump** operator: $[\cdot] = \cdot^+ - \cdot^-$
  - **Mean** operator: $\langle \cdot \rangle = \frac{\cdot^+ + \cdot^-}{2}$

- Continuity is weakly enforced, such that the method
  - Is consistent
  - Is stable
  - Has the optimal convergence rate

- Can be used to weakly enforce higher discontinuities
Computational homogenization for foamed materials

- Second-order FE2 method
  - Macro-scale second order continuum
    \[
    \overline{P}(\overline{X}) \cdot \nabla_0 - \overline{Q}(\overline{X}) : (\nabla_0 \otimes \nabla_0) = 0
    \]
  - Requires $C^1$ shape functions on the mesh
  - The $C^1$ can be weakly enforced using the DG method

\[
a(\overline{u}, \delta \overline{u}) = a^{\text{bulk}}(\overline{u}, \delta \overline{u}) + a^{\text{PI}}(\overline{u}, \delta \overline{u}) + a^{\text{QI}}(\overline{u}, \delta \overline{u}) = b(\delta \overline{u})
\]
Computational homogenization for foamed materials

- **Second-order FE2 method**
  - Macro-scale second order continuum
    \[ \overline{P}(\overline{X}) \cdot \mathbf{V}_0 - \overline{Q}(\overline{X}): (\mathbf{V}_0 \otimes \mathbf{V}_0) = 0 \]
  - Requires $C^1$ shape functions on the mesh
  - The $C^1$ can be weakly enforced using the DG method

\[ a(\mathbf{u}, \delta \mathbf{u}) = a^{\text{bulk}}(\mathbf{u}, \delta \mathbf{u}) + a^{\text{PI}}(\mathbf{u}, \delta \mathbf{u}) + a^{\text{QI}}(\mathbf{u}, \delta \mathbf{u}) = b(\delta \mathbf{u}) \]

- **Usual volume terms**

\[ a^{\text{bulk}}(\mathbf{u}, \delta \mathbf{u}) = \int_{V} [\overline{P}(\mathbf{u}): (\delta \mathbf{u} \otimes \mathbf{V}_0) + \overline{Q}(\overline{X}): (\delta \mathbf{u} \otimes \mathbf{V}_0 \otimes \mathbf{V}_0)]dV \]
Computational homogenization for foamed materials

- **Second-order FE2 method**
  - Macro-scale second order continuum
    \[
    \bar{P}(\bar{X}) \cdot \nabla_0 - \bar{Q}(\bar{X}) : (\nabla_0 \otimes \nabla_0) = 0
    \]
  - Requires $C^1$ shape functions on the mesh
  - The $C^1$ can be weakly enforced using the DG method

\[
a(\bar{u}, \delta \bar{u}) = a^{bulk}(\bar{u}, \delta \bar{u}) + a^{PI}(\bar{u}, \delta \bar{u}) + a^{QI}(\bar{u}, \delta \bar{u}) = b(\delta \bar{u})
\]

- **Weak enforcement of the $C^0$**
  - Continuity
  - Consistency
  - Stability
  between the finite elements

\[
a^{PI}(\bar{u}, \delta \bar{u}) = \int_{\partial_1 V} \left[ [\delta \bar{u}] \cdot (\bar{P} - \bar{Q} \cdot \nabla_0) \cdot \bar{N} + [\bar{u}] \cdot (\bar{P}(\delta \bar{u}) - \bar{Q}(\delta \bar{u}) \cdot \nabla_0) \cdot \bar{N} + [\bar{u}] \otimes \bar{N} : \left( \frac{\beta_P}{h_s} C^0 \right) : [\delta \bar{u}] \otimes \bar{N} \right] dV
\]
  - Allows efficient parallelization as elements are disjoint
Computational homogenization for foamed materials

• Second-order FE2 method
  – Macro-scale second order continuum

\[ \bar{P}(\bar{X}) \cdot \nabla_0 - \bar{Q}(\bar{X}) : (\nabla_0 \otimes \nabla_0) = 0 \]

  – Requires $C^1$ shape functions on the mesh
  – The $C^1$ can be weakly enforced using the DG method

\[ a(\bar{u}, \delta \bar{u}) = a^{bulk}(\bar{u}, \delta \bar{u}) + a^{PI}(\bar{u}, \delta \bar{u}) + a^{QI}(\bar{u}, \delta \bar{u}) = b(\delta \bar{u}) \]

• Weak enforcement of the $C^1$
  – Continuity
  – Consistency
  – Stability
  between the finite elements

\[ a^{QI}(\bar{u}, \delta \bar{u}) = \int_{\partial_i V} \left[ \left[ \delta \bar{u} \otimes \nabla_0 \right] \cdot \langle \bar{Q} \rangle \cdot \bar{N} + \left[ \bar{u} \otimes \nabla_0 \right] \cdot \langle \bar{Q}(\delta \bar{u}) \rangle \cdot \bar{N} + \right] dV \]

  – Allows efficient parallelization as elements are disjoint
Computational homogenization for foamed materials

- Capturing instabilities
  - Macro-scale: localization bands
    - Path following method on the applied loading
      \[ a(\bar{u}, \delta \bar{u}) = \bar{\mu} b(\delta \bar{u}) \]
    - Arc-length constraint on the load increment
      \[ \bar{h}(\Delta \bar{u}, \Delta \bar{\mu}) = \frac{\Delta \bar{u} \cdot \Delta \bar{u}}{X_0^2} + \Delta \bar{\mu}^2 - \Delta L^2 = 0 \]
Computational homogenization for foamed materials

- Capturing instabilities
  - Macro-scale: localization bands
    - Path following method on the applied loading
      \[ a(\bar{u}, \delta \bar{u}) = \bar{\mu} b(\delta \bar{u}) \]
    - Arc-length constraint on the load increment
      \[ h(\Delta \bar{u}, \Delta \bar{\mu}) = \frac{\Delta \bar{u} \cdot \Delta \bar{u}}{\bar{X}_0^2} + \Delta \bar{\mu}^2 - \Delta \bar{l}^2 = 0 \]
  - Micro-scale
    - Path following method on the applied boundary conditions
      \[ C \bar{u}_b - g(\bar{F}, \bar{F} \otimes \nabla_0) = 0 \]
      \[ \begin{cases} \bar{F} = \bar{F}_0 + \mu \Delta \bar{F} \\ \bar{F} \otimes \nabla_0 = (\bar{F} \otimes \nabla_0)_0 + \mu \Delta (\bar{F} \otimes \nabla_0) \end{cases} \]
    - Arc-length constraint on the load increment
      \[ h(\Delta u, \Delta \mu) = \frac{\Delta u \cdot \Delta u}{X_0^2} + \Delta \mu^2 - \Delta \bar{l}^2 = 0 \]
Computational homogenization for foamed materials

• Compression of an hexagonal honeycomb
  – Elasto-plastic material

• Comparison of different solutions

Full direct simulation

Multiscale with different macro-meshes
Computational homogenization for foamed materials

- Compression of an hexagonal honeycomb (2)
  - Captures of the softening onset
  - Captures the softening response
  - No macro-mesh size effect
Computational homogenization for foamed materials

- Compression of an hexagonal honeycomb plate with a centered hole
  - Results given by full and multi-scale models are comparable
Conclusions

• Non-local damage-enhanced mean-field-homogenization
  – MFH with damage model for the matrix material
  – Non-local implicit formulation
  – Can capture the strain softening
  – More in
    • 10.1016/j.ijsolstr.2013.07.022
    • 10.1016/j.ijplas.2013.06.006
    • 10.1016/j.cma.2012.04.011
    • 10.1007/978-1-4614-4553-1_13

• Computational homogenization for foamed materials
  – Second-order FE² method
  – Micro-buckling propagation
  – General way of enforcing PBC
  – More in
    • 10.1016/j.cma.2013.03.024
    • 10.1016/j.commatsci.2011.10.017
    • 10.1016/j.ijsolstr.2014.02.029

• Open-source software
  – Implemented in GMSH
    • http://geuz.org/gmsh/
QC method for grain-boundary sliding

DG-based fracture framework

Stiction failure in a MEMS sensor

DG-based fracture framework

SVE size effect on meso-scale properties