About the Multifractal Nature of Cantor's Bijection

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Introduction

In 1878, Cantor constructed a bijection between $[0,1]$ and $[0,1]^2$, bijection defined via continued fractions.

G. Cantor, *Ein Beitrag zur Mannigfaltigkeitslehre*, Journal für die reine und angewandte Mathematik (Crelle's Journal), **Vol. 84**, 242-258, 1878.

Contents of this presentation

1. Continued Fractions
2. Cantor's Bijection
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Notations

$$E = [0,1], \qquad D = E \cap \mathbb{Q} \qquad \text{and} \qquad I = E \setminus D.$$

Finite Continued Fractions

A. Ya. Khintchine, *Continued Fractions*, P. Noordhoff, 1963.

Let $\boldsymbol{a} = (a_j)_{j\in\{1,\ldots,n\}}$ a finite sequence of positive real numbers ($n \in \mathbb{N}$); the expression $[a_1, \ldots, a_n]$ is recursively defined as follows:

$$[a_1] = \frac{1}{a_1} \quad \text{and} \quad [a_1, \ldots, a_m] = \frac{1}{a_1 + [a_2, \ldots, a_m]},$$

for any $m \in \{2, \ldots, n\}$. If $\boldsymbol{a} \in \mathbb{N}^n$, we say that

$$[a_1, \ldots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

is a (simple) **finite continued fraction**.

Proposition

For any $\boldsymbol{a} \in \mathbb{N}^n$ ($n \in \mathbb{N}$), $[a_1, \ldots, a_n]$ belongs to D. Conversely, for any $x \in D$, there exists a natural number n and a sequence $\boldsymbol{a} \in \mathbb{N}^n$ such that $x = [a_1, \ldots, a_n]$.

Infinite Continued Fractions

Let $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$. We can show that the sequence $x_j = [a_1, \ldots, a_j]$ ($j \in \mathbb{N}$) converges. The limit is called an **infinite continued fraction** and is denoted $[a_1, \ldots]$.

If the real number $x \in E$ is equal to $[a_1, \ldots]$, we say that $[\boldsymbol{a}] = [a_1, \ldots]$ is a continued fraction corresponding to x.

Theorem – Representation of the real numbers (of E)

Any element of D can be expressed as a finite continued fraction. We have $x \in I$ if and only if there exists an infinite continued fraction corresponding to x; moreover, this infinite continued fraction is unique.

Remark

An element x of I is a quadratic number if and only if the corresponding sequence $\boldsymbol{a}$ such that $[\boldsymbol{a}] = x$ is ultimately periodic, i.e. there exist $k, J \in \mathbb{N}$ such that $a_{j+k} = a_j$ for any $j \geq J$.

Metric Theory of Continued Fractions

Let $x = [\boldsymbol{a}] \in I$; for $n \in \mathbb{N}$, we set

$$I_n(x) = \big\{ y = [\boldsymbol{b}] \in I : b_j = a_j \text{ if } j \in \{1, \ldots, n\} \big\}.$$

For any $n \in \mathbb{N}$, $I_n(x)$ is an "irrational subinterval" of I, $I_{n+1}(x) \subset I_n(x)$ and

$$\lim_{n \to +\infty} I_n(x) = \{x\}.$$

The properties of $I_n(x)$ will be useful to study the regularity of Cantor's bijection.

Cantor's Bijection

G. Cantor, *Ein Beitrag zur Mannigfaltigkeitslehre*, Journal für die reine und angewandte Mathematik (Crelle's Journal), **Vol. 84**, 242-258, 1878.

If $x = [\boldsymbol{a}] \in I$, we set

$$f_1(x) = [a_1, a_3, \ldots, a_{2j+1}, \ldots] \quad \text{and} \quad f_2(x) = [a_2, a_4, \ldots, a_{2j}, \ldots].$$

The application

$$f : I \to I^2 \; ; x \mapsto (f_1(x), f_2(x))$$

is the **Cantor's Bijection** on I.

Remarks

- If Q denotes the quadratic numbers of I, f is a one-to-one mapping between Q to Q^2.
- Since the cardinals of E and I are equal, f can be extended to a one-to-one mapping from E to E^2.
- For any $n \in \mathbb{N}$ and any $x \in I$, f_1 maps the interval $I_n(x)$ to $I_m(f_1(x))$, where $m = n/2$ if n is even and $m = (n+1)/2$ if n is odd. This shows that f_1 is a continuous function.

Cantor's Bijection

G. Cantor, *Ein Beitrag zur Mannigfaltigkeitslehre*, Journal für die reine und angewandte Mathematik (Crelle's Journal), **Vol. 84**, 242-258, 1878.

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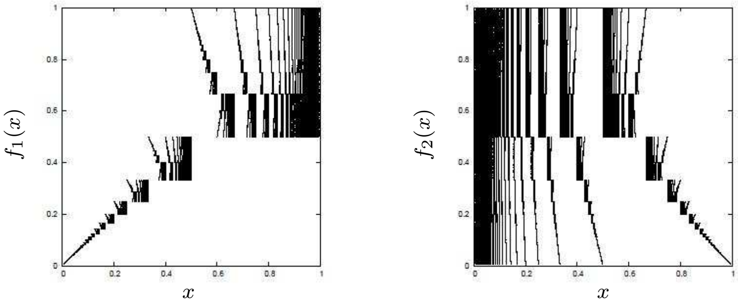
Cantor's Bijection

Representations of the functions f_1 (left panel) and f_2 (right panel)



For example, if $x \in (1/2, 1]$, then $x = [1, a_2, a_3, \ldots]$, $f_1(x) = [1, a_3, \ldots]$ and $f_1(x) \in (1/2, 1]$.

Cantor's Bijection

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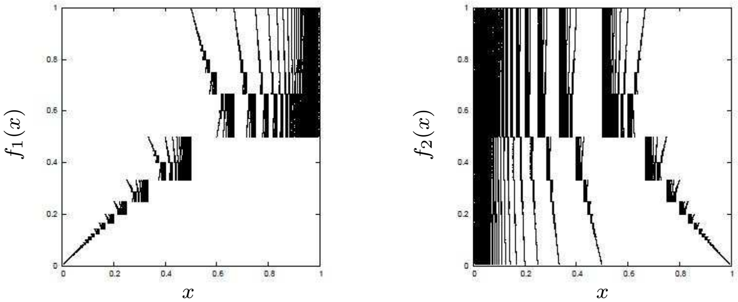

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Continuity of Cantor's Bijection

For all $x \in I$, we write $\varphi(x) = \boldsymbol{a}$ if $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ satisfies $x = [\boldsymbol{a}]$.

A usual distance on $\mathbb{N}^{\mathbb{N}}$ is given by

$$d(\boldsymbol{a}, \boldsymbol{b}) = \sum_{j=1}^{\infty} 2^{-j} \frac{|a_j - b_j|}{|a_j - b_j| + 1}$$

if $\boldsymbol{a} = (a_j)_{j \in \mathbb{N}}$ and $\boldsymbol{b} = (b_j)_{j \in \mathbb{N}}$ are two elements of $\mathbb{N}^{\mathbb{N}}$. We implicitly consider that $\mathbb{N}^{\mathbb{N}}$ is equipped with this distance. The sets I, D and E are endowed with the Euclidean distance.

Proposition

The application φ is an homeomorphism between I and $\mathbb{N}^{\mathbb{N}}$. In particular, Cantor's bijection f is an homeomorphism between I and I^2.

Remark

Since $(\mathbb{N}^{\mathbb{N}}, d)$ is a separable complete metric space, the space I is a Polish space.

Continuity of Cantor's Bijection

Netto's theorem

Any bijective map $g : E \to E^2$ is necessarily discontinuous.

H. Sagan, *Space-filling curves*, Universitext, New-York : Springer-Verlag, 1994.

Then, Cantor's bijection f can not be extended to a continuous bijection from E to E^2.

Proposition

Any extension of Cantor's bijection to E is discontinuous at any rational number.

Hölder Regularity

S. Jaffard, *Wavelet Techniques in Multifractal Analysis*, In Proceedings of Symposia in Pure Mathematics, **Vol. 72**, 91-152, 2004.

Let $\alpha \in [0, 1]$. A continuous and bounded real function g defined on $A \subset \mathbb{R}$ belongs to the Hölder space $C^{\alpha}(x)$ with $x \in A$ if there exists a constant $C > 0$ such that

$$|g(x) - g(y)| \leq C|x - y|^{\alpha},$$

for any $y \in A$. The Hölder exponent $h_g(x)$ of g at x is defined as follows:

$$h_g(x) = \sup\{\alpha \in [0, 1] : g \in C^{\alpha}(x)\}.$$

Remark

We have

$$h_g(x) = \liminf_{\substack{y \to x \\ y \in A}} \frac{\log |g(y) - g(x)|}{\log |y - x|}$$

Hölder Regularity of Cantor's Bijection

Surrounding Theorem

Let $n \in \mathbb{N}$. If $x = [\boldsymbol{a}]$ belongs to I and y belongs to $I_n(x) \setminus I_{n+1}(x)$, then we have

$$\frac{\frac{1}{n}\sum_{j=1}^{\lceil n/2 \rceil} \log(a_{2j-1})}{\frac{1}{n}\sum_{j=1}^{n+3} \log(a_j+1) + \frac{C_1(n)}{n}} \leq \frac{\log|f_1(x) - f_1(y)|}{\log|x-y|} \leq \frac{\frac{1}{n}\sum_{j=1}^{\lceil n/2 \rceil + 3} \log(a_{2j-1}+1) + \frac{C_2(n)}{2n}}{\frac{1}{n}\sum_{j=1}^{n} \log(a_j)}$$

with

$$C_1(n) = \frac{\log(2)}{2} + \log\left(\max\left\{\frac{a_{n+2}+2}{a_{n+2}+1}, \frac{a_{n+3}+2}{a_{n+3}+1}\right\}\right)$$

and

$$C_2(n) = \frac{\log(2)}{2} + \log\left(\max\left\{\frac{a_{2\lceil n/2 \rceil+3}+2}{a_{2\lceil n/2 \rceil+3}+1}, \frac{a_{2\lceil n/2 \rceil+5}+2}{a_{2\lceil n/2 \rceil+5}+1}\right\}\right).$$

There is a similar result for f_2.

Hölder Regularity of Cantor's Bijection

Ergodic Theorem

For any $k \in \mathbb{N} \cup \{0\}$, almost every sequence $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ satisfies

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log(a_j + k) = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log(a_{2j} + k)$$
$$= \lim_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log(a_{2j-1} + k) = \log(K_k),$$

where $K_k = \prod_{j=1}^{\infty} \left(1 + \frac{1}{j(j+2)}\right)^{\log(j+k)/\log(2)}$.

C. Ryll-Nardzewski, *On the Ergodic Theorems (II): Ergodic Theory of Continued Fractions*, Studia Mathematica **12**, 74-79, 1950.

R. Nair, *On the Metrical of Continued Fractions*, In Proceedings of the Amercian Mathematicae Society **120**, 1994.

Hölder Regularity of Cantor's Bijection

Ergodic Theorem

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$$\begin{aligned}\lim_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log(a_j + k) &= \lim_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log(a_{2j} + k) \\ &= \lim_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log(a_{2j-1} + k) = \log(K_k),\end{aligned}$$

where $K_k = \prod_{j=1}^{\infty} \left(1 + \frac{1}{j(j+2)}\right)^{\log(j+k)/\log(2)}$.

We say that a property P concerning sequences of $\mathbb{N}^{\mathbb{N}}$ holds almost everywhere if for almost every $x \in I$ (with respect to the Lebesgue measure), the sequence $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ such that $x = [\boldsymbol{a}]$ satisfies P.

Hölder Regularity of Cantor's Bijection

By combining the surrounding theorem and the ergodicity theorem, we obtain the following result.

Theorem

For almost every $x \in I$, we have

$$h_{f_1}(x), h_{f_2}(x) \in \left[\frac{\log(K_0)}{2\log(K_1)}, \frac{\log(K_1)}{2\log(K_0)} \right].$$

Then, $h_{f_1}(x)$ and $h_{f_2}(x)$ are included between 0.35 and 0.72.

S. Nicolay, L. Simons, *On the Multifractal Nature of Cantor's Bijection*, 2013, submitted.

Hölder Regularity of Cantor's Bijection

Remark

Let $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ be the sequence defined by

$$a_j = \begin{cases} 2^j & \text{if } j \text{ is even} \\ 1 & \text{if } j \text{ is odd} \end{cases},$$

for any $j \in \mathbb{N}$ and set $x = [\boldsymbol{a}]$. For this particular point, we have $h_{f_1}(x) = 0$, so that f_1 is a multifractal function.

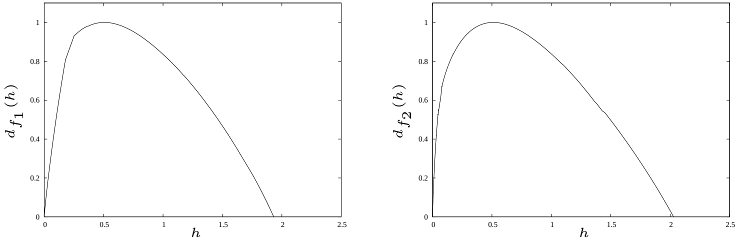
The function f_2 is also multifractal.

Corollary

Cantor's bijection f is multifractal.

Hölder Regularity of Cantor's Bijection

Numerical estimation of the spectrum of singularities of f_1 and f_2
with the wavelet leaders method



- We can hope that $h_{f_1}(x)$ and $h_{f_2}(x)$ are equal to $1/2$ for almost every $x \in I$.
- The functions f_1 and f_2 could be differentiable at some particular points of I.