A new multifractal formalism based on wavelet leaders: detection of non concave and non increasing spectra (Part I)

Céline Esser
Celine.Esser@ulg.ac.be

University of Liège

Fractal Geometry and Stochastics V
Tabarz – March 25, 2014

Joint work with
F. Bastin, S. Jaffard, T. Kleyntssens, S. Nicolay

Introduction

Let f be a locally bounded function.

- The Hölder exponent of f at x is $h_f(x) = \sup\left\{\alpha \;:\; f \in C^\alpha(x)\right\}$.
- The iso-Hölder sets of f are $E_h = \{x : h_f(x) = h\}$.

Definition

The spectrum of singularities d_f of f is defined by

$$d_f(h) = \dim_{\mathcal{H}} E_h \quad \forall h \geq 0.$$

A multifractal formalism is a formula which is expected to yield the spectrum of singularities of a function, from "global" quantities which are numerically computable.

Several multifractal formalisms based on a decomposition of $f \in L^2([0,1])$ in a wavelet basis

$$f = \sum_{j \in \mathbb{N}_0} \sum_{k=0}^{2^j-1} c_{j,k} \psi_{j,k}$$

have been proposed to estimate d_f.

A function f is uniformly Hölder if there is $\varepsilon > 0$ and $C > 0$ such that $|c_{j,k}| \leq C 2^{-\varepsilon j}$ for every j, k.

Hölder regularity and wavelet coefficients

If f is uniformly Hölder and if ψ is “smooth enough”, the Hölder exponent of f at x is

$$h_f(x) = \liminf_{j \to +\infty} \inf_k \frac{\log(|c_{j,k}|)}{\log(2^{-j} + |k2^{-j} - x|)}.$$

Advantage: easy to compute and relatively stable from a numerical point of view.

- The Frisch-Parisi formalism (1985) and the classical use of Besov spaces leads to a loss of information (only concave hull and increasing part of spectra can be recovered).
- Wavelet Leader Method (S. Jaffard, 2004): Modification of the Frisch-Parisi formalism using the wavelet leaders of the function and Oscillation spaces.
  $\longrightarrow$ Detection of decreasing part of concave spectra.
- Introduction of spaces of type S^{ν} (J.M. Aubry, S. Jaffard, 2005)
  $\longrightarrow$ Detection of non concave increasing part of spectra.
- More recently, introduction of spaces of the same type but based on the wavelet leaders of the signal.
  $\longrightarrow$ Detection of non concave and non increasing part of spectra.

- The Frisch-Parisi formalism (1985) and the classical use of Besov spaces leads to a loss of information (only concave hull and increasing part of spectra can be recovered).
- Wavelet Leader Method (S. Jaffard, 2004): Modification of the Frisch-Parisi formalism using the wavelet leaders of the function and Oscillation spaces.
  $\longrightarrow$ Detection of decreasing part of concave spectra.
- Introduction of spaces of type S^ν (J.M. Aubry, S. Jaffard, 2005)
  $\longrightarrow$ Detection of non concave increasing part of spectra.
- More recently, introduction of spaces of the same type but based on the wavelet leaders of the signal.
  $\longrightarrow$ Detection of non concave and non increasing part of spectra.

- The Frisch-Parisi formalism (1985) and the classical use of Besov spaces leads to a loss of information (only concave hull and increasing part of spectra can be recovered).
- Wavelet Leader Method (S. Jaffard, 2004): Modification of the Frisch-Parisi formalism using the wavelet leaders of the function and Oscillation spaces.
  $\longrightarrow$ Detection of decreasing part of concave spectra.
- Introduction of spaces of type $\mathcal{S}^{\nu}$ (J.M. Aubry, S. Jaffard, 2005)
  $\longrightarrow$ Detection of non concave increasing part of spectra.
- More recently, introduction of spaces of the same type but based on the wavelet leaders of the signal.
  $\longrightarrow$ Detection of non concave and non increasing part of spectra.

- The Frisch-Parisi formalism (1985) and the classical use of Besov spaces leads to a loss of information (only concave hull and increasing part of spectra can be recovered).
- Wavelet Leader Method (S. Jaffard, 2004): Modification of the Frisch-Parisi formalism using the wavelet leaders of the function and Oscillation spaces.
  $\longrightarrow$ Detection of decreasing part of concave spectra.
- Introduction of spaces of type $\mathcal{S}^{\nu}$ (J.M. Aubry, S. Jaffard, 2005)
  $\longrightarrow$ Detection of non concave increasing part of spectra.
- More recently, introduction of spaces of the same type but based on the wavelet leaders of the signal.
  $\longrightarrow$ Detection of non concave and non increasing part of spectra.

Wavelet leaders

Standard notation: For $j \in \mathbb{N}_0,\ k \in \{0, \ldots, 2^j - 1\}$,

$$\lambda(j,k) := \{x \in \mathbb{R} : 2^j x - k \in [0,1[\} = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right[,$$

and for all $j \in \mathbb{N}_0$, Λ_j denotes the set of all dyadic intervals (of $[0,1[$) of length 2^{-j}.
If $\lambda = \lambda(j,k)$, we use both notations $c_{j,k}$ or c_λ to denote the wavelet coefficients.

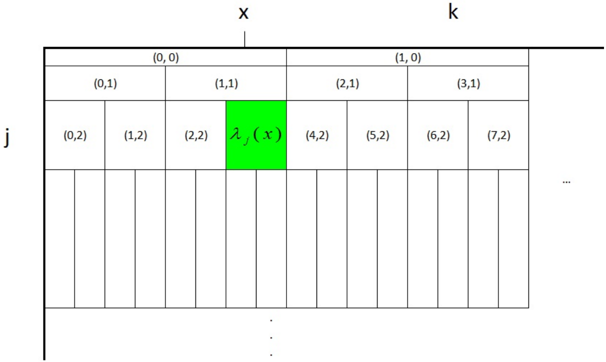
Definition

The wavelet leaders of a signal $f \in L^2([0,1])$ are defined by

$$d_\lambda := \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|, \quad \lambda \in \Lambda_j,\ j \in \mathbb{N}_0\,.$$

If $x \in [0,1]$, let $\lambda_j(x)$ denote the dyadic interval of length 2^{-j} which contains x. Then, we set

$$d_j(x) := d_{\lambda_j(x)} = \sup_{\lambda' \subset 3\lambda_j(x)} |c_{\lambda'}|.$$



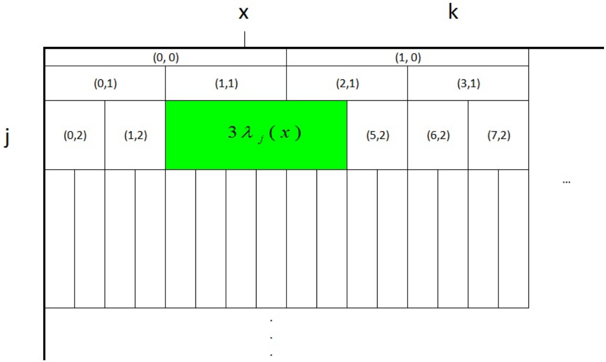
Hölder regularity and wavelet leaders

If f is uniformly Hölder, the Hölder exponent of f at x is given by

$$h_f(x) = \liminf_{j \to +\infty} \frac{\log d_j(x)}{\log 2^{-j}}.$$

Interpretation:

$$d_j(x) \sim 2^{-h_f(x) j}$$



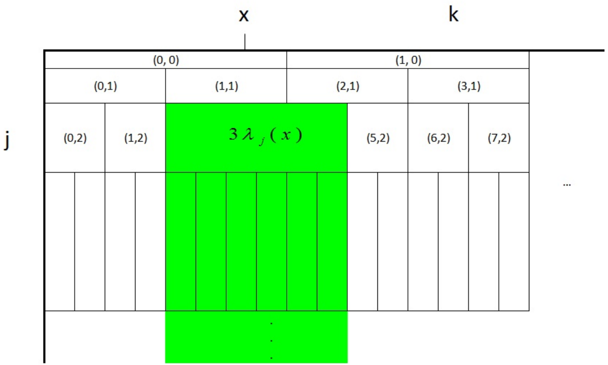
Hölder regularity and wavelet leaders

If f is uniformly Hölder, the Hölder exponent of f at x is given by

$$h_f(x) = \liminf_{j \to +\infty} \frac{\log d_j(x)}{\log 2^{-j}}.$$

Interpretation:

$$d_j(x) \sim 2^{-h_f(x) j}$$



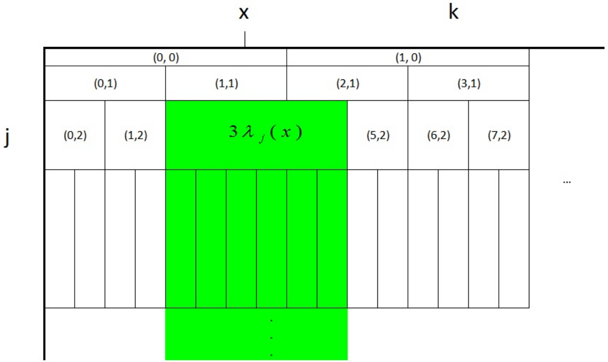
Hölder regularity and wavelet leaders

If f is uniformly Hölder, the Hölder exponent of f at x is given by

$$h_f(x) = \liminf_{j \to +\infty} \frac{\log d_j(x)}{\log 2^{-j}}.$$

Interpretation:

$$d_j(x) \sim 2^{-h_f(x)j}$$



Hölder regularity and wavelet leaders

If f is uniformly Hölder, the Hölder exponent of f at x is given by

$$h_f(x) = \liminf_{j \to +\infty} \frac{\log d_j(x)}{\log 2^{-j}}.$$

Interpretation:

$$d_j(x) \sim 2^{-h_f(x)j}$$

Wavelet Leader Method

The leader scaling function of a locally bounded function f is defined for every $p \in \mathbb{R}$ by

$$\tau_f(p) = \liminf_{j \to +\infty} \frac{\log 2^{-j} \sum^*_{\lambda \in \Lambda_j} d_\lambda^p}{\log 2^{-j}},$$

where $\sum^*_{\lambda \in \Lambda_j}$ means that the sum is taken over the $\lambda \in \Lambda_j$ such that $d_\lambda \neq 0$. The wavelet leader spectrum is then given by

$$L_f(h) = \inf_{p \in \mathbb{R}} \{hp - \tau_f(p)\} + 1.$$

Properties:

- L_f is independent of the chosen wavelet basis.
- If f is uniformly Hölder, $d_f(h) \leq L_f(h)$ for all $h \geq 0$.
- L_f is a concave function.

Wavelet Leader Method

The leader scaling function of a locally bounded function f is defined for every $p \in \mathbb{R}$ by

$$\tau_f(p) = \liminf_{j \to +\infty} \frac{\log 2^{-j} \sum^*_{\lambda \in \Lambda_j} d_\lambda^p}{\log 2^{-j}},$$

where $\sum^*_{\lambda \in \Lambda_j}$ means that the sum is taken over the $\lambda \in \Lambda_j$ such that $d_\lambda \neq 0$. The wavelet leader spectrum is then given by

$$L_f(h) = \inf_{p \in \mathbb{R}} \{hp - \tau_f(p)\} + 1.$$

Properties:

- L_f is independent of the chosen wavelet basis.
- If f is uniformly Hölder, $d_f(h) \leq L_f(h)$ for all $h \geq 0$.
- L_f is a concave function.

$\mathcal{S}^\nu$ spaces method

The wavelet profile ν_f of a locally bounded function f is defined by

$$\nu_f(h) = \lim_{\varepsilon \to 0^+} \limsup_{j \to +\infty} \frac{\log \# \{ \lambda \in \Lambda_j \ : \ |c_\lambda| \geq 2^{-(h+\varepsilon)j} \}}{\log 2^j}, \ h \in \mathbb{R} .$$

Interpretation:

- There are approximatively $2^{\nu_f(h)j}$ coefficients greater in modulus than 2^{-hj}.

Properties:

- ν_f is a right-continuous increasing function.
- ν_f is independent of the chosen wavelet basis.
- If f is uniformly Hölder,

$$d_f(h) \leq d^{\nu_f}(h) := \min \left\{ h \sup_{h' \in]0;h]} \frac{\nu_f(h')}{h'}, 1 \right\} \quad \forall h \geq 0.$$

$\mathcal{S}^\nu$ spaces method

The wavelet profile ν_f of a locally bounded function f is defined by

$$\nu_f(h) = \lim_{\varepsilon\to 0^+} \limsup_{j\to+\infty} \frac{\log \#\left\{\lambda \in \Lambda_j \ : \ |c_\lambda| \geq 2^{-(h+\varepsilon)j}\right\}}{\log 2^j}, \ h \in \mathbb{R}\,.$$

Interpretation:

- There are approximatively $2^{\nu_f(h)j}$ coefficients greater in modulus than 2^{-hj}.

Properties:

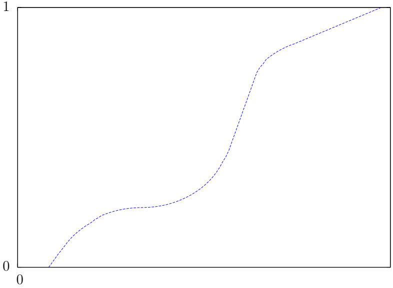
- ν_f is a right-continuous increasing function.
- ν_f is independent of the chosen wavelet basis.
- If f is uniformly Hölder,

$$d_f(h) \leq d^{\nu_f}(h) := \min\left\{h \sup_{h'\in]0;h]} \frac{\nu_f(h')}{h'}, 1\right\} \quad \forall h \geq 0.$$

Definition

Take $0 \leq a < b \leq +\infty$. A function $g : [a, b] \mapsto \mathbb{R}^+$ is with increasing-visibility if g is continuous at a and $\sup_{y \in]a,x]} \frac{g(y)}{y} \leq \frac{g(x)}{x}$ for all $x \in]a, b]$.

In other words, a function g is with increasing-visibility if for all $x \in]a, b]$, the segment $[(0,0), (x, g(x))]$ lies above the graph of g on $]a, x]$.



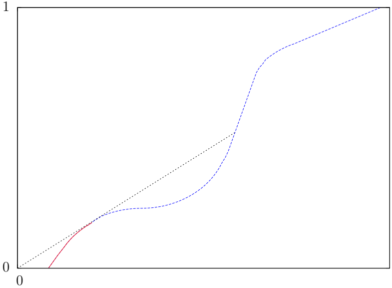
Example of ν_f (---) and $d_f^{\nu_f}$ (—)

The passage from ν_f to d^{ν_f} transforms the function ν_f into a function with increasing-visibility.

Definition

Take $0 \leq a < b \leq +\infty$. A function $g : [a,b] \mapsto \mathbb{R}^+$ is with increasing-visibility if g is continuous at a and $\sup_{y \in]a,x]} \frac{g(y)}{y} \leq \frac{g(x)}{x}$ for all $x \in]a,b]$.

In other words, a function g is with increasing-visibility if for all $x \in]a,b]$, the segment $[(0,0),(x,g(x))]$ lies above the graph of g on $]a,x]$.



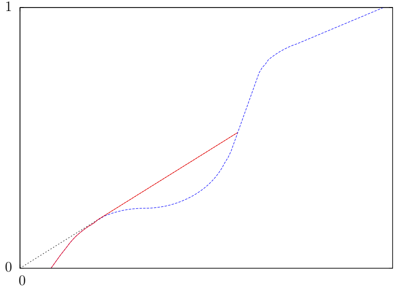
Example of ν_f (---) and $d_f^{\nu_f}$ (—)

The passage from ν_f to d^{ν_f} transforms the function ν_f into a function with increasing-visibility.

Definition

Take $0 \leq a < b \leq +\infty$. A function $g : [a,b] \mapsto \mathbb{R}^+$ is with increasing-visibility if g is continuous at a and $\sup_{y \in]a,x]} \frac{g(y)}{y} \leq \frac{g(x)}{x}$ for all $x \in]a,b]$.

In other words, a function g is with increasing-visibility if for all $x \in]a,b]$, the segment $[(0,0),(x,g(x))]$ lies above the graph of g on $]a,x]$.



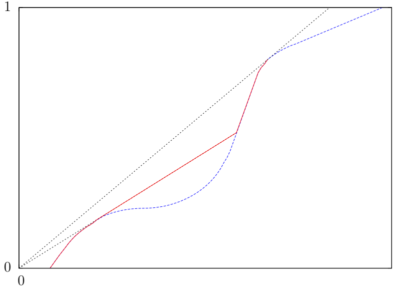
Example of ν_f (---) and $d_f^{\nu_f}$ (—)

The passage from ν_f to d^{ν_f} transforms the function ν_f into a function with increasing-visibility.

Definition

Take $0 \leq a < b \leq +\infty$. A function $g : [a,b] \mapsto \mathbb{R}^+$ is with increasing-visibility if g is continuous at a and $\sup_{y \in]a,x]} \frac{g(y)}{y} \leq \frac{g(x)}{x}$ for all $x \in]a,b]$.

In other words, a function g is with increasing-visibility if for all $x \in]a,b]$, the segment $[(0,0),(x,g(x))]$ lies above the graph of g on $]a,x]$.



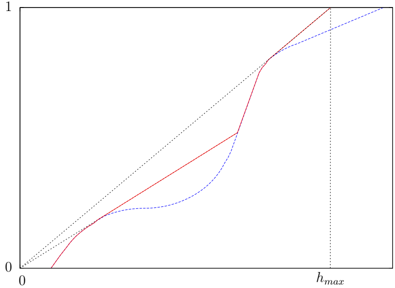
Example of ν_f (---) and $d_f^{\nu_f}$ (—)

The passage from ν_f to d^{ν_f} transforms the function ν_f into a function with increasing-visibility.

Definition

Take $0 \leq a < b \leq +\infty$. A function $g : [a,b] \mapsto \mathbb{R}^+$ is with increasing-visibility if g is continuous at a and $\sup_{y\in]a,x]} \frac{g(y)}{y} \leq \frac{g(x)}{x}$ for all $x \in]a,b]$.

In other words, a function g is with increasing-visibility if for all $x \in]a,b]$, the segment $[(0,0),(x,g(x))]$ lies above the graph of g on $]a,x]$.



Example of ν_f (---) and $d_f^{\nu_f}$ (—)

The passage from ν_f to d^{ν_f} transforms the function ν_f into a function with increasing-visibility.

Particular case

Assumption: f is a function whose wavelet coefficients are given by $c_\lambda = \mu(\lambda)$ where μ is a finite Borel measure on $[0,1]$.

Notation: Let f_β denotes the function whose wavelet coefficients are given by $c_\lambda^\beta = 2^{-\beta j} c_\lambda$.

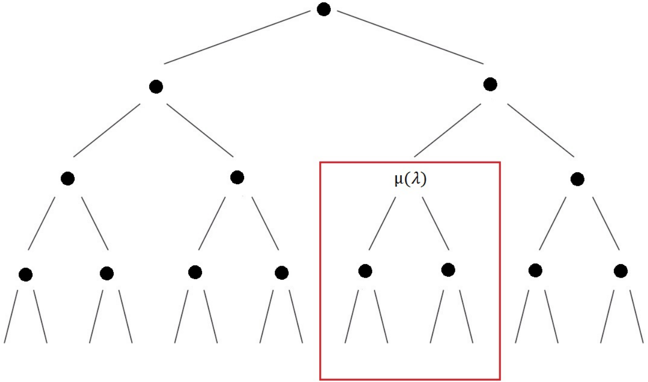
In this case, one has

- $d_{f_\beta}(h) = d_f(h-\beta)$ for all $h \geq 0$.
- $\nu_{f_\beta}(h) = \nu_f(h-\beta)$ for all $h \geq 0$.

Moreover, if

$$\inf\left\{\frac{\nu_f(x)-\nu_f(y)}{x-y} : x,y \in [h_{\min}, h'_{\max}],\ x<y\right\} > 0,$$

where $h_{\min} = \inf\{\alpha : \nu_f(\alpha) \geq 0\}$, $h'_{\max} = \inf\{\alpha : \nu_f(\alpha) = 1\}$, then there exists $\beta > 0$ such that the function ν_{f_β} is with increasing-visibility on $[h_{\min}, h'_{\max}]$. In this case, $d^{\nu_{f_\beta}} = \nu_{f_\beta}$ approximates d_{f_β}. Therefore the increasing part of d_f can be approximated by ν_f.



There is a tree-structure in the repartition of the wavelet coefficients

Large deviation-type argument

The wavelet leader density of f is defined for every $\alpha \in \mathbb{R}$ by

$$\widetilde{\rho}_f(h) = \lim_{\varepsilon \to 0^+} \limsup_{j \to +\infty} \frac{\log \#\{\lambda \in \Lambda_j : 2^{-(h+\varepsilon)j} \leq d_\lambda < 2^{-(h-\varepsilon)j}\}}{\log 2^j}.$$

Interpretation: There are approximatively $2^{\widetilde{\rho}_f(h)j}$ coefficients of size 2^{-hj}.

Heuristic argument: We consider the points x such that $h_f(x) = h$.

- $d_j(x) \sim 2^{-hj}$ and there are about $2^{\widetilde{\rho}_f(h)j}$ such dyadic intervals.
- If we cover each singularity x by dyadic intervals of size 2^{-j}, from the definition of the Hausdorff dimension, there are about $2^{d_f(h)j}$ such intervals.

$$\Longrightarrow \widetilde{\rho}_f(h) = d_f(h)$$

Problem: $\widetilde{\rho}_f$ may depend on the chosen wavelet basis!

Large deviation-type argument

The wavelet leader density of f is defined for every $\alpha \in \mathbb{R}$ by

$$\widetilde{\rho}_f(h) = \lim_{\varepsilon \to 0^+} \limsup_{j \to +\infty} \frac{\log \#\{\lambda \in \Lambda_j : 2^{-(h+\varepsilon)j} \leq d_\lambda < 2^{-(h-\varepsilon)j}\}}{\log 2^j}.$$

Interpretation: There are approximatively $2^{\widetilde{\rho}_f(h)j}$ coefficients of size 2^{-hj}.

Heuristic argument: We consider the points x such that $h_f(x) = h$.

- $d_j(x) \sim 2^{-hj}$ and there are about $2^{\widetilde{\rho}_f(h)j}$ such dyadic intervals.
- If we cover each singularity x by dyadic intervals of size 2^{-j}, from the definition of the Hausdorff dimension, there are about $2^{d_f(h)j}$ such intervals.

$$\Longrightarrow \widetilde{\rho}_f(h) = d_f(h)$$

Problem: $\widetilde{\rho}_f$ may depend on the chosen wavelet basis!

Large deviation-type argument

The wavelet leader density of f is defined for every $\alpha \in \mathbb{R}$ by

$$\widetilde{\rho}_f(h) = \lim_{\varepsilon \to 0^+} \limsup_{j \to +\infty} \frac{\log \#\{\lambda \in \Lambda_j : 2^{-(h+\varepsilon)j} \leq d_\lambda < 2^{-(h-\varepsilon)j}\}}{\log 2^j}.$$

Interpretation: There are approximatively $2^{\widetilde{\rho}_f(h)j}$ coefficients of size 2^{-hj}.

Heuristic argument: We consider the points x such that $h_f(x) = h$.

- $d_j(x) \sim 2^{-hj}$ and there are about $2^{\widetilde{\rho}_f(h)j}$ such dyadic intervals.
- If we cover each singularity x by dyadic intervals of size 2^{-j}, from the definition of the Hausdorff dimension, there are about $2^{d_f(h)j}$ such intervals.

$$\Longrightarrow \widetilde{\rho}_f(h) \geq d_f(h)$$

Problem: $\widetilde{\rho}_f$ may depend on the chosen wavelet basis!

$\mathcal{L}^\nu$ spaces Method

The wavelet leader profile of f is defined by

$$\widetilde{\nu}_f(h) = \begin{cases} \lim\limits_{\varepsilon\to 0^+} \limsup\limits_{j\to+\infty} \dfrac{\log \#\{\lambda \in \Lambda_j \ : \ d_\lambda \geq 2^{-(h+\varepsilon)j}\}}{\log 2^j} & \text{if } h < h_s, \\ \lim\limits_{\varepsilon\to 0^+} \limsup\limits_{j\to+\infty} \dfrac{\log \#\{\lambda \in \Lambda_j \ : \ d_\lambda < 2^{-(h-\varepsilon)j}\}}{\log 2^j} & \text{if } h \geq h_s, \end{cases}$$

where h_s is the smallest positive real such that $\widetilde{\nu}_f(h) = 1$.

Properties:

- $\widetilde{\nu}_f$ is independent of the chosen wavelet basis.
- If f is uniformly Hölder, $d_f(h) \leq \widetilde{\nu}_f(h)$ for all $h \geq 0$.

$\mathcal{L}^\nu$ spaces Method

The wavelet leader profile of f is defined by

$$\widetilde{\nu}_f(h) = \begin{cases} \lim\limits_{\varepsilon \to 0^+} \limsup\limits_{j \to +\infty} \dfrac{\log \#\left\{\lambda \in \Lambda_j \ : \ d_\lambda \geq 2^{-(h+\varepsilon)j}\right\}}{\log 2^j} & \text{if } h < h_s, \\ \lim\limits_{\varepsilon \to 0^+} \limsup\limits_{j \to +\infty} \dfrac{\log \#\left\{\lambda \in \Lambda_j \ : \ d_\lambda < 2^{-(h-\varepsilon)j}\right\}}{\log 2^j} & \text{if } h \geq h_s, \end{cases}$$

where h_s is the smallest positive real such that $\widetilde{\nu}_f(h) = 1$.

Properties:

- $\widetilde{\nu}_f$ is independent of the chosen wavelet basis.
- If f is uniformly Hölder, $d_f(h) \leq \widetilde{\nu}_f(h)$ for all $h \geq 0$.

Comparison of the formalisms

With the Wavelet Leader Method

If f is uniformly Hölder and if $\widetilde{\nu}_f$ is compactly supported, then

$$d_f(h) \leq \widetilde{\nu}_f(h) \leq L_f(h)$$

for every $h \in \mathbb{R}$ and L_f is the concave hull of $\widetilde{\nu}_f$.

With the S^ν Spaces Method

If f is uniformly Hölder, we have

$$d_f(h) \leq \widetilde{\nu}_f(h) \leq d^{\nu_f}(h)$$

for every $h \geq 0$. Moreover, the two methods coincide on $[h_{\min}, h_s]$ if and only if $\widetilde{\nu}_f$ is with increasing-visibility on $[h_{\min}, h_s]$.

Comparison of the formalisms

With the Wavelet Leader Method

If f is uniformly Hölder and if $\widetilde{\nu}_f$ is compactly supported, then

$$d_f(h) \leq \widetilde{\nu}_f(h) \leq L_f(h)$$

for every $h \in \mathbb{R}$ and L_f is the concave hull of $\widetilde{\nu}_f$.

With the S^ν Spaces Method

If f is uniformly Hölder, we have

$$d_f(h) \leq \widetilde{\nu}_f(h) \leq d^{\nu_f}(h)$$

for every $h \geq 0$. Moreover, the two methods coincide on $[h_{\min}, h_s]$ if and only if $\widetilde{\nu}_f$ is with increasing-visibility on $[h_{\min}, h_s]$.

To be continued...

In “A new multifractal formalism based on wavelet leaders: detection of non concave and non increasing spectra (Part II)”, T. Kleyntssens will present an **implementation of the formalism** based on $\mathcal{L}^{\nu}$ spaces.

References

J.M. Aubry, F. Bastin and S. Dispa, *Prevalence of multifractal functions in S^ν spaces*, J. Fourier Anal. Appl., **13**, 2, 2007, 175–185.

J.M. Aubry and S. Jaffard, *Random Wavelet Series*, Comm. Math. Phys, **227**, 2002, 483–514.

F. Bastin, C. Esser and S. Jaffard, *Beyond S^ν spaces with wavelet leaders*, preprint.

C. Esser, T. Kleyntssens, S. Jaffard and S. Nicolay, *A multifractal formalism for non concave and non increasing spectra: the L^ν spaces approach*, preprint.

S. Jaffard, *On the Frisch-Parisi conjecture*, J. Math. Pures Appl., **79**, 6, 2000,525-?552.

S. Jaffard, *Wavelet techniques in multifractal analysis*, Fractal Geometry and Applications : A Jubilee of Benoit Mandelbrot, Proc. Symp. Pure Maths., **72**, 2004, 91–151.