Pointwise regularity of some "historical non-differentiable functions"

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Techniques Fractales (Orléans)

Hölder-regularity

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a locally bounded function, $x \in \mathbb{R}$ and $\alpha > 0$; $f \in C^\alpha(x)$ if there exist $R, C > 0$ and a polynomial P_x of degree less than α such that

$$|h| < R \Rightarrow |f(x+h) - P_x(h)| \leq C|h|^\alpha. \qquad (*)$$

A function f belongs to C^α if there exists $C > 0$ such that $(*)$ holds for all x with $R = \infty$.

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If the Hölder exponent is unique $\forall x$, then f is called a monoHölder function.
If the Hölder exponent takes only one finite value, f is called a monofractal function.

The p-adic distance

Notation

Let $p \in \mathbb{N}$, $p > 1$. For a sequence of integers satisfying $0 \leq x_j < p$, we will use the following notation

$$(0; x_1, \ldots, x_j, \ldots)_p$$

to denote one of the expansions of the real number

$$x = \sum_{k=1}^{\infty} \frac{x_k}{p^k}. \qquad (*)$$

If there is no j such that $x_i = p - 1 \ \forall i \geq j$, $(*)$ is the proper expansion of x in basis p.

The p-adic distance

Notation

$$x = (0; x_1, \ldots, x_j, \ldots)_p$$

denotes one of the expansions in basis p of the real number x.

Definition

Let $s = (0; s_1, \ldots)_p$, $t = (0; t_1, \ldots)_p$ be the proper expansions of the real numbers s and t respectively and $\delta_p(s,t) = \inf\{k : s_k \neq t_k\} - 1$. The p-adic distance between s and t is

$$d_p(s,t) = p^{-\delta_p(s,t)}$$

This distance is an ultrametric distance.

p-adic Hölder spaces

Definition

Let $\alpha > 0$. A locally bounded function $f : [0,1] \to \mathbb{R}$ belongs to $C_p^\alpha(x)$ if there exists $C > 0$ and a polynomial P_x of degree less than α such that

$$|f(x+h) - P_x(h)| \leq C d_p(x, x+h)^\alpha. \qquad (*)$$

A function belongs to C_p^α if there exists $C > 0$ such that $(*)$ holds for any x.
The p-adic Hölder exponent of f at x is

$$h_p(x) = \sup\{\alpha : f \in C_p^\alpha(x)\}.$$

A sufficient condition to belong to $C_p^\alpha(x)$

Proposition

Let f be a bounded function explicitly defined on $D \subset [0,1]$ by

$$f : D \to \mathbb{R}^+ x = (0; x_1, \dots)_p \mapsto f(x) = (\dots, y_0; y_1, \dots)_{p'}$$

and such that there exist ϵ and a function g such that

$$d_p(s,t) < \epsilon \Rightarrow \begin{cases} d_{p'}(f(s), f(t)) < 1 \\ \delta_{p'}(f(s), f(t)) = g(s, \delta_p(s,t)) \end{cases} .$$

If

$$\alpha = \liminf_{u \to \infty} \frac{g(x_0, u)}{u} \frac{\log p'}{\log p} \leq 1, \qquad (*)$$

$f \in C_p^\alpha(x_0)$.
In particular, if $(*)$ does not depend on x_0, $f \in C_p^\alpha$.

p-adic monoHölderianity and monoHölderianity

Let F be defined as follows

$$F : [0,1] \to [0,1]^2 \quad x \mapsto \left(\begin{array}{c} f_1(x) \\ f_2(x) \end{array} \right)$$

where $F(1) = (1,1)$ and, if $(0; x_1, \ldots)_2$ is the proper expansion of x,

$$f_1(x) = (0; x_1, x_3, \ldots, x_{2j-1}, \ldots)_2$$

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A direct application of the previous proposition shows that $f_1, f_2 \in C_2^{1/2}$. Moreover, it is easy to check that F is a 2-adic monoHölder function with 2-adic Hölder exponent $1/2$.

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A direct application of the previous proposition shows that $f_1, f_2 \in C_2^{1/2}$. Moreover, it is easy to check that F is a 2-adic monoHölder function with 2-adic Hölder exponent $1/2$. However, F is not continuous.

Let us drop the p-adic distance

Notations

If $(0; s_1, \ldots)_p$ is the proper expansion of s in base p, let $\theta_p(s) = \inf\{k : s_k \neq 0\} - 1$ and, if $t = (0; t_1, \ldots)_p$,

$$\gamma_p(s, t) = \theta_p(|s - t|).$$

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$$\gamma_p(s,t) = \theta_p(|s-t|).$$

However, $p^{-\gamma_p(s,t)}$ does not define a distance.
To compute $\gamma_p(s,t)$, we have to compute $\delta_p(s,t)$ and to check that s and t are not of the form

$$s = (0; s_1, \ldots, s_{n-1}, s_n, 0, \ldots, s_m, \ldots)_p,$$

$$t = (0; s_1, \ldots, s_{n-1}, s_n - 1, p-1, \ldots, t_m, \ldots)_p.$$

If it is the case, $\gamma_p(s,t) > \delta_p(s,t)$; otherwise we can suppose that $\gamma_p(s,t) = \delta_p(s,t)$.

Strongly MonoHölder functions

Definition

Let $\alpha \in (0,1)$. A function f belongs to $I^\alpha(x_0)$ if

$$\exists C,\ R>0,\ \forall r \le R,\ \sup_{x,y \in B(x_0,r)} |f(x)-f(y)| \ge Cr^\alpha;$$

f belongs to I^α if

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Definition

Let $\alpha \in (0,1)$. A function is strongly monoHölder of exponent α ($f \in SM_\alpha$) if $f \in C^\alpha \cap I^\alpha$.

A sufficient condition for monoHölderianity

Let f be a bounded function explicitly defined on $D \subset [0,1]$ by

$$f : D \to \mathbb{R}^+ x = (0; x_1, \ldots)_p \mapsto f(x) = (\ldots, y_0; y_1, \ldots)_{p'}$$

and such that there exist $\eta > 0$ and a function g such that

$$\gamma_p(s,t) > \eta \Rightarrow \begin{cases} \gamma_{p'}(f(s), f(t)) > 0 \\ \gamma_{p'}(f(s), f(t)) = g(s, \gamma_p(s,t)) \end{cases} .$$

If

$$\alpha = \liminf_{u \to \infty} \frac{g(x_0, u)}{u} \frac{\log p'}{\log p} < 1, \qquad (*)$$

$h(x_0) = \alpha$ and $f \in I^\alpha(x_0)$. If $\alpha = 1$, $f \in C^\alpha(x_0)$.
In particular, if (*) does not depend on x_0, $f \in SM_\alpha$. If $\alpha = 1$, $f \in C^\alpha$.

Preliminary remarks and notations

The triadic Cantor set K is the set of real numbers x that can be written $x = (0; x_1, \ldots)_3$, with $x_j \in \{0, 2\}$ $\forall j$.

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Let f be a continuous function defined on a closed bounded subset $A \subset \mathbb{R}$. The linear extension of f is the continuous function g defined on $[\inf A, \sup A]$ by

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- $g(x) = f(x)$ if $x \in A$,
- g is linear otherwise.

if $f \in C^\alpha$, with $\alpha \in (0, 1)$, its linear extension also belong to C^α.

The Devil's staircase

The Devil's staircase is defined on K as follows,

$$D : K \longrightarrow [0,1] \qquad (0; x_1, \ldots, x_j, \ldots)_3 \mapsto (0; \frac{x_1}{2}, \ldots, \frac{x_j}{2}, \ldots)_2$$

and can be linearly extended on $[0,1]$

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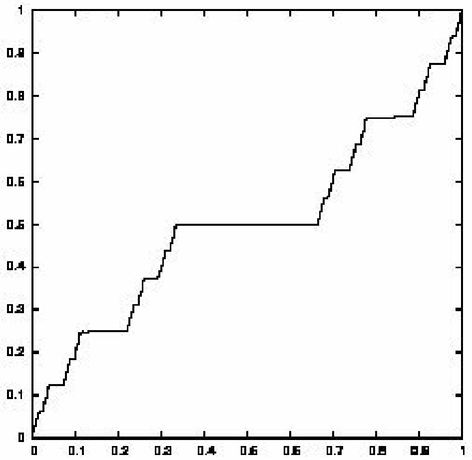
The Devil's staircase is a monofractal function with (finite) Hölder exponent $\log 2 / \log 3$.

Representation of the Devil’s staircase



The Takagi functions

The Takagi functions are defined as follows,

$$f : x \mapsto \sum_{k=0}^{\infty} \frac{(2^{mk} x)}{2^{nk}},$$

with $m, n \in \mathbb{N}$, $m \geq n$ and where $(x) = \mathrm{dist}(x, \mathbb{Z})$.

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The Takagi functions are monoHölder functions with Hölder exponent n/m.

The Hölder exponent of the Takagi functions

With the notation

$$\tilde{x}_j = \begin{cases} x_j & \text{if } x_{mk+1} = 0 \\ 1 - x_j & \text{if } x_{mk+1} = 1 \end{cases},$$

we have

$$f : (0; x_1, \ldots)_2 \mapsto \sum_{k=0}^{\infty} (0; \overbrace{0, \ldots, \tilde{x}_{mk+1}}^{nk+1}, \tilde{x}_{mk+2}, \ldots)_2.$$

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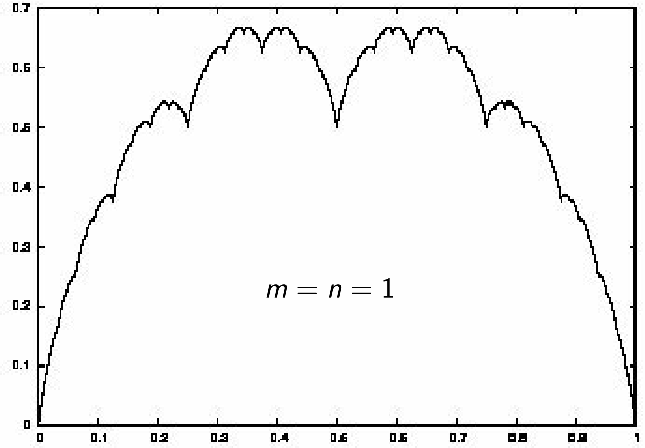
If $n = m$, one has to show that f is not differentiable.

Examples of Takagi functions



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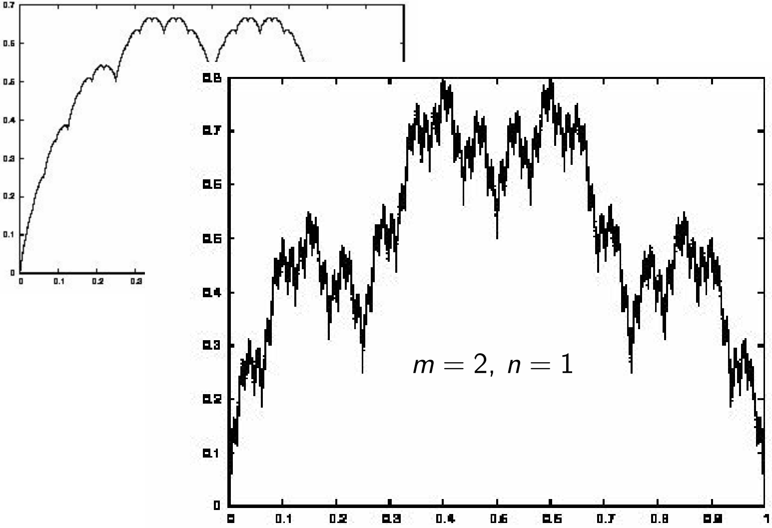

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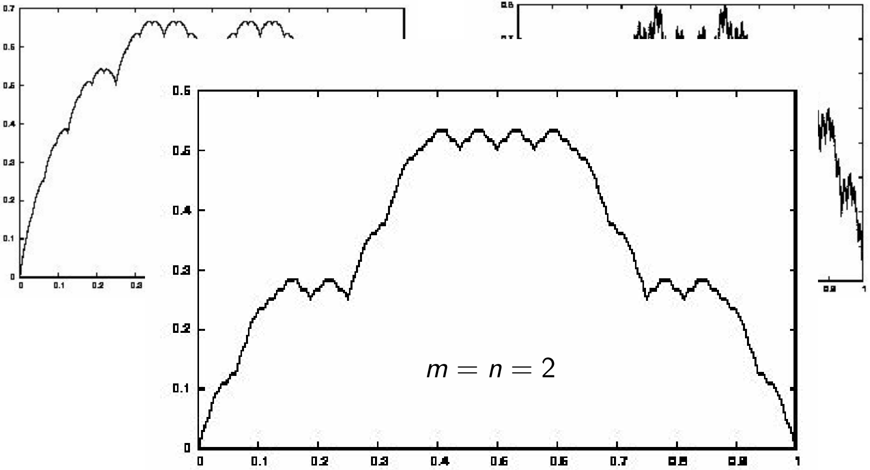

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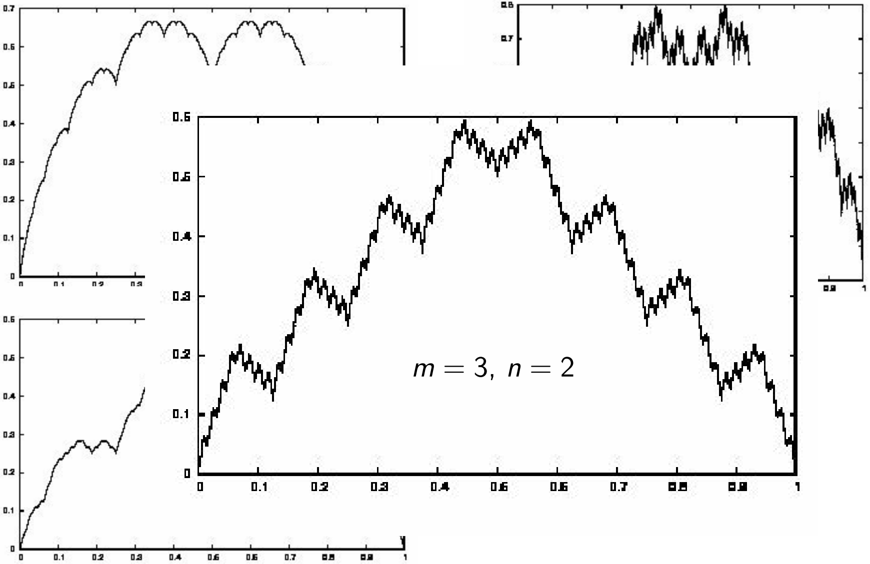

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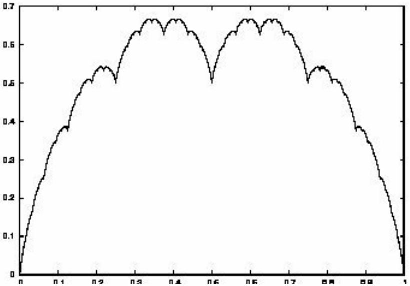

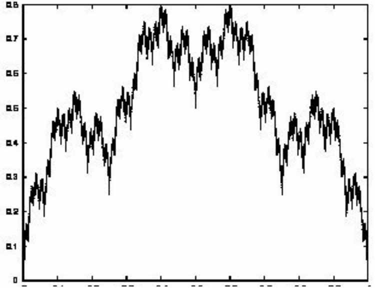

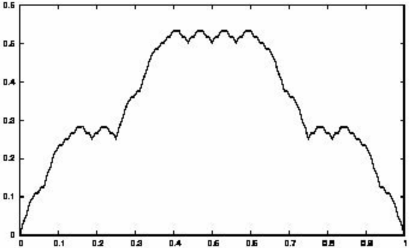

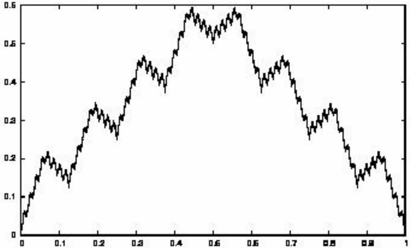

The Wunderlich function

If $x = (0; x_1, \ldots)_3$, let us set $x_0 = y_0 = 0$ and define the sequence $(y_j)_{j \in \mathbb{N}}$ recursively,

$$y_j = \begin{cases} y_{j-1} & \text{if } x_j = x_{j-1} \\ 1 - y_{j-1} & \text{else} \end{cases} .$$

The Wunderlich function is defined as follows,

$$f : [0,1] \to [0,1] \qquad (0; x_1, \ldots)_3 \mapsto (0; y_1, \ldots)_2.$$

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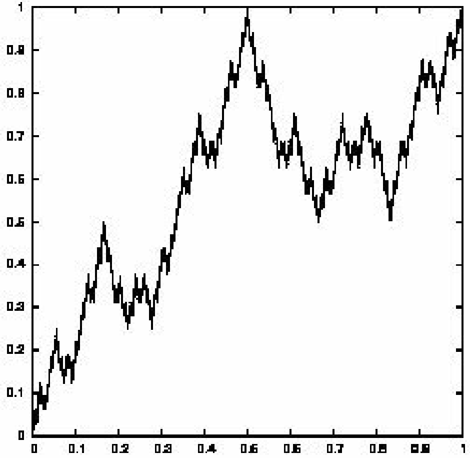
$$f : [0,1] \to [0,1] \qquad (0; x_1, \dots)_3 \mapsto (0; y_1, \dots)_2.$$

It is easy to check that this function is a monoHölder function with Hölder exponent $\log 2 / \log 3$.

Representation of the Wunderlich function



The Petr function

If $x = (0; x_1, \ldots)_{10}$, let $(b_j)_j$ denote the following sequence: $b_1 = 1$ and

$$b_j = \begin{cases} b_{j-1} & \text{if } x_{j-1} \text{ is even or equal to 9} \\ -b_{j-1} & \text{else} \end{cases} .$$

The sequences $(y_j)_j$ and $(z_j)_j$ are then defined as follows,

$$y_j = \begin{cases} x_j \bmod 2 & \text{if } b_j > 0 \\ 0 & \text{otherwise} \end{cases}, z_j = \begin{cases} 0 & \text{if } b_j > 0 \\ x_j \bmod 2 & \text{otherwise} \end{cases},$$

and lead to the definition of the Petr function

$$f : [0,1] \to [0,1] \qquad x \mapsto (0; y_1, \ldots)_2 - (0; z_1, \ldots)_2.$$

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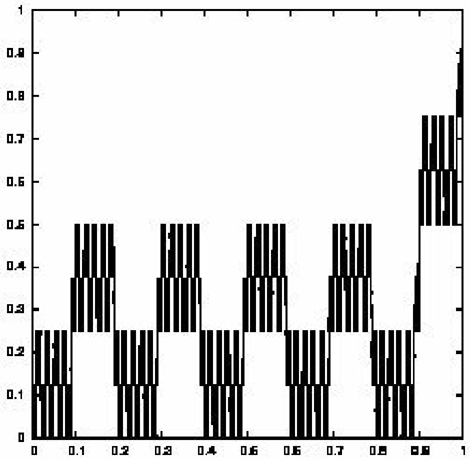
$$f : [0,1] \to [0,1] \qquad x \mapsto (0; y_1, \ldots)_2 - (0; z_1, \ldots)_2.$$

This function is a monoHölder function with Hölder exponent $\log 2 / \log 10$.

Representation of the Petr function



The Peano function

Let K the application defined by $Kj = 2 - j$ and set $K^0 j = j$. The Peano function is defined as follows,

$$P : [0,1] \longrightarrow [0,1] \qquad x \mapsto \begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix},$$

where

$$p_1((0; x_1, \ldots)_3) = (0; x_1, K^{x_2} x_3, \ldots, K^{\sum_{k=1}^{j-1} x_{2k}} x_{2j-1}, \ldots)_3$$

and

$$p_2((0; x_1, \ldots)_3) = (0; K^{x_1} x_2, \ldots, K^{\sum_{k=0}^{j-1} x_{2k+1}} x_{2j}, \ldots)_3,$$

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The Peano function is a monoHölder function with Hölder exponent $1/2$.

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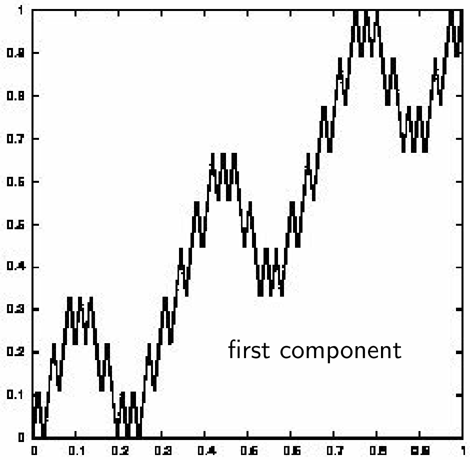

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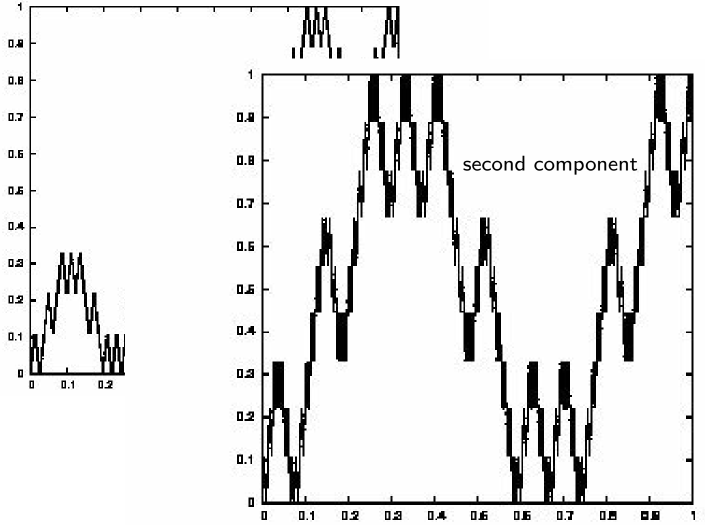

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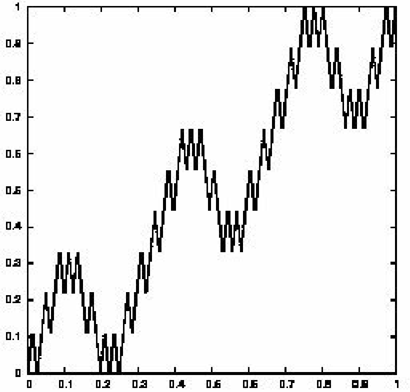

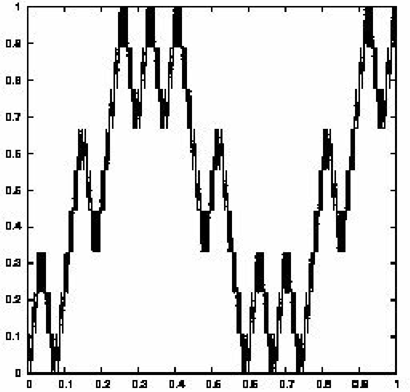

The Lebesgue function

The Lebesgue function is defined on K as follows

$$L: \; K \longrightarrow [0,1]^2$$

$$(0; x_1, x_2, \ldots, x_{2j-1}, x_{2j}, \ldots)_3 \longmapsto \left(\begin{array}{l} (0; \frac{x_1}{2}, \frac{x_3}{2}, \ldots, \frac{x_{2j-1}}{2}, \ldots)_2 \\ (0; \frac{x_2}{2}, \frac{x_4}{2}, \ldots, \frac{x_{2j}}{2}, \ldots)_2 \end{array} \right)$$

and can be linearly extended to $[0,1]$

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The Lebesgue function is a monofractal function with (finite) Hölder exponent $\log 2/(2\log 3)$.

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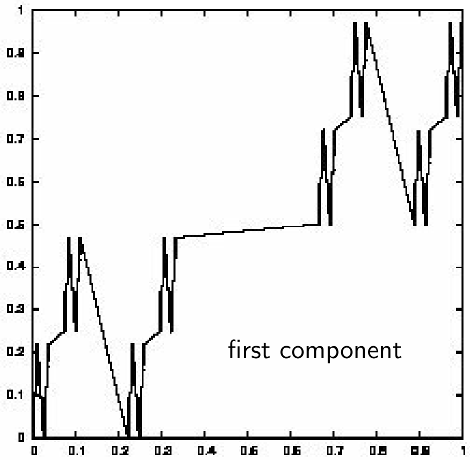
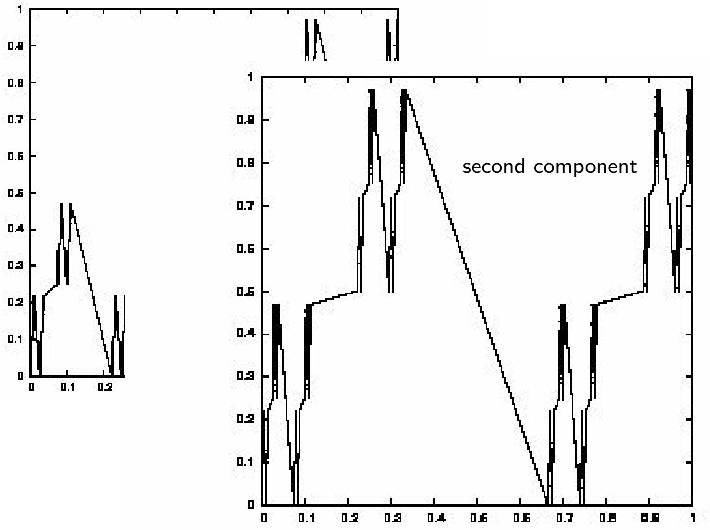


Representation of the Lebesgue function



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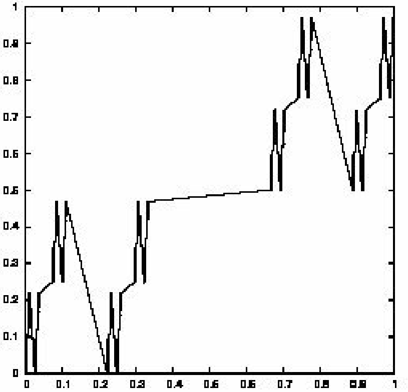

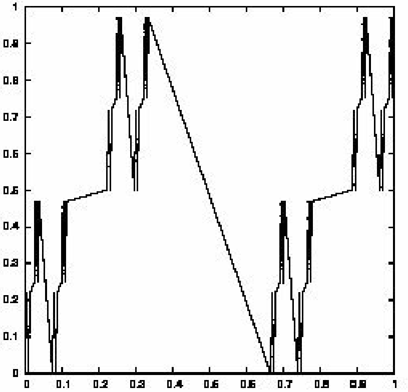

The Sierpiński function

Let Θ and τ be 1-periodic functions defined on $[0,1]$ as follows

$$\Theta(x) = \begin{cases} -1 & \text{if } x \in [\frac{1}{4}, \frac{3}{4}) \\ 1 & \text{otherwise} \end{cases},$$

and

$$\tau(x) = \begin{cases} \frac{1}{8} + 4x & \text{if } x \in [0, \frac{1}{4}) \cup [\frac{1}{2}, \frac{3}{4}) \\ \frac{1}{8} - 4x & \text{otherwise} \end{cases}.$$

The Sierpiński function is explicitly defined by

$$f(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} \prod_{j=0}^{k} \Theta(\tau^j(x)),$$

where we have set $\tau^0(x) = x$.

The Hölder exponent of the Sierpiński function

Indeed, if $x = (0; x_1, \ldots)_2$,

$$\Theta(x) = \begin{cases} -1 & \text{if } \begin{cases} x_1 = 0 \\ x_2 = 1 \end{cases} \text{ or } \begin{cases} x_1 = 1 \\ x_2 = 0 \end{cases} \\ \\ 1 & \text{if } \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \text{ or } \begin{cases} x_1 = 1 \\ x_2 = 1 \end{cases} \end{cases},$$

and

$$\tau(x) = \begin{cases} \frac{1}{2^3} + (0; x_3, \ldots)_2 & \text{if } x_2 = 0 \\ \frac{1}{2^3} - (0; x_3, \ldots)_2 & \text{if } x_2 = 1 \end{cases}.$$

If we set $p(x, k) = \prod_{j=0}^{k} \Theta(\tau^j(x))$,

$$f(x) = \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k \frac{p(x, k)}{2^k}$$

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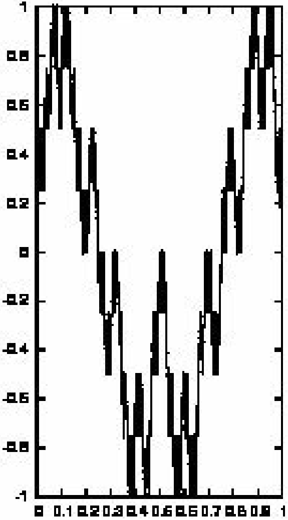
Finally, one can show that f is a monoHölder function with Hölder exponent $1/2$.

Representation of the Sierpiński function



The Schoenberg function

Let g be the 2-periodic even function which satisfies

$$g(t) = \begin{cases} 0 & \text{if } t \in [0, \frac{1}{3}] \\ 3t - 1 & \text{if } t \in [\frac{1}{3}, \frac{2}{3}] \\ 1 & \text{if } t \in [\frac{2}{3}, 1] \end{cases} .$$

The Schoenberg function is defined by

$$S : [0,1] \to [0,1]^2 \qquad x \mapsto \begin{pmatrix} s_1(x) \\ s_2(x) \end{pmatrix},$$

where

$$s_1(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{g(3^{2k} x)}{2^k}$$

and

$$s_2(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{g(3^{2k+1} x)}{2^k}.$$

The Hölder exponent of the Schoenberg function

If $x = (0; x_1, \ldots)_3$, we have

$$g(3^{2k}x) = \begin{cases} g_1((0; x_{2k+1}, \ldots)_3) & \text{if } (x_1, \ldots, x_{2k})_3 \text{ is even} \\ g_2((0; x_{2k+1}, \ldots)_3) & \text{if } (x_1, \ldots, x_{2k})_3 \text{ is odd} \end{cases}$$

where

$$g_1((0; x_{2k+1}, \ldots)_3) = \begin{cases} 0 & \text{if } x_{2k+1} = 0 \\ 1 & \text{if } x_{2k+1} = 2 \\ (0; x_{2k+2}, \ldots) & \text{if } x_{2k+1} = 1 \end{cases}$$

and

$$g_2((0; x_{2k+1}, \ldots)_3) = \begin{cases} 0 & \text{if } x_{2k+1} = 2 \\ 1 & \text{if } x_{2k+1} = 0 \\ 1 - (0; x_{2k+2}, \ldots) & \text{if } x_{2k+1} = 1 \end{cases},$$

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This allows to show that the Schoenberg function is monoHölder with Hölder exponent $\log 2/(2\log 3)$.

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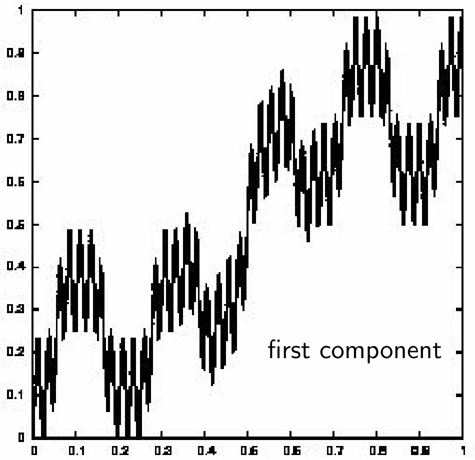

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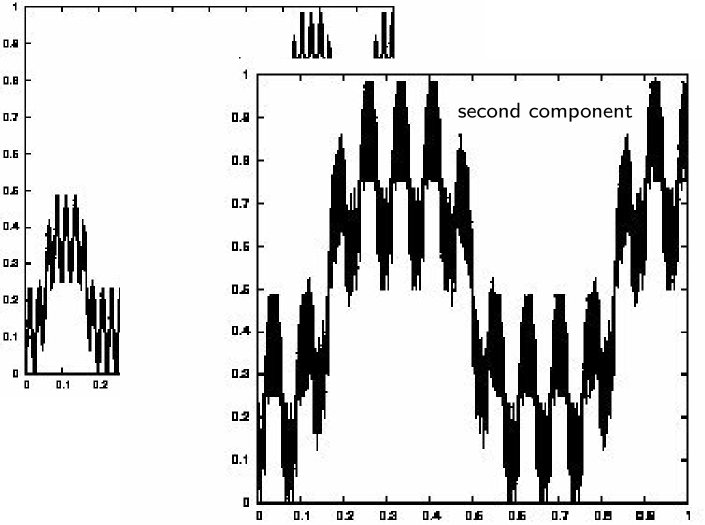

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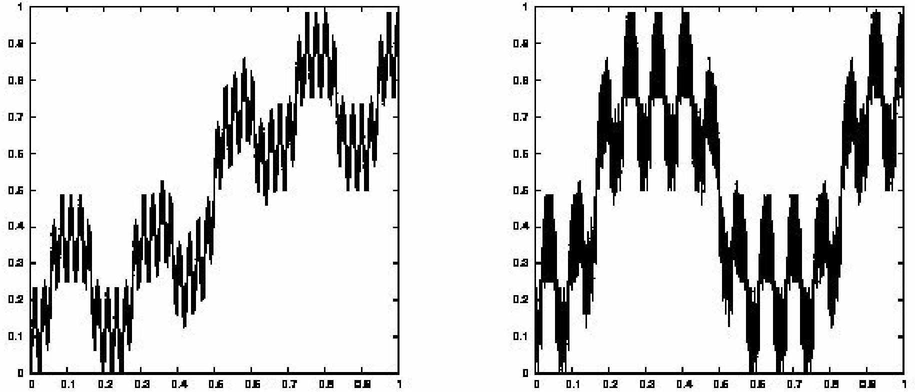

The generic Lebesgue functions

Let E_p is the set of real numbers whose one of the expansions in base $2p-1$ is $x = (0; x_1, \ldots)_{2p-1}$ with x_j even $\forall j$. In these settings, let

$$L_p : E_p \to [0,1]^2 \qquad x \mapsto \begin{pmatrix} l_1(x) \\ l_2(x) \end{pmatrix},$$

where

$$l_1((0; x_1, \ldots)_{2p-1}) = (0; \frac{x_1}{2}, \frac{x_3}{2}, \ldots, \frac{x_{2j-1}}{2}, \ldots)_p,$$

and

$$l_2((0; x_1, \ldots)_{2p-1}) = (0; \frac{x_2}{2}, \frac{x_4}{2}, \ldots, \frac{x_{2j}}{2}, \ldots)_p.$$

This function can be linearly extended to $[0,1]$.

The Hölder exponent of the generic Lebesgue function

Proposition

The restriction of L_p to E_p is an onto function.

The Hölder exponent of the generic Lebesgue function

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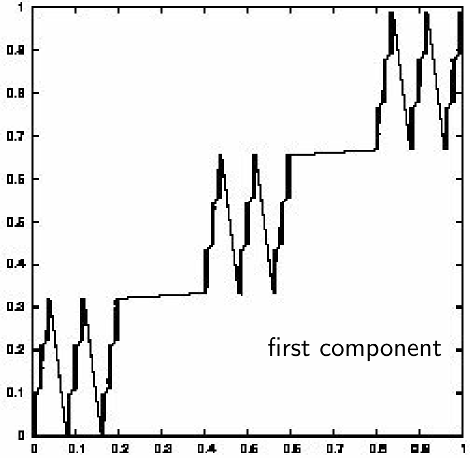
The restriction of L_p to E_p is an onto function.

The function L_p is a monofractal function with (finite) Hölder exponent $\log p/(2\log 2p - 1)$.

Representation of the generic Schoenberg function L_3



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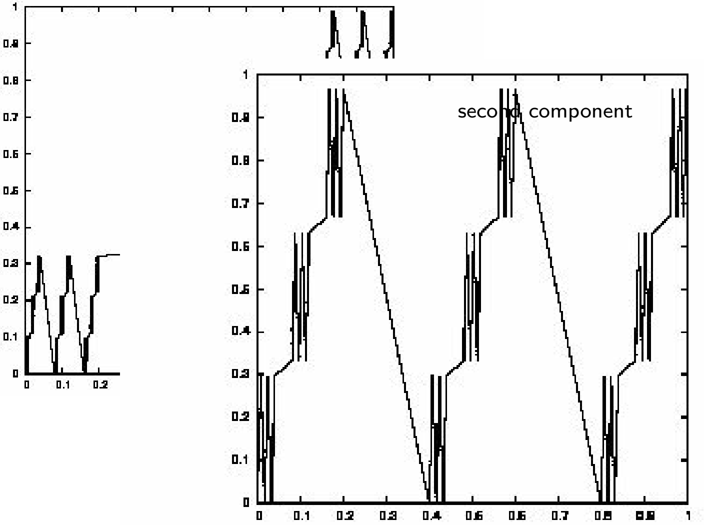

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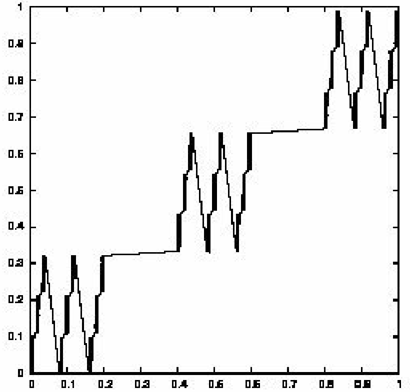

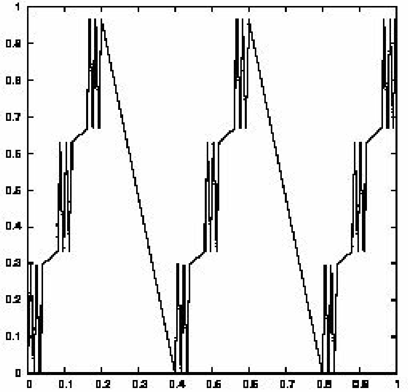

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Let g_p be the 2 periodic even function which satisfies

$$g_p(x) = \begin{cases} k & \text{if } x \in [\frac{2k}{2p-1}, \frac{2k+1}{2p-1}] \quad (0 \le k \le p-1) \\ (2p-1)x - k - 1 & \text{if } x \in]\frac{2k+1}{2p-1}, \frac{2k+2}{2p-1}[\quad (0 \le k < p-1) \end{cases}$$

and set

$$S_p : [0,1] \longrightarrow [0,1]^2 \qquad x \mapsto \begin{pmatrix} s_1(x) \\ s_2(x) \end{pmatrix},$$

where

$$s_1(x) = \frac{1}{p} \sum_{k=0}^{\infty} \frac{g_p((2p-1)^{2k} x)}{p^k}$$

and

$$s_2(x) = \frac{1}{p} \sum_{k=0}^{\infty} \frac{g_p((2p-1)^{2k+1} x)}{p^k}.$$

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The function S_p is a monoHölder function with Hölder exponent $\log p/(2\log 2p-1)$.

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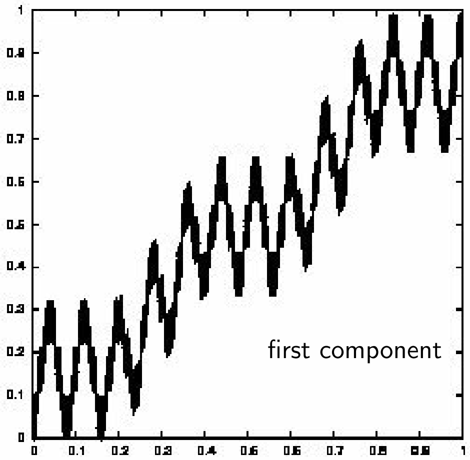

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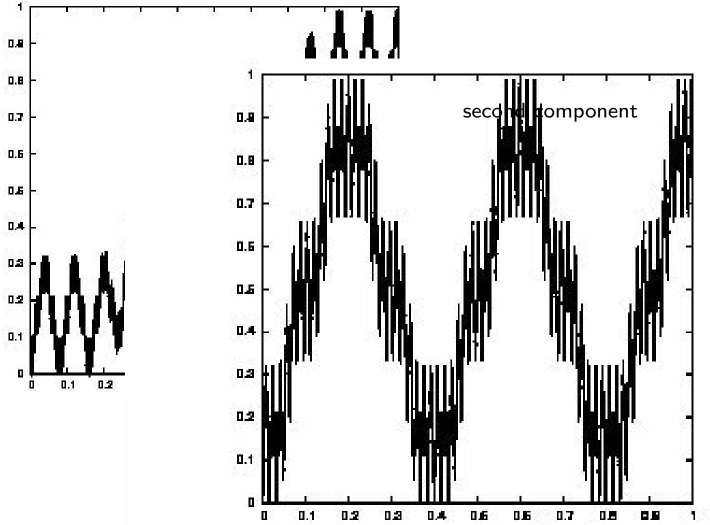

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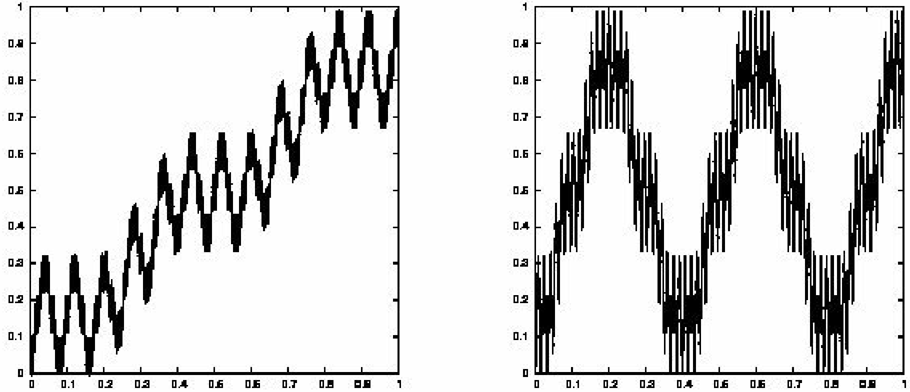

Definition of D_p and F_p

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$$L(x) = F \circ D(x) \qquad \forall x \in K^* = E_2^*.$$

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$$F_p : [0,1] \to [0,1]^2 \qquad x \mapsto \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix},$$

where $F_p(1) = (1,1)$ and, if $x = (0; x_1, \ldots)_p$ is the proper expansion of x

$$f_1(x) = (0; x_1, x_3, \ldots, x_{2j-1}, \ldots)_p$$

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One has

$$L_p(x) = F_p \circ D_p(x) \qquad \forall x \in E_p^*$$

Definition of the spectrum of singularities

Definition

Let f be a locally bounded function; its isoHölder sets are the sets

$$E_H = \{x : h(x) = H\}.$$

The spectrum of singularities of f is the function

$$d : \mathbb{R}^+ \cup \{\infty\} \to \mathbb{R}^+ \cup \{-\infty\} \qquad H \mapsto \dim_H(E_H),$$

where $\dim_H$ denotes the Hausdorff dimension, using the standard convention $\dim_H(\emptyset) = -\infty$.

About the regularity of F_p

Proposition

The function F_p has the following expansion

$$\begin{cases} f_1(x) = \sum_{n=0}^{\infty} a_n\{p^n x\}, & \text{where } a_{2n} = p^{-l} \text{ and } a_{2n+1} = -p^{-l-1} \\ f_2(x) = \sum_{n=1}^{\infty} a_n\{p^n x\}, & \text{where } a_{2n} = -p^{-l} \text{ and } a_{2n+1} = p^{-l-1} \end{cases}.$$

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Corollary

The spectrum of singularities of the function F_p is given by

$$d(H) = \begin{cases} 2H & \text{if } 0 \leq H \leq 1/2 \\ -\infty & \text{else} \end{cases}.$$

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However, the restriction of F_p to E_p^* is a monofractal function with Hölder exponent $1/2$.

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It is easy to check that F_p is a p-adic monoHölder function with Hölder exponent $1/2$.