About the regularity and the irregularity of some nondifferentiable functions

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Hölder-regularity

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a locally bounded function, $x \in \mathbb{R}$ and $\alpha \geq 0$; $f \in C^\alpha(x)$ if there exist $R, C > 0$ and a polynomial P_x of degree less than α such that

$$|h| < R \Rightarrow |f(x+h) - P_x(h)| \leq C|h|^\alpha. \qquad (*)$$

A function f belongs to C^α if there exists $C > 0$ such that $(*)$ holds for all x with $R = \infty$.

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The Hölder exponent of f at x is $\underline{h}(x) = \sup\{\alpha : f \in C^\alpha(x)\}$

Hölder-irreguarity

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Let $f : \mathbb{R} \to \mathbb{R}$ be a locally bounded function, $x \in \mathbb{R}$ and $0 \leq \alpha < 1$; $f \in I^\alpha(x)$ if there exist $R, C > 0$ such that

$$\sup_{y,y' \in B(x,r)} |f(y) - f(y')| \geq C r^\alpha, \quad \forall r \leq R,$$

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Definition

The upper Hölder exponent of f at x is $\overline{h}(x) = \inf\{\alpha : f \in I^\alpha(x)\}$.

A function f is strongly monoHölder of exponent α ($f \in SM^\alpha$) if $\overline{h}(x) = \underline{h}(x) = \alpha$, $\forall x$.

Representation of real numbers in base p

Notation

Let $p \in \mathbb{N}$, $p > 1$. For a sequence of integers satisfying $0 \leq x_j < p$, we will use the following notation

$$(0; x_1, \ldots, x_j, \ldots)_p$$

to denote one of the expansions of the real number

$$x = \sum_{k=1}^{\infty} \frac{x_k}{p^k}. \qquad (*)$$

If there is no j such that $x_i = p - 1 \ \forall i \geq j$, $(*)$ is the proper expansion of x in base p.

The p-adic distances

Let $s = (0; s_1, \ldots)_p$, $t = (0; t_1, \ldots)_p$ be the proper expansions of the real numbers s and t respectively and

$$\delta_p(s,t) = \inf\{k : s_k \neq t_k\} - 1.$$

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Definition

The p-adic distance between s and t is

$$d_p(s, t) = p^{-\delta_p(s,t)}$$

This distance is an ultrametric distance.

p-adic Hölder spaces

Definition

Let $\alpha > 0$. A locally bounded function $f : [0,1] \to \mathbb{R}$ belongs to $C_p^\alpha(x)$ if there exists $C > 0$ and a polynomial P_x of degree less than α such that

$$|f(x+h) - P_x(h)| \leq C d_p(x, x+h)^\alpha. \qquad (*)$$

A function belongs to C_p^α if there exists $C > 0$ such that $(*)$ holds for any x.
The p-adic Hölder exponent of f at x is

$$h_p(x) = \sup\{\alpha : f \in C_p^\alpha(x)\}.$$

A sufficient condition to belong to $C_p^\alpha(x)$

Let f be a bounded function explicitly defined on $D \subset [0,1]$ by

$$f : D \to \mathbb{R}^+ x = (0; x_1, \dots)_p \mapsto f(x) = (\dots, y_0; y_1, \dots)_{p'}$$

and such that there exist ϵ and a function g such that

$$d_p(s,t) < \epsilon \Rightarrow \delta_{p'}(f(s), f(t)) = g(s, \delta_p(s,t)).$$

If

$$\alpha = \liminf_{u \to \infty} \frac{g(x_0, u)}{u} \frac{\log p'}{\log p} < 1, \qquad (*)$$

then $h_p(x_0) = \alpha$.

p-adic monoHölderianity and monoHölderianity

Let F be defined as follows

$$F : [0,1] \to [0,1]^2 \quad x \mapsto \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

where $F(1) = (1,1)$ and, if $(0; x_1, \ldots)_2$ is the proper expansion of x,

$$f_1(x) = (0; x_1, x_3, \ldots, x_{2j-1}, \ldots)_2$$

and

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A direct application of the previous proposition shows that F is a dyadic monoHölder function with dyadic Hölder exponent $1/2$. However, F is not continuous.

A proximity index

Notations

If $(0; s_1, \ldots)_p$ is the proper expansion of s in base p, let $\theta_p(s) = \inf\{k : s_k \neq 0\} - 1$ and, if $t = (0; t_1, \ldots)_p$,

$$\gamma_p(s, t) = \theta_p(|s - t|).$$

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$$\gamma_p(s,t) = \theta_p(|s-t|).$$

However, $p^{-\gamma_p(s,t)}$ does not define a distance.
To compute $\gamma_p(s,t)$, we have to compute $\delta_p(s,t)$ and to check that s and t are not of the form

$$s = (0; s_1, \ldots, s_{n-1}, s_n, 0, \ldots, s_m, \ldots)_p,$$

$$t = (0; s_1, \ldots, s_{n-1}, s_n - 1, p-1, \ldots, t_m, \ldots)_p.$$

If it is the case, $\gamma_p(s,t) > \delta_p(s,t)$; otherwise we can suppose that $\gamma_p(s,t) = \delta_p(s,t)$.

The β expansion of a number

A number x can be expressed in a real basis $\beta > 1$ in the following way. Let $\epsilon(x) = \beta(x - \lfloor x \rfloor)$. The sequence $(x_n)_n$ defined as follows

$$x_1 = \lfloor \epsilon(x) \rfloor, \qquad x_n = \lfloor \epsilon \circ \cdots \circ \epsilon(x) \rfloor$$

gives the expansion of x in base β.

The greedy algorithm condition

These representations can be generalized in the following way. Let $\beta > 1$ a real number and $V_\beta = \{0\} \cup [1, \lceil \beta \rceil - 1]$. A sequence $(x_n)_n$ defined on V_β satisfies the greedy algorithm condition if the following inequalities hold,

$$\sum_{n=N}^{\infty} \frac{x_n}{\beta^n} < \beta^{1-N} \quad \forall N.$$

The β-sequences

Definition

A β-sequence is a sequence $(x_n)_n$ defined on V_β satisfying the greedy algorithm condition. A β-sequence defines a real positive number x in the following way,

$$x = \sum_{n=1}^{\infty} \frac{x_n}{\beta^n}.$$

Conversely, a real positive number can be represented in this fashion, using for example, the greedy algorithm.

The β-sequences and γ

One can define $\theta_\beta(s)$ and $\gamma_\beta(s,t)$ in the same way as before.

The β-sequences and γ

One can define $\theta_\beta(s)$ and $\gamma_\beta(s,t)$ in the same way as before. It is easy to see that these quantities do not depend on the β-sequences chosen to represent the numbers.

A criterion for the regularity

Let f be a bounded function explicitly defined on $D \subset [0,1]$ by

$$f : D \to \mathbb{R}^+ x = (0; x_1, \ldots)_\beta \mapsto f(x) = (\ldots, y_0; y_1, \ldots)_{\beta'}$$

and such that there exist $\eta > 0$ and a function g such that

$$\gamma_\beta(s,t) > \eta \Rightarrow \gamma_{\beta'}(f(s), f(t)) = g(s, \gamma_\beta(s,t)).$$

If

$$\underline{\alpha} = \liminf_{n\to\infty} \frac{g(x_0,n)}{n} \frac{\log \beta'}{\log \beta} < 1$$

and

$$\overline{\alpha} = \limsup_{n\to\infty} \frac{g(x_0,n)}{n} \frac{\log \beta'}{\log \beta} < 1,$$

then $\underline{h}(x_0) = \underline{\alpha}$ and $\overline{h}(x_0) = \overline{\alpha}$.

The Devil’s staircase

The triadic Cantor set K is the set of real numbers x that can be written $x = (0; x_1, \dots)_3$, with $x_j \in \{0, 2\}$ $\forall j$.

The Devil’s staircase is defined on K as follows,

$$D : K \to [0, 1] \qquad (0; x_1, \dots, x_j, \dots)_3 \mapsto (0; \frac{x_1}{2}, \dots, \frac{x_j}{2}, \dots)_2$$

and can be linearly extended on $[0, 1]$.

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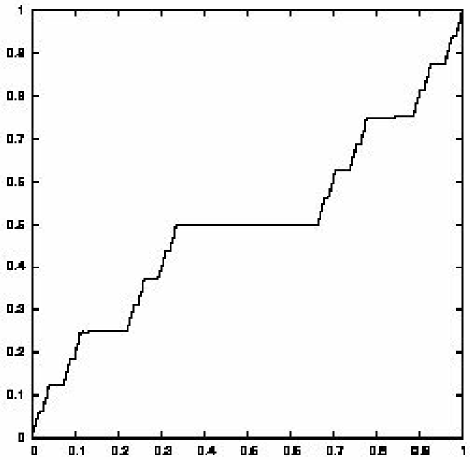
The Devil's staircase is a monofractal function with (finite) Hölder exponent $\log 2 / \log 3$.

Representation of the Devil's staircase



The Peano function (1890)

Let K the application defined by $Kj = 2 - j$ and set $K^0 j = j$. The Peano function is defined as follows,

$$P : [0,1] \to [0,1] \qquad x \mapsto \begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix},$$

where

$$p_1((0; x_1, \ldots)_3) = (0; x_1, K^{x_2} x_3, \ldots, K^{\sum_{k=1}^{j-1} x_{2k}} x_{2j-1}, \ldots)_3$$

and

$$p_2((0; x_1, \ldots)_3) = (0; K^{x_1} x_2, \ldots, K^{\sum_{k=0}^{j-1} x_{2k+1}} x_{2j}, \ldots)_3,$$

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The Peano function is a strongly monoHölder function with Hölder exponent $1/2$.

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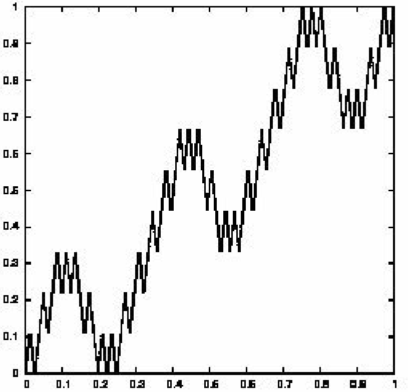

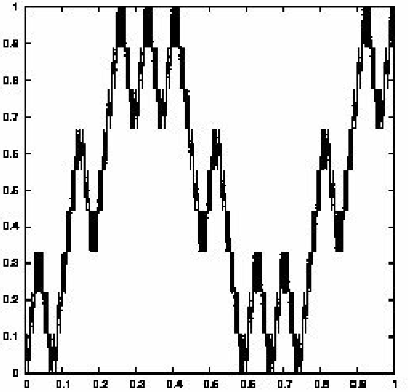

The Lebesgue function (1904)

The Lebesgue function is defined on K as follows

$$L: \; K \longrightarrow [0,1]^2$$

$$(0; x_1, x_2, \ldots, x_{2j-1}, x_{2j}, \ldots)_3 \mapsto \begin{pmatrix} (0; \frac{x_1}{2}, \frac{x_3}{2}, \ldots, \frac{x_{2j-1}}{2}, \ldots)_2 \\ (0; \frac{x_2}{2}, \frac{x_4}{2}, \ldots, \frac{x_{2j}}{2}, \ldots)_2 \end{pmatrix}$$

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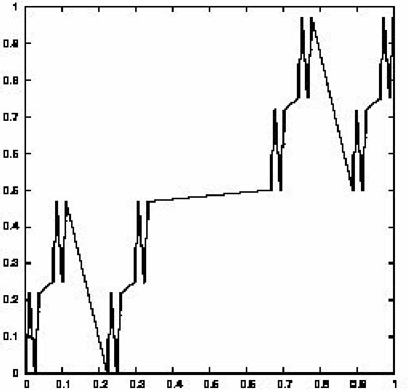
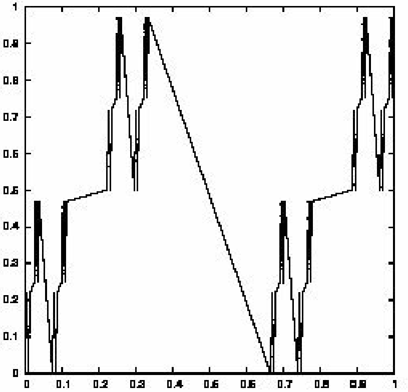
The Lebesgue function is a monofractal function with (finite) Hölder exponent $\log 2/(2\log 3)$.

Representation of the Lebesgue function





The Takagi functions (1903)

The Takagi functions are defined as follows,

$$f : x \mapsto \sum_{k=0}^{\infty} \frac{(2^{mk}x)}{2^{nk}},$$

with $m, n \in \mathbb{N}$, $m \geq n$ and where $(x) = \text{dist}(x, \mathbb{Z})$.

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The Takagi functions are strongly monoHölder functions with Hölder exponent n/m.

The Hölder exponent of the Takagi functions

With the notation

$$\tilde{x}_j = \begin{cases} x_j & \text{if } x_{mk+1} = 0 \\ 1 - x_j & \text{if } x_{mk+1} = 1 \end{cases},$$

we have

$$f : (0; x_1, \ldots)_2 \mapsto \sum_{k=0}^{\infty} (0; \overbrace{0, \ldots, \tilde{x}_{mk+1}}^{nk+1}, \tilde{x}_{mk+2}, \ldots)_2.$$

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This implies that, if $m > n$, f is a monoHölder function with Hölder exponent n/m.

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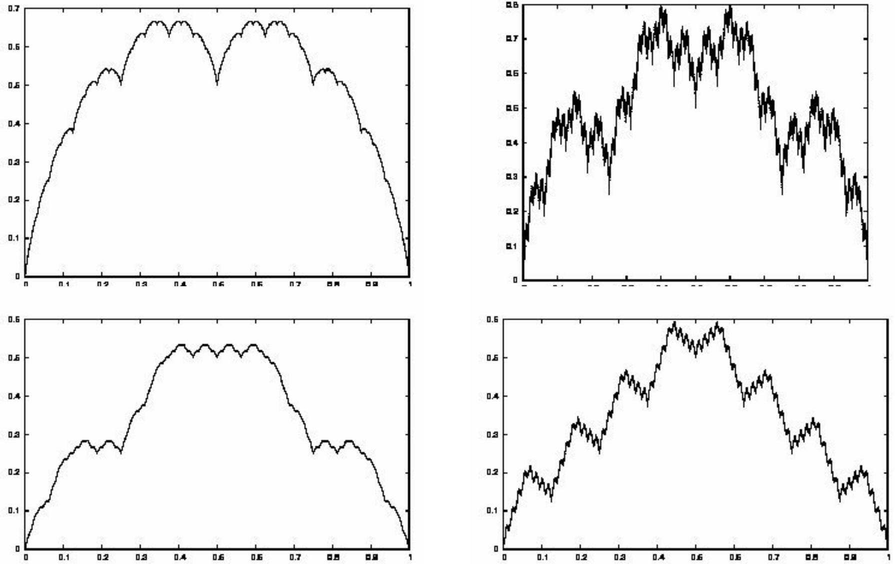
If $n = m$, one has to show that f is not differentiable.

Examples of Takagi functions



The Schoenberg function (1938)

Let g be the 2-periodic even function which satisfies

$$g(t) = \begin{cases} 0 & \text{if } t \in [0, \frac{1}{3}] \\ 3t - 1 & \text{if } t \in [\frac{1}{3}, \frac{2}{3}] \\ 1 & \text{if } t \in [\frac{2}{3}, 1] \end{cases} .$$

The Schoenberg function is defined by

$$S : [0,1] \longrightarrow [0,1]^2 \qquad x \mapsto \begin{pmatrix} s_1(x) \\ s_2(x) \end{pmatrix},$$

where

$$s_1(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{g(3^{2k} x)}{2^k}$$

and

$$s_2(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{g(3^{2k+1} x)}{2^k}.$$

The Hölder exponent of the Schoenberg function

If $x = (0; x_1, \ldots)_3$, we have

$$g(3^{2k}x) = \begin{cases} g_1((0; x_{2k+1}, \ldots)_3) & \text{if } (x_1, \ldots, x_{2k})_3 \text{ is even} \\ g_2((0; x_{2k+1}, \ldots)_3) & \text{if } (x_1, \ldots, x_{2k})_3 \text{ is odd} \end{cases}$$

where

$$g_1((0; x_{2k+1}, \ldots)_3) = \begin{cases} 0 & \text{if } x_{2k+1} = 0 \\ 1 & \text{if } x_{2k+1} = 2 \\ (0; x_{2k+2}, \ldots)_3 & \text{if } x_{2k+1} = 1 \end{cases}$$

and

$$g_2((0; x_{2k+1}, \ldots)_3) = \begin{cases} 0 & \text{if } x_{2k+1} = 2 \\ 1 & \text{if } x_{2k+1} = 0 \\ 1 - (0; x_{2k+2}, \ldots)_3 & \text{if } x_{2k+1} = 1 \end{cases},$$

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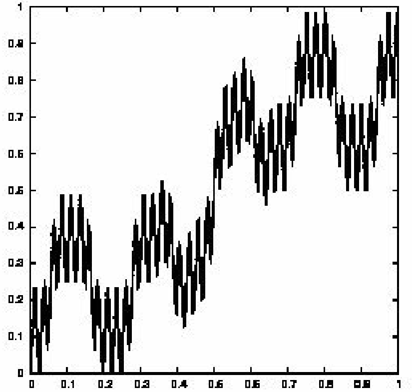
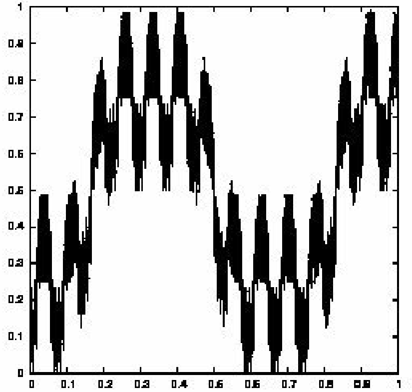
This allows to show that the Schoenberg function is strongly monoHölder with Hölder exponent $\log 2/(2\log 3)$.

Representation of the Schoenberg function





Other examples

One also can show that

- The Sierpiński space filling function (1912) is strongly monoHölder with Hölder exponent $1/2$
- The Petr function (1920) is strongly monoHölder with Hölder exponent $\log 2/\log 10$
- The Wunderlich function (1952) is strongly monoHölder with Hölder exponent $\log 2/\log 3$
- ...

Definition of F

Let F be (re)defined as follows

$$F : [0,1] \to [0,1]^2 \quad x \mapsto \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

where $F(1) = (1,1)$ and, if $(0; x_1, \ldots)_2$ is the proper expansion of x,

$$f_1(x) = (0; x_1, x_3, \ldots, x_{2j-1}, \ldots)_2$$

and

$$f_2(x) = (0; x_2, x_4, \ldots, x_{2j}, \ldots)_2.$$

Preliminaries

Let $x = (0; x_1, \dots)_p$ be a non p-adic rational number and $\delta(k) = \sup\{l : \forall l' \leq l, x_{k+l'} = x_k\} - 1$.
The sequence $(m_j)_j$ is defined recursively:
$m_1 = \inf\{l : x_l = 0 \text{ or } x_l = p-1\}$, and
$m_j = \inf\{l > m_{j-1} + \delta(m_{j-1}) : x_l = 0 \text{ or } x_l = p-1\}$. One also defines the sequence $(\delta_j)_j$ by $\delta_j = \delta(m_j)$.
Finally,

$$\rho_p(x) = \limsup_{k \to \infty} \delta_k / m_k;$$

if x is a p-adic rational, one sets $\rho_p(x) = \infty$.

About the approximation of a non p-adic rational by p-adic rationals

Proposition

If x is not a p-adic rational, the equation

$$|x - \frac{k}{p^l}| \leq (\frac{1}{p^l})^{\phi} \qquad (k \wedge p = 1)$$

has an infinity of solutions if and only if $\phi \leq \rho_p(x) + 1$.

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In what follows, we will denote by $\phi(x)$ the critical exponent $\phi(x) = \rho_2(x) + 1$.

About the pointwise regularity of F

Proposition

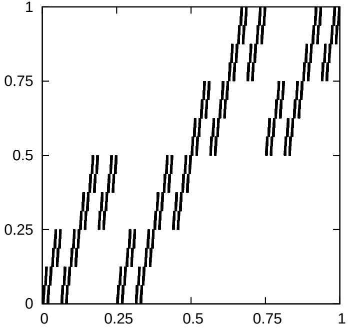
One has

$$\underline{h}(x) = \frac{1}{2\phi(x)}$$

and

$$\overline{h}(x) = \begin{cases} 1/2 & \text{if } x \text{ is not a dyadic rational} \\ 0 & \text{otherwise} \end{cases} .$$

Representation of $f_1(x)$



The Levy function

Let $\{x\} = x - \lfloor x \rfloor - 1/2$. Levy's function is defined as follows,

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Proposition

On has

$$\underline{h}(x) = \frac{1}{\phi(x)}$$

and

$$\overline{h}(x) = \begin{cases} 1 & \text{if } x \text{ is not a dyadic rational} \\ 0 & \text{otherwise} \end{cases}.$$

The Hausdorff measure

Let

$$H_\epsilon^\delta(E) = \inf\{\sum_{k=1}^{\infty} |E_k|^\delta : E \subset \bigcup_{k=1}^{\infty} E_k,\ |E_k| \leq \epsilon\}.$$

Clearly, H_ϵ^δ is an outer measure.

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Definition

The outer measure H^δ defined as

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is a metric outer measure. Its restriction to the σ-algebra of the H^δ-measurable sets defines the Hausdorff measure.

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Since the outer measure H^δ is metric, the algebra includes the Borelian sets.

The Hausdorff dimension

The Hasudorff measure is decreasing as δ goes to infinity. Moreover $H^\delta(E) > 0$ implies $H^{\delta'}(E) = \infty$ if $\delta' < \delta$. The following definition is thus meaningful.

Definition

The Hausdorff dimension $\dim_H(E)$ of a set $E \subset \mathbb{R}^d$ is defined as follows

$$\dim_H(E) = \sup\{\delta : H^\delta(E) = \infty\}.$$

With this definition, $\dim_H(\emptyset) = -\infty$.

Hausdorff dimension and p-adic approximations

Proposition

For any integer base, we have

$$\dim_H(\{x : \phi(x) = \alpha\}) = \frac{1}{\alpha}.$$

Multifractal analysis

The lower isoHölder sets are the sets

$$\underline{E}_H = \{x : \underline{h}(x) = H\}$$

Definition

The lower spectrum of singuarities of f is the function

$$\underline{d} : \mathbb{R}^+ \cup \{\infty\} \to \mathbb{R}^+ \cup \{-\infty\} \quad H \mapsto \dim_H(\underline{E}_H).$$

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The upper spectrum of singularities can be defined in a totally analogous ways.

Some results

Proposition

For the function F, we have

$$\underline{d}(H) = \begin{cases} 2H & \text{if } H \in [0, 1/2] \\ -\infty & \text{else} \end{cases}$$

and for the function L,

$$\underline{d}(H) = \begin{cases} H & \text{if } H \in [0, 1] \\ -\infty & \text{else} \end{cases} .$$

The upper spectra of singularities are trivial.