A new multifractal formalism based on wavelet leaders: detection of non concave and non increasing spectra (Part II)

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Joint work with C. Esser, S. Jaffard and S. Nicolay

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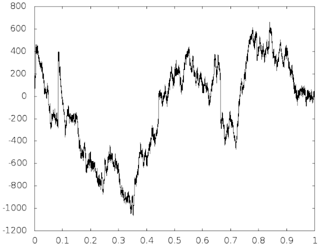
Fractal Geometry and Stochastics V

Introduction



- Study of **irregular functions** :
  $h_f(x_0) = \sup\{\alpha \geq 0 : f \in C^\alpha(x_0)\}$

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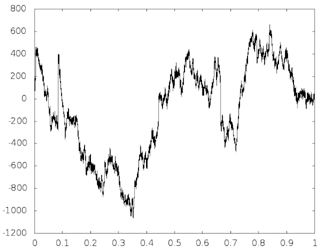

- Study of **irregular functions** :
  $h_f(x_0) = \sup\{\alpha \geq 0 : f \in C^{\alpha}(x_0)\}$
- Determination of the **size** of the set of points which share the same irregularity :
  $d_f : h \geq 0 \mapsto \dim_{\mathcal{H}}(\{x : h_f(x) = h\})$

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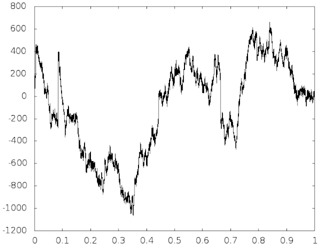

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In practice, we use a numerically computable function which approximates d_f, i.e. a **multifractal formalism**.

Plan

1. A new multifractal formalism based on wavelet leaders
2. Implementation of this new multifractal formalism
3. Numerical Simulations on Theoretical Examples
4. Conclusion

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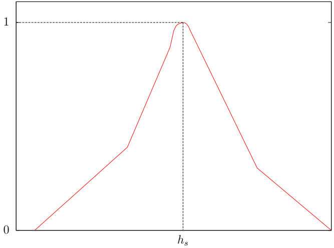
(Bastin, Esser, Jaffard, 2014 [4])
An approximation of spectrum of singularities of a function f is given by

$$\widetilde{\nu}_f(h) = \begin{cases} \lim_{\epsilon\to 0^+} \limsup_{j\to+\infty} \dfrac{\log \#\{k : d_{j,k} \geq C2^{-(h+\epsilon)j}\}}{\log 2^j} & \text{if } h \leq h_s \\ \lim_{\epsilon\to 0^+} \limsup_{j\to+\infty} \dfrac{\log \#\{k : d_{j,k} < C2^{-(h-\epsilon)j}\}}{\log 2^j} & \text{otherwise} \end{cases}$$

where $(d_{j,k})_{j\in\mathbb{N},k\in\{0,\dots,2^j-1\}}$ are the **wavelet leaders** of f, h_s is the smallest positive real such that $\widetilde{\nu}_f(\beta) = 1$ and C is a positive constant.



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Proposition

The constant C in the definition of $\widetilde{\nu}_f$ is arbitrary.

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- We use the Daubechies wavelets and the Mallat algorithm to compute a finite number of wavelet coefficients $(c_{j,k})_{j,k}$ (Daubechies, 1992 [5]) ;

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- We determine a **finite number** of wavelet leaders $(d_{j,k})_{j,k}$.
- Denote by $E_j^{\geq}(C,\alpha)(f) = \{k : d_{j,k} \geq C2^{-\alpha j}\}$. We must compute

$$\widetilde{\nu}_f(h) = \lim_{\epsilon\to 0^+} \limsup_{j\to+\infty} \frac{\log \# E_j^{\geq}(1, h+\epsilon)(f)}{\log 2^j}$$

for all $h \in [0; h_s]$ where h_s is the smallest $h \geq 0$ such that $\widetilde{\nu}_f(h) = 1$ and

$$\widetilde{\nu}_f(h) = \lim_{\epsilon\to 0^+} \limsup_{j\to+\infty} \frac{\log(\# E_j^{<}(1, h-\epsilon)(f)}{\log 2^j}$$

for all $h > h_s$.

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for all $h \in [0; h_s]$.
This means that $\# E_j^{\geq}(C,h)(f) \sim 2^{-\widetilde{\nu}_f(h)j}$ for j "large enough".
So, we can approximate $\widetilde{\nu}_f(h)$ by the **slope** of

$$j \in \mathbb{N} \mapsto \frac{\log \# E_j^{\geq}(C,h)(f)}{\log 2}$$

for j "large enough". This slope is denote by $\widetilde{\nu}_f^C(h)$.

For a fixed h, the main problem is to determine a good constant C because we have only a **finite number** of wavelet leaders :

- If C is too small, the detected value of $\widetilde{\nu}_f^C(h)$ will be 1 ;
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We construct the function

$$C > 0 \mapsto \widetilde{\nu}_f^C(h).$$

In theory, the constant is arbitrary, so, in practice, this function must stabilize if $h \geq h_{min} = \inf\{h \geq 0 : \widetilde{\nu}_f(h) = 0\}$.

Plan

Lévy process

A Lévy process (associated to an index $\beta \in [0;2]$) can be decomposed as the sum of a (possibly vanishing) Brownian part and an independent pure jump process.

Theorem (Jaffard, 1999 [7])

- If the Lévy process with index β has no Brownian part, then almost surely, the spectrum is given by

  $$d_f(h) = \beta h$$

  for all $h \in [0;1/\beta]$.

- If the Lévy process with index β has a Brownian part, then almost surely, the spectrum is given by

  $$d_f(h) = \begin{cases} \beta h & \text{if } h \in [0;1/2) \\ 1 & \text{if } h = 1/2 \end{cases} .$$

Levy process without Brownian part ($\beta = 1.3$)

$$\text{Function } j \in \mathbb{N} \mapsto \frac{\log \#\{k : d_{j,k} \geq C\ 2^{-hj}\}}{\log 2}$$

Levy process without Brownian part ($\beta = 1.3$)

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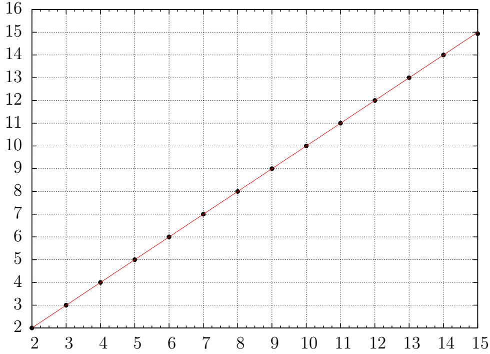
Levy process without Brownian part ($\beta = 1.3$)

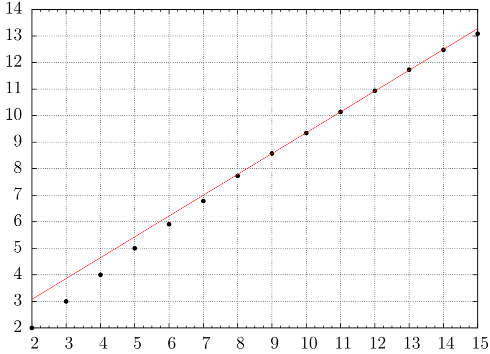
Function $j \in \mathbb{N} \mapsto \dfrac{\log \#\{k : d_{j,k} \geq 100\ 2^{-0.55 j}\}}{\log 2}$



slope $= \widetilde{\nu}_f^{100}(0.55) = 1$

Levy process without Brownian part ($\beta = 1.3$)

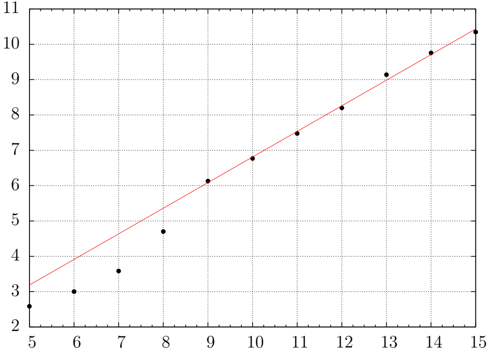
Function $j \in \mathbb{N} \mapsto \dfrac{\log \#\{k : d_{j,k} \geq 600\ 2^{-0.55 j}\}}{\log 2}$



$$\text{slope} = \widetilde{\nu}_f^{600}(0.55) = 0.784862$$

Levy process without Brownian part ($\beta = 1.3$)

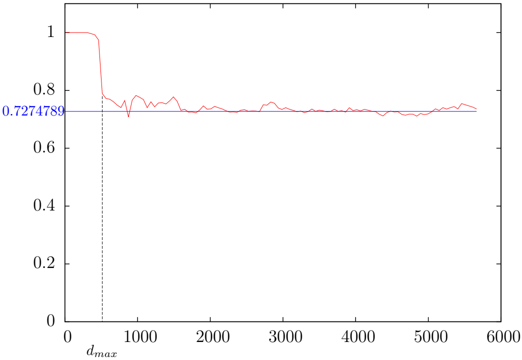
Function $j \in \mathbb{N} \mapsto \dfrac{\log \#\{k : d_{j,k} \geq 2500\ 2^{-0.55j}\}}{\log 2}$



slope $= \widetilde{\nu}_f^{2500}(0.55) = 0.724723$

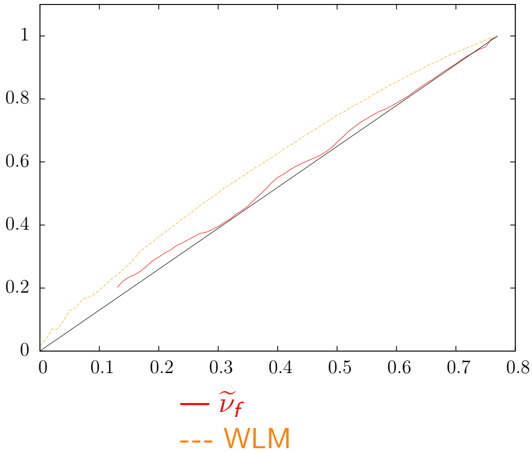
Levy process without Brownian part ($\beta = 1.3$)

Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.55)$



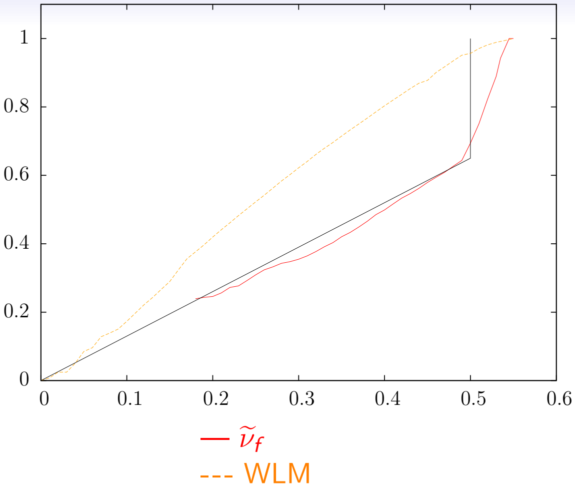
$$\widetilde{\nu}_f(0.55) = 0.715$$

Levy process without Brownian part ($\beta = 1.3$)



The results are the averages over 100 realizations of length 2^{17}.

Lévy process with a Brownian part ($\beta = 1.3$)



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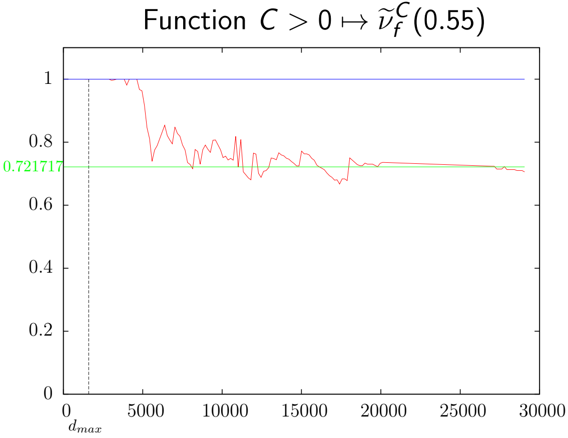

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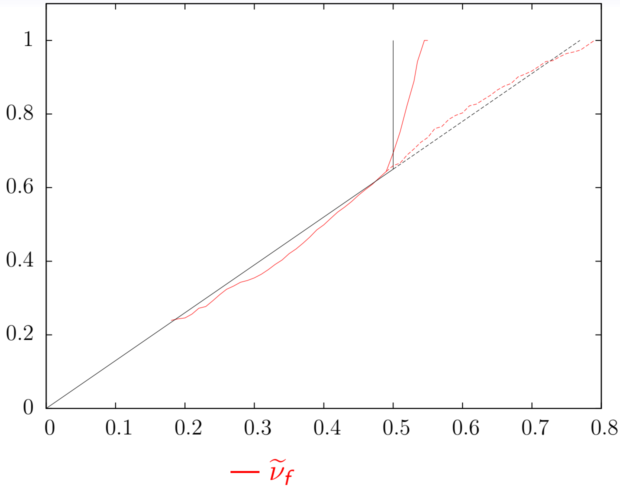
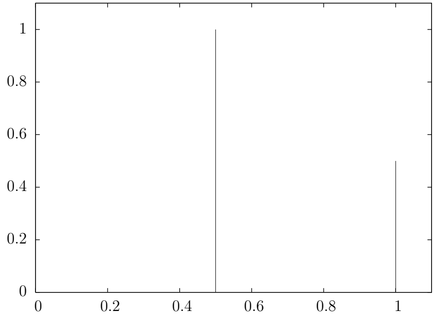


The square of a Brownian motion

Theorem (Abry, Jaffard, Wendt, 2012 [1])

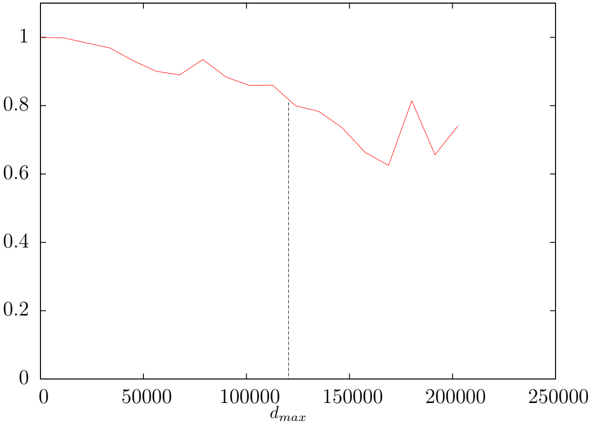
The spectrum of the square of a Brownian motion is given by

$$d(h) = \begin{cases} 1 & \text{if } h = 0.5 \\ 0.5 & \text{if } h = 1 \end{cases} .$$



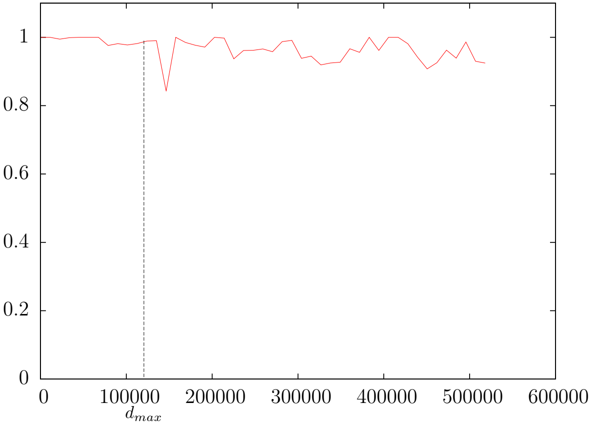
The square of a Brownian motion

Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.4)$



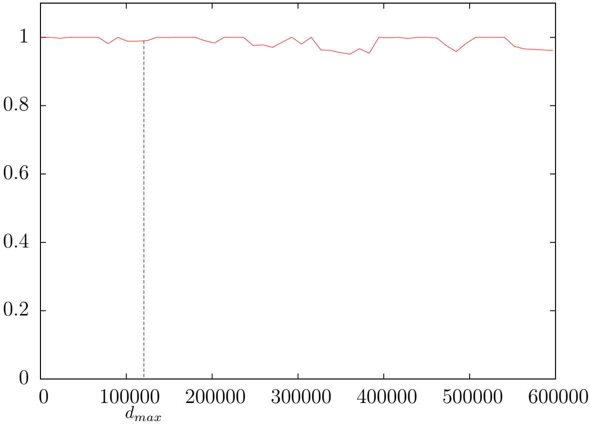
The square of a Brownian motion

Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.48)$



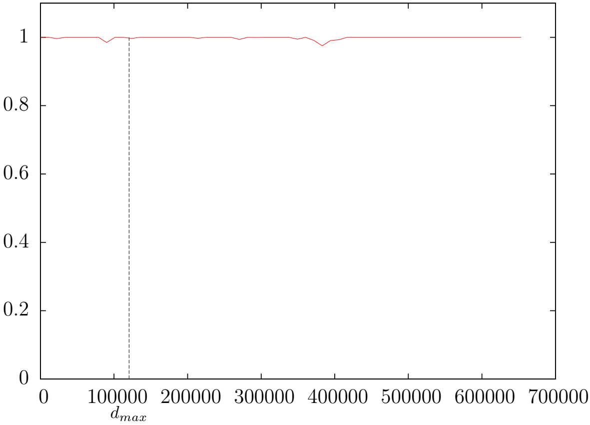
The square of a Brownian motion

Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.49)$



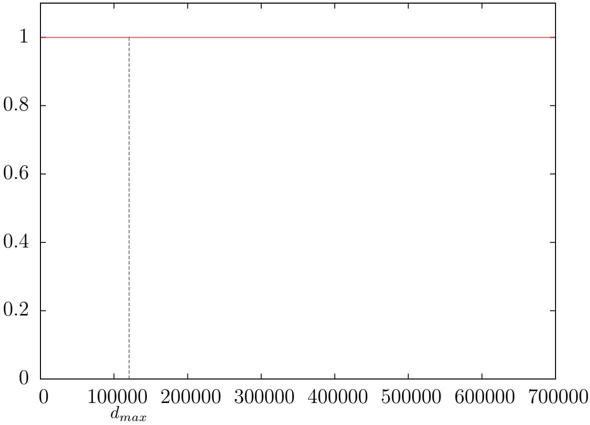
The square of a Brownian motion

Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.5)$



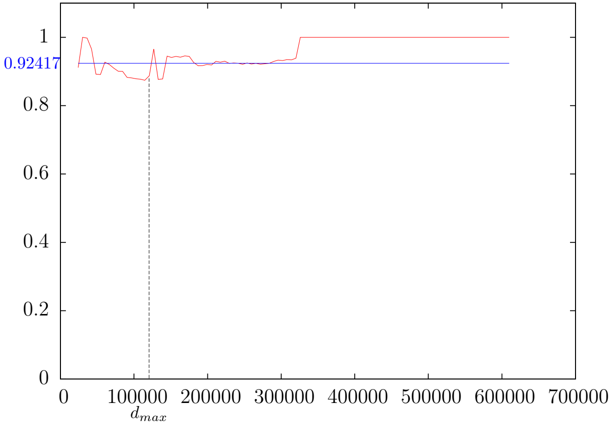
The square of a Brownian motion

Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.51)$



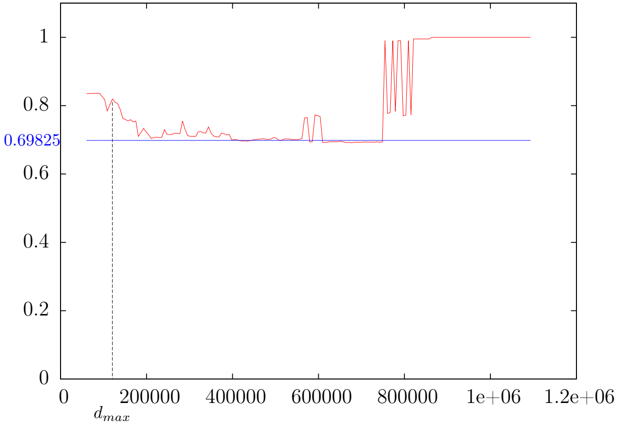
The square of a Brownian motion

Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.6)$



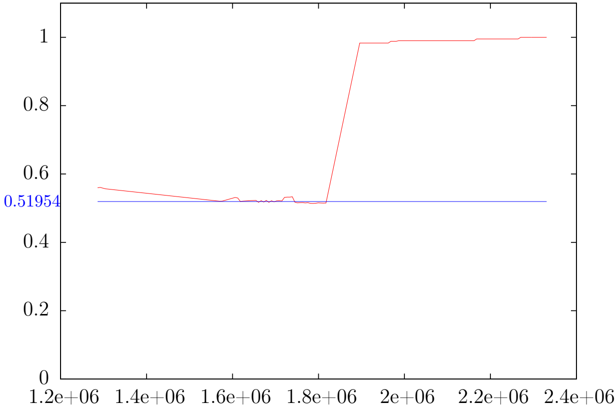
The square of a Brownian motion

Function $C > 0 \mapsto \widetilde{\nu}_f^C(0.8)$

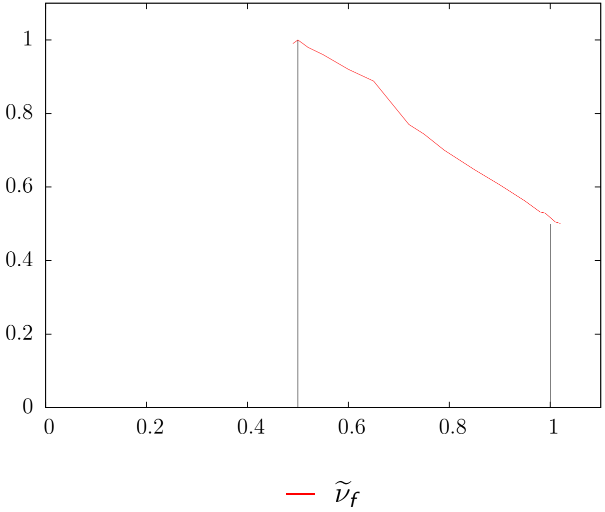


The square of a Brownian motion

Function $C > 0 \mapsto \widetilde{\nu}_f^C(1)$



The square of a Brownian motion



Cascades of Mandelbrot

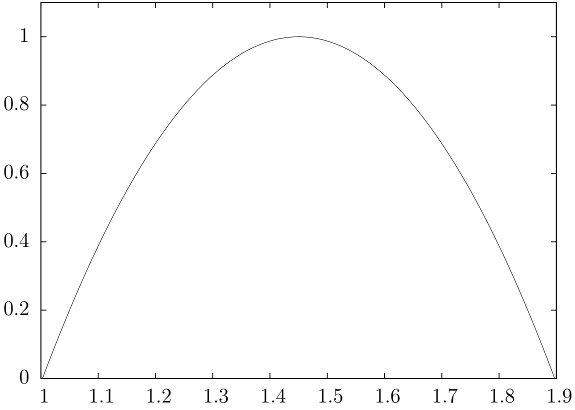
Take W a positive random variable such that $E[W] = 1$ and $W_{jk} \sim^{iid} W$ for all $j \in \mathbb{N}, k \in \{0, \ldots, 2^j - 1\}$. Almost surely, the following construction defines a Borel measure μ on $[0; 1]$:

$$
\begin{tikzcd}
 & & & c_{0,0} := 1 \arrow[dll, "W_{1,0}"'] \arrow[drr, "W_{1,1}"] & & & \\
 & c_{1,0} := \frac{c_{0,0}W_{1,0}}{2} \arrow[dl, "W_{2,0}"'] \arrow[dr, "W_{2,1}"] & & & & c_{1,1} := \frac{c_{0,0}W_{1,1}}{2} \arrow[dl, "W_{2,2}"'] \arrow[dr, "W_{2,3}"] & \\
c_{2,0} := \frac{c_{1,0}W_{2,0}}{2} & & c_{2,1} := \frac{c_{1,0}W_{2,1}}{2} & & c_{2,2} := \frac{c_{1,1}W_{2,2}}{2} & & c_{2,3} := \frac{c_{1,1}W_{2,3}}{2}
\end{tikzcd}
$$

with $\mu([k2^{-j}; (k+1)2^{-j}[) = c_{j,k}$. Moreover, $(c_{j,k})_{j,k}$ can be considered as wavelet coefficients of a function.

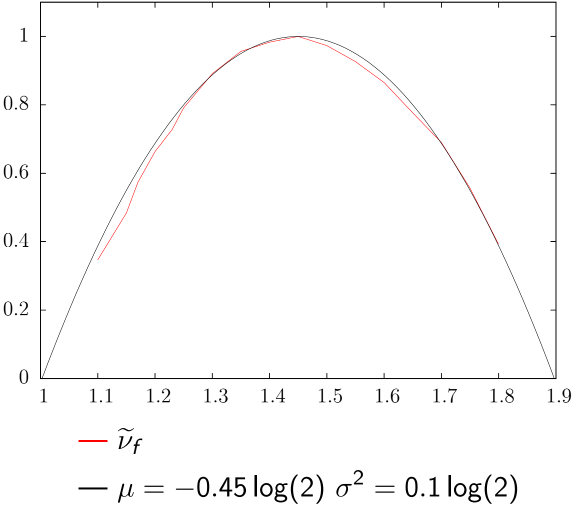
Cascades of Mandelbrot - log-normal

(Arneodo, Bacry, Muzy, 1998 [2] - Barral, Seuret, 2005 [3])



— $\mu = -0.45\log(2)$ $\sigma^2 = 0.1\log(2)$

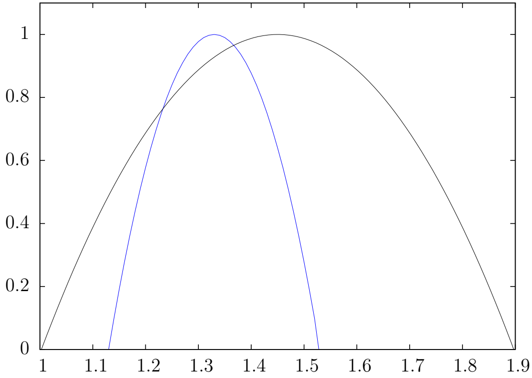
Cascades of Mandelbrot - log-normal



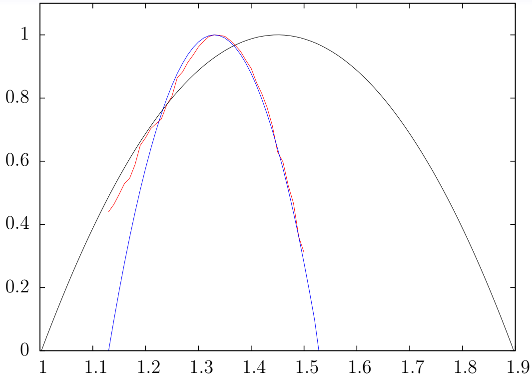
Cascades of Mandelbrot - sum of log-normal

Denote by μ_1 and μ_2 two borel measures on $[0;1]$. We can prove that

$$h_{\mu_1+\mu_2}(x) = \min\{h_{\mu_1}(x), h_{\mu_2}(x)\}.$$



Cascades of Mandelbrot - sum of log-normal



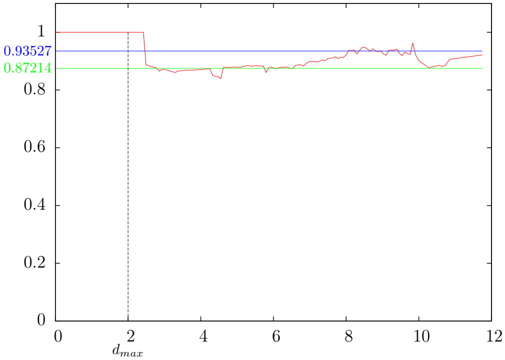
— $\widetilde{\nu}_f$

— $\mu = -0.33\log(2)\ \sigma^2 = 0.02\log(2)$

— $\mu = -0.45\log(2)\ \sigma^2 = 0.1\log(2)$

Cascades of Mandelbrot - sum of log-normal

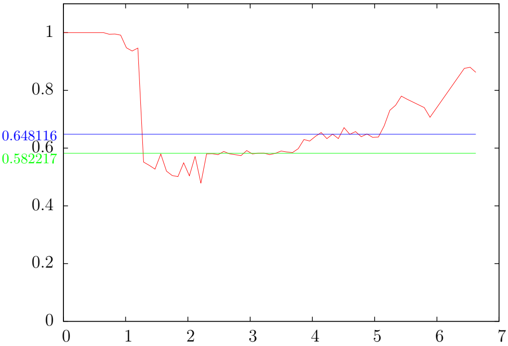
Function $C > 0 \mapsto \widetilde{\nu}_f^C(1.29)$



$$\mu = -0.33\log(2)\ \sigma^2 = 0.02\log(2) \rightarrow \widetilde{\nu}_f(1.29) = 0.96$$
$$\mu = -0.45\log(2)\ \sigma^2 = 0.1\log(2) \rightarrow \widetilde{\nu}_f(1.29) = 0.872$$

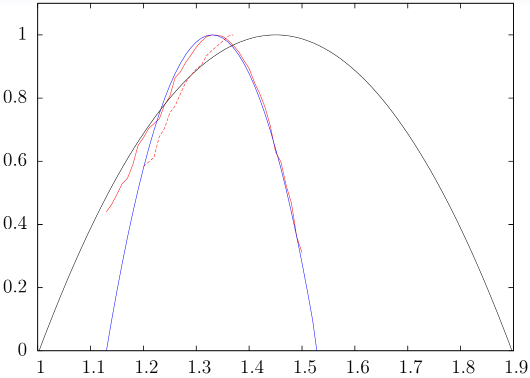
Cascades of Mandelbrot - sum of log-normal

Function $C > 0 \mapsto \widetilde{\nu}_f^C(1.2)$



$$\mu = -0.33\log(2)\ \sigma^2 = 0.02\log(2) \rightarrow \widetilde{\nu}_f(1.2) = 0.5775$$
$$\mu = -0.45\log(2)\ \sigma^2 = 0.1\log(2) \rightarrow \widetilde{\nu}_f(1.2) = 0.6875$$

Cascades of Mandelbrot - sum of log-normal



$\widetilde{\nu}_f$

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In the future,

- it should also be tested on real-life signals (cardiac signals, image processing and so on) ;
- ...

Thank you for your attention !

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