About pointwise smoothness of some nondifferentiable functions

S. Nicolay

Department of Mathematics, University of Liège
Partially joint work with M. Clausel and S. Jaffard

Fractal Geometry and Stochastics 4

Hölder-regularity

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a locally bounded function, $x \in \mathbb{R}$ and $\alpha \geq 0$; $f \in C^\alpha(x)$ if there exist $R, C > 0$ and a polynomial P_x of degree less than α such that

$$|h| < R \Rightarrow |f(x+h) - P_x(h)| \leq C|h|^\alpha. \qquad (*)$$

A function f belongs to C^α if there exists $C > 0$ such that $(*)$ holds for all x with $R = \infty$.

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The Hölder exponent of f at x is $\underline{h}(x) = \sup\{\alpha : f \in C^\alpha(x)\}$

Hölder irreguarity

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Let $f:\mathbb{R}\to\mathbb{R}$ be a locally bounded function, $x\in\mathbb{R}$ and $0\leq\alpha<1$; $f\in I^\alpha(x)$ if there exist $R, C>0$ such that

$$\sup_{y,y'\in B(x,r)} |f(y)-f(y')| \geq Cr^\alpha, \quad \forall r\leq R,$$

Hölder irreguarity

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$$\sup_{y,y' \in B(x,r)} |f(y) - f(y')| \ge C r^\alpha, \quad \forall r \le R,$$

Definition

The upper Hölder exponent of f at x is $\overline{h}(x) = \inf\{\alpha : f \in I^\alpha(x)\}$. A function f is strongly monoHölder of exponent α ($f \in SM^\alpha$) if $f \in C^\alpha \cap I^\alpha$.

Representation of real numbers in base p

Notation

Let $p \in \mathbb{N}$, $p > 1$. For a sequence of integers satisfying $0 \leq x_j < p$, we will use the following notation

$$(0; x_1, \ldots, x_j, \ldots)_p$$

to denote one of the expansions of the real number

$$x = \sum_{k=1}^{\infty} \frac{x_k}{p^k}. \qquad (*)$$

If there is no j such that $x_i = p - 1 \ \forall i \geq j$, $(*)$ is the proper expansion of x in base p.

Representation of real numbers in base p

Notation

$$x = (0; x_1, \ldots, x_j, \ldots)_p$$

denotes one of the expansions in base p of the real number x.

Definition

Let $s = (0; s_1, \ldots)_p$, $t = (0; t_1, \ldots)_p$ be the proper expansions of the real numbers s and t respectively and $\delta_p(s,t) = \inf\{k : s_k \neq t_k\} - 1$. The p-adic distance between s and t is

$$d_p(s,t) = p^{-\delta_p(s,t)}$$

This distance is an ultrametric distance.

Notations

If $(0; s_1, \ldots)_p$ is the proper expansion of s in base p, let $\theta_p(s) = \inf\{k : s_k \neq 0\} - 1$ and, if $t = (0; t_1, \ldots)_p$,

$$\gamma_p(s,t) = \theta_p(|s-t|).$$

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$$\gamma_p(s,t) = \theta_p(|s-t|).$$

However, $p^{-\gamma_p(s,t)}$ does not define a distance.
To compute $\gamma_p(s,t)$, we have to compute $\delta_p(s,t)$ and to check that s and t are not of the form

$$s = (0; s_1, \ldots, s_{n-1}, s_n, 0, \ldots, s_m, \ldots)_p,$$

$$t = (0; s_1, \ldots, s_{n-1}, s_n - 1, p-1, \ldots, t_m, \ldots)_p.$$

If it is the case, $\gamma_p(s,t) > \delta_p(s,t)$; otherwise we can suppose that $\gamma_p(s,t) = \delta_p(s,t)$.

A sufficient condition to belong to SM^α

Let f be a bounded function explicitly defined on $D \subset [0,1]$ by

$$f : D \to \mathbb{R}^+ x = (0; x_1, \dots)_p \mapsto f(x) = (\dots, y_0; y_1, \dots)_{p'}$$

and such that there exist $\eta > 0$ and a function g such that

$$\gamma_p(s,t) > \eta \Rightarrow \begin{cases} \gamma_{p'}(f(s), f(t)) > 0 \\ \gamma_{p'}(f(s), f(t)) = g(s, \gamma_p(s,t)) \end{cases} .$$

If

$$\underline{\alpha} = \liminf_{n\to\infty} \frac{g(x_0, n)}{n} \frac{\log p'}{\log p} < 1$$

and

$$\overline{\alpha} = \limsup_{n\to\infty} \frac{g(x_0, n)}{n} \frac{\log p'}{\log p} < 1,$$

then $\underline{h}(x_0) = \underline{\alpha}$ and $\overline{h}(x_0) = \overline{\alpha}$. If $\overline{\alpha} = \underline{\alpha}$ does not depend on x_0, $f \in SM^\alpha$.

The Devil’s staircase

The triadic Cantor set K is the set of real numbers x that can be written $x = (0; x_1, \dots)_3$, with $x_j \in \{0, 2\}\ \forall j$.

The Devil’s staircase is defined on K as follows,

$$D : K \to [0, 1] \qquad (0; x_1, \dots, x_j, \dots)_3 \mapsto (0; \frac{x_1}{2}, \dots, \frac{x_j}{2}, \dots)_2$$

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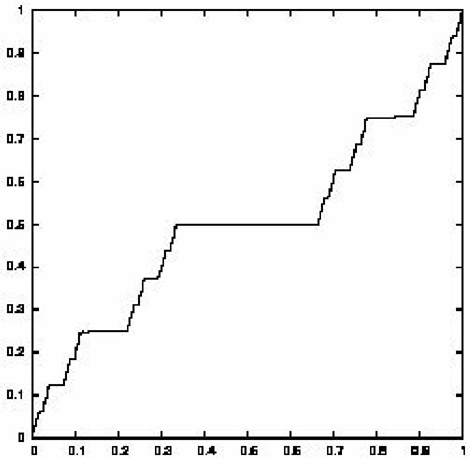
The Devil's staircase is a monofractal function with (finite) Hölder exponent $\log 2 / \log 3$.

Representation of the Devil's staircase



The Peano function (1890)

Let K the application defined by $Kj = 2 - j$ and set $K^0 j = j$. The Peano function is defined as follows,

$$P : [0,1] \to [0,1] \qquad x \mapsto \begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix},$$

where

$$p_1((0; x_1, \ldots)_3) = (0; x_1, K^{x_2} x_3, \ldots, K^{\sum_{k=1}^{j-1} x_{2k}} x_{2j-1}, \ldots)_3$$

and

$$p_2((0; x_1, \ldots)_3) = (0; K^{x_1} x_2, \ldots, K^{\sum_{k=0}^{j-1} x_{2k+1}} x_{2j}, \ldots)_3,$$

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The Peano function is a strongly monoHölder function with Hölder exponent $1/2$.

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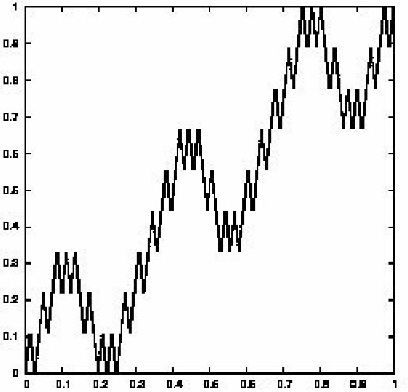

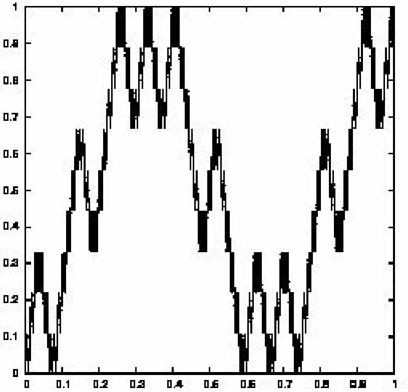

The Lebesgue function (1904)

The Lebesgue function is defined on K as follows

$$L: \; K \longrightarrow [0,1]^2$$

$$(0; x_1, x_2, \ldots, x_{2j-1}, x_{2j}, \ldots)_3 \longmapsto \begin{pmatrix} (0; \frac{x_1}{2}, \frac{x_3}{2}, \ldots, \frac{x_{2j-1}}{2}, \ldots)_2 \\ (0; \frac{x_2}{2}, \frac{x_4}{2}, \ldots, \frac{x_{2j}}{2}, \ldots)_2 \end{pmatrix}$$

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The Lebesgue function is a monofractal function with (finite) Hölder exponent $\log 2/(2 \log 3)$.

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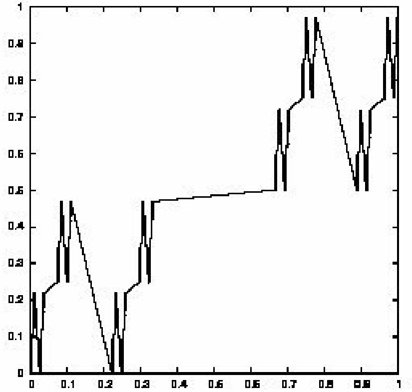

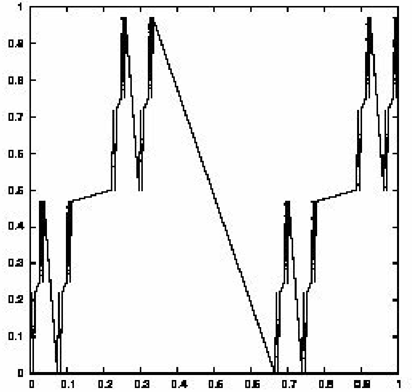

The Takagi functions (1903)

The Takagi functions are defined as follows,

$$f : x \mapsto \sum_{k=0}^{\infty} \frac{(2^{mk} x)}{2^{nk}},$$

with $m, n \in \mathbb{N}$, $m \geq n$ and where $(x) = \mathrm{dist}(x, \mathbb{Z})$.

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The Takagi functions are strongly monoHölder functions with Hölder exponent n/m.

The Hölder exponent of the Takagi functions

With the notation

$$\tilde{x}_j = \begin{cases} x_j & \text{if } x_{mk+1} = 0 \\ 1 - x_j & \text{if } x_{mk+1} = 1 \end{cases},$$

we have

$$f : (0; x_1, \ldots)_2 \mapsto \sum_{k=0}^{\infty} (0; \overbrace{0, \ldots, \tilde{x}_{mk+1}}^{nk+1}, \tilde{x}_{mk+2}, \ldots)_2.$$

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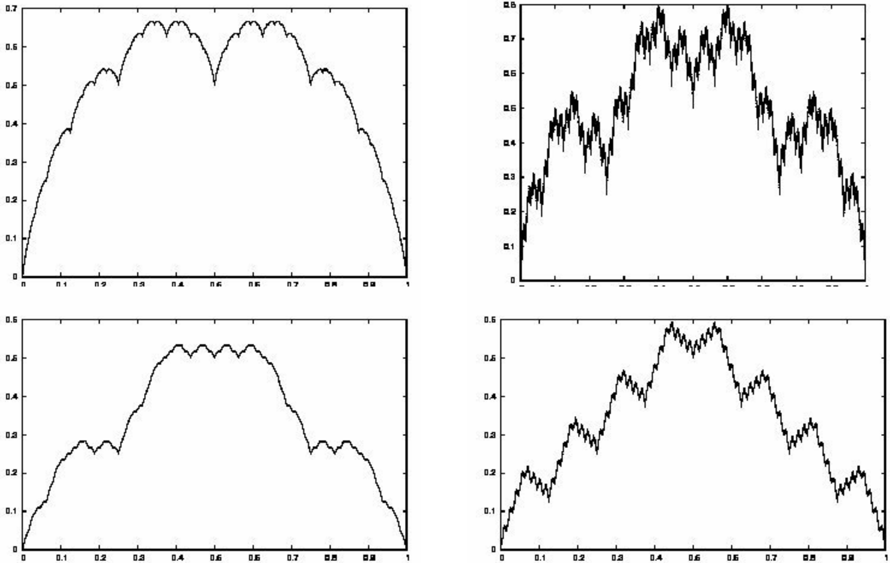
If $n = m$, one has to show that f is not differentiable.

Examples of Takagi functions



The Schoenberg function (1938)

Let g be the 2-periodic even function which satisfies

$$g(t) = \begin{cases} 0 & \text{if } t \in [0, \frac{1}{3}] \\ 3t - 1 & \text{if } t \in [\frac{1}{3}, \frac{2}{3}] \\ 1 & \text{if } t \in [\frac{2}{3}, 1] \end{cases} .$$

The Schoenberg function is defined by

$$S : [0,1] \longrightarrow [0,1]^2 \qquad x \mapsto \begin{pmatrix} s_1(x) \\ s_2(x) \end{pmatrix},$$

where

$$s_1(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{g(3^{2k}x)}{2^k}$$

and

$$s_2(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{g(3^{2k+1}x)}{2^k}.$$

The Hölder exponent of the Schoenberg function

If $x = (0; x_1, \ldots)_3$, we have

$$g(3^{2k}x) = \begin{cases} g_1((0; x_{2k+1}, \ldots)_3) & \text{if } (x_1, \ldots, x_{2k})_3 \text{ is even} \\ g_2((0; x_{2k+1}, \ldots)_3) & \text{if } (x_1, \ldots, x_{2k})_3 \text{ is odd} \end{cases}$$

where

$$g_1((0; x_{2k+1}, \ldots)_3) = \begin{cases} 0 & \text{if } x_{2k+1} = 0 \\ 1 & \text{if } x_{2k+1} = 2 \\ (0; x_{2k+2}, \ldots)_3 & \text{if } x_{2k+1} = 1 \end{cases}$$

and

$$g_2((0; x_{2k+1}, \ldots)_3) = \begin{cases} 0 & \text{if } x_{2k+1} = 2 \\ 1 & \text{if } x_{2k+1} = 0 \\ 1 - (0; x_{2k+2}, \ldots)_3 & \text{if } x_{2k+1} = 1 \end{cases},$$

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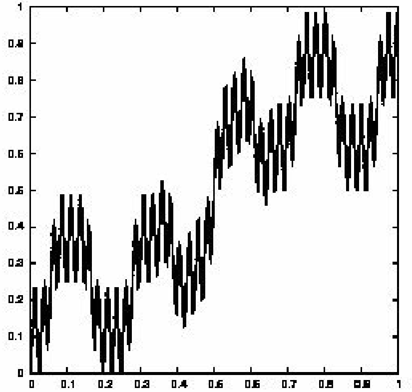
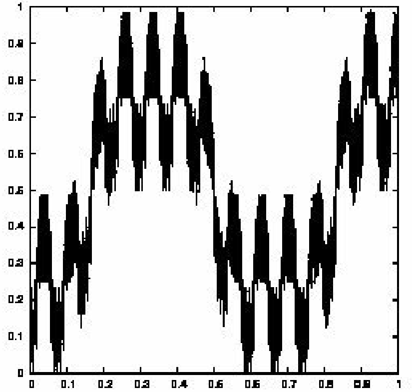
This allows to show that the Schoenberg function is strongly monoHölder with Hölder exponent $\log 2/(2\log 3)$.

Representation of the Schoenberg function





Other examples

One also can show that

- The Sierpiński space filling function (1912) is strongly monoHölder with Hölder exponent $1/2$
- The Petr function (1920) is strongly monoHölder with Hölder exponent $\log 2/\log 10$
- The Wunderlich function (1952) is strongly monoHölder with Hölder exponent $\log 2/\log 3$
- ...

Definition of f

Let $\{x\}$ denote the sawtooth function

$$\{x\} = x - [x] - 1/2$$

and define

$$f(x) = \begin{pmatrix} \frac{1}{2} + \sum_{k=0}^{\infty} a_k \{2^k x\} \\ \frac{1}{2} + \sum_{k=1}^{\infty} b_k \{2^k x\} \end{pmatrix},$$

where $a_{2k} = 2^{-k}$, $a_{2k+1} = -2^{-k-1}$ and $b_k = -a_k$.

A proposition

Proposition

Let $x \in [0,1)$ and $x = (0; x_1, \ldots)_2$ be its proper expansion in the binary system. One has

$$f(x) = \begin{pmatrix} (0; x_1, x_3, \ldots, x_{2k+1}, \ldots)_2 \\ (0; x_2, x_4, \ldots, x_{2k}, \ldots)_2 \end{pmatrix}.$$

Preliminaries

Let $x = (0; x_1, \ldots)_p$ be a non p-adic rational number and $\delta(k) = \sup\{l : \forall l' \leq l, x_{k+l'} = x_k\} - 1$.
The sequence $(m_j)_j$ is defined recursively:
$m_1 = \inf\{l : x_l = 0 \text{ or } x_l = p-1\}$, and
$m_j = \inf\{l > m_{j-1} + \delta(m_{j-1}) : x_l = 0 \text{ or } x_l = p-1\}$. One also defines the sequence $(\delta_j)_j$ by $\delta_j = \delta(m_j)$.
Finally,

$$\rho_p(x) = \limsup_{j \to \infty} \delta_k / m_k;$$

if x is a p-adic rational, one sets $\rho_p(x) = \infty$.

About the approximation of a non p-adic rational by p-adic rationals

Proposition

If x is not a p-adic rational, the equation

$$|x - \frac{k}{p^l}| \leq (\frac{1}{p^l})^{\phi} \qquad (k \wedge p = 1)$$

has an infinity of solutions if and only if $\phi \leq \rho_p(x) + 1$.

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In what follows, we will denote by $\phi(x)$ the critical exponent $\phi(x) = \rho_2(x) + 1$.

About the pointwise regularity of f

Proposition

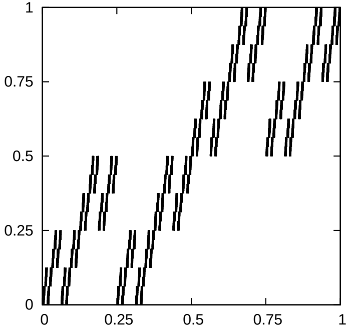
One has

$$\underline{h}(x) = \frac{1}{2\phi(x)}$$

and

$$\overline{h}(x) = \begin{cases} 1/2 & \text{if } x \text{ is not a } p\text{-adic rational} \\ 0 & \text{otherwise} \end{cases} .$$

Representation of $f(x)$



For Further Reading

S. Jaffard and S. Nicolay,
Pointwise smoothness of space-filling functions,
ACHA, to appear.

S. Nicolay,
A sufficient condition for a function to be strongly Hölderian,
submitted.

M. Clausel and S. Nicolay,
A multifractal formalism for pointwise anti-Hölderian irregularity,
submitted.