A Weak Local Regularity Property in S^ν spaces

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Fractal Geometry and Stochastics V

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Definition: pointwise Hölder spaces

A function $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to $C^\alpha(x_0)$ if there exists a polynomial P of degree at most α s.t.

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha,$$

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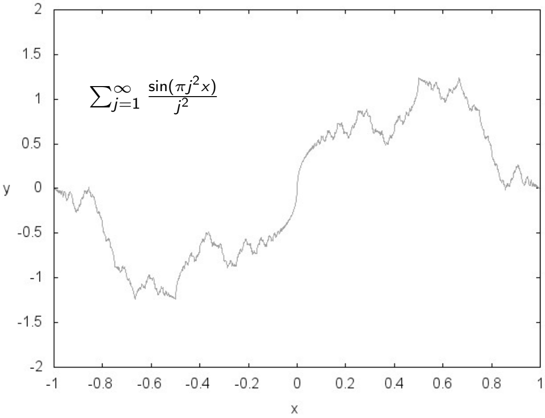
One has $C^{\alpha+\epsilon}(x_0) \subset C^\alpha(x_0)$ for any $\epsilon > 0$ and

Definition: Hölder exponent

$$h_f(x_0) = \sup\{\alpha \geq 0 : f \in C^\alpha(x_0)\}$$

is a pointwise notion of regularity called the Hölder exponent of f at x_0.

Given an "irregular function" f, obtaining the associated function h_f is often an insuperable task since the behavior of h_f can be very erratic.



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The Hölder spectrum of f is the function

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A multifractal formalism is a method that leads to an estimation of d. If when applied to f, it yields the exact spectrum, the multifractal formalism is said to hold for f.

Some classical multifractal formalisms (for functions):

- The box-counting method
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- The Wavelet Transform Modulus Maxima (WTMM) method
  One follows lines of maxima through the scales of the continuous wavelet transform
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- The Wavelet Leaders Method (WLM)
  Similar to the WTMM but with a discrete wavelet transform
  Proposed by Jaffard (2004)

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To overcome this problem, Jaffard introduced the S^ν spaces, which can lead to the non-concave increasing part of the spectrum.

Under some general conditions, there exist a function ϕ and $2^n - 1$ functions $\psi^{(i)}$ called wavelets s.t.

$$\{\phi(\cdot - k) : k \in \mathbb{Z}^n\} \bigcup \{\psi^{(i)}(2^j \cdot - k) : k \in \mathbb{Z}^n, j \in \mathbb{N}, 1 \leq i < 2^n\}$$

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Any function $f \in L^2(\mathbb{R}^n)$ can be decomposed as follows

$$f(x) = \sum_{k \in \mathbb{Z}^n} C_k \phi(x - k) + \sum_{j \in \mathbb{N}, k \in \mathbb{Z}^n, 1 \leq i < 2^n} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k),$$

with

$$C_k = \int f(x)\phi(x - k)\, dx, \quad c_{j,k}^{(i)} = 2^{dj} \int f(x)\psi^{(i)}(2^j x - k)\, dx.$$

We assume will assume:

- $\phi, \psi^{(i)} \in C^p(\mathbb{R}^n)$ with $p \geq \alpha + 1$,
- $D^\beta \phi$, $D^\beta \psi^{(i)}$ $(|\beta| \leq n)$ have fast decay,
- We will deal with functions f defined on the Torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$.

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We set

$$\Lambda = \{(i,j,k) : 1 \leq i < 2^n, j \in \mathbb{N}, k \in \{0, \ldots, 2^j - 1\}^n\}.$$

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If $(i, j, k) \in \Lambda$, the periodized wavelets

$$\psi_{j,k}^{(i)} = \sum_{l \in \mathbb{Z}^n} \psi^{(i)}(2^j(\cdot - l) - k)$$

form a basis of the one-periodic functions of $L^2([0, 1]^n)$.

For a sequence $c = (c_\lambda)_{\lambda \in \Lambda}$, $C > 0$ and $\alpha \in \mathbb{R}$, let us set

$$E_j(C, \alpha)[c] = \{(i, k) : |c_{j,k}^{(i)}| \geq C2^{-j\alpha}\}.$$

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Definition: wavelet profile

The wavelet profile ν_c of c is defined as

$$\nu_c(\alpha) = \lim_{\epsilon \to 0^+} \limsup_{j \to \infty} \frac{\log_2 \# E_j(1, \alpha + \epsilon)[c]}{j},$$

with $\alpha \in \mathbb{R}$.

It gives, in some way, the asymptotic behavior of the number of coefficients of c that have a given order of magnitude.

If c represents the wavelet coefficients of a function f, one sets $E_j(C,\alpha)[f] = E_j(C,\alpha)[c]$ and $\nu_f = \nu_c$.
This definition is independent of the chosen wavelet basis.

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Clearly, ν_f is non-decreasing, right-continuous and there exists $h_{\min} > 0$ s.t.

$$\nu_f(h) \in \begin{cases} \{-\infty\} & \text{if } h < h_{\min} \\ [0, n] & \text{if } h \geq h_{\min} \end{cases} .$$

A function with these properties is called an admissible profile.

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Definition: S^ν spaces

Given an admissible profile ν, the space S^ν is defined as

$$S^\nu = \{f \in L^2([0,1]^n) : \nu_f(\alpha) \leq \nu(\alpha),\ \alpha \in \mathbb{R}\}.$$

Let us write

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The multifractal formalism based on the S^ν spaces is defined as follows:

Definition: S^ν-based multifractal formalism

$$d_\nu(h) = \begin{cases} h \sup_{h' \in (0,h]} \dfrac{\nu(h')}{h'} & \text{if } h \leq h_{\max} \\ 1 & \text{else} \end{cases}.$$

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We have the following result:

Theorem

For any $f \in S^\nu$ and $h \geq 0$, $d(h) \leq d_\nu(h)$.

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Christensen introduced a notion of "almost everywhere" for such spaces (1972):

Definition: prevalent sets

Let E be a complete metric vector space. A borel set $B \subset E$ is Haar-null if there exists a probability measure μ strictly positive on some compact set s.t. $\mu(B + x) = 0$ for any $x \in E$.

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The complement of a Haar-null set is a prevalent set.

We have the following results:

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The sets

$$\{f \in S^\nu : d(h) = \begin{cases} d_\nu(h) & \text{if } h \leq h_{\max} \\ -\infty & \text{otherwise} \end{cases} \}$$

and

$$\{f \in S^\nu : h_f(x) = h_{\max} \text{ for almost every } x\}$$

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The "typical elements" (from the prevalence point of view) of S^ν are expected to be multifractal.

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as it is the case in the uniform Hölder spaces:

Theorem

If $H < 1$, the set

$$\{f \in C^H(\mathbb{R}^n) : \text{ the above inequalities are satisfied}\}$$

is prevalent.

Let Ω be an open subset of $\mathbb{R}^n$ and $0 \leq \alpha < 1$. One sets $\Omega_h = \{x \in \mathbb{R}^n : [x, x+h] \subset \mathbb{R}^n\}$.

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Definition: uniform Hölder spaces

A bounded function f belongs to $C^{\alpha}(\Omega)$ if for some $C, R > 0$ we have

$$r < R \Rightarrow \sup_{|h| \leq r} \|f(x+h) - f(x)\|_{L^{\infty}(\Omega_h)} \leq C r^{\alpha}.$$

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For $\alpha \geq 1$, $f(x+h) - f(x)$ must be replaced with $\Delta_h^{[\alpha]+1} f$.

Definition: weak uniform Hölder spaces

Let Ω be an open subset of $\mathbb{R}^n$ and $\alpha \geq 0$; a bounded function f belongs to $C_w^\alpha(\Omega)$ if for some $C > 0$ there exists a sequence (r_j) decreasing to zero s.t.

$$\sup_{|h| \leq r_j} \|f(x+h) - f(x)\|_{L^\infty(\Omega)} \leq C r_j^\alpha,$$

for any j.
Such a function is said to be weakly uniformly Hölder with exponent α on Ω.

Definition: weak uniform Hölder spaces

Let Ω be an open subset of $\mathbb{R}^n$ and $\alpha \geq 0$; a bounded function f belongs to $C^\alpha_w(\Omega)$ if for some $C > 0$ there exists a sequence (r_j) decreasing to zero s.t.

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for any j.
Such a function is said to be weakly uniformly Hölder with exponent α on Ω.

Definition: upper uniform Hölder exponent

The upper uniform Hölder exponent of a bounded function f on an open set Ω is defined as

$$\mathcal{H}_f(\Omega) = \sup\{\alpha \geq 0 : f \in C^\alpha_w(\Omega)\}.$$

Definition: decreasing sequence of open sets

A sequence (Ω_j) of open subsets of $\mathbb{R}^n$ is decreasing to $x_0 \in \mathbb{R}^n$ if

- $j_1 < j_2$ implies $\Omega_{j_2} \subset \Omega_{j_1}$,
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Definition: local irregularity exponent

The local irregularity exponent of a bounded function f at x_0 is

$$\mathcal{H}_f(x_0) = \sup_j \mathcal{H}_f(\Omega_j).$$

This quantity is well defined.

One sets $m = [\alpha] + 1$. For an open subset Ω of $\mathbb{R}^n$ let

$$\Lambda_j(\Omega) = \{(i,k) : \mathrm{supp}(\psi^{(i)}_{j,k}) \subset \Omega\} \quad \text{and} \quad \|c\|^j_\Omega = \sup_{\Lambda_j(\Omega)} |c_\lambda|.$$

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Theorem

Let $\alpha > 0$ and $f \in C^\alpha(\Omega)$. If there exist $C > 0$ and $\gamma > 1$ s.t.

$$\max\{\sup_{j \le l \le j + \log_2 j} \|c\|^l_\Omega, 2^{-jm} \sup_{j - \log_2 j \le l \le j} (2^{lm} \|c\|^l_\Omega)\} \ge C 2^{-j\alpha} j^\gamma,$$

for any $j \ge 0$, then $f \in I^\alpha(\Omega)$.

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for any $j \ge 0$, then $f \in I^\alpha(\Omega)$.
If $f \in I^\alpha(\Omega)$, there exist $C > 0$ and $\beta \in (0,1)$ s.t., for any $j \ge 0$,

$$\max\{\sup_{j \le l \le j+\log_2 j} \|c\|^l_\Omega, 2^{-jm} j^\beta \sup_{j-\log_2 j \le l \le j} (2^{lm}\|c\|^l_\Omega)\} \ge C 2^{-j\alpha}.$$

Theorem

If $f \in C^\alpha(\Omega)$ and $x_0 \in \mathbb{R}^n$, then $\mathcal{H}_f(x_0) = \alpha$ iff

$$\lim_{r \to 0} \lim_{j \to \infty} -1/j \log_2 \max\{ \sup_{j \leq l \leq j + \log_2 j} \|c\|^l_{B(x_0,r)},$$
$$2^{-jm} \sup_{j - \log_2 j \leq l \leq j} (2^{lm} \|c\|^l_{B(x_0,r)})\} = \alpha.$$

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Theorem

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is prevalent.

In other words, the most irregular point stands near x for any $x \in \mathbb{T}^n$ "almost surely".

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In other words, the most irregular point stands near x for any $x \in \mathbb{T}^n$ “almost surely”.

Corollary

If ν is not reduced to one point, the set

$$\{f \in S^\nu : h_{\max} = h_f(x) \neq \mathcal{H}_f(x) = h_{\min} \text{ for almost every } x\}$$

is prevalent.

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