Generalized pointwise Hölder spaces

D. Kreit & S. Nicolay

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A function $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ belongs to $\Lambda^s(x_0)$ if there exists a polynomial of degree at most s s.t.

$$\sup_{|h| \leq 2^{-j}} |f(x_0+h) - P(h)| \leq C 2^{-js},$$

for j sufficiently large.

Is it possible to be sharper and replace the sequence $(2^{-js})_j$ with a more general sequence $\sigma = (\sigma_j)_j$:
$f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ belongs to $\Lambda^{\sigma,M}(x_0)$ if there exists a polynomial of degree at most M s.t.

$$\sup_{|h| \leq 2^{-j}} |f(x_0+h) - P(h)| \leq C \sigma_j,$$

for j sufficiently large.

$$
\begin{tikzcd}
B^s_{p,q}(\mathbb{R}^d) \arrow[r, "\text{generalization}"] & B^{1/\sigma}_{p,q}(\mathbb{R}^d), \text{ where } \sigma \text{ is some sequence} \\
\Lambda^s(\mathbb{R}^d) = B^s_{\infty,\infty}(\mathbb{R}^d) \arrow[u, "\text{generalization}"] \arrow[r, "\text{generalization}"] & \Lambda^\sigma(\mathbb{R}^d) = B^{1/\sigma}_{\infty,\infty} \arrow[u, "\text{generalization}"]
\end{tikzcd}
$$

$$
\Lambda^\sigma(\mathbb{R}^d) \xrightarrow{\text{pointwise version}} \Lambda^\sigma(x_0)
$$

A sequence of real positive numbers is called admissible if

$$\frac{\sigma_{j+1}}{\sigma_j}$$

is bounded.
For such a sequence, we set

$$\underline{s}(\sigma) = \lim_j \frac{\log_2(\inf_{k\in\mathbb{N}} \frac{\sigma_{j+k}}{\sigma_j})}{j}$$

and

$$\overline{s}(\sigma) = \lim_j \frac{\log_2(\sup_{k\in\mathbb{N}} \frac{\sigma_{j+k}}{\sigma_j})}{j}.$$

- the open unit ball centered at the origin is denoted B,
- the set of polynomials of degree at most n is denoted $\mathbf{P}[n]$,
- $[s] = \sup\{n \in \mathbb{Z} : n \leq s\}$,
- if f is defined on $\mathbb{R}^d$,

$$\Delta_h^1 f(x) = f(x+h) - f(x)$$

and

$$\Delta_h^{n+1} f(x) = \Delta_h^1 \Delta_h^n f(x),$$

for any $x, h \in \mathbb{R}^d$

Definition

Les $s > 0$ and σ an admissible sequence ; a function $f \in L^\infty(\mathbb{R}^d)$ belongs to $\Lambda^{\sigma,M}(\mathbb{R}^d)$ iff there exists $C > 0$ s.t.

$$\sup_{|h|\leq 2^{-j}} \|\Delta_h^{[M]+1} f\|_\infty \leq C\sigma_j$$

Proposition

Les $s > 0$ and σ an admissible sequence ; a function $f \in L^\infty(\mathbb{R}^d)$ belongs to $\Lambda^{\sigma,s}(\mathbb{R}^d)$ iff there exists $C > 0$ s.t.

$$\inf_{P\in \mathbf{P}_{[M]}} \|f - P\|_{L^\infty(2^{-j}B+x_0)} \leq C\sigma_j,$$

for any $x_0 \in \mathbb{R}^d$ and any $j \in \mathbb{N}$.

Definition

A function $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ belongs to $\Lambda^{\sigma,M}(x_0)$ iff there exists $C > 0$ and $J \in \mathbb{N}$ s.t.

$$\inf_{P \in \mathbf{P}[M]} \|f - P\|_{L^\infty(2^{-j}B + x_0)} \leq C\sigma_j,$$

for any $j \geq J$.

Definition

A function $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ belongs to $\Lambda^{\sigma,M}(x_0)$ iff there exists $C > 0$ and $J \in \mathbb{N}$ s.t. for any $j \geq J$, there exists $P_j \in \mathbf{P}[M]$ for which

$$\sup_{|h| < 2^{-j}} |f(x_0 + h) - P_j(x_0 + h)| \leq C\sigma_j.$$

A function $f \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^d)$ belongs to $\Lambda^s(x_0)$ ($s \in \mathbb{R}$) iff there exists $C > 0$, a polynomial P of degree less than s and $J \in \mathbb{N}$ s.t. for any $j \geq J$,

$$\sup_{|h|<2^{-j}} |f(x_0+h) - P(x_0+h)| \leq C 2^{-js}.$$

There is one polynomial, independant from the scale.

Lemma

If $M < \underline{s}(\sigma^{-1})$, the sequence of polynomials occuring in the definition of $\Lambda^{\sigma,M}(x_0)$ satisfies

$$\|D^\beta P_k - D^\beta P_j\|_{L^\infty(x_0+2^{-k}B)} \leq C2^{j|\beta|}\sigma_j,$$

for any multi-index β s.t. $|\beta| \leq M$ and $k \geq j \geq J$.

In particular, $(D^\beta P(x_0))_j$ is a Cauchy sequence.

Lemma

If $M < \underline{s}(\sigma^{-1})$, and $(P_j)_j$ is a sequence of polynomials in the definition of $\Lambda^{\sigma,M}(x_0)$, for any multi-index β s.t. $|\beta| \leq M$, the limit

$$f_\beta(x_0) = \lim_j D^\beta P_j(x_0)$$

is independant of the chosen sequence $(P_j)_j$.

$f_\beta(x_0)$ is the Peano derivative of order β of f at x_0.

Theorem

If $M < \underline{s}(\sigma^{-1})$, then $f \in \Lambda^{\sigma,M}(x_0)$ iff there exist $C > 0$ and a polynomial $P \in \mathbf{P}[M]$ s.t.

$$\|f - P\|_{L^\infty(x_0+2^{-j}B)} \leq C\sigma_j,$$

for j sufficiently large. The polynomial is unique.

One has

$$P(x) = \sum_{|\beta|\leq M} f_\beta(x_0)\frac{(x-x_0)^\beta}{|\beta|!}.$$

For $s \in (0, \infty)$, let

- $\sigma_j = 2^{-js}$
- $M = [\underline{s}(\sigma^{-1})] = [s]$ if $s \notin \mathbb{N}$
- $M = s - 1$ if $s \in \mathbb{N}$

We have

$$\Lambda^s(x_0) = \Lambda^{\sigma,M}(x_0).$$

Corollary

If $M < \underline{s}(\sigma^{-1})$, one has

$$\Lambda^{\sigma,M}(x_0) \subset \Lambda^M(x_0).$$

Let

$$B_h^M(x_0, j) = \{x : [x, x + (M+1)h] \subset x_0 + 2^{-j}B\}.$$

Proposition

Let $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$; one has $f \in \Lambda^{\sigma,M}(x_0)$ iff there exist $C, J > 0$ s.t.

$$\sup_{h \in B_j} \|\Delta_h^{M+1} f\|_{L^\infty(B_h^M(x_0,j))} \leq C\sigma_j,$$

for any $j \geq J$.

Let ρ a radial function s.t. $\rho \in C_c^\infty(B)$, $\rho(B) \subset [0,1]$ and $\|\rho\|_1 = 1$. One sets, for any $j \in \mathbb{N}_0$,

$$\rho_j = 2^{-jd}\rho(\cdot/2^j).$$

Lemma

Let $N \in \mathbb{N}_0$; if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfies

$$\sup_{k \geq j} \|f * \rho_k - f\|_{L^\infty(x_0 + 2^{-j}B)} \leq C\sigma_j,$$

for $j \geq J$, then, for any multi-index β s.t. $|\beta| \leq N$, one has

$$\|D^\beta(f * \rho_j - f * \rho_{j-1})\|_{L^\infty(x_0+2^{-j}B)} \leq C2^{jN}\sigma_j,$$

for any $j \geq J$.

Proposition

If $f \in \Lambda^{\sigma,M}(x_0)$, then there exists $\Phi \in C_c^\infty(\mathbb{R}^d)$ s.t.

$$\sup_{k \geq j} \|f - f * \Phi_k\|_{L^\infty(x_0 + 2^{-j}B)} \leq C\sigma_j,$$

for j sufficiently large.
Conversely, if $\sigma \to 0$, $f \in \Lambda^\epsilon(\mathbb{R}^d)$ for some $\epsilon > 0$ and f satisfies the previous relation for some function $\Phi \in C_c^\infty(\mathbb{R}^d)$, then $f \in \Lambda^{\sigma,M}(x_0)$ for any M s.t. $M + 1 > \overline{s}(\sigma^{-1})$.

Under some general conditions, there exist a function ϕ and $2^d - 1$ functions $\psi^{(i)}$ called wavelets s.t.

$$\{\phi(\cdot - k) : k \in \mathbb{Z}^d\} \bigcup \{\psi^{(i)}(2^j \cdot -k) : k \in \mathbb{Z}^d, j \in \mathbb{N}_0\}$$

forms an orthogonal basis of $L^2(\mathbb{R}^d)$.
Any function $f \in L^2(\mathbb{R}^d)$ can be decomposed as follows,

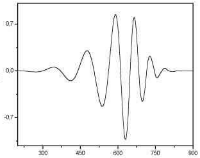
$$f(x) = \sum_{k \in \mathbb{Z}^d} C_k \phi(x-k) + \sum_{j \geq 0, k \in \mathbb{Z}^d, 1 \leq i < 2^d} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k),$$

with

$$C_k = \int f(x)\phi(x-k)\, dx, \quad c_{j,k}^{(i)} = 2^{dj} \int f(x)\psi^{(i)}(2^j x - k)\, dx.$$

We assume

- $\phi, \psi^{(i)} \in C^n(\mathbb{R}^d)$ with $n > M$,
- $D^\beta \phi$, $D^\beta \psi^{(i)}$ $(|\beta| \leq n)$ have fast decay,
- $\mathrm{supp}(\psi^{(i)}) \subset 2^{-j_0} B$ for some j_0.



We set

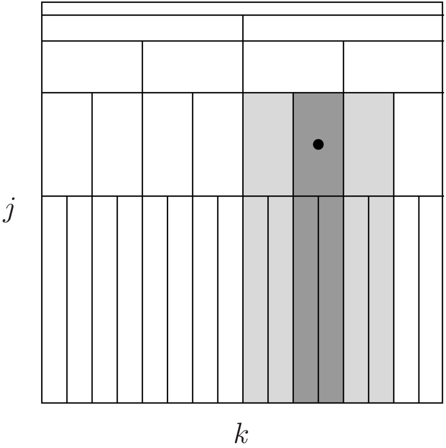
- $\lambda = \lambda(i, j, k) = \frac{k}{2^j} + \frac{i}{2^{j+1}} + [0, \frac{1}{2^{j+1}})^d$
- $c_\lambda = c_{j,k}^{(i)}$
- $\psi_\lambda = \psi^{(i)}(2^j \cdot -k)$.

The wavelet leaders are defined by

$$d_\lambda = \sup_{\lambda' \subset \lambda} |c_{\lambda'}|$$

If 3λ denotes the 3^d dyadic cubes adjacent to λ and $\lambda_j(x_0)$ the dyadic cube of length 2^{-j} containing x_0, one sets

$$d_j(x_0) = \sup_{\lambda \subset 3\lambda_j(x_0)} d_\lambda$$


j
k

Theorem

If $f \in \Lambda^{\sigma,M}(x_0)$, then there exists $C > 0$ s.t.

$$d_j(x_0) \leq C\sigma_j,$$

for j sufficiently large.
Conversely, if $\sigma \to 0$, $f \in \Lambda^{\epsilon}(\mathbb{R}^d)$ for some $\epsilon > 0$ and f satisfies the previous relation, then $f \in \Lambda^{\tau,M}(x_0)$, where

- τ is the sequence defined by $\tau_j = \sigma_j |\log_2 \sigma_j|$,
- M is any number satisfying $M + 1 > \overline{s}(\sigma^{-1})$.

If, for any $s > 0$, $\sigma^{(s)}$ is an admissible sequence, the application

$$\sigma^{(\cdot)} : s > 0 \mapsto \sigma^{(s)}$$

is called a family of admissible sequences.
A family of admissible sequences is decreasing for x_0 if

$$s < t \Rightarrow \Lambda^{\sigma^{(t)},[t]}(x_0) \subset \Lambda^{\sigma^{(s)},[s]}(x_0).$$

Let $\sigma^{(\cdot)}$ a family of decreasing sequences for x_0 and $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$; the Hölder exponent of f at x_0 for $\sigma^{(\cdot)}$ is

$$h_f^{\sigma^{(\cdot)}}(x_0) = \sup\{s > 0 : f \in \Lambda^{\sigma^{(s)},[s]}(x_0)\}.$$

Let

$$\overline{\Theta}^{(m)} = \sup_{k\in\mathbb{N}} \frac{\sigma_{k+1}^{(m)}}{\sigma_k^{(m)}}, \quad \underline{\Theta}^{(m)} = \inf_{k\in\mathbb{N}} \frac{\sigma_{k+1}^{(m)}}{\sigma_k^{(m)}},$$

Proposition

A family of admissible sequences is decreasing for x_0 if it satisfies the following conditions :

- if $m \leq s < t < m+1$ with $m \in \mathbb{N}_0$, $\sigma_j^{(t)} \leq C\sigma_j^{(s)}$ for j sufficiently large
- for any $m \in \mathbb{N}$, at least one of the following conditions is satisfied : there exists $\epsilon_0 > 0$ s.t. for any $\epsilon \in (0, \epsilon_0)$,

$\sigma_j^{(m)} \leq C\sigma_j^{(m-\epsilon)}$	$2^{-jm} \leq C\sigma_j^{(m-\epsilon)}$
if $1 < 2^m\overline{\Theta}^{(m)}$: $(\overline{\Theta}^{(m)})^j \leq C\sigma_j^{(m-\epsilon)}$	if $1 < 2^m\underline{\Theta}^{(m)}$: $\sigma_j^{(m)} \leq C\sigma_j^{(m-\epsilon)}$
if $1 > 2^m\overline{\Theta}^{(m)}$: $2^{-jm} \leq C\sigma_j^{(m-\epsilon)}$	if $1 > 2^m\underline{\Theta}^{(m)}$: $\sigma_j^{(m)}(2^m\underline{\Theta}^{(m)})^{-j} \leq C\sigma_j^{(m-\epsilon)}$
if $1 = 2^m\overline{\Theta}^{(m)}$: $j2^{-jm} \leq C\sigma_j^{(m-\epsilon)}$	if $1 = 2^m\underline{\Theta}^{(m)}$: $j\sigma_j^{(m)} \leq C\sigma_j^{(m-\epsilon)}$