ON THE APPROXIMATION OF INCOMPRESSIBLE MATERIALS IN THE DISPLACEMENT METHOD

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SUMMARY

A mathematical description of the numerical approximation to incompressibility with nearly incompressible displacement finite elements is presented. By the way of functional analysis, it leads to a condition of convergence to the incompressible problem when ν is increased up to 0·5, which can be identified with Fried's K_1 criterion.

INTRODUCTION

The displacement finite element method does not allow the study of incompressible structures because the stiffness matrix is no longer definite. From an engineering point of view it seems reasonable to expect that the nearly compressible solution will not be very different from the exact incompressible one. However, some numerical experiments with this approach have led to somewhat discouraging results.

A first solution was given by Herrmann *et al.*,[1,7,9] which derived a mixed displacement–pressure variational principle. Herrmann's solution, however, is not satisfactory in all cases,[10,16] so that other solutions were investigated.[2,3,8] But the question of what is the fundamental problem was not solved until Fried's major contribution.[2] Fried's proof was of a purely algebraic nature. The purpose of this paper is to present the problem by the way of functional analysis. It leads to a criterion of convergence when $\nu \to 0{\cdot}5$, which is equivalent to Fried's one, but it appears in the most natural way and leads to a very simple interpretation.

ANALYSIS OF THE INCOMPRESSIBILITY PROBLEM

The strain energy can be written in the general form

$$W(\varepsilon) = G\left\{\varepsilon_{ij}\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{ii}\varepsilon_{ii}\right\} \tag{1}$$

where $\varepsilon_{ij} = \frac{1}{2}(D_i u_j + D_j u_i)$, ν is Poisson's ratio and G Coulomb's modulus. When $\nu \to 0{\cdot}5$, the term $\nu/(1-2\nu)$ tends to infinity, and the energy density is no longer definite. It is now well known that in some particular cases the method consisting of setting $\nu = 0{\cdot}5 - \varepsilon$, ε being an arbitrary small position constant, may lead to very bad results. This problem has already been studied by Fried[2] using an algebraic method. Our purpose is to study it by the way of functional analysis.

For the sake of simplicity we shall restrict ourselves to the case where no displacement is fixed to any other value than zero. It is in fact no real restriction, since in the general case, the

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inhomogeneous kinematical boundary conditions can be reinterpreted in terms of applied loads.

For isostatic problems at least, the bilinear functional

$$(\mathbf{u}, \mathbf{v}) = \int_V 2G\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}V \tag{2}$$

defines a scalar product on the space H of kinematically admissible displacements. Consider now the subspace $I \subset H$ of displacement modes which verify the incompressibility equation

$$\operatorname{div} \mathbf{u} = 0 \tag{3}$$

Introduccing the bilinear functional

$$b(\mathbf{u}, \mathbf{v}) = \int_V 2G \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}\,\mathrm{d}V \tag{4}$$

it appears clearly that an element $\mathbf{u}$ of H belongs to I if and only if

$$b(\mathbf{u}, \mathbf{u}) = 0 \tag{5}$$

Let us next consider a closed linear subspace $S \subset H$ (which will be the finite element subspace). The quantity

$$e(S) = \inf_{\substack{\mathbf{u}\in S\\ \mathbf{u}\neq 0}} \frac{\int_v 2G(\operatorname{div}\mathbf{u})^2\,\mathrm{d}V}{\|\mathbf{u}\|^2} = \inf_{\substack{\mathbf{u}\in S\\ \|\mathbf{u}\|=1}} b(\mathbf{u}, \mathbf{u}) \tag{6}$$

has the following fundamental property:

$$\left.\begin{aligned} &e(S) > 0 \text{ if and only if } S \cap I = \{0\} \\ &e(S) = 0 \text{ if } S \cap I \neq \{0\} \end{aligned}\right\} \tag{7}$$

Recall that the set of linear functionals $f(\mathbf{u})$ which are defined on a Hilbert space T and verify the inequality

$$\|f\|_{T'} = \sup_{\substack{\mathbf{u}\in T\\ \mathbf{u}\neq 0}} \frac{|f(\mathbf{u})|}{\|\mathbf{u}\|} < \infty \tag{8}$$

is called dual of T|5|. It will be denoted T'. The quantity $\|f\|_{T'}$ is the norm of f in T'. As it appears immediately, if R and T are two linear subspaces of H such that $R \subset T \subset H$, the following inequality is verified for all f belonging to H:

$$\|f\|_{R'} \leqslant \|f\|_{T'} \leqslant \|f\|_{H'} \tag{9}$$

Consider now the elastic problem. The exact incompressible solution $\mathbf{u}$ is such that

$$\tfrac{1}{2}\|\mathbf{u}\|^2 - f(\mathbf{u}) = \inf_{\mathbf{v}\in I} \{\tfrac{1}{2}\|\mathbf{v}\|^2 - f(\mathbf{v})\} \tag{10}$$

This solution can also be characterized by the fact that the first variation of the functional must be zero:

$$(\mathbf{u}, \mathbf{v}) = f(\mathbf{v}) \quad \text{for all } \mathbf{v} \in I \tag{11}$$

so that $\mathbf{u}$ is Riesz's 'representator' of the functional f in I. As a consequence, one has

$$f(\mathbf{u}) = \|\mathbf{u}\|^2 = \left(\sup_{\substack{\mathbf{v}\in I\\ \mathbf{v}\neq 0}} \frac{(\mathbf{u}, \mathbf{v})}{\|\mathbf{v}\|}\right)^2 = \|f\|_{I'}^2 \tag{12}$$

The finite element method consists to select a finite dimensional subspace $S \subset H$ and to minimize the functional

$$\tfrac{1}{2}\|u\|^2 + \frac{\nu/2}{1-2\nu} b(u, u) - f(u) \tag{13}$$

on this subspace. Let S_I be the set of incompressible elements of S:

$$S_I = S \cap I \tag{14}$$

The orthogonal complement of S_I will be denoted S_c. Any element of S is thus of the form

$$\mathbf{u} = \mathbf{u}_I + \mathbf{u}_c, \ \mathbf{u}_I \in S_I, \ \mathbf{u}_c \in S_c \tag{15}$$

This decomposition leads to the following expression of the functional (13):

$$\tfrac{1}{2}\|\mathbf{u}_I\|^2 + \tfrac{1}{2}\|\mathbf{u}_c\|^2 + \frac{\nu/2}{1-2\nu} b(\mathbf{u}_c, \mathbf{u}_c) - f(\mathbf{u}_I) - f(\mathbf{u}_c) \tag{16}$$

so that the original minimum problem can split into the following ones:

(i) Minimize $\tfrac{1}{2}\|\mathbf{u}_I\| - f(\mathbf{u}_I)$ on S_I

(ii) Minimize $\tfrac{1}{2}\|\mathbf{u}_c\| + \dfrac{\nu/2}{1-2\nu} b(\mathbf{u}_c, \mathbf{u}_c) - f(\mathbf{u}_c)$ on S_c

Problem (i) is the discretization of the exact problem (10). Following the same argument so as to obtain (12), one has

$$\|\mathbf{u}_I\| = \|f\|_{S_I'} \tag{17}$$

By a similar method, problem (ii) leads to

$$\|u_c\|^2 + \frac{\nu}{1-2\nu} b(\mathbf{u}_c, \mathbf{u}_c) = f(\mathbf{u}_c) \tag{18}$$

But the definitions (6) and (8) imply

$$|f(\mathbf{u}_c)| \leq \|f\|_{S_c'}\|\mathbf{u}_c\|$$

$$|b(\mathbf{u}_c, \mathbf{u}_c)| \geq \|u_c\|^2 \, e(S_c)$$

so that

$$\|u_c\|^2 \left(1 + \frac{\nu}{1-2\nu} e(S_c)\right) \leq \|f\|_{S_c'} \|\mathbf{u}_c\|$$

and finally,

$$\|\mathbf{u}_c\| \leq \frac{\|f\|_{S_c'}}{1 + \dfrac{\nu}{1-2\nu} e(S_c)} \tag{19}$$

When $\nu \to 0{\cdot}5$, one has $\nu/(1-2\nu) \to \infty$, and this implies $\|u_c\| \to 0$. The greater $e(S_c)$, the faster the convergence to zero. *A priori*, this result is somewhat strengthening, since it means that when $\nu \to 0{\cdot}5$, the solution tends to be incompressible. But it is also the cause of possible failures. In fact, if the subspace S is such that $S_I = S, I = \{0\}$, and this happens when $e(S) \neq 0$, the finite element solution being necessarily contained in S_c. As a result, *it can only converge to zero when* $\nu \to 0{\cdot}5$.

Our criterion based on the definition of $e(S)$, is in fact equivalent to saying that Fried's K_1 matrix[2] is not singular, but our way of obtaining it is quite different.

The only way to remedy this situation is to replace the subspace S_l by a larger one, namely $\tilde{S}_l$, obtained by setting

$$\tilde{S}_l = \{\mathbf{u} \in S \mid \tilde{b}(\mathbf{u}, \mathbf{u}) = 0\} \tag{20}$$

$\tilde{b}(\mathbf{u}, \mathbf{u})$ being a suitable approximation of $b(\mathbf{u}, \mathbf{u})$. The quantity $e(S)$ has then to be reinterpreted as

$$\tilde{e}(S) = \inf_{\substack{\mathbf{u} \in S \\ \mathbf{u} \neq 0}} \frac{\tilde{b}(\mathbf{u}, \mathbf{u})}{\|\mathbf{u}\|^2} \tag{21}$$

and a correct choice of $\tilde{b}(\mathbf{u}, \mathbf{u})$ can hopefully lead to the result

$$\tilde{e}(S) = 0$$

The approximate problem is then to minimize the following functional:

$$\tfrac{1}{2}\|\mathbf{u}\|^2 + \frac{\nu/2}{1-2\nu}\tilde{b}(u, u) - f(u) \tag{22}$$

in place of the functional (13).

In particular, Fried's method, consisting of underintegrating the compressibility condition, leads to

$$\tilde{b}(\mathbf{u}, \mathbf{u}) = \sum_{\text{elts}} \sum_{l} W_l\,[(\operatorname{div}\mathbf{u})^2]_{P_l} \tag{23}$$

P_l and W_l being, respectively, the Gauss points and Gauss weights of a relatively simple formula.

Another method, proposed by Nagtegaal *et al.*[3] is essentially to set

$$\tilde{b}(\mathbf{u}, \mathbf{u}) = \sum_{\text{elts}} \left(\int_{V_{\text{elt}}} \operatorname{div}\mathbf{u}\,\mathrm{d}V\right)^2 \tag{24}$$

As pointed out in Reference 4 and in the very interesting papers by Hughes and Malkus,[6,13–15] these methods are equivalent to certain mixed elements.

REFERENCES

1. L. R. Herrmann, 'Elasticity equations for incompressible and nearly incompressible materials by a variational theorem', *AIAA J.* **3**, 1896–1900 (1965).
2. I. Fried, 'Finite element analysis of incompressible material by residual energy balancing', *Int. J. Sol. Struct.* **10**, 993–1002 (1974).
3. J. C. Nagtegaal, D. M. Parks and J. R. Rice, 'On numerically accurate finite element solutions in the fully plastic range', *Computer Meth. Appl. Mech. Eng*, **4**, 153–177 (1974).
4. J.-F. Debongnie, 'A new look at Herrmann's formulation of incompressibility', *Proc. Symp Computer Methods in Engineering, University of Southern California, Los Angeles* (1977).
5. C. Berge, *Espaces topologiques. Fonctions multivoques*, Dunod, Paris, 1959.
6. D. S. Malkus and T. J. R. Hughes, 'Mixed finite element methods—reduced and selective integration technique: a unification of concepts', *Computer Meth. Appl. Mech. Engng*, **15**, 63–81 (1978).
7. R. L. Taylor, K. Pister, L. R. Herrmann, 'On a variational theorem for incompressible and nearly-incompressible orthotropic elasticity', *Int. J. Sol. Struct.* **4**, 875–883 (1968).

8. Pin Tong, 'An assumed stress hybrid finite element method for an incompressible and near-incompressible material', *Int. J. Sol. Struct.*, **5**, 445–461 (1969).
9. S. W. Key, 'A variational principle for incompressible and nearly-incompressible anisotropic elasticity', *Int. J. Sol. Struct*, **5**, 951–964 (1969).
10. T. J. R. Hughes and H. Allik, 'Finite elements for compressible and incompressible continua', *Proc. Symp. Civil Engineering, Vanderbild University* (1969).
11. D. J. Naylor, 'Stresses in nearly incompressible materials by finite element with application to the calculation of excess pure pressures', *Int. J. num. Meth. Engng*, **8**, 443–460 (1974).
12. E. G. Thompson, 'Average and complete incompressibility in the finite element method', *Int. J. num. Meth. Engng*, **9**, 925–932 (1975).
13. D. S. Malkus, "A finite element displacement model valid for any value of the compressibility, *Int. J. Sol. Struct.* **12**, 731–738 (1976).
14. T. J. R. Hughes, 'Equivalence of finite elements for nearly-incompressible elasticity'. *Report LBL-5237*, Univ. California (1976).
15. T. J. R. Hughes, 'Equivalence of finite elements for nearly incompressible elasticity', *J. Appl. Mech.* pp. 181–183 (1977).
16. J. F. Debongnie, 'Sur la formulation de Herrmann pour l'étude des solides incompressibles', *J. de Mécanique*, **17**(4), (1978).