An introduction to Reliability Analysis

Vincent DENOEL

University of Liege, ArGEnCo, MS^{2}F
Departement of Architecture, Geology, Environment and Construction
- Solid, Structures and Fluid Mechanics Division -

January 2007

This redaction of this document and the development of the illustrations could be realized thanks to Prof. H. KUSAMA, School of Design and Architecture, University of Nagoya City, Japan. The author is grateful to the University of Nagoya City, its University Board of Directors and the concerned Faculty and Department Meetings. The redaction of this original document (text and figures) has been completed between November 23^{rd}, 2006 and January 25^{th}, 2007, during an invited stay of the author at the University of Nagoya City. It is warmly acknowledged for this invitation.

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Chapter 1

Introduction

Since the early times of civil engineering the design of structures has been perfomed in a deterministic way, i.e. under the assumption of given loads acting on structures with given properties, which results in unique displacements and internal forces. During the whole life of the structure it should however be accepted that the loading is not unique and that the material properties are not accurately determined in advance. For these reasons a certain variability in the loading as well as in structural properties could have to be taken into account. This results in a probabilistic design.

Usually the design of a structure consists in two successive steps: first the *analysis* and then the *verifications*:

$$\text{Design} = \text{Analysis} + \text{Verifications}$$

In a deterministic design these two steps come naturally the one after the other: the purpose of the analysis is to compute the structural displacements and internal forces (or stresses); then the verifications aim at checking some design criteria as resistance or displacement exeedance. These two steps can easily be separated because the data to be conveyed from the *analysis* to the *verifications* are simple deterministic values: unique displacements and stresses.

In a stochastic design, the separation between the two steps is not necessarily obvious. Indeed since the input data of the problem is not deterministic anymore, the purpose of a probabilistic analysis is to provide a probabilistic description of the structral response instead of a unique deterministic value. This is illustrated at Figure 1.1: the loads acting on the structure, the structural behaviour and eventually the geometry are all represented by given probability density functions (Fig. 1.1-a). The results of the analysis are the probability density functions of the displacements and internal stresses at each point of the structure (Fig. 1.1-b).

These results can be achieved:

- by analytical developements and hand calculations in some simple cases,
- by a Monte Carlo Simulation (MCS) technique,
- by means of fuzzy arithmetics,
- by means of a stochastic finite element approach.

The MCS technique is the most time-consuming method but leads, without any hypothesis, to the most complete results. It is able to provide the full probability density function of the response and hence any subsequent result. Once the deterministic theories are well understood its application is straightforward. Concerning the method based on fuzzy arithmetics it simply consists in common operations (+, -, *, /) but applied on fuzzy numbers. This methods allows thus computing the fuzziness of the response (i.e. its variability which is equivalent to the probability density function).

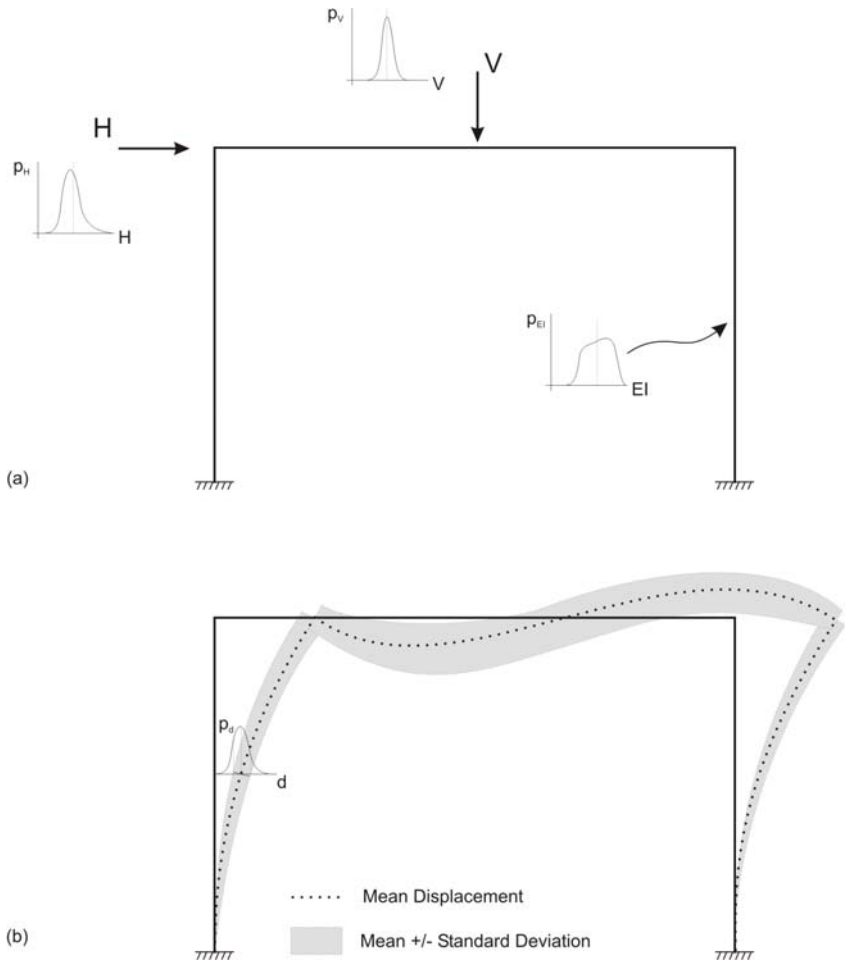

Figure 1.1: Probabilistic Analysis (a) The random loading and random material properties are specified by their probability density fucntions (b) The result of the analysis is the probability density functions of the displacements

Because of the evident complexity of some problems, the analytical approach can't be applied in any case; because of their evident time demand and mainly because of the huge quantity of information coming out from the analysis, the MCS technique and the fuzzy arithmetics are rather used for some localized checkings.

For these reasons, in civil engineering applications, the stochastic analysis and the stochastic verifications are generally proceeded at once in a so-called reliability analysis[1]. The use of this method is also explained by the fact that, at the design stage, the designer is seldomly interested in the whole pdf's of structural displacements but well in the upper tails of them. Indeed the upper tails of the pdf's only will be used for the design. The idea is thus to focus on this area only and not on the whole pdf's: the analysis consists in evaluating the reliability of the structure, i.e. the probability that, given some probabilistic loading and structural parameters, the load applied on the structure overcomes its resistance. This definition shows that this procedure involve both analysis and verifications at the same time. Compared to a Monte Carlo simulation, its application is very fast. As illustrated below it can indeed be seen as computing one single point of the whole pdf whereas,in its most basic formulation, the MCS technique establishes it all.

A reliability analysis consists in both the analysis and verification of the structure. Exactly as for the verification stage of a deterministic design, a set of checking conditions have to be provided. In order to simplify the problem, in this document, these conditions are considered one by one and the structure is said to be safe if all the conditions are fulfilled. For the sake of simplicity one condition only will be considered in this report.

Altough in a deterministic approach the checking condition can be proved to be fulfilled or not –i.e. to the question "Is the resistance larger than the load?", the answer is *yes* or *no*–, in a stochastic procedure, this condition can't be assessed otherwise than with a probabilistic measure: the probability of failure. The answer is of this kind: "yes the resistance is larger than the load with a probability equal to 95%". In the very simple context of two variables, resistance R and loading S, the failure condition is:

$$Z = R - S \tag{1.1}$$

The probability of failure is expressed by:

$$p_f = prob(Z < 0) = \iint\limits_{(R,S)|Z<0} p_{RS}\left(R,S\right) dRdS \tag{1.2}$$

and its computation is the major aim of any reliability analysis.

[1] Because it combines both analysis and verifications it should be rather called a "reliability-based design"

Chapter 2

Reliability analysis

In a deterministic design, and in the context of a two-variable problem, resistance R and loading S, the failure condition is assessed by the relation:

$$S > R \tag{2.1}$$

If this condition is not fulfilled, the failure is said to occur. Within the context of such a deterministic approach, this inequality can also be written with several equivalent formulations. For instance:

$$R - S < 0 \quad or \quad \frac{R}{S} - 1 < 0 \tag{2.2}$$

are strictly equivalent to Eq. (2.1). Although in a deterministic approach the checking condition can be proved to be fulfilled or not –i.e. to the question "Is the resistance larger than the load?", the answer is *yes* or *no*–, in a stochastic procedure, this condition can't be assessed otherwise than with a probabilistic measure: the probability of failure. The answer is then of this kind: "yes the resistance is larger than the load with a probability equal to 95%". In a reliability analysis, a failure condition has to be defined too. It is directly derived from the deterministic relation and the failure condition is defined by a new random variable G. For example the random variables corresponding to the deterministic relations in (2.2) are:

$$G_1(R,S) = R - S \quad \text{or} \quad G_2(R,S) = \frac{R}{S} - 1 \tag{2.3}$$

In this simple context of a two-variable problem the failure condition is:

$$G(R,S) < 0 \tag{2.4}$$

and the probability of failure is thus expressed by:

$$p_f = prob(G < 0) = \iint\limits_{(R,S)|G<0} p_{RS}(R,S)\, dRdS \tag{2.5}$$

where $p_{RS}(R,S)$, the joint probability density function between R and S, has to be integrated on the domain on which the failure condition is fulfilled (where the resistance is smaller than the loading). This is illustrated at Fig. 2.1.

In practical applications the resistance of the structure is given as a function of several parameters depending on the considered problem (stiffness, cross-section area, bending modulus, structural dimensions, etc.). Then in a more general context a failure condition is given as a scalar implicit relation of more than two parameters:

$$G(\{x\}) = G(x_1, x_2, ..., x_N) \tag{2.6}$$

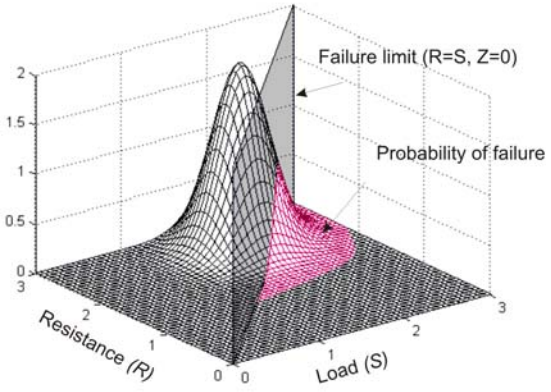

Figure 2.1: Illustration of the reliability analysis on 2-D variable problem.

the purpose of a reliability method consists in estimating the probability of failure as a result of the integral:

$$p_f = prob\left(G(\{x\}) \leq 0\right) = \underset{\{x\}|G(\{x\})\leq 0}{\int \cdots \int} p_x(x_1, x_2, ..., x_N) dx_1 dx_2 \cdots dx_N \tag{2.7}$$

where $p_x(x_1, x_2, ..., x_N)$ is the joint probability density function between all the variables.

In case of uncorrelated, gaussian variables and linear failure function the results of this integral can be obtained in close form. However in practical applications, these three conditions are very seldom satisfied together. The next sections will present how approximate methods have been developed in order to give estimations of this integral. The First Order Reliability Methods (FORM) consists in a large set of available methods and some of them are presented. The most basic one, the First Order Second Moment, is presented and its lack of rigour is enlightened. Then Advanced First Order Second Moment (AFOSM) are introduced as a result of the Hasofer-Lind theory. At this stage the simple theory valid for Gaussian Processes as well as the Equivalent Gaussian Method is presented.

2.1 First Order Second Moment (FOSM)

Let us assume that the load S and resistance R are two uncorrelated gaussian variables. They are characterized by their mean values (μ_S and μ_R) and standard deviations (σ_S and σ_R).Let us consider again the failure condition of Eq. (2.3):

$$Z_1(R,S) = R - S \quad \text{or} \quad Z_2(R,S) = \frac{R}{S} - 1 \tag{2.8}$$

where G has been re-written Z because Z_1 is a gaussian variable. Indeed, the theory of probability states that this linear combination of two gaussian variables is still gaussian. Z_1 is then very easy to handle: its mean and standard deviation are given by:

$$\mu_Z = \mu_R - \mu_S \tag{2.9}$$

$$\sigma_Z = \sqrt{\sigma_R^2 + \sigma_S^2} \tag{2.10}$$

The probability of failure, defined as the probability that Z_1 is negative (Eq. (2.5)), is represented by the shaded area in Fig. 2.2 and can be computed by:

$$p_f = \Phi\left(\frac{-\mu_Z}{\sigma_Z}\right) = 1 - \Phi\left(\frac{\mu_Z}{\sigma_Z}\right) \tag{2.11}$$

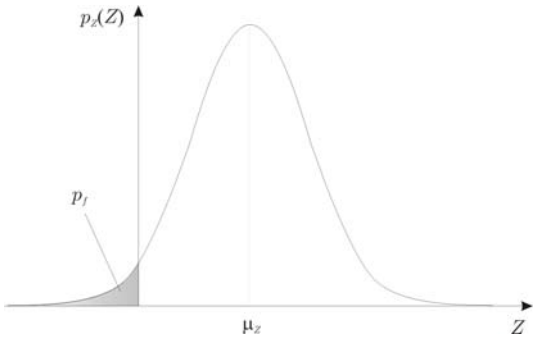

Figure 2.2: Probability density function of $Z_1 = R - S$

where Φ is the normal cumulative distribution function. This relation shows that it is very convenient to define a *reliability index* as:

$$\beta = \frac{\mu_Z}{\sigma_Z} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \tag{2.12}$$

so that the probability of failure is expressed by:

$$p_f = \Phi(\beta) = 1 - \Phi(\beta) \tag{2.13}$$

Historically the first idea of the reliability analysis was to compute this reliability index as the ratio of the mean failure condition and its standard deviation, and then to express the probability of failure by Eq. (2.13). This development shows that this way to estimate the probability of failure is rigorous, i.e. returns the exact probability of failure, in case of uncorrelated gaussian variables and linear failure function. Because it was developed from the consideration of a linear failure function (First Order) and Gaussian processes (represented by their first two statistical moments), this method is called the First-Order Second-Moment method (FOSM).

This procedure for the computation of the probability of failure can be used for the second "equivalent" failure condition $Z_2 = \frac{R}{S} - 1 < 0$. In this case, the mean value of the failure condition μ_Z and its standard deviation σ_Z are not easy to compute is close form. They can however be estimated easily if the failure condition is linearized around the mean value:

$$Z_2(R,S) \simeq Z_2(\mu_R, \mu_S) + (R - \mu_R) \left.\frac{\partial Z_2}{\partial R}\right|_{\substack{R=\mu_R \\ S=\mu_S}} + (S - \mu_S) \left.\frac{\partial Z_2}{\partial S}\right|_{\substack{R=\mu_R \\ S=\mu_S}} \tag{2.14}$$

The expectation of this relation, taken side-by-side, gives:

$$\mu_{Z_2} = Z_2(\mu_R, \mu_S) = \frac{\mu_R}{\mu_S} - 1 \tag{2.15}$$

and the variance of Z_2 is expressed by:

$$\begin{aligned} \sigma_{Z_2}^2 &= E\left[\left((R - \mu_R) \left.\frac{\partial Z_2}{\partial R}\right|_{\substack{R=\mu_R \\ S=\mu_S}} + (S - \mu_S) \left.\frac{\partial Z_2}{\partial S}\right|_{\substack{R=\mu_R \\ S=\mu_S}}\right)^2\right] \\ &= \sigma_R^2 \left.\frac{\partial Z_2}{\partial R}\right|_{\substack{R=\mu_R \\ S=\mu_S}} + \sigma_S^2 \left.\frac{\partial Z_2}{\partial S}\right|_{\substack{R=\mu_R \\ S=\mu_S}} \\ &= \frac{\sigma_R^2}{\mu_S} - \frac{\mu_R \sigma_S^2}{\mu_S^2} \end{aligned} \tag{2.16}$$

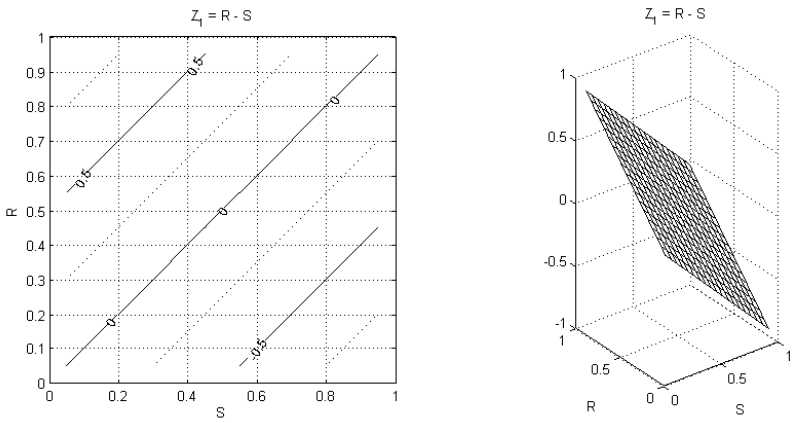

Figure 2.3: Example of linear failure function and thus linear failure condition

The reliability index, as defined in the FOSM method, is:

$$\beta = \frac{\mu_{Z_2}}{\sigma_{Z_2}} = \frac{\frac{\mu_R}{\mu_S} - 1}{\sqrt{\frac{\sigma_R^2}{\mu_S} - \frac{\mu_R \sigma_S^2}{\mu_S^2}}} \tag{2.17}$$

The probability of failure related to this second failure condition is then expressed by $p_f = \Phi(\beta) = 1 - \Phi(\beta)$. Simply because the reliability indices are different (Eqs. (2.12) and (2.17)) the probability of failure is different than the one obtained with the first failure condition. This result indicates that the FOSM method has to be used with an extreme care. Indeed it would be expected that several failure condition giving the same result in a deterministic approach give also the same result in a probabilistic approach. This example shows that it is not the case with the FOSM method. This is known as the lack of invariance property and is illustrated in section 3.1.5.

Definition 1 *A reliability analysis method presents the invariance property if any transformation of a linear failure function returns the same reliability index. It is important to notice that the invariance property is related to a linear failure condition and not necessarily to a linear failure function. This difference seems to be subtle but is very important. Indeed a non linear failure function, e.g.* $Z_2(R,S) = R/S - 1$*, can be associated to a linear failure condition (*$Z_2 = 0 \Rightarrow R - S = 0$ *which is linear).This is illustrated at Figs 2.3 to 2.5. The first two are linear and non linear failure functions but leading to the same linear failure condition:* $Z = 0$ *is the same straight line on both graphs. Typically non linear failure function related to linear conditions (Fig. 2.4) can be obtained by a transformation of a linear failure function. The invariance property concerns these functions. Fig. 2.5 illustrates a non linear function with a non linear failure condition. In this case the limit condition* $Z = 0$ *is not linear.*

The FOSM method is rigorous in case of linear failure function only (Fig. 2.3 only) and not for any linear failure condition. Therefore it does not fulfil the invariance property. The FOSM method consists in replacing the exact failure function by a linearized relation, this linearization being done at the mean point. In case of linear failure function, this linearization returns the same result no matter the point around which the linearization is performed. But Fig. 2.4 shows clearly that if the linearization is performed on a point which does not lye on $Z = 0$, the resulting hyperplane is different (because level lines are not parallel). This is the exact reason for which the FOSM presents the lack of invariance.

Despite this major drawback the FOSM method has been widely used for many years. Indeed if the random variables are Gaussian and if the failure function is linear this method returns the

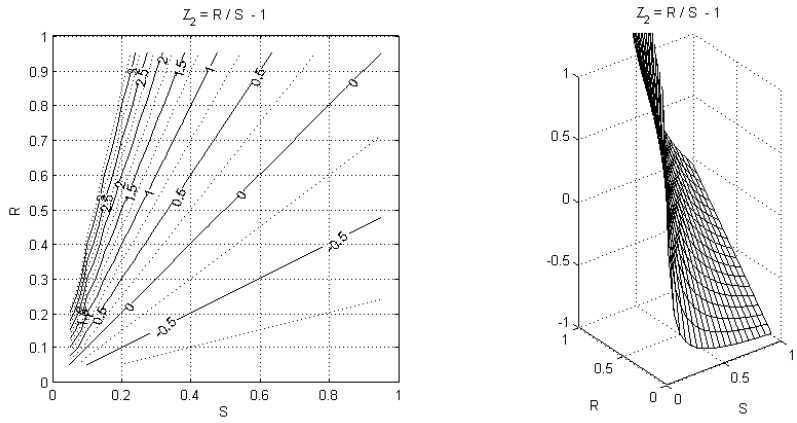

Figure 2.4: Example of non linear failure function but linear failure condition

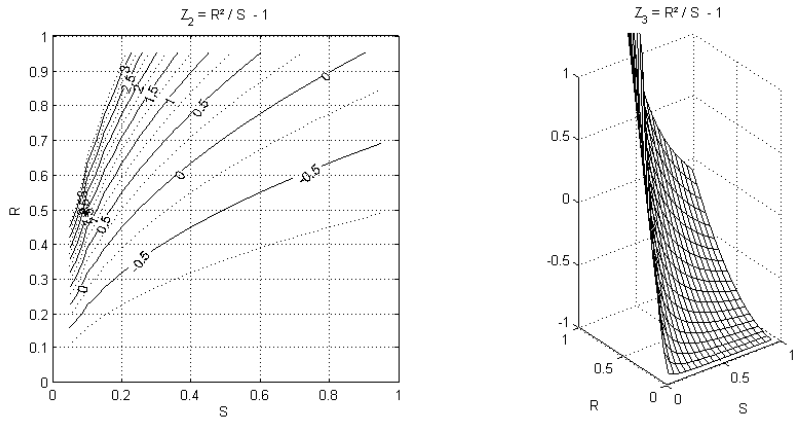

Figure 2.5: Example of non linear failure function and non linear failure condition

exact probability of failure. If the probability distribution are slightly different than the gaussian distribution or if the failure condition is slightly non linear, this method can however be used to determine estimations of the exact probability of failure. The mathematical definition of the FOSM method can be generalized in case of several random variables.

Definition 2 . *FOSM: Let us suppose a non linear relation of random variables:*

$$G(\{x\}) = G(x_1, x_2, ..., x_N) \tag{2.18}$$

describing the failure condition $G(\{x\}) = 0$. *The reliability index related to this failure condition is defined as the ratio of the mean and standard deviation of the failure function:*

$$\beta = \frac{\mu_G}{\sigma_G} \tag{2.19}$$

where the mean and standard deviation are estimated by linearizing the failure condition around the mean variable $\{\mu_x\}$ *:*

$$G(\{x\}) \simeq G\left(\{\mu_x\}\right) + \left(\{x\} - \{\mu_x\}\right)^T \left.\frac{\partial G}{\partial \{x\}}\right|_{\{x\}=\{\mu_x\}} \tag{2.20}$$

which gives:

$$\mu_G = G(\{\mu_x\}) \tag{2.21}$$

$$\sigma_G^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} \left.\frac{\partial G}{\partial x_i}\right|_{\{x\}=\{\mu_x\}} \left.\frac{\partial G}{\partial x_j}\right|_{\{x\}=\{\mu_x\}} cov_{X_i X_j} \tag{2.22}$$

where $cov_{X_i X_j}$ *represents the covariance between variables* X_i *and* X_j. *The subsequent probability of failure is rigorous in case of gaussian variable and linear failure function only.*

2.2 Advanced First Order Second Moment (AFOSM)

2.2.1 Presentation of the method

Because of its lack of invariance the First Order Second Moment method has been upgraded to an advanced method. The idea is then to develop a reliability method able to return the exact probability of failure in case of any linear failure condition and not only linear function (e.g as in Fig. 2.4).

The major reproach that can be made to the previous development is that it considers from the beginning the linearity of the failure function and not only the linearity of the failure condition. Instead of focusing on the probability distribution of $G(\{x\})$ and mainly its mean and standard deviation, other methods rather target the estimation of the probability of failure as a direct result of the multiple integral (Eq. (2.5)). As it is presented in the following this lead to reliability methods having the invariance property. They are known as Advanced First Order Second Moment (AFOSM) reliability analyses.

Similarly to the developments of the previous section, the most simple AFOSM is presented in the context of uncorrelated Gaussian variables and linear failure condition.First it is convenient to transform the physical variables into zero-mean and unit standard deviation variables (Hasofer-Lind transformation). New reduced variables are defined by:

$$\widehat{x}_i = \frac{x_i - \mu_{x_i}}{\sigma_{x_i}} \tag{2.23}$$

which have now the interesting property to lead to an axisymmetric joint probability density function (because the initial variables are supposed to be uncorrelated). The failure condition is supposed to be linear, i.e. it can be written in this form:

$$G(\{x\}) = 0 \quad \Leftrightarrow \quad a_1 x_1 + a_2 x_2 + ... + a_N x_N = a_0 \quad \Leftrightarrow \quad \{a\}^T \{x\} = a_0 \tag{2.24}$$

This failure condition can also be written as a function of the new variables as:

$$\widehat{G}(\{\widehat{x}\}) = G(x_1(\widehat{x}_1), x_2(\widehat{x}_2), ..., x_N(\widehat{x}_N)) \tag{2.25}$$

Since the original failure condition was linear the linear transformation keeps the linearity of the limits of the domain of integration. Then $\widehat{G}(\{\widehat{x}\})$ can be written:

$$\widehat{G}(\{\widehat{x}\}) = 0 \quad \Leftrightarrow \quad \widehat{a}_1\widehat{x}_1 + \widehat{a}_2\widehat{x}_2 + ... + \widehat{a}_N\widehat{x}_N = \widehat{a}_0 \quad \Leftrightarrow \quad \{\widehat{a}\}^T\{\widehat{x}\} = \widehat{a}_0 \tag{2.26}$$

This is schematically illustrated on the 2-variable problem in Fig. 2.6. Because the variables are uncorrelated, the principal axes of the joint probability denstiy function between R and S are parallel to axes R and S (Fig. 2.6-(a)). The shaded area represents the domain of integration, i.e. the zone in which $G < 0$, or again the domain in which the resistance R is smaller than the loading S. With the new reduced variables (Fig; 2.6-(b)), the joint probability density function is represented by concentric circular level curves and the failure condition $\widehat{Z} = 0$ is still a linear relation of $\widehat{R}$ and $\widehat{S}$.

Furthermore in the context of uncorrelated variables the joint probability density function can be factorized in its marginal probability density functions as:

$$p_f = \int_{\{x\}|G(\{x\})\leq 0} \cdots \int p_{x_1}(x_1)p_{x_2}(x_2)...p_{x_N}(x_N)dx_1dx_2\cdots dx_N \tag{2.27}$$

or as a function of the reduced variables:

$$p_f = \int_{\{\widehat{x}\}|\widehat{G}(\{\widehat{x}\})\leq 0} \cdots \int p_{\widehat{x}_1}(\widehat{x}_1)p_{\widehat{x}_2}(\widehat{x}_2)...p_{\widehat{x}_N}(\widehat{x}_N)d\widehat{x}_1d\widehat{x}_2\cdots d\widehat{x}_N \tag{2.28}$$

where $p_{\widehat{x}_i}(\widehat{x}_i) = p_{x_i}(x_i(\widehat{x}_i))$, $i \in [1, N]$ are the marginal probability density functions of the reduced variables, i.e. the normal distributions because the distribution of the physical variables is gaussian. By noting the normal probability density function as $p_z(x)$, Eq. (2.28) can thus also be written:

$$p_f = \int_{\{\widehat{x}\}|\widehat{G}(\{\widehat{x}\})\leq 0} \cdots \int p_z(\widehat{x}_1)p_z(\widehat{x}_2)...p_z(\widehat{x}_N)d\widehat{x}_1d\widehat{x}_2\cdots d\widehat{x}_N \tag{2.29}$$

The factorization of the joint probability density functions reduces the complexity of the definite integral to the shape of its domain of integration only. This shape is however not really complex because its limit, represented by the failure condition, is a hyperplane (a line in Fig. 2.6-(b)). In order to make it simpler again, let us define another transformation:

$$\{\widetilde{x}\} = [R]\{\widehat{x}\} \tag{2.30}$$

where $[R]$ is the adequate rotation matrix transforming the domain of integration into a domain with axes parallel to the reduced directions. It can be understood intuitively that the probability of failure can be expressed as:

$$p_f = \int_{-\infty}^{+\infty} p_z(\widetilde{x}_1)d\widetilde{x}_1 \cdots \int_{-\infty}^{+\infty} p_z(\widetilde{x}_{N-1})d\widetilde{x}_{N-1} \int_{\beta}^{+\infty} p_z(\widetilde{x}_N)d\widetilde{x}_N \tag{2.31}$$

where the domain of integration is running from $-\infty$ to $+\infty$ for each variable except for the last one which has to cover the interval $[\beta, +\infty]$. Because of Kolmogorov's first axiom and because the marginal probability density functions are normal, the probability of failure is thus finally reduced to:

$$p_f = \int_{\beta}^{+\infty} p_z(\widetilde{x}_N)d\widetilde{x}_N = 1 - \Phi(\beta) = \Phi(-\beta) \tag{2.32}$$

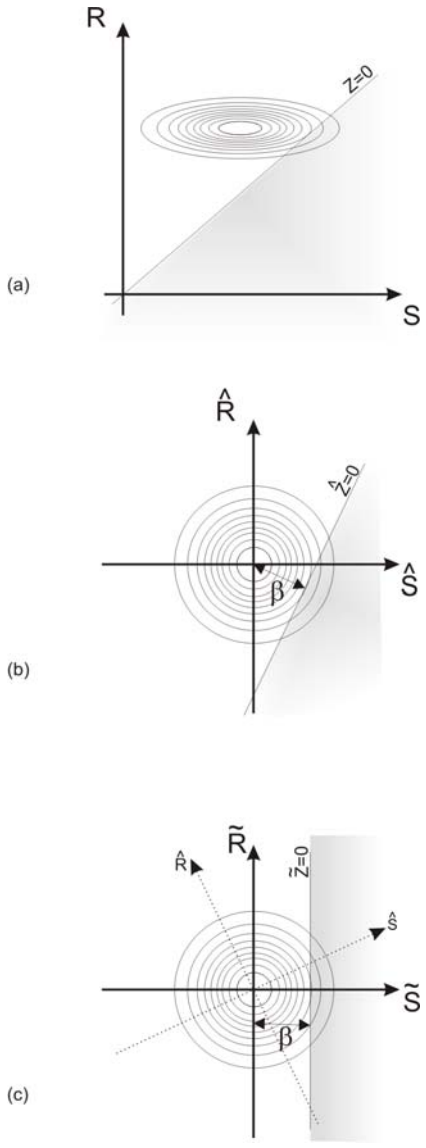

Figure 2.6: Successive transformations: (a) Initial physical variables - (b) Reduced variables with zero mean and unit variance - (c) Rotated reduced variables

where Φ is the normal cumulative density function and β is the reliability index defined as the shortest distance, in the reduced space (Fig. 2.6-(b)), between the origin and the failure condition. On the simple two-variable problem with gaussian variables and linear failure condition, this geometric definition of the reliability index is equivalent to what was given in the previous section (FOSM). It is however different in case of a non linear failure condition as illustrated in secition 3.1.5.

Both the FOSM and AFOSM give the same reliability indices and hence probability of failure in case of gaussian processes and linear failure functions. If the failure function is non linear but the failure condition is linear the AFOSM return also the exact probability of failure. This can be understood intuitively since the AFOSM method is based on a geometric approach and the domain of integration is anyway limited by a hyperplane in case of linear condition (no matter the linearity or not of the function). Furthermore the linearity of the failure function is not sought at all in the AFOSM whereas it was the FOSM is based on this hypothesis.

In case of non linear conditions, the AFOSM method gives however estimations only of the exact probability of failure. Because of the geometric definition of the reliability index, the AFOSM method can be applied but it should be kept in mind that the resulting error in the probability of failure increases as the non linearity of the condition increases.

Definition 3 . *AFOSM: Let us suppose a non linear relation of uncorrelated Gaussian variables:*

$$G(\{x\}) = G(x_1, x_2, ..., x_N) \tag{2.33}$$

describing the failure condition. A set of reduced (zero-mean unit-variance) variables is defined by:

$$\widehat{x}_i = \frac{x_i - \mu_{x_i}}{\sigma_{x_i}} \tag{2.34}$$

and the failure condition, expressed with these new variables is:

$$\widehat{G}(\{\widehat{x}\}) = G(x_1(\widehat{x}_1), x_2(\widehat{x}_2), ..., x_N(\widehat{x}_N)) = 0 \tag{2.35}$$

The reliability index related to this failure condition is defined as the shortest distance in the reduced space between the origin and this hypersurface.

Theorem 4 *To find the minimum of a scalar function $F(\{\widehat{x}\})$ under the condition $G(\{\widehat{x}\}) = 0$ is a usual mathematical problem. It is usually solved by introducing Lagrange multipliers. Indeed for an extremum of F to exist on G the gradient of F must line up with the gradient of G. One is then the multiple of the other*

$$\nabla F = -\lambda \nabla G \tag{2.36}$$

where λ is called the Lagrange multiplier. If these two vectors are equal then each of their components are also equal:

$$\frac{\partial F}{\partial x_i} + \lambda \frac{\partial G}{\partial x_i} = 0 \qquad \text{for } i = 1, ...N \tag{2.37}$$

Together with $G(\{\widehat{x}\}) = 0$, these N relations form a set of $N+1$ equations with $N+1$ unknowns. The optimum point is obtained by the resolution of this set of non linear equations.

The AFOSM requires the computation of the shortest distance between a given hypersurface and the origin. This can be performed by introducing a Lagrange multiplier. The distance between any point and the origin is expressed by:

$$\beta(\{\widehat{x}\}) = \sqrt{\{\widehat{x}\}^T \{\widehat{x}\}} \tag{2.38}$$

The application of Lagrange's theory to the squared distance lead to this set of equations:

$$\begin{cases} \frac{\partial}{\partial \widehat{x}_i}\left(\{\widehat{x}\}^T \{\widehat{x}\}\right) + \lambda \frac{\partial \widehat{G}}{\partial \widehat{x}_i} = 0 \\ \widehat{G}(\widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_N) = 0 \end{cases} \tag{2.39}$$

which gives, after simplification:

$$\begin{cases} \widehat{x}_i + \frac{\lambda}{2}\frac{\partial \widehat{G}}{\partial \widehat{x}_i} = 0 \\ \widehat{G}\left(\widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_N\right) = 0 \end{cases} \tag{2.40}$$

The resolution of this set of equations gives the design point (or most probable failure point) $\{\widehat{x}^*\}$. The corresponding reliability index can subsequently be estimated thanks to Eq. (2.38).

2.2.2 Link between the FOSM and AFOSM methods

The FOSM method is based on the linearization of the failure function at the mean variable point ($\{x\} = \{\mu_x\}$). Because this point does not necessarily lie on the failure limit, this reliability method does not have the invariant property. In order to bypass this problem the linearization of the function could be done at the design point ($\{x\} = \{\widehat{x}^*\}$) which lies in any case on the failure limit. The failure function is thus approached by:

$$G(\{x\}) \simeq (\{x\} - \{x^*\})^T \left.\frac{\partial G}{\partial \{x\}}\right|_{\{x^*\}} \tag{2.41}$$

in which $G\left(\{\widehat{x}^*\}\right) = 0$ has been taken into account. It is important to notice that the design point $\{\widehat{x}^*\}$ is a priori unknown.

Following the FOSM's procedure, the reliability index is expressed as the ratio of the mean value of the failure condition μ_G and its standard deviation σ_G. Thanks to the new linearization they are expressed by:

$$\mu_G = E\left[G(\{x\})\right] = (\{\mu_x\} - \{x^*\})^T \left.\frac{\partial G}{\partial \{x\}}\right|_{\{x^*\}} \tag{2.42}$$

and

$$\begin{aligned} \sigma_G^2 &= E\left[\left(G(\{x\}) - \mu_G\right)^2\right] \\ &= E\left[\left(\left(\{x\} - \{\mu_x\}\right)^T \left.\frac{\partial G}{\partial \{x\}}\right|_{\{x^*\}}\right)^2\right] \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} E\left[\left(x_i - \mu_{x_i}\right)\left(x_j - \mu_{x_j}\right)\right] \left.\frac{\partial G}{\partial x_i}\right|_{\{x^*\}} \left.\frac{\partial G}{\partial x_j}\right|_{\{x^*\}} \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} cov_{x_i x_j} \left.\frac{\partial G}{\partial x_i}\right|_{\{x^*\}} \left.\frac{\partial G}{\partial x_j}\right|_{\{x^*\}} \end{aligned}$$

where $cov_{X_i X_j}$ represents the covariance between variables X_i and X_j. In order to compare more easily with the development related to the AFOSM method the same development can be written in the reduced space ($\{\mu_x\} \to 0$, $cov_{x_i x_j} \to \delta_{ij}$, $x \to \widehat{x}$). The resulting reliability index is then expressed by:

$$\beta = \frac{\mu_G}{\sigma_G} = \frac{-\sum_{i=1}^{N} \widehat{x}_i^* \left.\frac{\partial \widehat{G}}{\partial \widehat{x}_i}\right|_{\{\widehat{x}^*\}}}{\sqrt{\sum_{i=1}^{N}\left(\left.\frac{\partial \widehat{G}}{\partial \widehat{x}_i}\right|_{\{\widehat{x}^*\}}\right)^2}} \tag{2.43}$$

In the AFOSM method the reliability index is defined as the shortest distance between the origin (in the reduced space) and the limit hyperplan. It has been shown that the resulting expression of the reliability index is (Eq. (2.38)):

$$\beta = \sqrt{\sum_{i=1}^{N} \widehat{x}_i^{*2}} \tag{2.44}$$

After some trivial transformations and using Eq. (2.40), this relation can also be written:

$$\begin{aligned}\beta &= \sqrt{\sum_{i=1}^{N} \widehat{x}_i^{*2}} = \frac{\frac{2}{\lambda}\sum_{i=1}^{N} \widehat{x}_i^{*2}}{\frac{2}{\lambda}\sqrt{\sum_{i=1}^{N} \widehat{x}_i^{*2}}} \\ &= \frac{-\sum_{i=1}^{N} \widehat{x}_i^{*}\left(-\frac{2}{\lambda}\widehat{x}_i^{*}\right)}{\sqrt{\sum_{i=1}^{N}\left(-\frac{2}{\lambda}\widehat{x}_i^{*}\right)^2}} = \frac{-\sum_{i=1}^{N} \widehat{x}_i^{*} \left.\frac{\partial \widehat{G}}{\partial \widehat{x}_i}\right|_{\{\widehat{x}^*\}}}{\sqrt{\sum_{i=1}^{N}\left(\left.\frac{\partial \widehat{G}}{\partial \widehat{x}_i}\right|_{\{\widehat{x}^*\}}\right)^2}}\end{aligned} \tag{2.45}$$

which results in the same relation as Eq. (2.43). This indicates clearly that the geometric definition of the reliability index (without requiring any linearization of the failure function!) is strictly equivalent to the linearization of the failure function around the design point. This demonstration establishes a direct and clear link between the FOSM method and the AFOSM method. A second definition can thus be given for the AFOSM method.

Definition 5 . *AFOSM: Let us suppose a non linear relation of random variables:*

$$G(\{x\}) = G(x_1, x_2, ..., x_N) \tag{2.46}$$

describing the failure condition $G(\{x\}) = 0$. The reliability index related to this failure condition is defined as the ratio of the mean and standard deviation of the failure function:

$$\beta = \frac{\mu_G}{\sigma_G} \tag{2.47}$$

where the mean and standard deviation are estimated by linearizing the failure condition around the design point $\{x^*\}$:

$$G(\{x\}) \simeq (\{x\} - \{x^*\})^T \left.\frac{\partial G}{\partial \{x\}}\right|_{\{x\}=\{x^*\}} \tag{2.48}$$

which gives:

$$\mu_G = (\{\mu_x\} - \{x^*\})^T \left.\frac{\partial G}{\partial \{x\}}\right|_{\{x\}=\{x^*\}} \tag{2.49}$$

$$\sigma_G^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} cov_{x_i x_j} \left.\frac{\partial G}{\partial x_i}\right|_{\{x\}=\{x^*\}} \left.\frac{\partial G}{\partial x_j}\right|_{\{x\}=\{x^*\}} \tag{2.50}$$

where $cov_{X_i X_j}$ represents the covariance between variables X_i and X_j. The subsequent probability of failure is rigorous in case of gaussian variable and linear failure condition only. Since the linearization is performed around an a priori unknown point (the design point), an iterative resolution scheme has to b adopted.

2.2.3 Details of the resolution

The previous developments related to the AFOSM show that this reliability method can be considered in two equivalent (but clearly distinct) ways:

- the geometric definition of the reliability index (Def. 3) leads to a set of non linear equations. This system has to be solved (2.40) with any numerical procedure. Usually a second-order iterative scheme, like Newton-Raphson method, is used.
- the reliability index can be obtained as the ratio of the mean to the standard deviation of the failure function, after having linearized it around the design point (Def. 5). Since the design point is a priori unknown, this procedure requires also an iterative technique for the computation of the reliability index.

The equivalence between these two methods is seldom presented and many author focus just on one or the other without any justification. In this report, the equivalence has been demonstrated (Section 2.2.2) and it is clear that both ways to estimate the reliability index will give the same result[1] . For this reason, in the following, the second way to consider the reliability index will be adopted.

In his lecture note on reliability analysis ([5]), Kusama adopts this alternative and applies a convenient iterative resolution procedure. The successive steps of this procedure can be summarized in an algorithmic manner.

Algorithm 6 *AFOSM in the physical space, for uncorrelated gaussian variables*

Step 1: Give an initial guess[2] *of the design point* $x_i^{(0)}$ *and start the iterations at the next step (with* $k = 0$*).*

Step 2: There is no reason for this point to lie on the failure condition. So, compute the failure condition at this point:

$$G^{(k)} = G\left(\left\{x^{(k)}\right\}\right) \tag{2.51}$$

Step 3: Compute the gradient of the failure function at this point:

$$n_i^{(k)} = \left.\frac{\partial G\left(\{x\}\right)}{\partial x_i}\right|_{\{x\}=\left\{x^{(k)}\right\}} \tag{2.52}$$

Step 4: Compute a new orientation for the design point:

$$\alpha_i^{(k)} = \frac{n_i^{(k)}\sigma_{x_i}}{\sqrt{\sum_{i=1}^{N}\left(n_i^{(k)}\sigma_{x_i}\right)^2}} \tag{2.53}$$

Step 5: Compute the estimated mean and standard deviation of the failure condition:

$$\mu_G = G^{(k)} + \sum_{i=1}^{N} n_i^{(k)}\left(\mu_{x_i} - x_i^{(k)}\right) \tag{2.54}$$

$$\sigma_G = \sum_{i=1}^{N} \alpha_i^{(k)} n_i^{(k)} \sigma_{x_i} \tag{2.55}$$

Step 6: Compute a new estimation of the reliability index:

$$\beta^{(k)} = \frac{\mu_G}{\sigma_G} \tag{2.56}$$

Step 7: Compute a better estimation of the design point:

$$x_i^{(k+1)} = \mu_{x_i} - \alpha_i^{(k)}\beta^{(k)}\sigma_{x_i} \tag{2.57}$$

increment k *by* 1 *and loop from Step 2 to Step 7 until the convergence is reached.*

[1] The major difference between both ways to consider the reliability index concerns the convergence of the iterative schemes. For some badly conditioned systems (i.e. some complex failure conditions), one method could be much faster than the other, or even converge correctly whereas the other diverges dramatically.

[2] The initial guess of the design point could simply be the mean value (the first iteration is then the exact application of the FOSM method). In Kusama's procedure the initial guess of the design point is given by initial guesses on $\beta^{(0)}$ and $\left\{\alpha^{(0)}\right\}$ where α_i represent the orientation of the design point with respect to the mean values. As a first guess, he recommends to choose $\beta^{(0)} = 3$ and $\alpha_i^{(0)} = \frac{1}{\sqrt{N}}$ which seems to be appropriate in some circumstances only. The initial design point is then computed by:
$x_i^{(0)} = \mu_{x_i} - \alpha_i^{(0)}\beta^{(0)}\sigma_{x_i}$

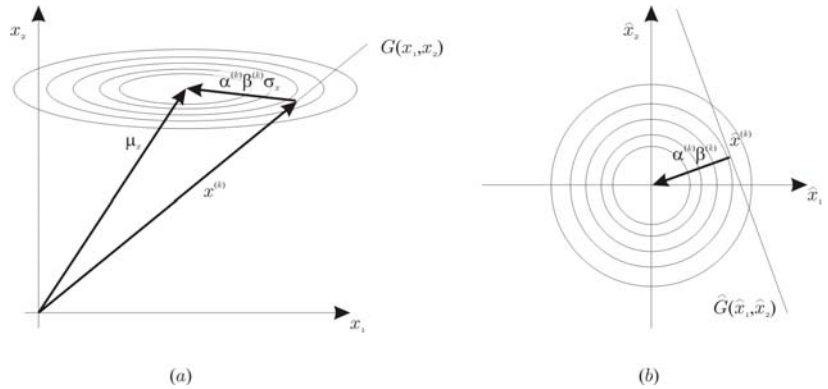

Figure 2.7: Illustration of the difference between: (a) the physical space (Algorithm 6) and (b) the reduced space.(Algorithm 7).

The developments of the next sections will indicate that the use of reduced variables is not really necessary but clearly convenient in case of correlated variables. For this reason, this modification is already adopted at this stage. Algorithm 6 is slightly modified in order to perform the reliability analysis in the reduced space (zero-mean and unit-variance variables).

Algorithm 7 *AFOSM in the reduced space, for uncorrelated gaussian variables*

Step 0.1: Define reduced variables:

$$\widehat{x}_i = \frac{x_i - \mu_i}{\sigma_i} \quad \Leftrightarrow \quad x_i = \mu_i + \sigma_i \widehat{x}_i \tag{2.58}$$

Step 0.2: Write the failure condition with the reduced variables:

$$\widehat{G}\left(\{\widehat{x}\}\right) = G\left(\{x\left(\{\widehat{x}\}\right)\}\right) \tag{2.59}$$

Step 1: Give an initial guess of the design point $\widehat{x}_i^{(0)}$ *and start the iterations at the next step (with* $k = 0$*).*

Step 2: There is no reason for this point to lie on the reduced failure condition. So, compute the reduced failure condition at this point:

$$\widehat{G}^{(k)} = \widehat{G}\left(\left\{\widehat{x}^{(k)}\right\}\right) \tag{2.60}$$

Step 3: Compute the gradient of the reduced failure function at this point:

$$n_i^{(k)} = \left.\frac{\partial \widehat{G}\left(\{\widehat{x}\}\right)}{\partial \widehat{x}_i}\right|_{\{\widehat{x}\}=\left\{\widehat{x}^{(k)}\right\}} \tag{2.61}$$

Step 4: Compute a new orientation for the design point:

$$\alpha_i^{(k)} = \frac{n_i^{(k)}}{\left|n^{(k)}\right|} = \frac{n_i^{(k)}}{\sqrt{\sum_{i=1}^{N}\left(n_i^{(k)}\right)^2}} \tag{2.62}$$

Step 5: Compute the estimated mean and standard deviation of the failure condition:

$$\mu_G = \widehat{G}^{(k)} - \sum_{i=1}^{N} n_i^{(k)} x_i^{(k)} \tag{2.63}$$

$$\sigma_G = \sum_{i=1}^{N} \alpha_i^{(k)} n_i^{(k)} \tag{2.64}$$

Step 6: Compute a new estimation of the reliability index:

$$\beta^{(k)} = \frac{\mu_G}{\sigma_G} \tag{2.65}$$

Step 7: Compute a better estimation of the design point:

$$\widehat{x}_i^{(k+1)} = -\alpha_i^{(k)} \beta^{(k)} \tag{2.66}$$

increment k by 1 and loop from Step 2 to Step 7 until the convergence is reached.

The selection of the first guess for Step 1 of this algorithm is subject to the same remarks as in Algorithm 6. However it should be added that if the first guess is taken as the mean physical variables (eventually reduced), Algorithms 6 and 7 correspond to the simple FOSM after the first step. This algorithm and this remark are presented by an illustrative example in section 3.1.4 (p. 34).

The application of this algorithm is rigorous in case of linear failure condition only. In the opposite case, the method can also be applied but it should be kept in mind that it gives approximate results only.

2.3 Advanced First Order Second Moment for Correlated variables (AFOSMC)

Under the three conditions considered in section 2.2 (uncorrelated, gaussian variables and linear failure condition) the probability of failure is directly related to the reliability index β through Eq. (2.32). If any one of these conditions is not fulfilled the reliability index can be computed (thanks to its geometrical definition) but can't be linked directly to the probability of failure. For this reason the reliability of a structure is often quantified by its reliability index rather than by its probability of failure. However the aim of many reliability analysis methods is to relate as accurately as possible the reliability index to the actual probability of failure of the structure. This means also that if any one of the three conditions considered in the context of Algorithm 7 is not fulfiled, this procedure can however be used but leads definitely to erroneous estimations of the actual reliability index and probability of failure. More appropriate resolution techniques have to be considered in this case. This section aims at presenting the modifications to be brought to the AFOSM method (Algorithm 7) in order to use it in the context of correlated variables. This results in the AFOSMC method.

When the physical variables (x_i) involved in the problem are not uncorrelated, the principal axes of the joint probability density function are no longer parallel to the axis (Fig.2.8-(a)). The simple reduction proposed by Eq. (2.23) does not transform the joint probability density function into an axisymmetric function. To this purpose, another transformation has to be introduced:

$$\{\widehat{x}\} = [A]\left(\{x\} - \{\mu_x\}\right) \tag{2.67}$$

where $\{\mu_x\} = E[\{x\}]$ is the vector of mean physical variables and $[A]$ is the transformation matrix which will produce uncorrelated reduced variables $\{\widehat{x}\}$. The mathematical expectation of this relation shows that this transformation provides zero-mean reduced variables. Furthermore their covariance matrix is expressed by:

$$[V_{\widehat{x}}] = E\left[\{\widehat{x}\}\{\widehat{x}\}^T\right] = [A]\, E\left[\left(\{x\} - \{\mu_x\}\right)\left(\{x\}^T - \{\mu_x\}^T\right)\right][A]^T = [A]\,[V_x]\,[A]^T \tag{2.68}$$

Since the aim of the reduction is to provide zero-mean and unit-variance variables, the matrix $[A]$ will be chosen such as to provide:

$$[V_{\widehat{x}}] = [A]\,[V_x]\,[A]^T = [I] \tag{2.69}$$

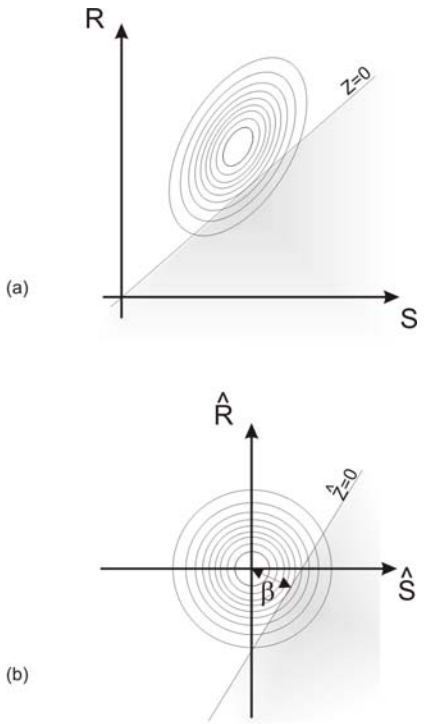

Figure 2.8:

where $[I]$ is the identity matrix. Hence $[A]$ can simply be estimated as the eigen vectors of the covariance matrix of the physical variables. The normalization of the eigen vectors is of course performed in such a way that Eq. (2.69) is satisfied.

In this section it is still supposed that the failure condition is a linear function of the physical parameters (Eq. 2.24). Since the transformation used to produce uncorrelated reduced variables is linear, the failure condition expressed with the new variables remains again linear:

$$\begin{aligned} G(\{x\}) = 0 \quad &\Leftrightarrow \quad \{a\}^T \{x\} = a_0 \\ \widehat{G}(\{\widehat{x}\}) = 0 \quad &\Leftrightarrow \quad \{a\}^T \left(\{\mu_x\} + [A]^{-1} \{\widehat{x}\} \right) = a_0 \\ &\Leftrightarrow \quad \underbrace{\{a\}^T [A]^{-1}}_{\{\widehat{a}\}^T} \{\widehat{x}\} = \underbrace{a_0 - \{a\}^T \{\mu_x\}}_{\widehat{a}_0} \end{aligned} \tag{2.70}$$

The reduced variables are now uncorrelated, their joint probability density function is axisymmetric and can be factorized. Then the same developments as presented in the previous section are still valid. In particular, Eq. (2.29) still holds, the reliability index with the same definition as previously can be determined and it is again linked rigorously to the probability of failure.

In view of these developments, the former algorithm of the AFOSM method can be slightly transformed to suit the context of correlated variables.

Algorithm 8 *AFOSMC, for gaussian variables*
Step 0.1: Define reduced variables:

$$\{\widehat{x}\} = [A] \left(\{x\} - \{\mu_x\} \right) \quad \Leftrightarrow \quad \{x\} = \{\mu_x\} + [A]^{-1} \{\widehat{x}\} \tag{2.71}$$

where $[A]$ *is such that* $[A] [V_x] [A]^T = [I]$ *and* $[V_x]$ *is the covariance matrix of the physical variables* $\{x\}$.

Step 0.2: Write the failure condition with the reduced variables:

$$\widehat{G}\left(\{\widehat{x}\}\right) = G\left(\{x\left(\{\widehat{x}\}\right)\}\right) = G\left(\{\mu_x\} + [A]^{-1}\{\widehat{x}\}\right) \tag{2.72}$$

Step 1: Give an initial guess of the design point $\widehat{x}_i^{(0)}$ *and start the iterations at the next step (with* $k = 0$*).*

Step 2: There is no reason for this point to lie on the reduced failure condition. So, compute the reduced failure condition at this point:

$$\widehat{G}^{(k)} = \widehat{G}\left(\left\{\widehat{x}^{(k)}\right\}\right) \tag{2.73}$$

Step 3: Compute the gradient of the reduced failure function at this point:

$$n_i^{(k)} = \left.\frac{\partial \widehat{G}\left(\{\widehat{x}\}\right)}{\partial \widehat{x}_i}\right|_{\{\widehat{x}\}=\left\{\widehat{x}^{(k)}\right\}} \tag{2.74}$$

Step 4: Compute a new orientation for the design point:

$$\alpha_i^{(k)} = \frac{n_i^{(k)}}{\left|n^{(k)}\right|} = \frac{n_i^{(k)}}{\sqrt{\sum_{i=1}^{N}\left(n_i^{(k)}\right)^2}} \tag{2.75}$$

Step 5: Compute the estimated mean and standard deviation of the failure condition:

$$\mu_G = \widehat{G}^{(k)} - \sum_{i=1}^{N} n_i^{(k)} x_i^{(k)} \tag{2.76}$$

$$\sigma_G = \sum_{i=1}^{N} \alpha_i^{(k)} n_i^{(k)} \tag{2.77}$$

Step 6: Compute a new estimation of the reliability index:

$$\beta^{(k)} = \frac{\mu_G}{\sigma_G} \tag{2.78}$$

Step 7: Compute a better estimation of the design point:

$$\widehat{x}_i^{(k+1)} = -\alpha_i^{(k)} \beta^{(k)} \tag{2.79}$$

increment k *by* 1 *and loop from Step 2 to Step 7 until the convergence is reached.*

The application of this algorithm is rigorous in case of linear failure condition only. In the opposite case, the method can also be applied but it should be kept in mind that it gives approximate results only. If the physical variables are uncorrelated, this Algorithm degenerates to Alg. 7 and the reduction matrix $[A]$ is a diagonal matrix composed of the inverses of the standard deviations of the random variables.

This Algorithm is illustrated through worked examples in section 3.2.

2.4 Second-Order Reliability Methods (SORM)

When the failure condition exhibits strong non linearities, the application of FORM methods can lead to significant erroneous results. Indeed as indicated in its name, a FOR method considers the first order representation of the limit state only, i.e. replaces it by a hyperplane. This is illustrated

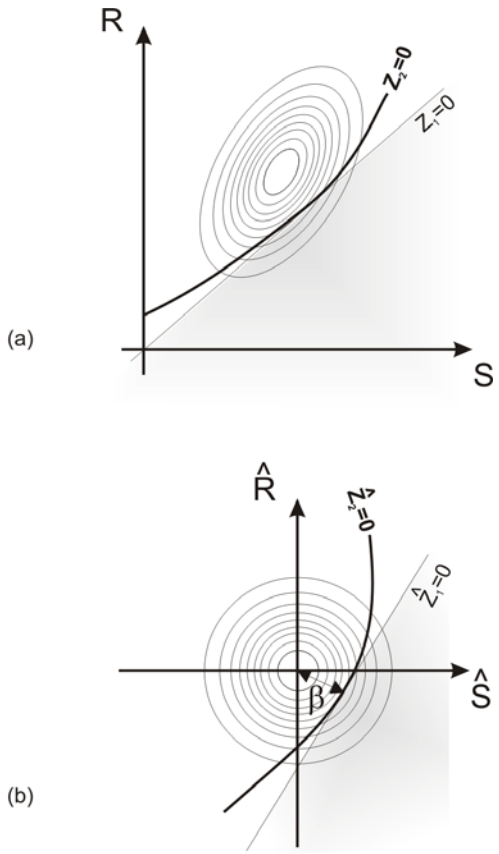

Figure 2.9: Illustration of a non linear failure function and the corresponding linearized function (at the design point)

in Fig. 2.9. Because of the definition of the reliability index in a geometric way, both failure conditions have the same reliability index. The probability of failure is rigorously related to the reliability index when the failure condition is linear and when the physical variables are normally distributed. For this reason, the security of usual structures against failure is often assessed by means of the reliability index only.

Fig. 2.9 shows also that computing the probability of failure associated to a non linear condition consists in replacing it by the tangent hyperplane at the design point (in this example both failure condition Z_1 and Z_2 have the same reliability index). The resulting probability of failure is thus underestimated if the curvature of the actual failure condition faces the origin (as in Fig. 2.9); on the opposite, a curvature towards infinity will results in a overestimation of the exact probability of failure.

The evaluation of the probability of failure in case of some simple non linear failure functions can be found in the literature but the cases in which this integral can be computed are very seldom. Anyway the explicit computation of the probability of failure for any quadratic failure function is not imaginable! For this reason, some approximate methods have been developed. The most usual is due to Breitung ([1], [2]) which gives a simple approximate expressions of the probability of failure, as a function of the reliability index (obtained with a FORM method):

$$p_f \simeq \Phi(-\beta) \prod_{i=1}^{n-1} \frac{1}{\sqrt{(1-\beta\kappa_i)}} \tag{2.80}$$

$$p_f \simeq \Phi(-\beta) \prod_{i=1}^{n-1} \frac{1}{\sqrt{\left(1-\frac{\phi(\beta)}{\Phi(-\beta)}\kappa_i\right)}} \tag{2.81}$$

The approximate results are obtained by asymptotical developments of the integrands. The probability of failure is also expressed as a function of the extrinsic curvatures of the failure function κ_i at the design point. It can be checked that, if the curvatures are all equal to 0 (linear failure condition), this result degenerates into $\Phi(-\beta)$ which is well correct if the failure condition is linear (FORM). The second relation seems to provide better results because it smoothens the singularity at $\beta\kappa_i = 1$. In this second expression $\phi(\beta)$ denotes the standard normal density function.

The development of such a method goes beyond the scope of this technical report and is therefore not illustrated in the following examples. The error associated to the use of a FORM in case of non linear failure function is however illustrated by comparison with a Monte Carlo Simulation (Sections 3.2.6, 3.3.2 and 3.3.3).

2.5 First-Order Gaussian Second Moment Method (FOGSM)

The Gaussian distribution is often used because of its convenience. However not any physical characteristic could be modelled with a Gaussian variable. The most arguable point is that a Gaussian variable can exhibit negative (even $-\infty$, but with a very small probability). For this reason, other random distributions have to be considered for the variables involved in a structural design.

Fig 2.10-(a) illustrates the same failure condition ($Z = R - S$) as previously considered, but in the context of non gaussian variables. In this case the level curves of the joint probability density function are not elliptic anymore. An example of the most general case is given in Fig. 2.10-(a). The previous developments related to the FOSM and AFOSM methods indicated that the probability of failure could be computed easily, provided the joint probability density function can be factorized, allowing then a reduction of the order of integration to 1 (and the definition of the reliability index). Since a huge quantity of developments has been performed with this usual definition of the reliability index, a common way to handle non gaussian variables is to transform the problem into the same form as previously. An adequate transformation (details given hereafter) is computed in such a way to transform the non gaussian variables into gaussian

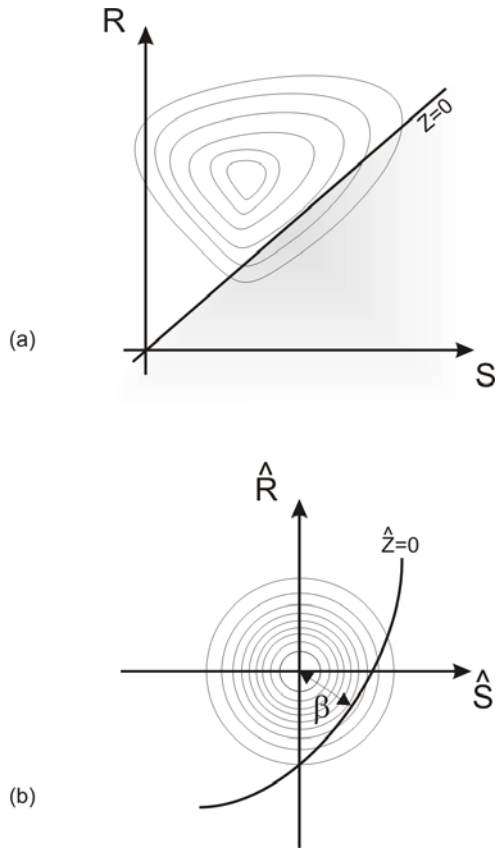

Figure 2.10:

ones. Typically this reduction is a non linear relation (because any linear function of a gaussian process keeps its linearity). The failure condition is transformed accordingly (Fig. 2.10-(b)). Even if the original failure condition was linear (in the physical space) the reduced failure condition exhibits a non linear shape because of this reduction.

Then when the considered random variables are non Gaussian, the reduced failure function is non linear. Adequate resolution techniques (e.g. SORM) should thus be used. However, provided the non linearity remains slight, first order methods (like AFOSM) can already give reasonable estimations of the probability of failure.

Theorem 9 *Transformation of a single random variable. Let x and y be random variables with their respective probability density functions $p_x(x)$ and $p_y(y)$. If y is expressed as a function of x by a monotonic relation $y = y(x)$ which leads to the univoque reverse relation $x = x(y)$, both probability density functions are related by:*

$$\begin{aligned} p_x(x) &= p_y(y)\left|\frac{dy(x)}{dx}\right| \\ p_y(y) &= p_x(x)\left|\frac{dx(y)}{dy}\right| \end{aligned} \tag{2.82}$$

The proof of this relation is straightforward when considering the cumulative density functions. For example, the cumulative density function of y is:

$$F_y(y_0) = prob(y < y_0) = prob(x < x(y_0)) = F_x(x(y_0)) \tag{2.83}$$

and the corresponding probability density function, obtained by simple derivation with respect to y_0, is:

$$p_y(y) = \frac{dF_y(y)}{dy} = \frac{dF_x(x)}{dx}\left|\frac{dx}{dy}\right| = p_x(x)\left|\frac{dx}{dy}\right| \tag{2.84}$$

This general relation can be used in two interesting context.

Theorem 10 *Tranformation from a uniform distribution. Let us suppose that x is uniformly distributed on $[0,1]$ and that $F_y(y)$ is a given cumulative density function. The function $y(x)$ transforming x to y and such that the resulting cumulative density function corresponds to $F_y(y)$ is:*

$$y(x) = F_y^{-1}(x) \tag{2.85}$$

If x is uniformly distributed on $[0,1]$, its cumulative density function is expressed by:

$$F_x(x_0) = \begin{cases} 0 & x_0 < 0 \\ x_0 & x_0 \in [0,1] \\ 1 & x_0 > 1 \end{cases} \tag{2.86}$$

Furthermore, from Eq. (2.83), it is possible to write:

$$x(y_0) = F_x^{-1}[F_y(y_0)] \Leftrightarrow y(x_0) = F_y^{-1}[F_x(x_0)] \tag{2.87}$$

which demonstrates the theorem ($F_x(x_0) = x_0$). This property is commonly used for the generation of non uniform random variables (Monte Carlo simulation techniques). Indeed, today's computers are all equipped with a random number generator. This generator is often limited to the generation of a uniform random variable (between 0 and 1). Thanks to the previous relation any random variable with a given cumulative density function (and hence probability density function) can be generated.

Theorem 11 *Transformation to a Gaussian distribution. Let us suppose that the probability distribution of x is known. The transformation $y(x)$ that provides a normal variable y is given by:*

$$y(x) = \Phi^{-1}[F_x(x)]$$

where Φ represents the gaussian cumulative density function.

The proof of this theorem is a straightforward application of Eq. (2.87).

Theorem 12 *Transformation of a multiple random variables. Let $\{x\}$ and $\{y\}$ be sets random variables with their respective joint probability density functions $p_x(\{x\})$ and $p_y(\{y\})$. If $\{y\}$ is univoquely expressed as a function of $\{x\}$ and reversely as well, both joint probability density functions are related by:*

$$\begin{aligned} p_x(\{x\}) &= p_y(\{y\})\left|\frac{d\{y(\{x\})\}}{d\{x\}}\right| = p_y(\{y\})\begin{vmatrix} \frac{dy_1}{dx_1} & \frac{dy_2}{dx_1} & \dots & \frac{dy_N}{dx_1} \\ \frac{dy_1}{dx_2} & \ddots & & \vdots \\ \vdots & & & \\ \frac{dy_1}{dx_N} & \dots & & \frac{dy_N}{dx_N} \end{vmatrix} \\ p_y(\{y\}) &= p_x(\{x\})\left|\frac{d\{x(\{y\})\}}{d\{y\}}\right| = p_x(\{x\})\begin{vmatrix} \frac{dx_1}{dy_1} & \frac{dx_2}{dy_1} & \dots & \frac{dx_N}{dy_1} \\ \frac{dx_1}{dy_2} & \ddots & & \vdots \\ \vdots & & & \\ \frac{dx_1}{dy_N} & \dots & & \frac{dx_N}{dy_N} \end{vmatrix} \end{aligned} \tag{2.88}$$

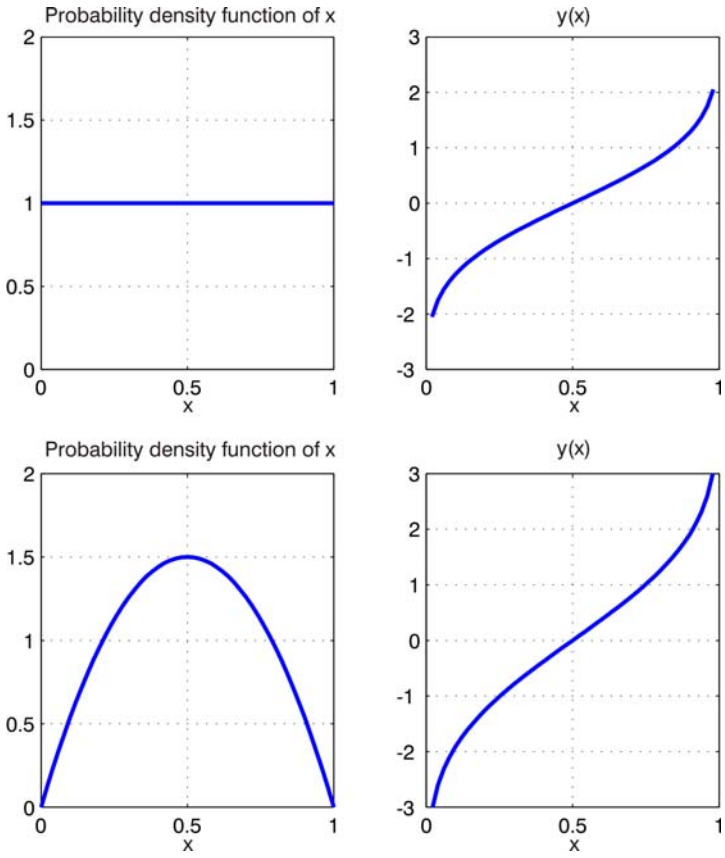

Figure 2.11: Illustration of transformation functions. (a) From uniform distribution to Gaussian, (b) From the standard beta $(2, 2)$ distribution to Gaussian

The demonstration of this relation is similar to the proof of Theorem 9.

In case of correlated non gaussian variables, advanced tranformation techniques (Rosenblatt Transformation, Nataf Transformation) have to be applied. This goes beyond the scope of this report. Then in the following developments and illustrations it will be supposed that the non gaussian random variables are not correlated. This allows reducing each of them separately with Th. 11. Depending on the original probability density function, the relation $y(x)$ which will provide a gaussian variable y has to be computed. This is illustrated at Fig. 2.11 for the uniform distribution and the standard beta distribution. Each physical variable leads then to the definition of a corresponding reduced variable. The computation of the reliability index is then performed as in the previous case. These developments can be applied to formalize the method in an algorithmic manner.

Algorithm 13 *FOGSM in the reduced space, for uncorrelated gaussian variables*

Step 0.1: Define reduced variables:

$$\widehat{x}_i = \Phi^{-1}\left[F_{x_i}\left(x_i\right)\right] \quad \Leftrightarrow \quad x_i = F_{x_i}^{-1}\left[\Phi\left(\widehat{x}_i\right)\right] \tag{2.89}$$

In almost any case this function has to be computed in a numerical way.

Step 1: Give an initial guess of the design point $\widehat{x}_i^{(0)}$ *and start the iterations at the next step (with* $k = 0$*).*

Step 2: There is no reason for this point to lie on the reduced failure condition. So, compute the reduced failure condition at this point:

$$\widehat{G}^{(k)} = \widehat{G}\left(\left\{\widehat{x}^{(k)}\right\}\right) = G\left(\left\{x^{(k)}\left(\left\{\widehat{x}^{(k)}\right\}\right)\right\}\right) \tag{2.90}$$

Step 3: Compute the gradient of the reduced failure function at this point:

$$n_i^{(k)} = \left.\frac{\partial \widehat{G}\left(\{\widehat{x}\}\right)}{\partial \widehat{x}_i}\right|_{\{\widehat{x}\}=\left\{\widehat{x}^{(k)}\right\}} \tag{2.91}$$

Step 4: Compute a new orientation for the design point:

$$\alpha_i^{(k)} = \frac{n_i^{(k)}}{\left|n^{(k)}\right|} = \frac{n_i^{(k)}}{\sqrt{\sum_{i=1}^{N}\left(n_i^{(k)}\right)^2}} \tag{2.92}$$

Step 5: Compute the estimated mean and standard deviation of the failure condition:

$$\mu_G = \widehat{G}^{(k)} - \sum_{i=1}^{N} n_i^{(k)} x_i^{(k)} \tag{2.93}$$

$$\sigma_G = \sum_{i=1}^{N} \alpha_i^{(k)} n_i^{(k)} \tag{2.94}$$

Step 6: Compute a new estimation of the reliability index:

$$\beta^{(k)} = \frac{\mu_G}{\sigma_G} \tag{2.95}$$

Step 7: Compute a better estimation of the design point:

$$\widehat{x}_i^{(k+1)} = -\alpha_i^{(k)} \beta^{(k)} \tag{2.96}$$

increment k *by* 1 *and loop from Step 2 to Step 7 until the convergence is reached.*

This algorithm is applied in section 3.3.2.

2.6 First-Order Gaussian Approximation Method (FOGAM)

The developments of the previous section aimed at presenting the rigorous procedure for the computation of the reliability index in case of non gaussian processes. It was shown that the transformation to the reduced space leads systematically to a non linear function, even in case of linear failure condition. For this reason, the application of a FORM method will thus lead to approximate results.

Furthermore the first step of this method (transformation to uncorrelated, zero-mean, unit-variance variables) is not necessarily easy to perform without the appropriate numerical tools. Since the FOGSM does not provide the exact results, a simplified version of this method is sometimes applied. In this version the complex transformation is avoided. The key idea of this method, the First-Order Gaussian Approximation Method, lies in the replacement of the actual probability density functions by equivalent Gaussian probability distribution. There are several ways to perform this equivalence but the most usual is presented in this technical report. Since the most important region concerns the failure domain, the equivalent gaussian variables are chosen such as to provide the same probability density function and cumulative density function at the design point. For each variable two equations allow thus the determination of the two necessary parameters (mean and variance) for the full characterisation of the equivalent Gaussian process. Once equivalent Gaussian variable are established the problem is formatted as in the AFOSM method and the same resolution procedure can be applied.

Compared to the FOGSM, this new method is a bit more time-consuming. Indeed, the computation of the equivalent gaussian variables depends on the location of the design point and has thus to be performed at each iteration. Once the

Algorithm 14 *FOGAM in the reduced space, for uncorrelated gaussian variables*

Step 0.a: Give an estimation of the design point $x_i^{(0)}$ *and start the iterations at the next step (with* $k = 0$*).*

Step 0.b: Define equivalent Gaussian variables ($\mu_i^{(0)}$, $\sigma_i^{(0)}$)

Step 0.1: Define reduced variables:

$$\widehat{x}_i = \frac{x_i - \mu_i^{(0)}}{\sigma_i^{(0)}} \quad \Leftrightarrow \quad x_i = \mu_i^{(0)} + \sigma_i^{(0)} \widehat{x}_i \tag{2.97}$$

Step 0.2: Write the failure condition with the reduced variables:

$$\widehat{G}\left(\{\widehat{x}\}\right) = G\left(\{x\left(\{\widehat{x}\}\right)\}\right) \tag{2.98}$$

Step 1: Compute the corresponding reduced design point $\widehat{x}_i^{(0)}$

Step 2: There is no reason for this point to lie on the reduced failure condition. So, compute the reduced failure condition at this point:

$$\widehat{G}^{(k)} = \widehat{G}\left(\left\{\widehat{x}^{(k)}\right\}\right) \tag{2.99}$$

Step 3: Compute the gradient of the reduced failure function at this point:

$$n_i^{(k)} = \left.\frac{\partial \widehat{G}\left(\{\widehat{x}\}\right)}{\partial \widehat{x}_i}\right|_{\{\widehat{x}\}=\left\{\widehat{x}^{(k)}\right\}} \tag{2.100}$$

Step 4: Compute a new orientation for the design point:

$$\alpha_i^{(k)} = \frac{n_i^{(k)}}{\left|n^{(k)}\right|} = \frac{n_i^{(k)}}{\sqrt{\sum_{i=1}^{N}\left(n_i^{(k)}\right)^2}} \tag{2.101}$$

Step 5: Compute the estimated mean and standard deviation of the failure condition:

$$\mu_G = \widehat{G}^{(k)} - \sum_{i=1}^{N} n_i^{(k)} x_i^{(k)} \tag{2.102}$$

$$\sigma_G = \sum_{i=1}^{N} \alpha_i^{(k)} n_i^{(k)} \tag{2.103}$$

Step 6: Compute a new estimation of the reliability index:

$$\beta^{(k)} = \frac{\mu_G}{\sigma_G} \tag{2.104}$$

Step 7: Compute a better estimation of the design point:

$$\widehat{x}_i^{(k+1)} = -\alpha_i^{(k)} \beta^{(k)} \tag{2.105}$$

increment k by 1 *and loop from Step 2 to Step 7 until the convergence is reached.*

This method is applied in section 3.3.3.

2.7 Summary

In the previous sections several techniques have been presented for the computation of the probability of failure, by means of a so-called reliability analysis. The simplest procedure (FOSM) based on the linearization of the failure function around the mean physical variables lead to erroneous results, even in case on non linear failure function but linear failure condition. The AFOSM method consists in performing this linearization in the neighbourhood of the design point, i.e. the most probable point of failure. It is possible to prove that this method provides the exact results (for the probability of failure) in case of gaussian variables and linear failure condition. If these conditions are not fulfilled, a reliability index can be computed (thank to a geometrical definition) but it can't be linked directly to the probability of failure.

A dedicated procedure, the AFOSMC, has been introduced for correlated variables. Basically its aim is strictly equivalent to the AFOSM method. It is therefore also valid for gaussian variables and linear failure conditions only.

In case of non Gaussian variables, the transformation to the reduced variables (as in the AFSOM and AFOSMC methods) is not linear anymore. It transforms then any linear failure condition into a non linear one. A procedure similar to the AFOSM has been presented (FOGSM). It practical application is not always convenient because of this non linear transformation. For this reason, another method, also presented in this document (the FOGAM) is sometimes applied. The simplicity of this method lies in the replacement of the non gaussian variables by equivalent gaussian ones.

Chapter 3

Illustrations

3.1 Bending model (uncorrelated variables)

3.1.1 Probabilistic analysis: analytical approach

In the bending problem a usual deterministic analysis would lead to this relation between the stiffness, the applied force and the rotation:

$$kL^2\theta = \frac{PL}{2} \tag{3.1}$$

In a deterministic design, all the quantities involved in this relation are perfectly known which allows the univoque determination of the rotation (analysis). Then this results has to be compared to an admissible value (verification) or eventually post-treated in any other way.

In a probabilistic context the quantities involved in Eq. (3.1) are defined by means of random variables. For the sake of simplicity, let us suppose temporarily that the beam length L is a deterministic given and that k and P are uncorrelated random variables specified by their own pdf's. The equivalent philosophy to what is done in a deterministic approach would be:

- to analyse the structure, i.e. compute the rotation θ, or more precisely its probability density function;
- to decide whether this result is satisfactory or not

In this very simple example, the probability density function of the rotation can be obtained in an analytical way. Indeed the cumulative distribution function of this variable is:

$$\begin{aligned} F_\theta(\theta_0) &= prob(\theta < \theta_0) \\ &= prob\left(\frac{P}{2kL} < \theta_0\right) \end{aligned} \tag{3.2}$$

By considering all the possible values that P could take, this relation can also be written:

$$\begin{aligned} F_\theta(\theta_0) &= \int_{-\infty}^{+\infty} prob(P_0 < P \leq P_0 + dP_0)\, prob\left(\frac{P_0}{2kL} < \theta_0\right) dP_0 \\ &= \int_{-\infty}^{+\infty} p_P(P_0)\left[1 - prob\left(k \leq \frac{P_0}{2L\theta_0}\right)\right] dP_0 \\ &= \int_{-\infty}^{+\infty} p_P(P_0)\left[1 - F_k\left(\frac{P_0}{2L\theta_0}\right)\right] dP_0 \qquad (3.3) \\ &= 1 - \int_{-\infty}^{+\infty} p_P(P_0)\, F_k\left(\frac{P_0}{2L\theta_0}\right) dP_0 \qquad (3.4) \end{aligned}$$

Then the probability density function of θ can be computed by simple derivation of the cumulative distribution function:

$$p_\theta(\theta_0) = \frac{dF_\theta}{d\theta_0} = \frac{1}{2L\theta_0^2}\int_{-\infty}^{+\infty} P_0 p_P(P_0)\, p_k\left(\frac{P_0}{2L\theta_0}\right) dP_0 \tag{3.5}$$

This relation is valid for any marginal probability density functions for k and P. For this application, let us suppose that they are described by gaussian variables with given means and standard deviations (μ_P, μ_k, σ_P, σ_k):

$$p_P(P_0) = \frac{1}{\sqrt{2\pi}\sigma_P} e^{-\frac{(P_0-\mu_P)^2}{2\sigma_P^2}} \tag{3.6}$$

$$p_k(k_0) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{(k_0-\mu_k)^2}{2\sigma_k^2}} \tag{3.7}$$

After some simplifications, the introduction of Eqs. (3.6) and (3.7) into Eq. (3.5) gives:

$$p_\theta(\theta_0) = \frac{2L}{\sqrt{2\pi}} \frac{2L\mu_P\theta_0\sigma_k^2 + \mu_k\sigma_P^2}{\left((2L\sigma_k\theta_0)^2 + \sigma_P^2\right)^{3/2}} e^{-\frac{1}{2}\left[\frac{(\mu_P - 2L\mu_k\theta_0)^2}{(2L\sigma_k\theta_0)^2+\sigma_P^2}\right]} \tag{3.8}$$

which indicates that the rotation θ is not a gaussian variable. This relation is the main result of the analysis and can be used to compute any subsequent information as the cumulative distribution function of the rotation:

$$F_\theta(\theta) = \int_{-\infty}^{\theta} p_\theta(\theta_0)\, d\theta_0 \tag{3.9}$$

This definite integral has no analytical formulation but could be estimated in a numerical way. For example, the probability of failure in the sense of the reliability theory, i.e. the probability that a given rotation is exceeded could be computed:

$$prob(\theta > \theta_L) = 1 - F_\theta(\theta_L) \tag{3.10}$$

In order to facilitate the comparison with the following numerical resolution, a numerical application is given. Let us suppose that $L = 1m$, and P and k are Gaussian variables with means and standard deviations respectively equal to:

$$\begin{aligned} \mu_P &= 10kN \quad ; \quad \sigma_P = 1kN \\ \mu_k &= 150kN/m \quad ; \quad \sigma_k = 20kN/m \end{aligned} \tag{3.11}$$

Fig. 3.1 represents the probability density function of the rotation resulting from Eq. (3.8). This example shows that the distribution is skewed to the right and hence that this distribution is not gaussian anymore. The cumulative distribution function ($F_\theta(\theta_L)$) can be established in a numerical way and the determination of the associated probability of failure is straightforward (Fig. 3.1). For the numerical application, it is supposed that the maximum allowed rotation is $\theta_L = 0.05rad$. Then the probability of failure, i.e. the probability that $\theta > \theta_L$ is:

$$p_f = 1 - F_\theta(\theta_L) = 0.012674 \tag{3.12}$$

Despite the remarkable simplicity of the data, the complexity of these results shows, as mentioned in section 1, that analytical developments are clearly limited to some simple cases. In the following the same example will be studied with a reliability method.

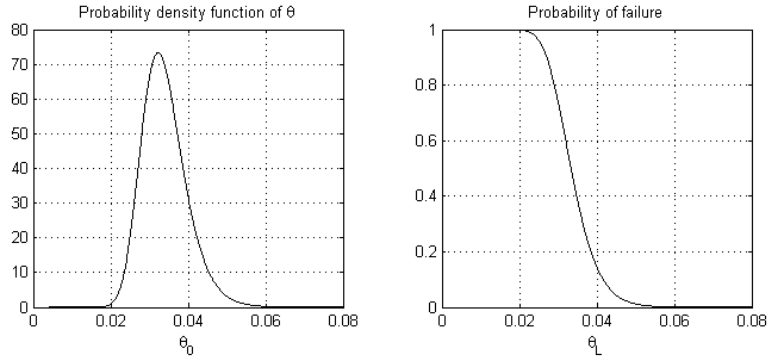

Figure 3.1: Probability density function of the rotation obtained by analytical developments

3.1.2 Reliability analysis: analytical approach

In this very simple 2-variable example, the reliability analysis can be performed in an analytical way. These developments will be first considered in order to prove the equivalence with the analytical developments of the previous section. Then the numerical resolution will be illustrated on the same example. Finally, this two-variable problem will also be used to illustrate the invariance principle.

Because it includes both the analysis and verification steps, the reliability analysis requires the definition of a failure condition, i.e. a maximum structural displacement which should not be overcome. In this case the failure is defined with respect to a maximum rotation θ_L. The reliability analysis consists in the computation of the probability that the failure won't be achieved, i.e. the probability that:

$$\frac{\frac{PL}{2}}{kL^2} < \theta_L \tag{3.13}$$

With notation of section 2, the failure condition can be defined by:

$$Z(P,k) = \theta_L - \frac{P}{2kL} \tag{3.14}$$

where the applied load as well as the structural stiffness are considered as probabilistic parameters. In this section it is considered that the marginal probability density functions are gaussian distributions with given means and standard deviations (μ_k, σ_k, μ_P, σ_P).

At first the random variables will be considered to be uncorrelated, i.e. $\rho_{kP} = 0$. This will allow the comparison with the analytical results presented in the introduction of this section. The reduction to zero-mean unit-variance variables is performed as in Eq. (2.23):

$$\widehat{k} = \frac{k - \mu_k}{\sigma_k}; \quad \widehat{P} = \frac{P - \mu_P}{\sigma_P} \tag{3.15}$$

and the failure condition is accordingly transformed:

$$\widehat{Z}(\widehat{P},\widehat{k}) = \theta_L - \frac{\mu_P + \sigma_P \widehat{P}}{2L\left(\mu_k + \sigma_k \widehat{k}\right)} \tag{3.16}$$

As expected the equation related to the failure condition is linear:

$$\widehat{Z}(\widehat{P},\widehat{k}) = 0 \Rightarrow 2\theta_L L\sigma_k \widehat{k} - \sigma_P \widehat{P} + 2\theta_L L\mu_k - \mu_P = 0 \tag{3.17}$$

The intercepts are:

$$a_1 = \frac{2\theta_L L\mu_k - \mu_P}{\sigma_P} \text{ and } a_2 = -\frac{2\theta_L L\mu_k - \mu_P}{2\theta_L L\sigma_k} \tag{3.18}$$

and the shortest distance from the origin to the failure condition, i.e. the reliability index, is:

$$\beta = \frac{|a_1 a_2|}{\sqrt{a_1^2 + a_2^2}} = \frac{|2\theta_L L \mu_k - \mu_P|}{\sqrt{(\sigma_P)^2 + (2\theta_L L \sigma_k)^2}} \tag{3.19}$$

Since the physical variables are gaussian and the failure condition is linear, the reliability index is rigorously linked to the probability of failure:

$$p_f = 1 - \Phi(\beta) = \Phi(-\beta) \tag{3.20}$$

The same numerical application as in the previous section can be considered. The successive estimation of the intercepts, the reliability index and probability of failure results in:

$$\begin{aligned} a_1 &= 5 \;\; ; \;\; a_2 = -2.5 \\ \beta &= 2.2361 \\ p_f &= 0.012674 \end{aligned} \tag{3.21}$$

which is exactly the same as obtained in the previous section. The analytical developments of the previous section are often seen as giving a more complete information about the probability distribution of the response. Indeed this function could be plotted as a direct result of the computation (Fig. 3.1)[1].

3.1.3 Reliability analysis: illustration of the numerical resolution (FOSM)

The first usual way to consider a reliability analysis consists the linearization of the failure function around the mean physical variables and then the computation of the mean and standard deviation of the failure condition. This is the FOSM method. Its definition (Def. 2, p. 11) is strictly applied in this section in order to illustrate the method on a simple example.

The considered failure function is the same as previously:

$$Z(P,k) = \theta_L - \frac{P}{2kL} \tag{3.25}$$

and the random variables, P and k have also the same probability density functions, i.e gaussian distribution characterized by means and standard deviations given in (3.11). The mean failure condition is expressed by Eq. (2.21):

$$\mu_Z = Z(\{\mu_x\}) = 0.05 - \frac{10}{2.150.1} = 0.01667 \tag{3.26}$$

[1] The use of a reliability analysis is often limited to the computation of the reliability index and probability of failure. This result should thus be considered as the most interesting result that can be provided by the reliability analysis. This should be seen as a drawback concerning this method because some other (of course more time consuming) like the MCS are able to provide a lot of supplementary information: the whole pdf of the result. Actually this result could also be provided by the reliability analysis. Indeed the probability of failure is closely linked to the cumulative distribution function (Eq. (3.10)). So if the failure condition is expressed as :

$$Z(x_1, x_2, ...x_N) = P - f(x_1, x_2, ...x_N) \tag{3.22}$$

and if the reliability analysis is repeated for several values of the parameter P, then the probability that $P - f(x_1, x_2, ...x_N)$ is smaller than zero is computed for several values of P. This is nothing else than the cumulative distribution function of P! In this application the failure condition has the same form as in Eq. (3.22) with $P = \theta_L$. The cumulative density function of the rotation angle is thus expressed by:

$$F_\theta(\theta_L) = \Phi(\beta(\theta_L)) \tag{3.23}$$

Then the probability density function of θ can be established:

$$p_\theta(\theta_L) = \frac{d\Phi(\beta(\theta_L))}{d\theta_L} = \left.\frac{d\Phi(\beta)}{d\beta}\right|_{\beta=\theta_L} \frac{d\beta(\theta_L)}{d\theta_L} \tag{3.24}$$

It can be checked that this relation is strictly equivalent to Eq. (3.8).

As a conclusion, even if a reliability-based design of often seen as including both the analysis and verification stages, this example showed that the results of the analysis only can be obtained by choosing the failure condition (verification condition) appropriately.

Concerning the estimation of the standard deviation of the failure condition (Eq. (2.22)), its expression can be slightly simplified because the physical variables, P and k, are independent. Indeed the double summation can be replaced by a simple one:

$$\sigma_Z^2 = \sum_{i=1}^{N} \left(\frac{\partial Z}{\partial x_i}\bigg|_{\{x\}=\{\mu_x\}} \right)^2 \sigma_{X_i}^2 \tag{3.27}$$

The derivatives appearing in this expression can be computed in close form:

$$\frac{\partial Z}{\partial P} = -\frac{1}{2kL} \rightarrow \frac{\partial Z}{\partial P}\bigg|_{\{x\}=\{\mu_x\}} = \frac{-1}{2.150.1} = -0.0033 \tag{3.28}$$

$$\frac{\partial Z}{\partial k} = \frac{P}{2k^2L} \rightarrow \frac{\partial Z}{\partial k}\bigg|_{\{x\}=\{\mu_x\}} = \frac{10}{2.150^2.1} = 2.222.10^{-4} \tag{3.29}$$

which gives, after injection into (3.27):

$$\sigma_Z = \sqrt{(-0.0033.1)^2 + (2.222.10^{-4}.20)^2} = 0.00556 \tag{3.30}$$

The estimation of the reliability index and the corresponding probability of failure is then straightforward (Eq. 2.19):

$$\beta = \frac{\mu_Z}{\sigma_Z} = \frac{0.01667}{0.00556} = 3 \tag{3.31}$$

$$p_f = 0.0013499 \tag{3.32}$$

This result is clearly different than what was obtained in the previous section. This illustrates that the FOSM method is unable to provide neither a good estimation of the reliability index nor the probability of failure in case of non linear failure condition.

3.1.4 Reliability analysis: illustration of the numerical resolution (AFOSM)

In this section the reliability analysis is performed in a numerical way. Since the physical variables are not correlated, the algorithms presented in section 2.2 can be applied. Furthermore, since the failure condition is linear, they should give the exact reliability index and probability of failure. For this example, the steps of Algorithm 7 (p. 18) are considered.

Step 0.1: The new reduced variables are:

$$\widehat{k} = \frac{k - 150}{20} \quad \text{and} \quad \widehat{P} = P - 10 \tag{3.33}$$

Step 0.2: The same failure condition as in the previous section is considered. Hence the reduced failure condition is:

$$\widehat{G}\left(\widehat{P}, \widehat{k}\right) = 0.05 - \frac{\widehat{P} + 10}{2\left(20\widehat{k} + 150\right)} \tag{3.34}$$

Step 1: The first guess of the design point is given as by Kusama ([5]). It is supposed that the reliability index and the direction of the design point are:

$$\beta = 3 \quad ; \quad \alpha = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \tag{3.35}$$

the initial design point is then computed by:

$$\widehat{x}^{(0)} = -\alpha\beta = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = (-2.1213, -2.1213) \tag{3.36}$$

This point is represented in Fig. 3.2-(a). This figure represents also the level curves of the joint probability density function of $\widehat{k}$ and $\widehat{P}$. Because of the variable reduction, they are concentric circles. The level curves of the reduced failure function are also represented. The reduced failure condition is the particular curve corresponding to a zero-level.

Step 2: This point clearly does not lie on the failure condition. Indeed the reduced failure condition at this point is equal to:

$$\widehat{G}^{(0)} = 0.05 - \frac{-2.1213 + 10}{2\left(20\left(-2.1213\right) + 150\right)} = 0.0134 \tag{3.37}$$

Step 3: If this point was really the design point then the origin would be in the direction of the gradient of the reduced failure function (see Lagrange's theory of the constrained minimisation). It is then useful to compute the gradient of the reduced failure function:

$$\frac{\partial \widehat{G}}{\partial \widehat{P}} = \frac{-1}{300 + 40\widehat{k}} \quad ; \quad \frac{\partial \widehat{G}}{\partial \widehat{k}} = \frac{40\left(\widehat{P} + 10\right)}{\left(300 + 40\widehat{k}\right)^2} \tag{3.38}$$

and to evaluate it at the current point ($\widehat{k} = -2.1213, \widehat{P} = -2.1213$):

$$n_1^{(0)} = \frac{\partial \widehat{G}}{\partial \widehat{P}} = -4.648.10^{-3} \quad ; \quad n_2^{(0)} = \frac{\partial \widehat{G}}{\partial \widehat{k}} = 6.808.10^{-3} \tag{3.39}$$

This vector is indicated by the small arrow starting at $\widehat{x}^{(0)}$ on Fig. 3.2-(a). As it is the gradient of the reduced failure function, it is perpendicular to its level curve at $\widehat{x}^{(0)}$.

Step 4: This vector can be used to define a new direction, i.e. a unit vector:

$$\left|n^{(0)}\right| = \sqrt{\left(-4.648.10^{-3}\right)^2 + \left(6.808.10^{-3}\right)^2} = 8.244.10^{-3} \tag{3.40}$$

$$\left\{\alpha^{(0)}\right\} = \frac{\left\{n^{(0)}\right\}}{\left|n^{(0)}\right|} = \left(\frac{-4.648.10^{-3}}{8.244.10^{-3}}, \frac{6.808.10^{-3}}{8.244.10^{-3}}\right) = (-0.5638, 0.8259) \tag{3.41}$$

This direction will be used for the next iteration in the other way, i.e. as the direction from the origin to the new design point. This is the basement of the algorithm. This direction can be linked to the original α used in Step 1. It can be seen as a better approximation of this direction. In this case the first design point was poorly chosen. Besides the direction of the new design point, the new distance to the origin (β) has to be provided.

Step 5: This distance is deduced from the following procedure. The reduced failure function is replaced by its tangent hyperplane at $\widehat{x}^{(0)}$. Since the new design point should lie on the zero-level curve, the intersection between this hyperplane and the horizontal plane $\widehat{G} = 0$ is sought. The distance between the resulting line and the current location of the design point $\widehat{x}^{(0)}$ is then used for the computation of the next distance from the origin. It can be shown that this methodology is equivalent to estimating the mean and standard deviation of the reduced failure condition by:

$$\begin{aligned} \mu_G &= \widehat{G}^{(0)} - \sum_{i=1}^{N} n_i^{(k)} \widehat{x}_i^{(k)} \\ &= 0.0134 - \left[\left(-4.648.10^{-3}\right)\left(-2.1213\right) + 6.808.10^{-3}\left(-2.1213\right)\right] \\ &= 0.01796 \end{aligned} \tag{3.42}$$

$$\begin{aligned} \sigma_G &= \alpha_1 \frac{\partial \widehat{G}}{\partial \widehat{P}} + \alpha_2 \frac{\partial \widehat{G}}{\partial \widehat{k}} \\ &= -0.5638.\left(-4.648.10^{-3}\right) + 0.8259.6.808.10^{-3} = 8.244.10^{-3} \end{aligned} \tag{3.43}$$

Note: another advantage of the computation in the reduced space is that σ_G is actually equal to $\left|n^{(k)}\right|$ and thus does not necessarily be estimated.

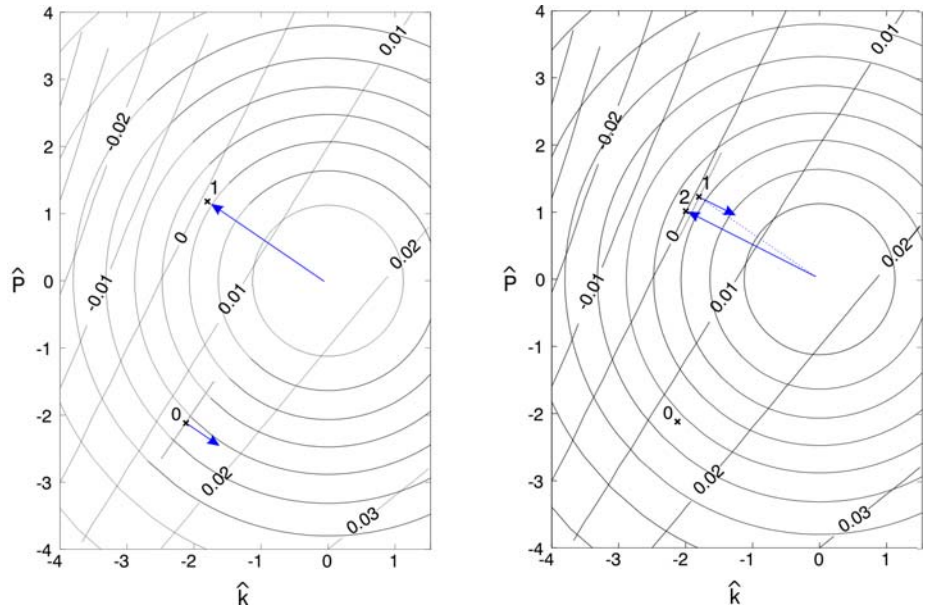

Figure 3.2: Illustration of the iterative resolution (AFOSM in reduced space with uncorrelated variables, Algorithm 7)

Step 6: ... and then computing the new distance from the origin, i.e. the reliability index, by:

$$\beta^{(0)} = \frac{\mu_G}{\sigma_G} = 2.1790 \tag{3.44}$$

Step 7: Finally the new approximation of the design point is obtained by:

$$\left\{\widehat{x}^{(1)}\right\} = -\left\{\alpha^{(0)}\right\}\beta^{(0)} = -\left(-0.5638, 0.8259\right).2.1790 \tag{3.45}$$

$$= \left(1.2286, -1.7996\right) \equiv \left(\widehat{P}^{(1)}, \widehat{k}^{(1)}\right) \tag{3.46}$$

This point is represented by the cross and point 1 on Fig. 3.2-(a). The iterations can now start with this new guess for the design point. So going back to Step 3, the gradient of the reduced failure function is estimated. It is represented on Fig. 3.2-(b) by the line starting from $x^{(1)}$. This gradient is used to compute a new direction (Step 4) and this direction will be used to determine the next design point. The next distance from the origin, i.e. the next reliability index, is obtained by replacing the actual reduced failure function by its tangent hyperplane at $\widehat{x}^{(1)}$ or, again, by the equivalent computation of μ_G and σ_G (Steps 6 and 7). This leads to a new estimation of the design point, $\widehat{x}^{(2)}$, represented on Fig. 3.2-(b) too. It can be observed that the convergence is very fast on this simple example.

Numerical values of some representative parameters are given in Table 3.1 for the first five iterations. The results in the first line correspond to the first iteration and are obtained as explained previously. The numerical values given in this table indicate that the convergence is faster for the reliability index (β) than the position of the design point ($\widehat{P}$, $\widehat{k}$). Indeed three iterations are enough to let the former converge whereas five iterations are necessary for the convergence of the latter. This illustrates the second-order convergence of the reliability index. It can be understood intuitively thanks to the geometrical definition of the reliability index.

Table 3.2 provides more detailed results for the first three iterations. The intermediate results for the six major steps of the AFOSM method (Alg.7, p. 18) are given. They should be useful to the reader who desires to develop himself the same iterative procedure.

It.	$\widehat{X}_1^* = \widehat{P}$	$\widehat{X}_2^* = \widehat{k}$	μ_G	σ_G	β	p_f
1	-2.1213	-2.1213	$1.796.10^{-2}$	$8.244.10^{-3}$	2.1790	$1.4666.10^{-2}$
2	$+1.2286$	-1.7996	$2.169.10^{-2}$	$9.688.10^{-3}$	2.2388	$1.2586.10^{-2}$
3	$+1.0134$	-1.9963	$2.272.10^{-2}$	$1.016.10^{-2}$	2.2361	$1.2674.10^{-2}$
4	$+0.9996$	-2.0002	$2.273.10^{-2}$	$1.016.10^{-2}$	2.2361	$1.2674.10^{-2}$
5	$+1.0000$	-2.0000	$2.273.10^{-2}$	$1.016.10^{-2}$	2.2361	$1.2674.10^{-2}$

Table 3.1: Summarized results of the AFOSM analysis

	Iteration	1	2	3
Step 1	$\widehat{X}_1^* = \widehat{P}$	-2.1213	$+1.2286$	$+1.0134$
	$\widehat{X}_2^* = \widehat{k}$	-2.1213	-1.7996	-1.9963
	P	$+7.8787$	$+11.229$	$+11.013$
	k	$+107.57$	$+114.01$	$+110.07$
Step 2	G	$+1.3380.10^{-2}$	$+7.5510.10^{-4}$	$-2.6977.10^{-5}$
Step 3	$\frac{\partial \widehat{G}}{\partial \widehat{P}}$	$-4.6480.10^{-3}$	$-4.3857.10^{-3}$	$-4.5424.10^{-3}$
	$\frac{\partial \widehat{G}}{\partial k}$	$+6.8084.10^{-3}$	$+8.6389.10^{-3}$	$+9.0896.10^{-3}$
Step 4	α_1	-0.5638	-0.4527	-0.4470
	α_2	$+0.8259$	$+0.8917$	$+0.8945$
Step 5	μ_G	$+1.7963.10^{-2}$	$+2.1690.10^{-2}$	$+2.2722.10^{-2}$
	σ_G	$+8.2436.10^{-3}$	$+9.6884.10^{-3}$	$+1.0161.10^{-2}$
Step 6	β	$+2.1790$	$+2.2388$	$+2.2361$
	p_f	$+1.4666.10^{-2}$	$+1.2586.10^{-2}$	$+1.2674.10^{-2}$
Step 1 – Guess design point –				
Step 2 – Compute Failure Condition –				
Step 3 – Compute the gradient of the Failure Condition –				
Step 4 – Compute the orientation of the next design point –				
Step 5 – Compute the mean and std of Failure Cond –				
Step 6 – Compute the reliability index and probability of failure –				

Table 3.2: Detailed results of the AFOSM analysis

3.1.5 Reliability analysis: illustration of the invariance principle (FOSM .vs. AFOSM)

The most simple reliability analysis is the FOSM technique. Section 2 presented the need for an upgraded version, the AFOSM method, necessary because of the lack of invariance (see Def. 1, p. 9). The invariance principle is illustrated in this section. To this purpose, three different failure functions are studied in the following:

$$\begin{aligned} Z_1(P,k) &= \theta_L - \frac{P}{2kL} \\ Z_2(P,k) &= 2kL\theta_L - P \\ Z_3(P,k) &= \ln \frac{2kL\theta_L}{P} \end{aligned} \tag{3.47}$$

It can be checked easily that the corresponding failure conditions are all equivalent:

$$Z_1(P,k) = 0 \Leftrightarrow Z_2(P,k) = 0 \Leftrightarrow Z_3(P,k) = 0 \tag{3.48}$$

For the sake of simplicity the same numerical application is again considered ($\theta_L = 0.05$, $L = 1m$, $P = N(10kN; 1kN)$, $k = N(150kN; 20kN)$). The resulting three functions are represented by level curves in Fig. 3.3. It can be checked that the failure condition represented by the thick zero level curve are all three equivalent. The concentric circles represent the joint probability density function between the reduced variables $\widehat{P}$ and $\widehat{k}$. They are of course identical because this is a given data. The reduced variables are obtained by the same relation as previously (3.15). Since the failure condition and the joint probability density function are the same for all three functions, they should return the same probability of failure. This is typically the invariance principle.

Table 3.3 gives the numerical results obtained with the FOSM and AFOSM methods for these three functions.

Failure function: $Z_1(P,k) = \theta_L - \frac{P}{2kL}$				
	FOSM		**AFOSM**	
		1^{st} iter.	2^{nd} iter.	3^{rd} iter.
Reliab Index (β)	3	2.179	2.2388	2.2361
Prob. failure (p_f)	0.0013499	0.014666	0.012586	0.012674
Failure function: $Z_2(P,k) = 2kL\theta_L - P$				
	FOSM		**AFOSM**	
		1^{st} iter.	2^{nd} iter.	3^{rd} iter.
Reliab Index (β)	2.2361	2.2361	2.2361	2.2361
Prob. failure (p_f)	0.012674	0.012674	0.012674	0.012674
Failure function: $Z_3(P,k) = \ln \frac{2kL\theta_L}{P}$				
	FOSM		**AFOSM**	
		1^{st} iter.	2^{nd} iter.	3^{rd} iter.
Reliab Index (β)	2.4328	1.9394	2.2476	2.2360
Prob. failure (p_f)	0.0074915	0.026229	0.0123	0.012675

Table 3.3: Illustration of the invariance principle

Concerning the FOSM method the same procedure as in section 3.1.3 is applied. The reliability indices are different for the three functions ($\beta = 3$, $\beta = 2.2361$ and $\beta = 2.4328$). This illustrates the lack of invariance. The exact result is actually $\beta = 2.2361$ which is precisely obtained for the second function (Z_2). In this case the failure function is linear, its level curves are parallel to each other and the FOSM method returns the exact result. The conditions under which the FOSM method returns the exact reliability index are thus:

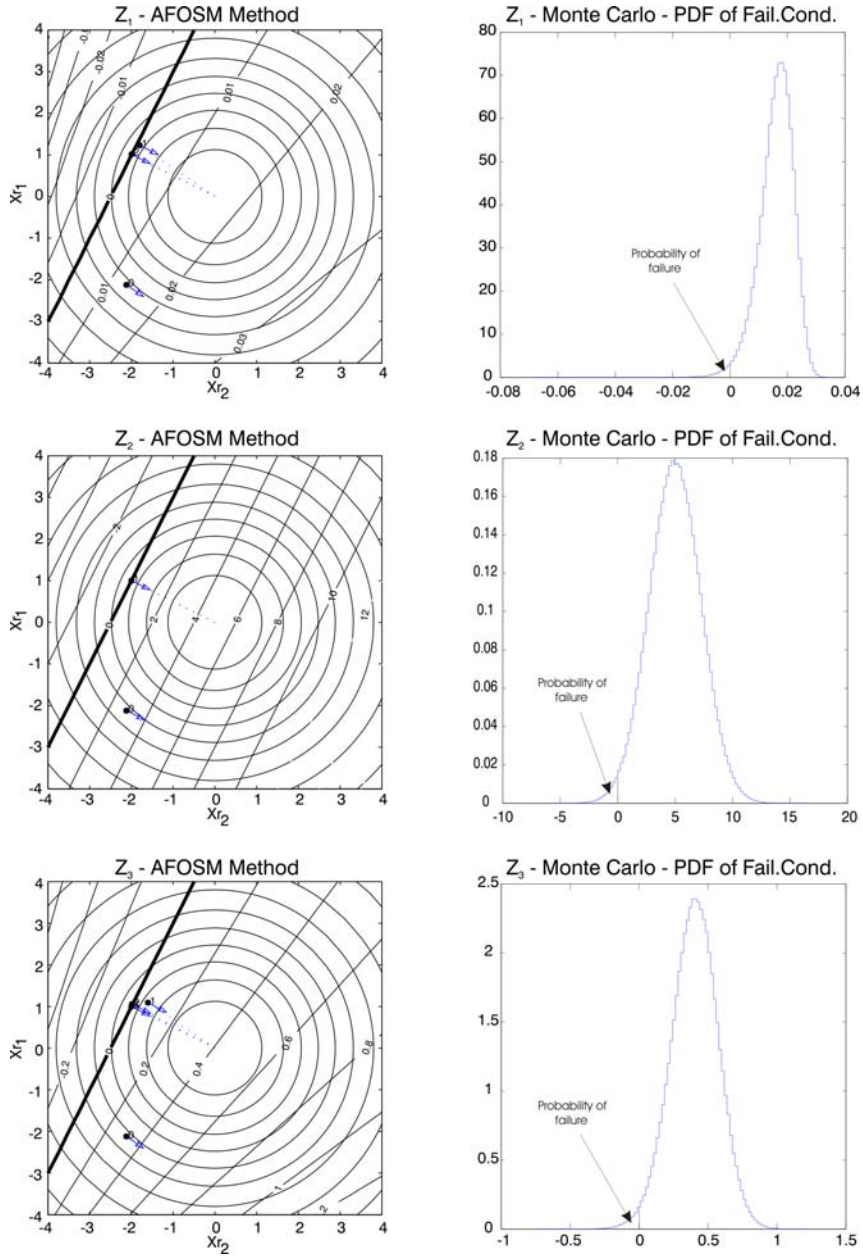

Figure 3.3: Illustration of the invariance principle. Z_1 and Z_3 are non linear failure function, Z_2 is a linear failure function but they all lead to the same linear failure condition.

- gaussian variables
- linear failure function

Concerning the AFOSM method the same iterative procedure as applied in section 3.1.4 is applied. The evolution of the reliability index and the corresponding probability of failure are given in Table 3.3 for the first three iterations. For all three functions the convergence is quite fast and after the third iteration the reliability index is almost equal to 2.2361 which is the exact value. The convergence can also be tracked on Fig. 3.3 where the successive estimations of the design point are represented by crosses. Anyway the AFOSM method provides the same result for these three functions, having the same failure condition. For this reason, the AFOSM method possesses the invariance property. This is easily understandable thanks to the geometrical definition of the reliability index. The conditions under which the AFOSM method returns the exact reliability index are thus:

- gaussian variables
- linear failure condition

It is also interesting to note that no iterations are actually required for function Z_2. The exact result is directly obtained at the first iteration, which means that any first guess results in the exact reliability index. As a particular case, the mean physical variable results in the right estimation, which justifies, in an other way, the reason for which the FOSM method provides the exact result in for this function.

In the right column Fig. 3.3 illustrates the invariance principle by means of a Monte Carlo simulation approach. A large number ($N = 10^6$) of couple (P_i, k_i) are generated in such a way that the histograms of the generated series coincides with the given probability density functions of P and k. For each couple, the corresponding value of the failure function is computed and this operation is repeated for all three failure functions. This results thus, for each function, in a series of ($N = 10^6$) values of the failure function. The probability density function of the failure function is approached by the histograms of these series. These results are represented in Fig. 3.3. The probability of failure is the area under the histograms and below $Z = 0$. They are identical for all three functions. It is also interesting to note that the probability density function resulting from function Z_2 is gaussian (which is due to the linearity of the failure function in this case).

3.1.6 Reliability analysis: 4-variable problem

In this section an application of the 1-DOF problem is considered, in which all four involved variables are supposed to be random. The failure function is:

$$Z(P, k, L, \theta_L) = \theta_L - \frac{P}{2kL} \tag{3.49}$$

where the four random variables are supposed to be gaussian variables characterized by:

$$\begin{aligned} P &= N(10kN; 1kN) \\ k &= N(150kN; 20kN) \\ L &= N(1m, 0.1m) \\ \theta_L &= N(0.05, 0.002) \end{aligned} \tag{3.50}$$

Back to the practical application it is easy to understand why the applied load P can be defined as a random variable. This is also the case for the stiffness k which can exhibit an significant variability in case of materials like timber or concrete. Concerning the beam length L the origin of its randomness is rather linked to manufacturing processes. Thanks to today's improved technologies, the standard deviation of a beam length is generally very small (compared to the mean length). The maximum allowable rotation θ_L can also exhibit a certain variability.

Indeed the ultimate rotation allowed in a section is sometimes linked to the ductility of the materials, which exhibits clearly an important variability.

The reliability analysis is first performed with the FOSM method (Def. 2, p. 11). The mean value of the failure function is approached by:

$$\mu_Z = Z(\{\mu_x\}) = 0.05 - \frac{10}{2.150.1} = 0.01667 \tag{3.51}$$

and, as in the previous section, the standard deviation is simplified to:

$$\sigma_Z^2 = \sum_{i=1}^{N} \left(\frac{\partial Z}{\partial x_i}\Big\rfloor_{\{x\}=\{\mu_x\}} \right)^2 \sigma_{X_i}^2 \tag{3.52}$$

where the derivatives of the failure function can be expressed in closed forms:

$$\begin{aligned}
\frac{\partial Z}{\partial P} &= -\frac{1}{2kL} \rightarrow \frac{\partial Z}{\partial P}\Big\rfloor_{\{x\}=\{\mu_x\}} = \frac{-1}{2.150.1} = -0.0033 \\
\frac{\partial Z}{\partial k} &= \frac{P}{2k^2L} \rightarrow \frac{\partial Z}{\partial k}\Big\rfloor_{\{x\}=\{\mu_x\}} = \frac{10}{2.150^2.1} = 2.222.10^{-4} \\
\frac{\partial Z}{\partial L} &= \frac{P}{2kL^2} \rightarrow \frac{\partial Z}{\partial L}\Big\rfloor_{\{x\}=\{\mu_x\}} = \frac{10}{2.150.1} = 0.0333 \\
\frac{\partial Z}{\partial \theta_L} &= 1 \rightarrow \frac{\partial Z}{\partial \theta_L}\Big\rfloor_{\{x\}=\{\mu_x\}} = 1
\end{aligned} \tag{3.53}$$

which results in:

$$\begin{aligned}
\sigma_Z &= \sqrt{\left(-0.0033.1\right)^2 + \left(2.222.10^{-4}.20\right)^2 + \left(0.0333.0.1\right)^2 + \left(1.0.002\right)^2} \\
&= 0.0067805
\end{aligned} \tag{3.54}$$

The estimated (unique) reliability index and the corresponding probability of failure are finally:

$$\beta = \frac{\mu_Z}{\sigma_Z} = \frac{0.01667}{0.0067805} = 2.458 \tag{3.55}$$

$$p_f = 0.0069851 \tag{3.56}$$

Despite its lack of rigour the FOSM method is so simple that is can illustrate simply some interesting features. For instance, in this case, if it is supposed that the standard deviations of the last two variables (L, θ_L) tend towards zero, Eq. (3.52) shows that the result tends to the same result as in the previous section, i.e. the result corresponding to the 2-variable problem. This *variance-continuity property* is for sure as important as the invariance principle. It could be used advantageously in a complex problem in order to reduce the number of random variables.

The AFOSM method for non correlated gaussian variables (Alg.7, p. 18) is applied to the same problem. The numerical results for the successive steps of the first iteration are detailed in the following.

Step 0.1: The new reduced variables are:

$$\widehat{k} = \frac{k - 150}{20} \qquad \widehat{P} = P - 10 \tag{3.57}$$

$$\widehat{L} = \frac{L - 1}{0.1} \qquad \widehat{\theta_L} = \frac{\theta_L - 0.05}{0.002} \tag{3.58}$$

Step 0.2: The reduced failure condition is:

$$\widehat{G}\left(\widehat{P}, \widehat{k}, \widehat{L}, \widehat{\theta_L}\right) = \left(0.002\widehat{\theta_L} + 0.05\right) - \frac{\widehat{P} + 10}{2\left(0.1\widehat{L} + 1\right)\left(20\widehat{k} + 150\right)} \tag{3.59}$$

Step 1: The first guess of the design point is given as by Kusama ([5]). It is supposed that the reliability index and the direction of the design point are:

$$\beta = 3 \quad ; \quad \alpha = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \tag{3.60}$$

and the initial design point is then computed by:

$$\widehat{x}^{(0)} = -\alpha\beta = (-1.5, -1.5, -1.5, -1.5) \tag{3.61}$$

Step 2: The failure condition at this point is equal to:

$$\widehat{G}^{(0)} = (0.002.(-1.5) + 0.05) - \frac{-1.5 + 10}{2\,(0.1\,(-1.5) + 1)\,(20\,(-1.5) + 150)} = 5.3333.10^{-3} \tag{3.62}$$

which indicates that this first candidate is not the actual design point.

Step 3: The direction of the gradient of the reduced failure function can be computed in closed form:

$$\frac{\partial \widehat{G}}{\partial \widehat{P}} = \frac{-1}{2\left(0.1\widehat{L} + 1\right)\left(20\widehat{k} + 150\right)} \tag{3.63}$$

$$\frac{\partial \widehat{G}}{\partial \widehat{k}} = \frac{20\left(\widehat{P} + 10\right)}{2\left(0.1\widehat{L} + 1\right)\left(20\widehat{k} + 150\right)^2} \tag{3.64}$$

$$\frac{\partial \widehat{G}}{\partial \widehat{L}} = \frac{0.1\left(\widehat{P} + 10\right)}{2\left(0.1\widehat{L} + 1\right)^2\left(20\widehat{k} + 150\right)} \tag{3.65}$$

$$\frac{\partial \widehat{G}}{\partial \widehat{\theta_L}} = 0.002 \tag{3.66}$$

and its evaluation at the current design point $(-1.5, -1.5, -1.5, -1.5)$ gives:

$$n_1^{(0)} = \frac{\partial \widehat{G}}{\partial \widehat{P}} = -4.902.10^{-3} \tag{3.67}$$

$$n_2^{(0)} = \frac{\partial \widehat{G}}{\partial \widehat{k}} = 6.944.10^{-3} \tag{3.68}$$

$$n_3^{(0)} = \frac{\partial \widehat{G}}{\partial \widehat{L}} = 4.902.10^{-3} \tag{3.69}$$

$$n_4^{(0)} = \frac{\partial \widehat{G}}{\partial \widehat{\theta_L}} = 2.10^{-3} \tag{3.70}$$

Step 4: This vector can be used to define the new direction (used for the next iteration), i.e. a unit vector:

$$\begin{aligned} \left|n^{(0)}\right| &= \sqrt{(-4.902.10^{-3})^2 + (6.808.10^{-3})^2 + (4.902.10^{-3})^2 + (2.10^{-3})^2} \\ &= 10.01.10^{-3} \end{aligned} \tag{3.71}$$

$$\begin{aligned} \left\{\alpha^{(0)}\right\} &= \frac{\left\{n^{(0)}\right\}}{\left|n^{(0)}\right|} = \left(\frac{-4.902.10^{-3}}{10.01.10^{-3}}, \frac{6.944.10^{-3}}{10.01.10^{-3}}, \frac{4.902.10^{-3}}{10.01.10^{-3}}, \frac{2.10^{-3}}{10.01.10^{-3}}\right) \\ &= (-0.4895, 0.6935, 0.4895, 0.1997) \end{aligned} \tag{3.72}$$

Step 5: The corresponding reliability index is deduced from estimations of the mean and standard deviation of the failure condition, which are both expressed by:

$$\begin{aligned}\mu_G &= \widehat{G}^{(0)} - \sum_{i=1}^{N} n_i^{(k)} x_i^{(k)} \\ &= 5.3333.10^{-3} - \left[\left(-4.902.10^{-3}\right)(-1.5) + 6.944.10^{-3}(-1.5)\right. \\ &\qquad \left. +4.902.10^{-3}(-1.5) + 2.10^{-3}(-1.5)\right] = 0.018750 \\ \sigma_G &= \left|n^{(0)}\right| = 10.01.10^{-3}\end{aligned} \tag{3.73}$$

Step 6: The reliability index is then obtained by:

$$\beta^{(0)} = \frac{\mu_G}{\sigma_G} = 1.8723 \tag{3.74}$$

Step 7: Finally the new approximation of the design point is obtained by means of the new direction $\alpha^{(0)}$ and this distance $\beta^{(0)}$:

$$\begin{aligned}\left\{\widehat{x}^{(1)}\right\} &= -\left\{\alpha^{(0)}\right\}\beta^{(0)} = -\left(-0.4895, 0.6935, 0.4895, 0.1997\right).1.8723 \\ &= (0.9165, -1.2984, -0.9165, -0.3739) \equiv \left(\widehat{P}^{(1)}, \widehat{k}^{(1)}, \widehat{L}^{(1)}, \widehat{\theta_L}^{(1)}\right)\end{aligned}$$

Exactly as for the previous example of application of the AFOSM method, the most important steps of the iterative procedure are summarized in Table 3.4 and given in more details in Table 3.5.

3.1.7 Reliability analysis: example of divergence

As any iterative procedure, the successive steps of the algorithm developed for the AFOSM (Alg.7, p. 18) could be unstable. An illustration of such a numerical divergence is illustrated on a simple 2-variable problem. The failure function is:

$$Z(P,k) = \theta_L - \frac{P}{2kL} \tag{3.75}$$

where the random variables P and k are supposed to be gaussian with means and standard deviations given by:

$$\begin{aligned}\mu_P &= 10kN \quad ; \quad \sigma_P = 1kN \\ \mu_k &= 150kN/m \quad ; \quad \sigma_k = 80kN/m\end{aligned} \tag{3.76}$$

It has to be underlined that the standard deviation of the stiffness is now four times larger than the value considered in the previous illustrations. The 3-D representation of the surface corresponding to the failure function is represented at Fig. 3.4. The failure condition, i.e. the intersection of this failure function with the horizontal plane is represented by two secant lines. More precisely one of these lines is not really part of the failure function because it corresponds to the annulation of the denominator in the failure function ($\widehat{k}_{\lim} = \frac{-\mu_k}{\sigma_k}$). It is however considered so in a numerical way because the sign of the function changes from one side of this line to the other ($+\infty$ for $\widehat{k} < \widehat{k}_{\lim}$, $-\infty$ for $\widehat{k} > \widehat{k}_{\lim}$).

The successive iterates obtained with the usual procedure are represented on the left side. The initial point (represented by 0) is chosen by imposing the reliability index $\beta = 3$ and the direction $\alpha = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. This point lies behind the singularity. It can be observed that the successive iterates lie on the same side of the line. Furthermore they are diverging very fast. On the other

It.	$\widehat{X}_1 = \widehat{P}$	$\widehat{X}_2 = \widehat{k}$	$\widehat{X}_3 = \widehat{L}$	$\widehat{X}_4 = \widehat{\theta_L}$	β	p_f
1	-1.5000	-1.5000	-1.5000	-1.5000	1.8723	$3.0579.10^{-2}$
2	$+0.9165$	-1.2984	-0.9165	-0.3739	1.9413	$2.6111.10^{-2}$
3	$+0.8099$	-1.4256	-0.9733	-0.3650	1.9398	$2.6205.10^{-2}$
4	$+0.8059$	-1.4342	-0.9651	-0.3535	1.9397	$2.6205.10^{-2}$
5	$+0.8057$	-1.4353	-0.9636	-0.3532	1.9397	$2.6205.10^{-2}$

Table 3.4: Summarized results of the AFOSM analysis

	Iteration	1	2	3
Step 1	$\widehat{X}_1^* = \widehat{P}$	-1.5000	$+0.9165$	$+0.8099$
	$\widehat{X}_2^* = \widehat{k}$	-1.5000	-1.2984	-1.4256
	$\widehat{X}_3 = \widehat{L}$	-1.5000	-0.9165	-0.9733
	$\widehat{X}_4 = \widehat{\theta_L}$	-1.5000	-0.3739	-0.3650
	P	$+8.5000$	$+10.917$	$+10.810$
	k	$+120.00$	$+124.03$	$+121.49$
	L	$+0.8500$	$+0.9083$	$+0.9027$
	θ_L	$+4.7000.10^{-2}$	$+4.9252.10^{-2}$	$+4.9270.10^{-2}$
Step 2	G	$+5.3333.10^{-3}$	$+8.0501.10^{-4}$	$-1.6135e-005$
Step 3	$\frac{\partial \widehat{G}}{\partial \widehat{P}}$	$-4.9020.10^{-3}$	$-4.4380.10^{-3}$	$-4.5594.10^{-3}$
	$\frac{\partial \widehat{G}}{\partial \widehat{k}}$	$+6.9444.10^{-3}$	$+7.8120.10^{-3}$	$+8.1137.10^{-3}$
	$\frac{\partial \widehat{G}}{\partial \widehat{L}}$	$+4.9020.10^{-3}$	$+5.3335.10^{-3}$	$+5.4600.10^{-3}$
	$\frac{\partial \widehat{G}}{\partial \widehat{\theta_L}}$	$+2.0000.10^{-3}$	$+2.0000.10^{-3}$	$+2.0000.10^{-3}$
Step 4	α_1	-0.4895	-0.4172	-0.4155
	α_2	$+0.6935$	$+0.7343$	$+0.7393$
	α_3	$+0.4895$	$+0.5014$	$+0.4975$
	α_4	$+0.1997$	$+0.1880$	$+0.1822$
Step 5	μ_G	$+1.8750.10^{-2}$	$+2.0652.10^{-2}$	$+2.1287.10^{-2}$
	σ_G	$+1.0014.10^{-2}$	$+1.0638.10^{-2}$	$+1.0974.10^{-2}$
Step 6	β	$+1.8723$	$+1.9413$	$+1.9398$
	p_f	$+3.0579.10^{-2}$	$+2.6111.10^{-2}$	$+2.6205.10^{-2}$
Step 1 – Guess design point –				
Step 2 – Compute Failure Condition –				
Step 3 – Compute the gradient of the Failure Condition –				
Step 4 – Compute the orientation of the next design point –				
Step 5 – Compute the mean and std of Failure Cond –				
Step 6 – Compute the reliability index and probability of failure –				

Table 3.5: Detailed results of the AFOSM analysis

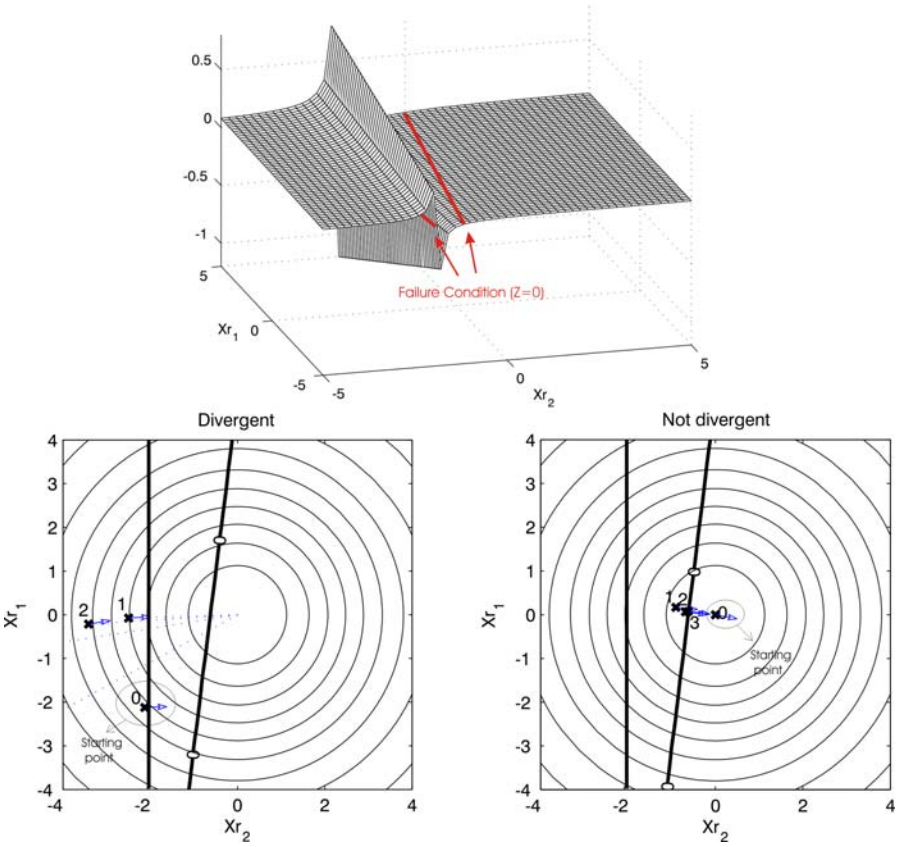

Figure 3.4: Example of divergent solution (left) and convergent solution (right)

hand, on the right side, the initial point is chosen at the origin of the axes in the reduced space. This point corresponds to the means physical variables. In this case, the convergence is reached as expected. This starting point is probably better than the former one because it always lies on the safe side of the failure condition. The problem is then much better conditioned and the risks for divergence are then decreased. Table 3.6 summarizes the numerical values of some representative variables obtained during the five iterations (with both starting points). The upper part of this table, related to the divergent solution, indicates clearly the divergence.

			Divergent solution			
It.	$\widehat{X}_1^* = \widehat{P}$	$\widehat{X}_2^* = \widehat{k}$	μ_G	σ_G	β	p_f
1	-2.1213	-2.1213	2.025	$0.8120e$	2.4943	$6.3094.10^{-3}$
2	-0.0779	-2.4931	0.5557	0.1626	3.4176	$3.1592.10^{-4}$
3	-0.2125	-3.4109	$1.791.10^{-1}$	$2.625.10^{-2}$	6.8250	$4.3960.10^{-12}$
4	-1.0581	-6.7425	$7.874.10^{-2}$	$2.686.10^{-3}$	29.3202	0.0000
5	-14.0180	-25.7521	$5.148.10^{-2}$	$2.654.10^{-4}$	193.9559	0.0000
			Convergent solution			
1	$+0.0000$	$+0.0000$	$1.667.10^{-2}$	$1.809.10^{-2}$	0.9214	$1.7841.10^{-1}$
2	0.1698	-0.9057	$4.679.10^{-2}$	$6.795.10^{-2}$	0.6885	$2.4556.10^{-1}$
3	$+0.0653$	-0.6854	$2.793.10^{-2}$	$4.476.10^{-2}$	0.6240	$2.6633.10^{-1}$
4	$+0.0732$	-0.6196	$2.497.10^{-2}$	$4.026.10^{-2}$	0.6202	$2.6757.10^{-1}$
5	$+0.0767$	-0.6154	$2.481.10^{-2}$	$4.001.10^{-2}$	0.6202	$2.6757.10^{-1}$

Table 3.6: Example of divergent solution : summarized results

3.1.8 Reliability analysis: a parametric study

An adequate choice of the initial point is important in some limit cases. The previous section illustrates a divergent series resulting from a large standard deviation of the stiffness compared to its mean value. Indeed if the standard deviation is large compared to the mean value, the occurrence of a very small stiffness becomes higher. This results in the occurrence of some very large rotations, i.e. a probability density function of the failure condition excessively skewed to the right, which means also that the Monte Carlo simulation are difficult to be performed properly in this case.

The reliability analysis is difficult to be performed too. Fig. 3.5 illustrate in a parametric way the divergence problems. For large values of the standard deviation, the starting point has to be chosen carefully (e.g. $(0, 0)$), otherwise the resulting reliability index can be drastically misevaluated.

Another important point to notice about this figure concerns the very fast increase of the probability of failure located around $\sigma_k = 20$. In this area the ratio of the mean to the standard deviation of the stiffness is equal to 4 ($\mu_k = 80$). This fast increase of the probability of failure is then due to the previously explained phenomenon: the occurrence of very small stiffness (even negative!) is much higher, which results immediately in very high rotations and hence probability of failure.

In practical applications, very small (or even negative) stiffnesses are not realistic. It could even be supposed that a minimum value is guaranteed. If the standard deviation of the stiffness is however large compared to the mean value, this reasoning indicates that the gaussian distribution is probably not the best for a stiffness parameter. In practical applications, when it is desired to impose lower values of some parameters (because of physical reasons), other more appropriate distributions like the lognormal or the beta probability density functions are rather adopted. The way to handle such non gaussian probability density functions is illustrated in section 3.2.

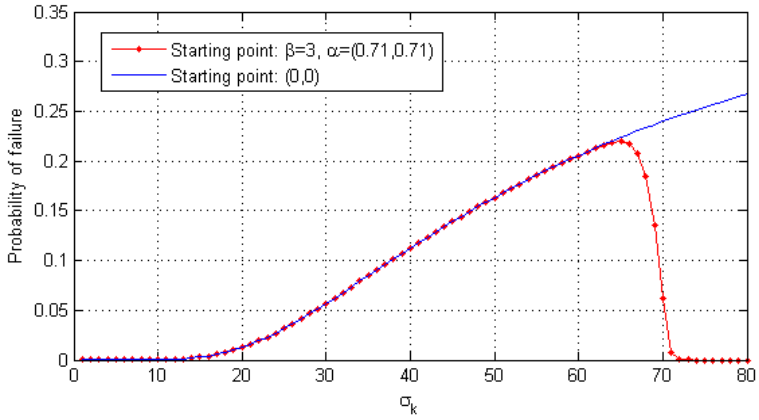

Figure 3.5: Probability of failure obtained for various standard deviations of the stiffness

3.2 Buckling model (correlated non gaussian variables)

3.2.1 Probabilistic analysis: analytical approach 1

In the buckling problem a usual deterministic analysis would lead to this relation between the stiffness, the applied force and the beam length:

$$\left(kL^2 - PL\right)\theta = 0 \tag{3.77}$$

from which the (eulerian) critical load $P_{cr} = kL$ can be computed. In a probabilistic context the applied load Q is given in a probabilistic way and the checking of the instability of the column requires that $Q - P_{cr} \geq 0$. In this section, it is considered that the applied load Q and the structural stiffness k are random parameters and that the length of the compressed element is given in a deterministic way. The safety against failure is expressed by means of a failure function:

$$Z\left(Q,k\right) = kL - Q \tag{3.78}$$

but, contrarily to what is done in a reliability analysis, the aim of the more general probabilistic analysis is the determine the probability density function of this failure function (instead of the probability of failure associated to the failure condition $Z = 0$). The cumulative density function of the failure function is expressed by:

$$\begin{aligned} F_Z\left(Z_0\right) &= prob\left(Z < Z_0\right) \\ &= prob\left(kL - Q < Z_0\right) \end{aligned} \tag{3.79}$$

Considering all the possible values that Q could take, this relation can also be written:

$$\begin{aligned} F_Z\left(Z_0\right) &= \int_{-\infty}^{+\infty} prob\left(Q_0 < Q \leq Q_0 + dQ_0\right) prob\left(kL < Q_0 + Z_0 \mid Q_0 < Q \leq Q_0 + dQ_0\right) dQ_0 \\ &= \int_{-\infty}^{+\infty} p_Q\left(Q_0\right) F_{k|Q}\left(\frac{Q_0 + Z_0}{L} \mid Q_0\right) dQ_0 \end{aligned} \tag{3.80}$$

where $F_{k|Q}\left(k_0 \mid Q_0\right)$ represents a conditional cumulative density function, i.e. the probability that k is smaller than k_0 provided Q lies in the interval $[Q_0; Q_0 + dQ_0]$. The probability density function of Z can be computed by simple derivation of its cumulative distribution function:

$$p_Z\left(Z_0\right) = \frac{dF_Z}{dZ_0} = \frac{1}{L}\int_{-\infty}^{+\infty} p_Q\left(Q_0\right) p_{k|Q}\left(\frac{Q_0 + Z_0}{L} \mid Q_0\right) dQ_0 \tag{3.81}$$

The conditional probability density function introduced in this relation $p_{k|Q}\left(k_0 \mid Q_0\right)$ can be expressed as a function of the joint and marginal probability density functions, thanks to the factorization theorem:

$$\begin{aligned} p_Z\left(Z_0\right) &= \frac{1}{L}\int_{-\infty}^{+\infty} p_Q\left(Q_0\right)\frac{p_{kQ}\left(\frac{Q_0+Z_0}{L},Q_0\right)}{p_Q\left(Q_0\right)}dQ_0 \\ &= \frac{1}{L}\int_{-\infty}^{+\infty} p_{kQ}\left(\frac{Q_0+Z_0}{L},Q_0\right)dQ_0 \end{aligned} \tag{3.82}$$

This relation is valid for any joint probability function between k and Q. In particular, if it is supposed that these random variables are described by correlated gaussian variables, their joint probability density function is:

$$p_{kQ}\left(k_0,Q_0\right) = \frac{1}{2\pi\sigma_k\sigma_Q\sqrt{1-\rho^2}}e^{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{k_0-\mu_k}{\sigma_k}\right)^2-2\rho\frac{(k_0-\mu_k)\left(Q_0-\mu_Q\right)}{\sigma_k\sigma_Q}+\left(\frac{Q_0-\mu_Q}{\sigma_Q}\right)^2\right]} \tag{3.83}$$

The introduction of this relation into Eq. (3.82) results, after some computations in:

$$p_Z\left(Z_0\right) = \frac{1}{\sqrt{2\pi}\sigma_Z}e^{\frac{-1}{2}\left[\left(\frac{Z_0-\mu_Z}{\sigma_Z}\right)^2\right]} \tag{3.84}$$

where $\mu_Z = L\mu_k - \mu_Q$ and $\sigma_Z^2 = \sigma_Q^2 - 2\rho\sigma_Q L\sigma_k + L^2\sigma_k^2$. This relation shows that the random variable Z is also a gaussian variable. This could have been expected in advance because Z is defined (Eq. (3.78)) as a linear combination of gaussian processes (correlated indeed but the gaussianity is the only condition to be fulfilled for this property). Any subsequent information resulting from the probability density function can be estimated. For instance the cumulative density function of Z can be computed:

$$F_Z\left(Z_0\right) = \Phi\left(\frac{Z_0-\mu_Z}{\sigma_Z}\right) \tag{3.85}$$

As a numerical example, these values are used as an illustration:

$$\begin{aligned} \mu_k = 150kN/m \quad &; \quad \sigma_k = \quad 20kN/m \\ \mu_Q = 100kN \quad &; \quad \sigma_Q = \quad 30kN \\ L = 1m \quad &; \quad \rho = \quad -0.6 \end{aligned} \tag{3.86}$$

From these it is possible to compute $\mu_Z = 50$ and $\sigma_Z = \sqrt{20^2 - 2\rho * 20 * 30 + 30^2} = \sqrt{1300 - 120\rho} = 44.94$ (for $\rho = -0.6$). The probability density function and the corresponding cumulative density function are represented at Fig. 3.6. As justified however hereover the gaussianity of the resulting random variable can also be observed on this figure.

The probabilistic analysis as performed so far provides more information than the reliability analysis. The results of a reliability analysis can be recovered as a particular case. Indeed by setting $Z_0 = 0$ in the previous expression of the cumulative density function, the probability of failure related to the failure condition is obtained:

$$p_f = 1 - F_Z\left(0\right) = 1 - \Phi\left(\frac{0-\mu_Z}{\sigma_Z}\right) = 1 - \Phi\left(-1.1125\right) = 0.1330 \tag{3.87}$$

The analytical developments indicate that the correlation coefficient affects the probability of failure through the standard deviation of the failure function (σ_Z). Its influence on this standard deviation is considerable. An idea of the range of variation is obtained by computing the standard deviations obtained for the extreme values ($\rho = -1$, $\rho = +1$) of the correlation coefficient:

$$\begin{aligned} \rho = -1 \quad &\rightarrow \quad \sigma_Z^2 = \sigma_Q^2 + 2\sigma_Q L\sigma_k + L^2\sigma_k^2 \quad \rightarrow \quad \sigma_Z = \sigma_Q + L\sigma_k \\ \rho = +1 \quad &\rightarrow \quad \sigma_Z^2 = \sigma_Q^2 - 2\sigma_Q L\sigma_k + L^2\sigma_k^2 \quad \rightarrow \quad \sigma_Z = \left|\sigma_Q - L\sigma_k\right| \end{aligned} \tag{3.88}$$

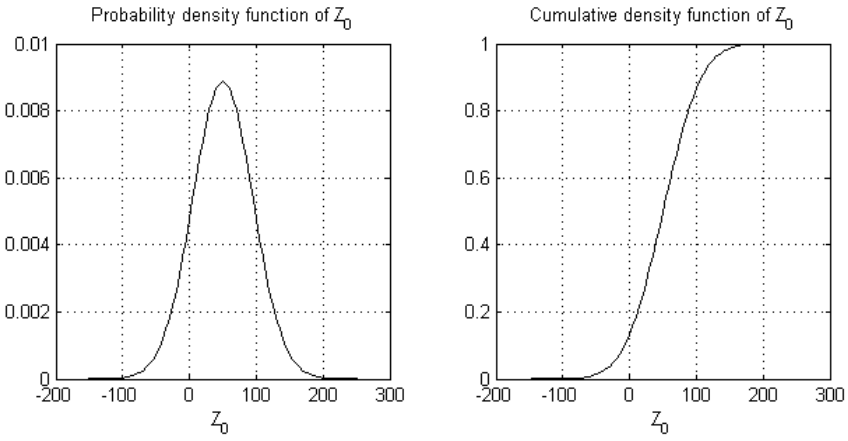

Figure 3.6: Probability density function and cumulative density function of the failure function (numerical application Eq. (3.86), $\rho = -0.6$)

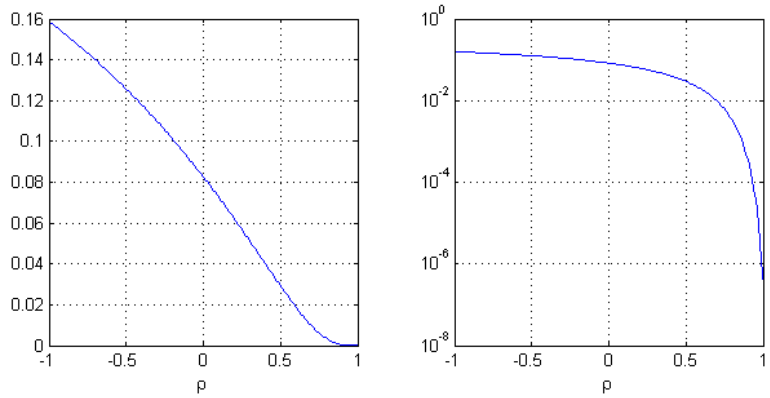

Figure 3.7: Probability of failure resulting from the probabilistic analysis (analytical developments) - Influence of the correlation coefficient

Since now the standard deviation of Z appears at the denominator of the argument of the Φ function (Eq. (3.87)) its influence of the resulting probability of failure is really significant. Fig. 3.7 represents the effect of the correlation coefficient on the probability of failure:

$$p_f(\rho) = 1 - \Phi\left(\frac{-\mu_Z}{\sigma_Z(\rho)}\right) = 1 - \Phi\left(\frac{-50}{\sqrt{1300 - 120\rho}}\right)$$

This function starts from approximately $p_f = 0.16$ for $\rho = -1$, decreases almost uniformly until $\rho = +0.9$ and then much faster from $\rho = +0.9$ to $\rho = +1.0$ (see logarithmic scale on the right side of the figure). The logarithmic representation of this function shows that a small error on the correlation coefficient affects significantly the estimated probability of failure (in the range $\rho \in [0.9; 1.0]$). For this reason a lot of effort is put nowadays in a more precise estimation of correlation coefficients involved in many civil engineering applications.

It is also interesting to notice, in this simple application, that the decrease of the probability of failure (as a function of the correlation coefficient) can be easily explained. Indeed a negative correlation coefficient means that the random variable Q and k have more often different signs (compared to their respective mean value) than the same ones. In other words, if Q is smaller than its mean value, then it is very likely that k is larger than its mean value, and reversely. This

indicates that the probability to have a large Q and a small k simultaneously is high, and hence that the probability of failure is high. On the contrary, if the correlation coefficient is positive, the likelihood to have Q and k both larger than their corresponding mean values is higher. It is then easy to understand that the probability of failure in then reduced in this case.

3.2.2 Reliability analysis: analytical approach

In order to proceed to developments comparable to those of the previous section, the problem is now solved from the beginning by means of a reliability analysis. The considered failure function is:

$$Z\left(Q,k\right) = kL - Q \tag{3.89}$$

The aim of the reliability analysis is to compute the probability of failure, i.e. the probability that $Z\left(Q,k\right)$ is smaller than 0. This quantity is expressed by means of the reliability index, defined as the shortest distance, in the reduced space, from the origin to the failure condition. This quantity is computed analytically in this section and with an iterative numerical method in section 3.2.3.

The random variables are supposed to be correlated gaussian processes. Their covariance matrix is:

$$\left[V_{Qk}\right] = \left[\begin{array}{cc} \sigma_Q^2 & \rho\sigma_Q\sigma_k \\ \rho\sigma_Q\sigma_k & \sigma_k^2 \end{array} \right] \tag{3.90}$$

A first reduction similar to the reduction performed in case of uncorrelated variables leads to the definition of:

$$\widetilde{Q} = \frac{Q-\mu_Q}{\sigma_Q} \quad ; \quad \widetilde{k} = \frac{k-\mu_k}{\sigma_k} \tag{3.91}$$

This relation can also be written in a matrix format:

$$\left\{ \begin{array}{c} \widetilde{Q} \\ \widetilde{k} \end{array} \right\} = \underbrace{\left[\begin{array}{cc} \frac{1}{\sigma_Q} & 0 \\ 0 & \frac{1}{\sigma_k} \end{array} \right]}_{[b]} \left(\underbrace{\left\{ \begin{array}{c} Q \\ k \end{array} \right\}}_{\{x\}} - \underbrace{\left\{ \begin{array}{c} \mu_Q \\ \mu_k \end{array} \right\}}_{\{\mu_x\}} \right) \tag{3.92}$$

which will be advantageously used in the following developments. The failure condition is accordingly transformed:

$$\widetilde{Z}(\widetilde{Q},\widetilde{k}) = \left(\mu_k + \sigma_k\widetilde{k}\right) L - \left(\mu_Q + \sigma_Q\widetilde{Q}\right) \tag{3.93}$$

The mathematical expectation of Eq. (3.91) shows that the new reduced variables have a zero mean. The computation of their covariance matrix leads to:

$$\left[\widetilde{V}_{\widetilde{Q}\widetilde{k}}\right] = \left[\begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array} \right] \tag{3.94}$$

which shows that they have also a unit variance but the same correlation coefficient as the physical variables. This indicates the need for another reduction aiming at providing uncorrelated variables. The theoretical developments of section 2.3 show that this reduction can be realized by means of the matrix $[a]$ defined in such a way that:

$$[a]\left[\widetilde{V}_{\widetilde{Q}\widetilde{k}}\right][a]^T = [I] \tag{3.95}$$

The eigen values of $\left[\widetilde{V}_{\widetilde{Q}\widetilde{k}}\right]$ and the corresponding eigen vectors are:

$$v_1 = 1+\rho \quad ; \qquad v_2 = 1-\rho \tag{3.96}$$

$$V_1 = \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} \quad ; \qquad V_2 = \left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\} \tag{3.97}$$

After having normalized these eigen vectors properly, the evaluation of matrix $[a]$ is then straightforward:

$$[a] = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{1+\rho}} & \frac{1}{\sqrt{1+\rho}} \\ \frac{1}{\sqrt{1-\rho}} & \frac{-1}{\sqrt{1-\rho}} \end{bmatrix} \quad \Leftrightarrow \quad [a]^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1+\rho} & \sqrt{1-\rho} \\ \sqrt{1+\rho} & -\sqrt{1-\rho} \end{bmatrix} \tag{3.98}$$

It can be checked that the injection of this expression of $[a]$ into Eq. (3.95) satisfies trivially the relation. The second reduction leads finally to the definition of new reduced variables:

$$\{\widehat{x}\} = [a]\{\widetilde{x}\} = \underbrace{[a][b]}_{[A]}(\{x\} - \{\mu_x\}) \tag{3.99}$$

which have now the required properties to be uncorrelated, with zero-mean and unit-variance. The computation of the reliability index can thus now be performed. By recasting both successive reductions together, the global reduction matrix $[A]$ is introduced (as in section 2.3). The expression of the failure function with the new variables is:

$$\widehat{Z}(\widehat{Q},\widehat{k}) = \left(\mu_k + \sigma_k \frac{\sqrt{1+\rho}\widehat{Q} + \sqrt{1-\rho}\widehat{k}}{\sqrt{2}}\right) L - \left(\mu_Q + \sigma_Q \frac{\sqrt{1+\rho}\widehat{Q} - \sqrt{1-\rho}\widehat{k}}{\sqrt{2}}\right) \tag{3.100}$$

This reduced failure function is still linear. After some reorganisation this more usual form is obtained:

$$\widehat{Z}(\widehat{Q},\widehat{k}) = \mu_k L - \mu_Q + (\sigma_k L - \sigma_Q)\sqrt{\frac{1+\rho}{2}}\widehat{Q} + (\sigma_k L + \sigma_Q)\sqrt{\frac{1-\rho}{2}}\widehat{k} \tag{3.101}$$

The failure condition, obtained by imposing a zero value to this function is then a plane. Its equation can be written in the form:

$$\{a\}^T\{x\} = a_0 \tag{3.102}$$

where

$$\begin{aligned} \{a\} &= \begin{Bmatrix} (\sigma_k L - \sigma_Q)\sqrt{\frac{1+\rho}{2}} \\ (\sigma_k L + \sigma_Q)\sqrt{\frac{1-\rho}{2}} \end{Bmatrix} \\ a_0 &= \mu_Q - \mu_k L \end{aligned} \tag{3.103}$$

The shortest distance β from the origin to a plane with the generic equation given by (3.102) can be shown to be $\beta = \frac{|a_0|}{|a|}$. This is exactly the relation required for the estimation of the reliability index:

$$\beta = \frac{\left|\mu_Q - \mu_k L\right|}{\sqrt{\frac{1+\rho}{2}(\sigma_k L - \sigma_Q)^2 + \frac{1-\rho}{2}(\sigma_k L + \sigma_Q)^2}} = \frac{\left|\mu_Q - \mu_k L\right|}{\sqrt{\sigma_Q^2 - 2\rho\sigma_Q L\sigma_k + L^2\sigma_k^2}} \tag{3.104}$$

The corresponding probability of failure is expressed by:

$$p_f = \Phi(-\beta) = 1 - \Phi(\beta) \tag{3.105}$$

which is strictly equivalent to the subsequent development of the probabilistic approach (Eq. (3.87)). If the numerical values given in Equ. (3.86) are considered, the reliability index and the probability of failure are:

$$\beta = 1.1125 \quad ; \quad p_f = \Phi(-\beta) = 0.1330 \tag{3.106}$$

3.2.3 Reliability analysis: illustration of numerical resolution (AFOSMC)

In this section, the numerical iterative method is used for the resolution of the same reliability analysis. Since the random variables are correlated, Alg. 8 (p. 20) is applied. The only difference between this algorithm and Alg. 7 consists in the reduction to uncorrelated variables.

As in the previous two sections the failure condition is expressed by:

$$Z\left(k,Q\right)=kL-Q \tag{3.107}$$

where Q and k are correlated Gaussian variable, characterized by their covariance matrix:

$$\left[V_{kQ}\right]=\left[\begin{array}{cc}\sigma_k^2 & \rho\sigma_k\sigma_Q\\ \rho\sigma_Q\sigma_k & \sigma_Q^2\end{array}\right] \tag{3.108}$$

The mean values and standard deviations of both random processes are given in Eq. (3.86);

Step 0.1: The first step consists in defining the reduced variables:

$$\left\{\widehat{x}\right\}=\left[A\right]\left(\left\{x\right\}-\left\{\mu_x\right\}\right)\quad\Leftrightarrow\quad\left\{x\right\}=\left\{\mu_x\right\}+\left[A\right]^{-1}\left\{\widehat{x}\right\} \tag{3.109}$$

where $[A]$ is such that $\left[A\right]\left[V_{Qk}\right]\left[A\right]^T=\left[I\right]$, i.e. is related to the eigen vectors of the covariance matrix. They are computed as follows: first the eigen values are computed as the roots of $\det\left(\left[V_{Qk}\right]-\lambda\left[I\right]\right)$:

$$\left[V_{kQ}\right]=\left[\begin{array}{cc}20^2 & -0.6*20*30\\ -0.6*20*30 & 30^2\end{array}\right]=\left[\begin{array}{cc}400 & -360\\ -360 & 900\end{array}\right] \tag{3.110}$$

$$\begin{aligned}\det\left(\left[V_{kQ}\right]-\lambda\left[I\right]\right) &= \left[\begin{array}{cc}400-\lambda & -360\\ -360 & 900-\lambda\end{array}\right]=0\\ &\Leftrightarrow\quad \lambda^2-1300\lambda+230400=0\\ &\Leftrightarrow\quad \lambda_1=211.71\quad\text{or}\quad\lambda_2=1088.29\end{aligned} \tag{3.111}$$

Then for each eigen value the corresponding eigen vector is obtained by:

$$\begin{aligned}\left(\left[V_{kQ}\right]-\lambda_1\left[I\right]\right)\left\{x_1\right\} &= 0\Leftrightarrow\left[\begin{array}{cc}400-211.71 & -360\\ -360 & 900-211.71\end{array}\right]\left\{x_1\right\}=0\\ &\Rightarrow\quad\left\{x_1\right\}=a_1\left\{\begin{array}{c}1.9119\\ 1\end{array}\right\}\\ \left(\left[V_{kQ}\right]-\lambda_2\left[I\right]\right)\left\{x_2\right\} &= 0\Leftrightarrow\left[\begin{array}{cc}400-1088.29 & -360\\ -360 & 900-1088.29\end{array}\right]\left\{x_2\right\}=0\\ &\Rightarrow\quad\left\{x_2\right\}=a_2\left\{\begin{array}{c}-0.5230\\ 1\end{array}\right\}\end{aligned}$$

where constants a_1 and a_2 can be chosen anyhow (except equal to zero). Both eigen vectors can be placed side-by-side into one matrix $\left[X\right]=\left[\left\{x_1\right\},\left\{x_2\right\}\right]$. The main property of the eigen vectors (and hence this eigen matrix) is to be able to diagonalize the covariance matrix:

$$\left[X\right]^T\left[V_{kQ}\right]\left[X\right]=\left[\begin{array}{cc}985.6a_1^2 & 0\\ 0 & 1386a_2^2\end{array}\right] \tag{3.112}$$

If a_1 and a_2 are chosen in such a way to provide unit element on the diagonal of this matrix, i.e. for example $a_1=-0.0319$ and $a_2=0.0269$, then it can be checked that the requested matrix $[A]$ is given by:

$$\left[A\right]=\left[X\right]^T=\left[\begin{array}{cc}1.9119*(-0.0319) & -0.5230*0.0269\\ -0.0319 & 0.0269\end{array}\right]^T=\left[\begin{array}{cc}-0.0609 & -0.0319\\ -0.0140 & 0.0269\end{array}\right] \tag{3.113}$$

This first step is not effortless but hopefully it must be performed only once because the iterative method loops from steps 2 to 7. The inverse of matrix $[A]$ is useful for the computation of the inverse relation and must also be computed once only:

$$[A]^{-1} = \begin{bmatrix} -12.893 & -15.290 \\ -6.743 & 29.232 \end{bmatrix} \tag{3.114}$$

The reduced variables are finally expressed by:

$$\{\widehat{x}\} = [A]\left(\{x\} - \{\mu_x\}\right)$$

$$\begin{aligned} \begin{Bmatrix} \widehat{k} \\ \widehat{Q} \end{Bmatrix} &= \begin{bmatrix} -0.0609 & -0.0319 \\ -0.0140 & 0.0269 \end{bmatrix} \begin{Bmatrix} k \\ Q \end{Bmatrix} - \begin{Bmatrix} 12.32 \\ 0.5787 \end{Bmatrix} \\ \Leftrightarrow \begin{Bmatrix} k \\ Q \end{Bmatrix} &= \begin{Bmatrix} 150 \\ 100 \end{Bmatrix} + \begin{bmatrix} -12.893 & -15.290 \\ -6.743 & 29.232 \end{bmatrix} \begin{Bmatrix} \widehat{k} \\ \widehat{Q} \end{Bmatrix} \end{aligned} \tag{3.115}$$

where $[A]$ is such that $[A]\,[V_x]\,[A]^T = [I]$ and $[V_x]$ is the covariance matrix of the physical variables $\{x\}$.

Step 0.2: Write the failure condition with the reduced variables:

$$\begin{aligned} \widehat{Z}\left(\{\widehat{x}\}\right) &= kL - Q = \left(150 - 12.893\widehat{k} - 15.290\widehat{Q}\right) L - \left(100 - 6.743\widehat{k} + 29.232\widehat{Q}\right) \\ &= 50 - 6.1496\widehat{k} - 44.522\widehat{Q} \end{aligned} \tag{3.116}$$

Step 1: Now the preliminary steps related to the reduction are done and the iterative procedure can start. An initial guess of the design point $\widehat{x}_i^{(0)}$ have to be provided. In this analysis the mean physical variable (i.e. $\widehat{Q} = 0$, $\widehat{k} = 0$) is considered.

Step 2: The reduced failure condition at this point is:

$$\widehat{Z}^{(0)} = 50 \tag{3.117}$$

which indicates that this guess is the design point does not lie on the failure condition and needs thus to be more accurately estimated.

Step 3: The gradient of the reduced failure function can be computed in closed form:

$$\begin{aligned} n_1^{(0)} &= \frac{\partial \widehat{Z}\left(\{\widehat{x}\}\right)}{\partial \widehat{k}} = -6.1496 \\ n_2^{(0)} &= \frac{\partial \widehat{Z}\left(\{\widehat{x}\}\right)}{\partial \widehat{Q}} = -44.522 \end{aligned} \tag{3.118}$$

In this particular example, they are constant because the failure function is linear.

Step 4: The orientation corresponding to the next design point is given by:

$$\begin{aligned} \alpha_1^{(0)} &= \frac{n_1^{(0)}}{\left|n^{(0)}\right|} = \frac{6.1498}{44.9447} = -0.1368 \\ \alpha_2^{(0)} &= \frac{n_2^{(0)}}{\left|n^{(0)}\right|} = \frac{-44.522}{44.9447} = -0.9906 \end{aligned} \tag{3.119}$$

Step 5: The estimated mean and standard deviation of the failure condition are:

$$\begin{aligned} \mu_G &= \widehat{G}^{(0)} - \sum_{i=1}^{N} n_i^{(0)} x_i^{(0)} = 50 \\ \sigma_G &= \left|n^{(0)}\right| = 44.9447 \end{aligned} \tag{3.120}$$

Step 6: Finally the new estimation of the reliability index is obtained by:

$$\beta^{(0)} = \frac{\mu_G}{\sigma_G} = \frac{50}{44.9447} = 1.1125 \tag{3.121}$$

Step 7: Compute a better estimation of the design point:

$$\begin{aligned} \widehat{k}^{(1)} &= -\alpha_1^{(0)}\beta^{(0)} = -\left(-0.1368\right) * 1.1125 = 0.1522 \\ \widehat{Q}^{(1)} &= -\alpha_2^{(0)}\beta^{(0)} = -\left(-0.9906\right) * 1.1125 = 1.1020 \end{aligned} \tag{3.122}$$

Since the failure function is linear, no iteration is actually required (see Section 3.1.5). Indeed it could be checked that going back to step 2 would indicate that this new design point lies on the failure condition. Furthermore the direction of the gradient is the same as the direction obtained at the previous step. As a conclusion it can be said that the application of the AFOSMC algorithm is exactly the same as the AFOSM algorithm excepted for step 0.2 which requires a little bit more computation for the former.

The same parametric study concerning the influence of the correlation coefficient on the reliability index and probability of failure can be performed. Figure 3.8 illustrates the results obtained with the AFOSMC algorithm, for several correlation coefficients. In the physical space (left subfigures) the same failure function is considered in each case. It is represented by the uniformly distributed parallel lines. The joint probability density function of variables $x_1 = k$ and $x_2 = Q$ are also represented by level curves. They are concentric ellipse inclined to the left for negative correlation coefficients and to the right for positive values. In each diagram the probability of failure is represented by the surface under this joint probability function, and behind the failure condition (represented by the thick line). The evolution from the diagram corresponding $\rho = -0.9$ to the diagram corresponding to $\rho = +0.9$ shows clearly that the probability of failure decreases with increasing correlation coefficients. In the reduced space (square subfigures) the reduced failure condition is different in each. Indeed it depends on the reduction matrix ($[A]$) which is a function of the correlation coefficient. These diagrams indicate also the decrease of the reliability index (and hence of the probability of failure) with increasing correlation coefficients.

Fig. 3.9 is a copy of Fig.3.7 in which dots have been added in order to represent the results obtained with the numerical iterative method (AFOSMC). It shows that both methods provide exactly the same results.

3.2.4 Probabilistic analysis: analytical approach 2

In the buckling problem a usual deterministic analysis would lead to this relation between the stiffness, the applied force and the beam length:

$$\left(kL^2 - PL\right)\theta = 0 \tag{3.123}$$

from which the (eulerian) critical load $P_{cr} = kL$ can be computed. In a probabilistic context the quantities involved in this relation (k, L) are defined by means of random variables. The probabilistic analysis of the structure requires to determine, given the statistical characteristics of k and L, the same information about the critical load P_{cr}. The computation of this general information is a probabilistic analysis. In this simple example, the major part of this analysis can be performed in an analytical way. These developments are presented in this section and will be compared in the next section to analytical and numerical reliability analyses.

It is supposed that the stiffness and the beam length are represented by correlated gaussian random variables characterized by their mean values (μ_k ,μ_L), their standard deviations (σ_k, σ_L) and the correlation coefficient ρ.The cumulative distribution function of the eulerian critical load is expressed by:

$$\begin{aligned} F_P\left(P_0\right) &= prob\left(P < P_0\right) \\ &= prob\left(kL < P_0\right) \end{aligned} \tag{3.124}$$

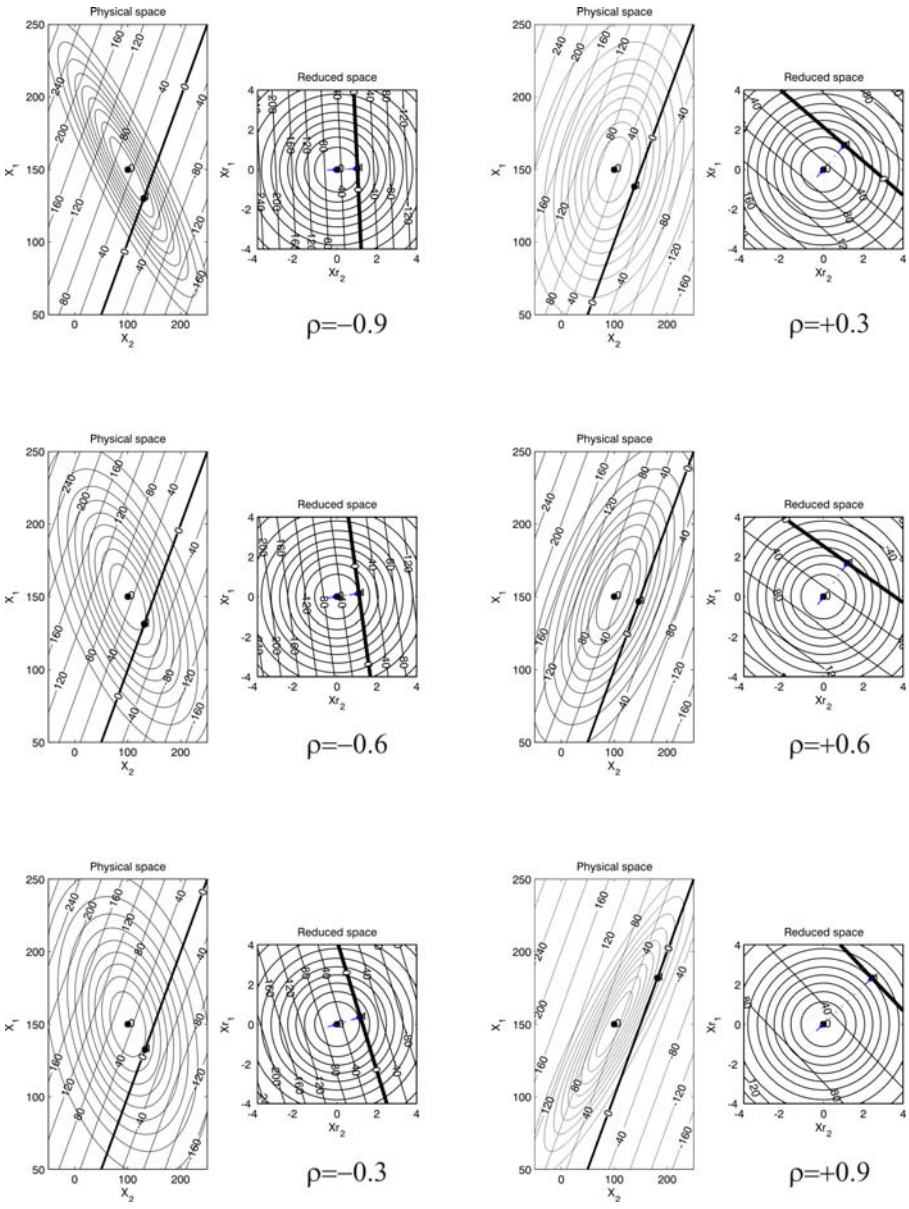

Figure 3.8: Illustration of the influence of the correlation coefficient on the probability of failure - Results of the reliability analysis (AFOSMC)

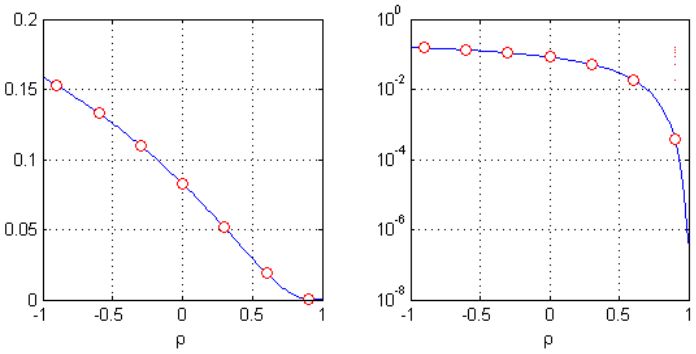

Figure 3.9: Probability of failure resulting from the probabilistic analysis (analytical developments, continuous line) and from the reliability analysis (numerical iterative method, dots) - Influence of the correlation coefficient

Considering all possible values that k could take, this relation can also be written:

$$\begin{aligned} F_P(P_0) &= \int_{-\infty}^{+\infty} prob(k_0 < k \leq k_0 + dk_0)\, prob(k_0 L < P_0 \mid k_0 < k \leq k_0 + dk_0)\, dk_0 \\ &= \int_{-\infty}^{+\infty} p_k(k_0)\, F_{L|k}\left(\frac{P_0}{k_0} \mid k_0\right) dk_0 \end{aligned}$$

where $F_{L|k}(L_0 \mid k_0)$ represents a conditional cumulative density function, i.e. the probability that L is smaller than L_0 provided k lies in the interval $[k_0; k_0 + dk_0]$. The probability density function of P can be computed by simple derivation of its cumulative distribution function:

$$\begin{aligned} p_P(P_0) &= \frac{dF_P}{dP_0} = \int_{-\infty}^{+\infty} \frac{p_k(k_0)}{k_0} p_{L|k}\left(\frac{P_0}{k_0} \mid k_0\right) dk_0 \\ &= \int_{-\infty}^{+\infty} \frac{p_k(k_0)}{k_0} \frac{p_{Lk}\left(\frac{P_0}{k_0}, k_0\right)}{p_k(k_0)} dk_0 \\ &= \int_{-\infty}^{+\infty} \frac{p_{Lk}\left(\frac{P_0}{k_0}, k_0\right)}{k_0} dk_0 \end{aligned} \tag{3.125}$$

where $p_{Lk}(L_0, k_0)$ is the joint probability density function between L and k. This relation is valid for any joint probability function between k and L. For this application, it is supposed that these random variables are described by correlated gaussian variables, then:

$$p_{Lk}(L_0, k_0) = \frac{1}{2\pi\sigma_L\sigma_k\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{L_0-\mu_L}{\sigma_L}\right)^2 - 2\rho\frac{(L_0-\mu_L)(k_0-\mu_k)}{\sigma_L\sigma_k} + \left(\frac{k_0-\mu_k}{\sigma_k}\right)^2\right]} \tag{3.126}$$

The introduction of this relation into (3.125) gives:

$$p_P(P_0) = \int_{-\infty}^{+\infty} \frac{1}{2\pi k_0\sigma_L\sigma_k\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{\frac{P_0}{k_0}-\mu_L}{\sigma_L}\right)^2 - 2\rho\frac{\left(\frac{P_0}{k_0}-\mu_L\right)(k_0-\mu_k)}{\sigma_L\sigma_k} + \left(\frac{k_0-\mu_k}{\sigma_k}\right)^2\right]} dk_0 \tag{3.127}$$

which is not possible to simplify explicitly. It is however easy to check that this function is well a probability density function, i.e. that $\int_{-\infty}^{+\infty} p_P(P_0)\, dP_0 = 1$. The cumulative density function of

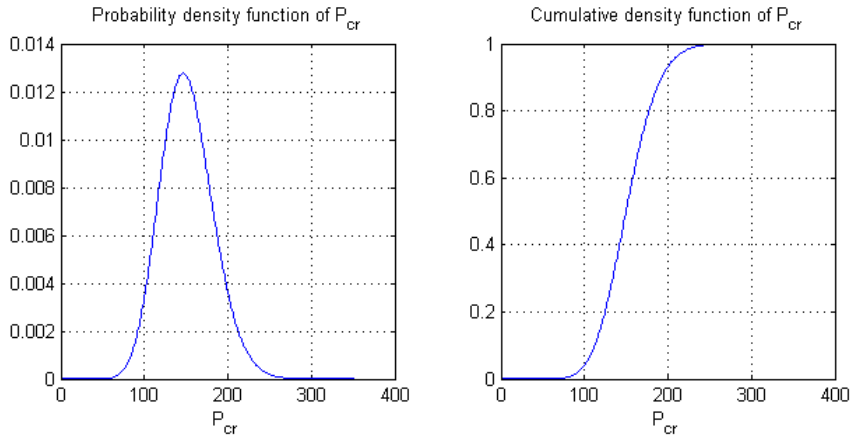

Figure 3.10: Probability density function and cumulative density function of the eulerian critical load ($P_{cr} = kL$) - Application of the analytical developments

P is obtained by integration of this last relation with respect to P_0:

$$F_P(P) = \int_{-\infty}^{P} p_P(P_0)\, dP_0 \tag{3.128}$$

which gives after some developments:

$$F_P(P_0) = \frac{1}{2} + \frac{1}{2\sqrt{2\pi}\sigma_k} \int_{-\infty}^{+\infty} e^{\frac{-(k_0-\mu_k)^2}{2\sigma_k^2}} \operatorname{erf} \frac{P_0\sigma_k + \rho k_0^2 \sigma_L - k_0(\mu_L\sigma_k + \rho\mu_k\sigma_L)}{\sqrt{2}k_0\sqrt{1-\rho^2}\sigma_k\sigma_L} dk_0 \tag{3.129}$$

where erf represents the error function, i.e. $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$. This example is the first analytical development associated to a non linear relation ($P_{cr} = kL$). The important complexity related to analytical developments is evident and this example clearly shows why numerical methods take a major place in reliability analysis. As a particular application the former relations are estimated for some chosen numerical values:

$$\begin{array}{rcl} \mu_k &=& 150kN/m \quad ; \quad \sigma_k = 20kN/m \\ \mu_L &=& 1m \quad ; \quad \sigma_L = 0.1m \\ \rho &=& 0.6 \end{array} \tag{3.130}$$

The probability density function (Eq. (3.127)) and the corresponding cumulative density function are estimated with appropriate numerical integrations. They are computed point-by-point. Fig. 3.10 represents the resulting functions. Clearly they show that the eulerian critical load is a non gaussian process, exhibiting a skewness to the right. The most probable eulerian critical load is approximately equal to $150kN$. From the cumulative density function of P_{cr} a probability of failure (as in a reliability context) can be determined. For example the probability that the critical load is smaller than 90 is equal to 0.016676. Again this subsequent result shows that a complete probabilistic analysis is certainly too rich and encapsulates the results of a probabilistic analysis. In other words, if the failure function:

$$Z(k, L) = kL - 90 \tag{3.131}$$

was considered, the resulting probability of failure should be: $p_f = 0.016676$.

The analytical approach of the probabilistic analysis is probably too complex for practical applications. It presents however the major advantage of providing exact results. They are used in the following as a comparison with the results of the reliability analysis, in order to quantify the precision obtained with this latter method.

3.2.5 Reliability analysis: analytical approach

In the context of a reliability analysis, the probability of failure concerning a given failure function is assessed. In order to proceed to developments comparable to those of the previous section, the considered failure function is:

$$Z(k,L) = kL - Q \tag{3.132}$$

The aim of the reliability analysis is to compute the probability of failure, i.e. the probability that $Z(k,L)$ is smaller than 0, which is also equivalent to the probability that $Q < kL$. This quantity is expressed by means of the reliability index, defined as the shortest distance, in the reduced space, from the origin to the failure condition. This quantity is computed analytically in this section and numerically in the next section (application of Alg. 8 (p. 20)).

As in the probabilistic approach of the previous section the random variables are supposed to be correlated gaussian processes. Their covariance matrix is:

$$[V_{kL}] = \begin{bmatrix} \sigma_k^2 & \rho\sigma_L\sigma_k \\ \rho\sigma_k\sigma_L & \sigma_L^2 \end{bmatrix} \tag{3.133}$$

The computation of the reliability index requires the reduction of the physical variables to zero-mean unit-variance uncorrelated variables. This operation has to be performed exactly as in Section 3.2.2. It is convenient to introduce two successive reductions:

$$\{\widehat{x}\} = [a]\{\widetilde{x}\} = \underbrace{[a][b]}_{[A]}(\{x\} - \{\mu_x\}) \tag{3.134}$$

where

$$[a] = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{1+\rho}} & \frac{1}{\sqrt{1+\rho}} \\ \frac{1}{\sqrt{1-\rho}} & \frac{-1}{\sqrt{1-\rho}} \end{bmatrix} \quad ; \quad [b] = \begin{bmatrix} \frac{1}{\sigma_k} & 0 \\ 0 & \frac{1}{\sigma_L} \end{bmatrix} \tag{3.135}$$

which leads then to:

$$[A] = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sigma_k\sqrt{1+\rho}} & \frac{1}{\sigma_L\sqrt{1+\rho}} \\ \frac{1}{\sigma_k\sqrt{1-\rho}} & \frac{-1}{\sigma_L\sqrt{1-\rho}} \end{bmatrix} \quad \Leftrightarrow \quad [A]^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} \sigma_k\sqrt{1+\rho} & \sigma_k\sqrt{1-\rho} \\ \sigma_L\sqrt{1+\rho} & -\sigma_L\sqrt{1-\rho} \end{bmatrix} \tag{3.136}$$

The reduced failure condition can be expressed by means of this reduction:

$$\widehat{Z}(\widehat{k},\widehat{L}) = \left(\mu_k + \sigma_k\frac{\sqrt{1+\rho}\widehat{k} + \sqrt{1-\rho}\widehat{L}}{\sqrt{2}}\right)\left(\mu_L + \sigma_L\frac{\sqrt{1+\rho}\widehat{k} - \sqrt{1-\rho}\widehat{L}}{\sqrt{2}}\right) - Q \tag{3.137}$$

After some modifications, this reduced failure condition can also be written:

$$\begin{aligned}\widehat{Z}(\widehat{k},\widehat{L}) &= \sigma_k\sigma_L\left(\frac{1+\rho}{2}\widehat{k}^2 - \frac{1-\rho}{2}\widehat{L}^2\right) \\ &\quad + (\mu_L\sigma_k + \mu_k\sigma_L)\sqrt{\frac{1+\rho}{2}}\widehat{k} + (\mu_L\sigma_k - \mu_k\sigma_L)\sqrt{\frac{1-\rho}{2}}\widehat{L} + \mu_k\mu_L - Q\end{aligned} \tag{3.138}$$

In this example the failure condition is non linear[2]. More precisely Eq. (3.138) indicates that the failure condition is a branch of hyperbola. For this reason the reliability index can be computed but can't be directly linked to the probability of failure. Furthermore when the failure condition is not linear the location of the closest point to the origin is much more complex to compute. For the sake of simplicity in the notations, let Eq. (3.138) be rewritten:

$$\widehat{Z}\left(\widehat{k},\widehat{L}\right) = a_1\widehat{k}^2 + a_2\widehat{L}^2 + a_3\widehat{k} + a_4\widehat{L} + a_5 \tag{3.139}$$

[2] This was already introduced in the previous section.

The gradient of this function is expressed by:

$$\overrightarrow{\nabla}\widehat{Z}(\widehat{k},\widehat{L}) = \left\{ \begin{array}{c} 2a_1\widehat{k} + a_3 \\ 2a_2\widehat{L} + a_4 \end{array} \right\} \tag{3.140}$$

The reliability index is found by the location of the point lying on this curve and situated the closest to the origin. This condition requires to find the minimum of:

$$\beta^2\left(\widehat{k},\widehat{L}\right) = \widehat{k}^2 + \widehat{L}^2 \tag{3.141}$$

Following Lagrange's theory, the position of this design point is obtained when the gradients of $\widehat{Z}\left(\widehat{k},\widehat{L}\right)$ and $\beta^2\left(\widehat{k},\widehat{L}\right)$ line up:

$$\overrightarrow{\nabla}\widehat{Z}(\widehat{k},\widehat{L}) = \left\{ \begin{array}{c} 2a_1\widehat{k} + a_3 \\ 2a_2\widehat{L} + a_4 \end{array} \right\} = \overrightarrow{\nabla}\beta^2(\widehat{k},\widehat{L}) = \lambda \left\{ \begin{array}{c} 2\widehat{k} \\ 2\widehat{L} \end{array} \right\} \tag{3.142}$$

The design point (indicated by symbol *) is then such that the following relations fulfilled:

$$\widehat{k}^* = \frac{a_3}{2\left(\lambda - a_1\right)} \tag{3.143}$$

$$\widehat{L}^* = \frac{a_4}{2\left(\lambda - a_2\right)} \tag{3.144}$$

Because the design point must also lie on the failure condition, these relations can be injected into Eq. (3.139), leading then to:

$$\frac{a_1 a_3^2}{4\left(\lambda - a_1\right)^2} + \frac{a_2 a_4^2}{4\left(\lambda - a_2\right)^2} + \frac{a_3^2}{2\left(\lambda - a_1\right)} + \frac{a_4^2}{2\left(\lambda - a_2\right)} + a_5 = 0 \tag{3.145}$$

The explicit resolution of this equation is not easy but its solution can be preformed with many usual solvers (after having rearranged this relation into a polynomial form). The resolution of this relation leads to the value of λ, which gives, after replacement into Eqs. (3.143) and (3.144) the location of the reduced design point. The estimation of the reliability index β is straightforward. A probability of failure can be computed but it should not be forgotten that this probability of failure is erroneous because the failure function is non linear.

As a numerical application, the values given in Eqs. (3.130) are considered. Their introduction into Eq. (3.138) leads to:

$$\begin{array}{rcl} a_1 &=& 1.6 \quad ; \quad a_2 = -0.4 \\ a_3 &=& 31.305 \quad ; \quad a_4 = 2.23607 \\ a_5 &=& 60 \end{array} \tag{3.146}$$

The resolution of Eq. (3.145) gives $\lambda = -5.73545$, which in turn allows computing the position of the reduced design point:

$$\widehat{k}^* = \frac{a_3}{2\left(\lambda - a_1\right)} = -2.13381 \tag{3.147}$$

$$\widehat{L}^* = \frac{a_4}{2\left(\lambda - a_2\right)} = -0.209548 \tag{3.148}$$

The reliability index is estimated now as the distance from this point to the origin:

$$\beta = \sqrt{\widehat{k}^{*2} + \widehat{L}^{*2}} = 2.1441 \tag{3.149}$$

For information an estimation of the probability of failure can be given:

$$p_f \simeq \Phi^{-1}(-\beta) = 1.6013.10^{-2} \tag{3.150}$$

This value is slightly different than the exact result obtained in the previous paragraph ($p_f = 0.016676$). The deviation from the exact result is however quite low. This is due to the slight curvature of the failure condition (as illustrated in the next section). Finally as an informative indication, the physical design variables can be computed. Their estimation requires first the computation of the transformation matrix $[A]$ (Eq. (3.136)). After some computation, this leads to:

$$\begin{aligned} k &= 109.96 \\ L &= 0.81851 \end{aligned} \tag{3.151}$$

It can be checked that kL is equal to 90. Among all the combinations (k, L) such that kL is equal to 90,(e.g. $90 * 1 = 90$, $60 * 1.5 = 90$, $30 * 3 = 90$, $900 * 0.1 = 90$,...), this combination of both parameters is the most probable (for the considered probability distributions of k and L, i.e. $\mu_k = 150$, $\mu_L = 1$, $\sigma_k = 20$, $\sigma_L = 0.1$, $\rho = 0.6$.

3.2.6 Reliability analysis: illustration of numerical resolution (AFOSMC)

In this section the same reliability analysis is performed, but in a numerical way. Algorithm 8 (p. 20)) is applied step-by-step in order to illustrate its application in the context of a non linear failure function.

Step 0.1: The new reduced variables are defined by:

$$\{\widehat{x}\} = [A](\{x\} - \{\mu_x\}) \quad \Leftrightarrow \quad \{x\} = \{\mu_x\} + [A]^{-1}\{\widehat{x}\} \tag{3.152}$$

where $[A]$ is such that $[A][V_x][A]^T = [I]$ and $[V_x]$ is the covariance matrix of the physical variables $\{x\}$. In this example, $[V_x]$ is equal to:

$$[V_x] = \begin{bmatrix} \sigma_k^2 & \rho\sigma_L\sigma_k \\ \rho\sigma_k\sigma_L & \sigma_L^2 \end{bmatrix} = \begin{bmatrix} 400 & 1.2 \\ 1.2 & 0.01 \end{bmatrix} \tag{3.153}$$

The eigen values and eigen vectors of $[V_x]$ are:

$$[V] = \begin{bmatrix} 0.003 & -1 \\ -1 & -0.003 \end{bmatrix} \quad ; \quad [D] = \begin{bmatrix} 0.0064 & 0 \\ 0 & 400.0036 \end{bmatrix} \tag{3.154}$$

They are such that $[V_x][V] = [V][D]$. It is easy to see that the transformation matrix $[A]$ can be computed as:

$$[A] = \left([V][D]^{-1/2}\right)^T = \begin{bmatrix} 0.0375 & -12.5 \\ -0.05 & -0.00015 \end{bmatrix} \tag{3.155}$$

where ^{T} denotes a matrix transposition. It can effectively be checked that $[A][V_x][A]^T$ is the 2-by-2 identity matrix. This transformation matrix can be computed once for all. It will be used as such for the subsequent iterations. The relation between the physical variables and the reduced ones are then:

$$\begin{cases} \widehat{k} = 0.0375k - 12.5L + 6.875 \\ \widehat{L} = -0.05k - 0.00015L + 7.5 \end{cases} \tag{3.156}$$

$$\begin{cases} k = 0.00024\widehat{k} - 20\widehat{L} + 150 \\ L = -0.08\widehat{k} - 0.06\widehat{L} + 1 \end{cases} \tag{3.157}$$

Step 0.2: The second preliminary step consists in computing the reduced failure condition. The introduction of Eq. (3.157) into the failure condition leads to:

$$\begin{aligned}\widehat{G}\left(\widehat{k},\widehat{L}\right) &= \left(0.00024\widehat{k}-20\widehat{L}+150\right)\left(-0.08\widehat{k}-0.06\widehat{L}+1\right)-90 \\ &= -1.92.10^{-5}\widehat{k}^2+1.2\widehat{L}^2+1.6000144\widehat{k}\widehat{L}-12.00024\widehat{k}-29\widehat{L}+60 \end{aligned} \quad (3.158)$$

Step 1: The first guess of the design point is supposed to correspond to the mean physical variables:

$$\widehat{x}^{(0)}=(0,0)\,,\ \text{i.e. } x^{(0)}=(150,1) \quad (3.159)$$

This point is represented in Fig. 3.11. This figure represents also the level curves of the joint probability density function of k and L (left side, physical space) and $\widehat{k}$ and $\widehat{L}$ (right side, reduced space). Because of the physical variables are correlated the principal axes of the joint probability density function are inclined. The reduction makes them equal. The level curves of the (ev. reduced) failure function are also represented. The failure condition is the particular curve corresponding to a zero-level. It is non linear in this case, as indicated by the previous relations. The representation in the reduced space (with squared axis) shows that the non linearity of the failure condition is not so high and then that the application of a First Order Reliability analysis could lead to an acceptable result.

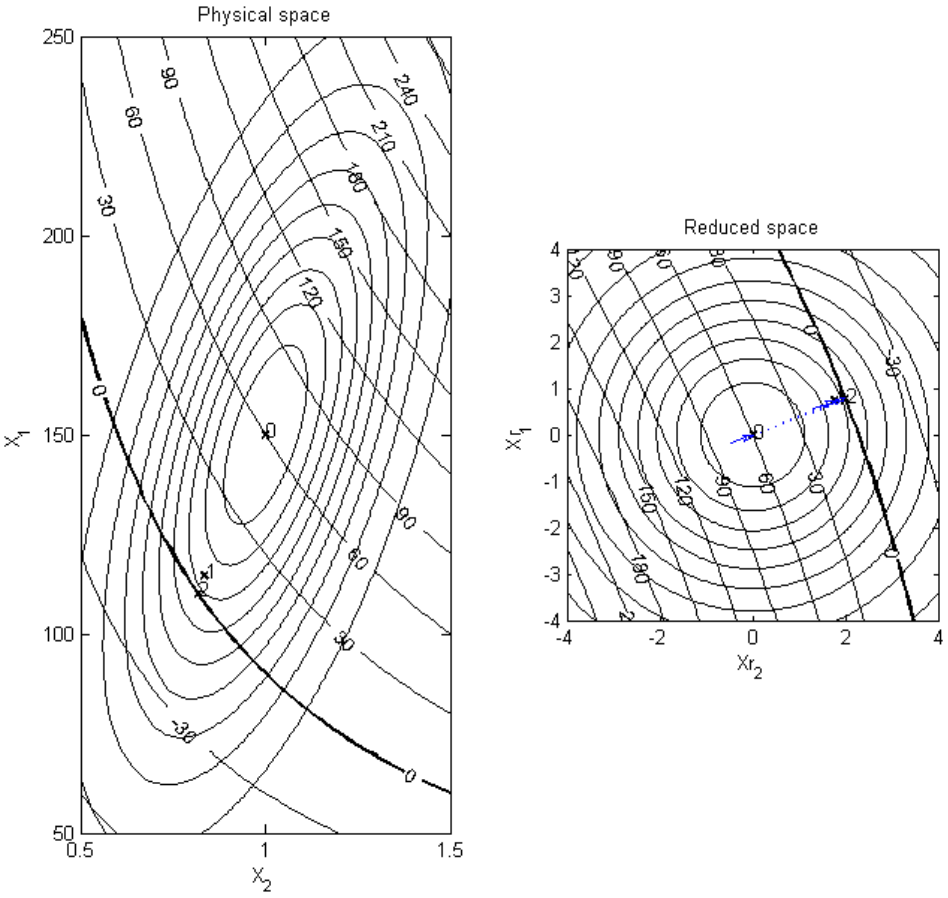

Figure 3.11: Illustration of the AFOSMC with a non linear failure condition

Step 2: This point clearly does not lie on the failure condition. Indeed the reduced failure condition at this point is equal to:

$$\widehat{G}^{(0)}(0,0) = 60 \tag{3.160}$$

Step 3: The next step consists in computing the gradient of the reduced failure function at this point, i.e.:

$$\frac{\partial \widehat{G}}{\partial \widehat{k}} = -3.84.10^{-5}\widehat{k} + 1.6000144\widehat{L} - 12.00024 \tag{3.161}$$

$$\frac{\partial \widehat{G}}{\partial \widehat{L}} = 2.4\widehat{L} + 1.6000144\widehat{L} - 29 \tag{3.162}$$

and to evaluate it at the current point ($\widehat{k}=0, \widehat{L}=0$):

$$n_1^{(0)} = \frac{\partial \widehat{G}}{\partial \widehat{k}} = -12.00024 \quad ; \quad n_2^{(0)} = \frac{\partial \widehat{G}}{\partial \widehat{L}} = -29 \tag{3.163}$$

This vector is indicated by the small at $\widehat{x}^{(0)}$ on Fig. 3.11 (right side). As it is the gradient of the reduced failure function, it is perpendicular to its level curve at $\widehat{x}^{(0)}$.

Step 4: This vector can be used to define a new direction, i.e. a unit vector:

$$\left|n^{(0)}\right| = \sqrt{(-12.00024)^2 + (-29)^2} = 31.385 \tag{3.164}$$

$$\left\{\alpha^{(0)}\right\} = \frac{\left\{n^{(0)}\right\}}{\left|n^{(0)}\right|} = \left(\frac{-12.00024}{31.385}, \frac{-29}{31.385}\right) = (-0.3823, -0.9240) \tag{3.165}$$

which is used to find the position of the next iterate.

Step 5: The approximate reliability index is obtained by the estimation of the mean failure condition and its standard deviation :

$$\mu_G = \widehat{G}^{(0)} - \sum_{i=1}^{N} n_i^{(k)} \widehat{x}_i^{(k)} = 60 \tag{3.166}$$

$$\sigma_G = \alpha_1 \frac{\partial \widehat{G}}{\partial \widehat{P}} + \alpha_2 \frac{\partial \widehat{G}}{\partial \widehat{k}} = \left|n^{(0)}\right| = 31.385 \tag{3.167}$$

Step 6: Their ratio simply gives:

$$\beta^{(0)} = \frac{\mu_G}{\sigma_G} = 1.9118 \tag{3.168}$$

An estimation of the probability of failure can be provided as well:

$$p_f = \Phi^{-1}\left(-\beta^{(0)}\right) = 2.7954.10^{-2} \tag{3.169}$$

Step 7: Finally the new approximation of the design point is obtained by:

$$\left\{\widehat{x}^{(1)}\right\} = -\left\{\alpha^{(0)}\right\}\beta^{(0)} = -(-0.3823, -0.9240).1.9118 \tag{3.170}$$

$$= (0.7309, 1.7665) \equiv \left(\widehat{k}^{(1)}, \widehat{L}^{(1)}\right) \tag{3.171}$$

This point is represented by the cross and point 1 on Fig. 3.11. As in the previous illustrations the iterations can now start. Figure 3.11 shows that the convergence is achieved quite fast. This is also a consequence of the slightness of the non linearity of the failure condition. Tables 3.7 and 3.8 provide either summarized either detailed results of this numerical application. The results obtained with the numerical procedure and the analytical developments of the previous section

do not coincide perfectly because the transformation matrix was not chosen exactly in the same way (it is not unique!). However, the resulting design point (Eq. (3.151)) is identical for both methods. The reliability indexes and the probability of failure are also exactly the same.

It is also interesting to notice that the result of the first iteration corresponds to the application of the simple FOSM method. Its application leads to a significantly wrong probability of failure ($p_f = 2.7954.10^{-2}$).

For information, a crude Monte Carlo simulation has been applied twice to this problem with $N = 100000$ samples. The statistical characteristics obtained for the failure condition are (in both cases):

$$\begin{aligned} \mu_G &= 61.2599 \quad 61.2086 \\ \sigma_G &= 31.4826 \quad 31.4915 \\ \gamma_{3_G} &= 0.29944 \quad 0.30117 \\ \gamma_{4_G} &= 0.12329 \quad 0.12303 \\ p_f &= 0.016902 \quad 0.016916 \end{aligned}$$

which indicates that this number of samples seems large enough to produce a confident result. This probability of failure is in a very good accordance with the results obtained with the probabilistic approach, $p_f = 1.6676.10^{-2}$ (section 3.2.4). The probability of failure obtained with the AFOSMC method ($p_f = 1.6013.10^{-2}$) is slightly lower because the curvature of the failure function in the reduced space is turned towards the origin.

3.3 Vibration model (non linear failure function)

3.3.1 Probabilistic analysis: analytical approach

The free vibrations of the single degree of freedom system are expressed by:

$$kL^2\theta + \frac{mL^2}{4}\ddot{\theta} = 0 \tag{3.172}$$

This relation is the basement of the dynamic analysis. An important information obtained from this relation is the natural circular frequency:

$$\omega = 2\sqrt{\frac{k}{m}} \tag{3.173}$$

In a deterministic design the stiffness and mass parameters are perfectly known. The analysis of the structure is straightforward because it simply consists in introducing the known values of k and m in this relation. In a probabilistic analysis, k and m are both characterized by their respective probability density functions. The aim of such an analysis is to determine the probabilistic distribution of the circular frequency.

In this section the non Gaussiannity of random variables is studied. In this simple analytical approach it is supposed that k and m are uncorrelated uniformly distributed variables. Their respective probability density functions are given by:

$$p_k(k) = \begin{cases} 0 & \text{for } k \notin [a_k; b_k] \\ \frac{1}{b_k - a_k} & \text{for } k \in [a_k; b_k] \end{cases} \tag{3.174}$$

$$p_m(m) = \begin{cases} 0 & \text{for } m \notin [a_m; b_m] \\ \frac{1}{b_m - a_m} & \text{for } m \in [a_m; b_m] \end{cases} \tag{3.175}$$

In order to avoid this conditional representation, the unit step function U is introduced. The function[3] $U(x)$ is equal to 0 for $x < 0$ and 1 for $x \geq 0$. The given probability density functions

[3] Three major properties of this function are used in these developments:

It.	$\widehat{X}_1^* = \widehat{k}$	$\widehat{X}_2^* = \widehat{L}$	μ_G	σ_G	β	p_f
1	+0.0000	+0.0000	60.00	31.38	1.9118	$2.7954.10^{-2}$
2	+0.7309	+1.7665	54.19	25.31	2.1409	$1.6142.10^{-2}$
3	+0.7759	+1.9953	52.75	24.60	2.1441	$1.6013.10^{-2}$
4	+0.7676	+2.0020	52.73	24.59	2.1441	$1.6013.10^{-2}$

Table 3.7: Summarized results of the AFOSMC analysis

	Iteration	1	2	3
Step 1	$\widehat{X}_1^* = \widehat{k}$	+0.0000	+0.7309	+0.7759
	$\widehat{X}_2^* = \widehat{L}$	+0.0000	+1.7665	+1.9953
	k	150.00	114.67	110.09
	L	1.0000	0.83553	0.81821
Step 2	G	+60.000	+5.8106	$+7.9291.10^{-2}$
Step 3	$\frac{\partial \widehat{G}}{\partial \widehat{k}}$	−12.000	−9.1733	−8.8072
	$\frac{\partial \widehat{G}}{\partial \widehat{L}}$	−29.000	−23.591	−22.970
Step 4	α_1	−0.3823	−0.3624	−0.3580
	α_2	−0.9240	−0.9320	−0.9337
Step 5	μ_G	+60.000	54.189	+52.745
	σ_G	+31.385	+25.312	+24.600
Step 6	β	+1.9118	+2.1409	+2.1441
	p_f	$+2.7954.10^{-2}$	$+1.6142.10^{-2}$	$+1.6013.10^{-2}$
Step 1 – Guess design point –				
Step 2 – Compute Failure Condition –				
Step 3 – Compute the gradient of the Failure Condition –				
Step 4 – Compute the orientation of the next design point –				
Step 5 – Compute the mean and std of Failure Cond –				
Step 6 – Compute the reliability index and probability of failure –				

Table 3.8: Detailed results of the AFOSMC analysis

are thus expressed by:

$$p_k(k) = \frac{1}{b_k - a_k}\left[U(k-a_k) - U(k-b_k)\right] \tag{3.176}$$

$$p_m(m) = \frac{1}{b_m - a_m}\left[U(m-a_m) - U(m-b_m)\right] \tag{3.177}$$

Exactly as for the probabilistic analyses of the previous sections, the probabilistic characteristics of the unknown (the circular frequency in this case) are obtained by first establishing its cumulative distribution function, i.e.:

$$\begin{aligned} F_\omega(\omega_L) &= prob(\omega < \omega_L) \\ &= \int_{-\infty}^{+\infty} prob(m_0 < m < m_0 + dm_0) prob\left(2\sqrt{\frac{k}{m_0}} < \omega_L\right) dm_0 \\ &= \int_{-\infty}^{+\infty} p_m(m_0) F_k\left(\frac{m_0\omega_L^2}{4}\right) dm_0 \end{aligned} \tag{3.178}$$

The corresponding probability density function is obtained by simple derivation with respect to ω. By considering that $p_m(m_0)$ is equal to 0 outside $[a_m, b_m]$, the probability density function of the circular frequency is expressed by:

$$p_\omega(\omega) = \frac{dF_\omega(\omega)}{d\omega} = \frac{\omega}{2(b_m - a_m)} \int_{a_m}^{b_m} m_0 p_k\left(\frac{m_0\omega^2}{4}\right) dm_0$$

It is advantageous to use the change of variables $k = \frac{m_0\omega^2}{4}$ in order to simplify the integrand to:

$$p_\omega(\omega) = \frac{8}{\omega^3(b_m - a_m)} \int_{\frac{a_m\omega^2}{4}}^{\frac{b_m\omega^2}{4}} k p_k(k)\, dk \tag{3.179}$$

The introduction of the actual probability density function of k (Eq. (3.176)) in this relation leads after some developments to:

$$p_\omega(\omega) = \frac{8}{\omega^3(b_m - a_m)(b_k - a_k)}\left[\frac{k^2 - a_k^2}{2}U(k-a_k) - \frac{k^2 - b_k^2}{2}U(k-b_k)\right]_{k=\frac{a_m\omega^2}{4}}^{k=\frac{b_m\omega^2}{4}} \tag{3.180}$$

where the final variation operation has no been performed for the sake of compactness in the notations. It can be checked that this relation corresponds well to a probability density function ($\int_{-\infty}^{+\infty} p_\omega(\omega)\, d\omega = 1$). Furthermore, this relation indicates that the probability distribution of the circular frequency is neither uniform nor Gaussian.

A numerical example is illustrated in Fig. 3.12. It is a simple application of the previous analytical developments with:

$$\begin{aligned} a_k &= 180kN/m \quad ; \quad b_k = 220kN/m \\ a_m &= 30kg \quad ; \quad b_m = 70kg \end{aligned} \tag{3.181}$$

- $\int U(x-x_0)\, dx = (x-x_0)\, U(x-x_0) + cst$
- $\int (x-x_0)\, U(x-x_0)\, dx = \frac{(x-x_0)^2}{2} U(x-x_0) + cst$
- $\int xU(x-x_0)\, dx = \frac{x^2 - x_0^2}{2} U(x-x_0) + cst$

The uniform probability density function of k and m are represented on the top line of this figure. As a numerical application of Eq. (3.180), the probability density function of ω is also represented. It is composed of three pieces of parabola. The corresponding cumulative distribution function is also represented (lower right graph). Because the definition intervals of the input variables (k and m) are limited, the probability density function of ω is strictly equal to zero outside a finite interval. This important characteristic relates a significant difference compared to a simulation with Gaussian variables. More precisely in the context of a reliability analysis where it is desired to handle with small failure probabilities and hence tails of distributions, the representation of the physical variables in their tail regions is very important.

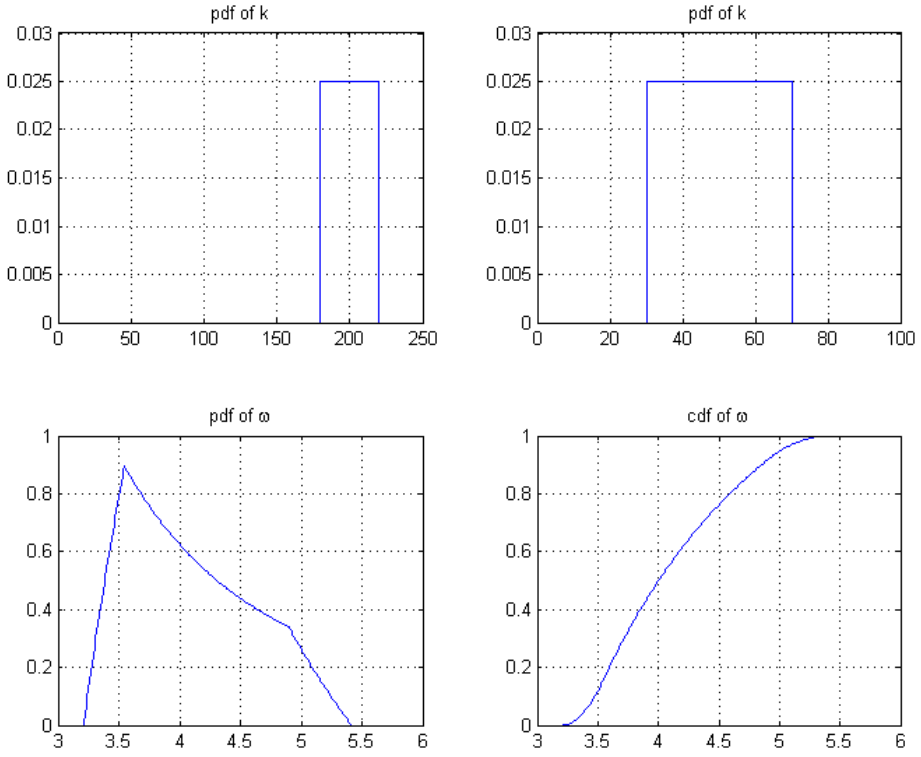

Figure 3.12: Numerical application of the probabilistic analysis (analytical developments) - Probability density functions of k, m and ω, and cumulative distribution function of ω.

In order to validate these analytical developments, a Monte Carlo simulation has been used in order to solve the same problem. A set of $N = 100000$ couples (k, m) is generated in order to match their respective probability density functions. For each set the corresponding circular frequency is computed and finally the histogram (using 31 bins) of this series of N values is computed. It is represented in Fig. 3.13 together with the corresponding cumulative distribution function. The good correspondence with the results of Fig. 3.12 validates the previous analytical developments of the probabilistic analysis.

In order to compare with forthcoming results obtained with a reliability analysis, the probability that the circular frequency is smaller than $\omega_L = 3.3$ can be computed (from Eq. (3.178)):

$$p_f = prob\left(\omega < \omega_L\right) = 0.01283 \tag{3.182}$$

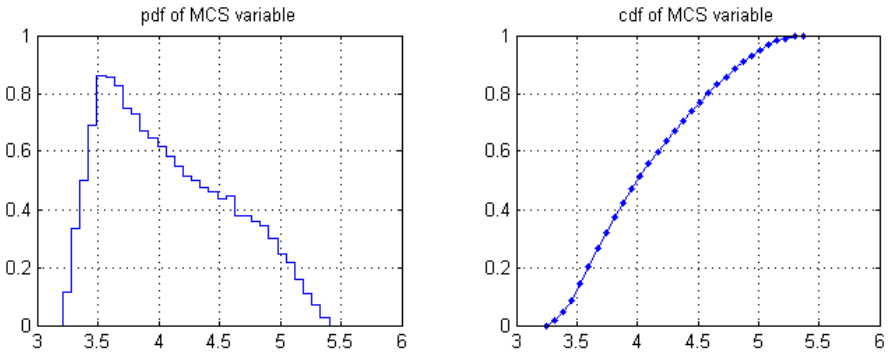

Figure 3.13: Results of the Monte Carlo Simulation ($N = 100000$ samples)

3.3.2 Reliability analysis: illustration of numerical resolution (FOGSM)

The same problem as in the previous section is now considered by means of a reliability analysis. Since the purpose of a structural design usually consists is checking that the circular frequency is larger than a critical value (contrarily to displacement and internal forces which have to be smaller than design values), the considered failure function is:

$$G(k,m) = 2\sqrt{\frac{k}{m}} - \omega_L \tag{3.183}$$

in which the limit circular frequency is supposed to be deterministically given ($\omega_L = 3.3 rad/s$). The structural stiffness k and mass m are supposed to be uniformly distributed with the same numerical parameters as in the previous section (Eq.(3.181)). The reliability analysis is first performed with the FOGSM and then with the FOGAM (next section).

For this application the algorithm of the FOGSM method is applied (Alg. 13, p. 27).

Step 0.1: The first step consists in computing the transformation functions, from the physical variables to the reduced (i.e. uncorrelated gaussian) variables. Since the physical variables are here uniformly distributed, these functions are simply expressed by:

$$\widehat{k} = \Phi^{-1}\left(\frac{k-a_k}{b_k-a_k}\right) = \Phi^{-1}\left(\frac{k-180}{40}\right) \equiv T_k(k) \tag{3.184}$$

$$\widehat{m} = \Phi^{-1}\left(\frac{m-a_m}{b_m-a_m}\right) = \Phi^{-1}\left(\frac{m-30}{40}\right) \equiv T_m(m) \tag{3.185}$$

In a hand calculation, they can be estimated from tabulated values of the normal cumulative density function. Table 3.9 gives the values of the transformation function for some selected values of k and m.

The inverse transformation relations are:

$$k = a_k + (b_k - a_k)\,\Phi\left(\widehat{k}\right) \tag{3.186}$$

$$m = a_m + (b_m - a_m)\,\Phi\left(\widehat{m}\right) \tag{3.187}$$

They are useful for the transformation of the failure function to its reduced form:

$$\widehat{G}(\widehat{k},\widehat{m}) = 2\sqrt{\frac{a_k + (b_k - a_k)\,\Phi\left(\widehat{k}\right)}{a_m + (b_m - a_m)\,\Phi\left(\widehat{m}\right)}} - \omega_L \tag{3.188}$$

This function is represented in Fig. 3.14 by its level curves. In the physical space (left side) the level curves are straight. The non linear transformation transforms them to non linear level

k	m	$T_k(k)$, $T_m(m)$	k	m	$T_k(k)$, $T_m(m)$
180	30	$-\infty$	202	52	0.1257
182	32	-1.6449	204	54	0.2533
184	34	-1.2816	206	52	0.3853
186	36	-1.0364	208	58	0.5244
188	38	-0.8416	210	60	0.6745
190	40	-0.6745	212	62	0.8416
192	42	-0.5244	214	64	1.0364
194	44	-0.3853	216	66	1.2816
196	46	-0.2533	218	68	1.6449
198	48	-0.1257	220	70	$+\infty$
200	50	0			

Table 3.9: Transformation functions from uniform to Gaussian distributions

curves in the reduced space (right side). As usual the most important level curve is the failure condition represented by thick lines. The explicit expression of the reduced failure function and it representation by level curves is not really necessary for the computation but provides a good illustration of the method. On this figure, the joint probability density function of k and m are represented by a rectangular unit step (its level curves are located on the limits of the graph) and a gaussian distribution, as usual, in the reduced space (concentric circles) This kind of graphical illustration has to be considered as an extra (non-required) information.

Step 1: After this preliminary step the complete information about the reduced space is known. The iterations start with the guess of an initial design point. It is chosen as the mean physical variable, i.e.

$$\begin{aligned} k^{(0)} &= 200 \;\; ; \;\; \widehat{k}^{(0)} = 0 \\ m^{(0)} &= 50 \;\; ; \;\; \widehat{m}^{(0)} = 0 \end{aligned} \tag{3.189}$$

Step 2: This point does not lie of the failure condition. The failure condition (or equivalently the reduced failure condition) is equal to:

$$\begin{aligned} G(k,m) &= 2\sqrt{\frac{k}{m}} - \omega_L = 2\sqrt{\frac{200}{50}} - 3.3 = 0.7 \\ \widehat{G}(\widehat{k},\widehat{m}) &= 2\sqrt{\frac{180 + 40\Phi(0)}{30 + 40\Phi(0)}} - 3.3 = 0.7 \end{aligned} \tag{3.190}$$

Step 3: The third step consist in computing the gradient of the reduced failure function at the design point. In this case these derivatives can be computed in closed form:

$$\frac{\partial \widehat{G}}{\partial \widehat{k}} = \frac{(b_k - a_k)\,\phi\left(\widehat{k}\right)}{\sqrt{\left(a_k + (b_k - a_k)\,\Phi\left(\widehat{k}\right)\right)\left(a_m + (b_m - a_m)\,\Phi(\widehat{m})\right)}} \tag{3.191}$$

$$\frac{\partial \widehat{G}}{\partial \widehat{m}} = -\sqrt{\frac{a_k + (b_k - a_k)\,\Phi\left(\widehat{k}\right)}{\left(a_m + (b_m - a_m)\,\Phi(\widehat{m})\right)^3}}\,(b_m - a_m)\,\phi(\widehat{m}) \tag{3.192}$$

where ϕ represents the standard normal probability density function ($\phi(t) = \exp\left[-t^2/2\right]/\sqrt{2\pi}$).

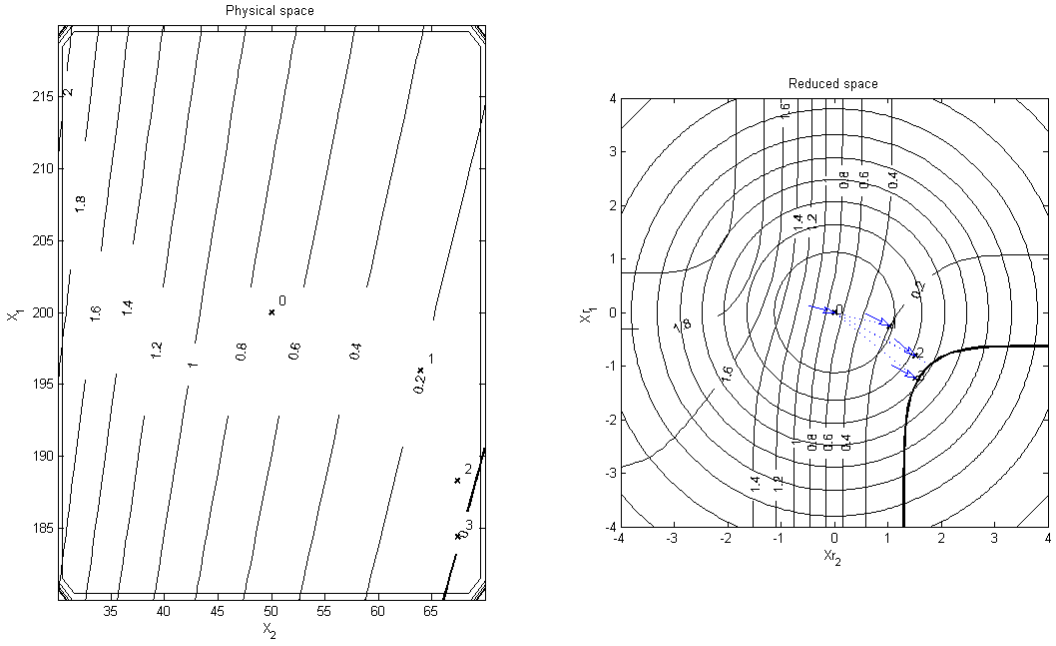

Figure 3.14: Illustration of the FOGSM method

The estimation of these derivatives at the design point gives[4]:

$$\frac{\partial \widehat{G}}{\partial \widehat{k}} = \frac{40\frac{1}{\sqrt{2\pi}}}{\sqrt{(180+40*0.5)(30+40*0.5)}} = \frac{2}{\sqrt{50\pi}} = 0.1596 \tag{3.193}$$

$$\frac{\partial \widehat{G}}{\partial \widehat{m}} = -\sqrt{\frac{180+40*0.5}{(30+40*0.5)^3}}40\frac{1}{\sqrt{2\pi}} = -\frac{4}{5}\sqrt{\frac{2}{\pi}} = -0.6383 \tag{3.194}$$

[4] This example shows that the computation of the gradient of the reduced failure function is not necessarily easy. In more complex applications, the explicit establishment of the reduced failure function itself ($\widehat{G}$) is more complex. Its gradient can be computed by decomposing the derivations as:

$$\frac{\partial \widehat{G}}{\partial \widehat{k}} = \frac{\partial G\left(k\left(\widehat{k}\right), m\left(\widehat{m}\right)\right)}{\partial \widehat{k}} = \left.\frac{\partial G(k,m)}{\partial k}\right|_{\substack{k=k\left(\widehat{k}\right)\\ m=m\left(\widehat{m}\right)}} \frac{\partial k\left(\widehat{k}\right)}{\partial \widehat{k}}$$

In this relation the derivative of G with respect to k is supposed to be computable (it depends on the initial failure function only). Concerning the second derivative, it could be more complex. Indeed, in the most general case, $k\left(\widehat{k}\right) = F_k^{-1}\left(\Phi\left(\widehat{k}\right)\right)$, in which F_k^{-1} could not have an explicit form. The derivation can however be simplified by the following considerations:

$$\begin{aligned}\frac{\partial k\left(\widehat{k}\right)}{\partial \widehat{k}} &= \frac{\partial F_k^{-1}\left(\Phi\left(\widehat{k}\right)\right)}{\partial \widehat{k}} = \left.\frac{\partial F_k^{-1}(c)}{\partial c}\right|_{c=\Phi\left(\widehat{k}\right)} \frac{\partial \Phi\left(\widehat{k}\right)}{\partial \widehat{k}} \\ &= \frac{\phi\left(\widehat{k}\right)}{\left.\frac{\partial F_k(k)}{\partial k}\right|_{k=F_k^{-1}\left(\Phi\left(\widehat{k}\right)\right)}} = \frac{\phi\left(\widehat{k}\right)}{p_k\left(F_k^{-1}\left(\Phi\left(\widehat{k}\right)\right)\right)}\end{aligned}$$

These considerations are useful in the context of hand calculations, since the derivative is expressed as a function of accessible quantities. In the worst case scenario, F_k^{-1} has to be found from tabulated values.

Step 4: The orientation for the next design point is obtained by:

$$\alpha_1^{(0)} = \frac{n_1^{(0)}}{\left|n^{(0)}\right|} = \frac{0.1596}{\sqrt{0.1596^2 + (-0.6383)^2}} = 0.2425$$

$$\alpha_2^{(0)} = \frac{n_2^{(0)}}{\left|n^{(0)}\right|} = \frac{-0.6383}{\sqrt{0.1596^2 + (-0.6383)^2}} = -0.9701$$

Step 5: The distance of this new point from the origin requires the estimation of the mean and standard deviation of the failure function:

$$\mu_G = \widehat{G}^{(0)} - n_1^{(0)}\widehat{k}^{(0)} - n_2^{(0)}\widehat{m}^{(0)} = -0.7 \tag{3.195}$$

$$\sigma_G = \left|n^{(0)}\right| = 0.65795 \tag{3.196}$$

Step 6: The next reliability index is obtained by:

$$\beta^{(0)} = \frac{\mu_G}{\sigma_G} = \frac{-0.7}{0.65795} = 1.0639$$

Since the reduced failure condition is not linear in this case, the related probability ($p_f = \Phi^{-1}(-\beta) = 0.1437$) of failure is not correct. Of course it can be computed its value should not be used.

Step 7: Finally the new estimation of the design point is obtained by:

$$\widehat{x}^{(1)} = -\alpha^{(0)}\beta^{(0)} = (-0.2580, 1.0321) = \left(\widehat{k}^{(1)}, \widehat{m}^{(1)}\right) \tag{3.197}$$

The iterative procedure requires to start back again at step 1 and repeat the operation until convergence. Tables 3.10 and 3.11 give summarized and detailed results of the application of the FOGSM method. They indicate that the convergence seems to be very slow in this application: six iterations are not sufficient for the stabilization of the reliability index.

It.	$\widehat{X}_1^* = \widehat{k}$	$\widehat{X}_2^* = \widehat{m}$	μ_G	σ_G	β	p_f
1	$+0.0000$	$+0.0000$	0.7000	0.6580	1.0639	0.14369
2	-0.2580	$+1.0321$	0.5006	0.2911	1.7199	$4.2727.10^{-2}$
3	-0.8147	$+1.5147$	0.3160	0.1616	1.9554	$2.5271.10^{-2}$
4	-1.2297	$+1.5203$	0.2771	0.1403	1.9743	$2.4173.10^{-2}$
5	-0.9453	$+1.7333$	0.2418	0.1247	1.9392	$2.6240.10^{-2}$
6	-1.4046	$+1.3370$	0.3170	0.1721	1.8419	$3.2741.10^{-2}$

Table 3.10: Summarized results of the FOGSM analysis

Fig. 3.14 provides also an overview of the results of the FOGSM method. The locations of the reduced and physical variables are represented for the first three iterations. As usual the small arrows represent the gradient of the reduced failure function at the considered point. Fig. 3.15 presents the same results up to the 30^{th} iteration. The successive positions of the reduced design point are represented by red squared and linked together with red lines. This figure illustrates the lack of convergence of the FOGSM method. This typical divergence is reported in many reference books and is comparable to the divergence of the Newton-Raphson iterative method in the vicinity of inflexion points.

As a solution to this problem, the design point could be considered as the point lying the closest to the reduced failure condition (minimum of $\left|\widehat{G}\right|$), i.e. point 4 in this application for which

	Iteration	1	2	3
Step 1	$\widehat{X}_1^* = \widehat{k}$	$+0.0000$	-0.2580	-0.8147
	$\widehat{X}_2^* = \widehat{m}$	$+0.0000$	$+1.0321$	$+1.5147$
	k	200.00	195.93	188.30
	m	50.000	63.960	67.403
Step 2	G	0.7000	0.20045	0.04289
Step 3	$\frac{\partial \widehat{G}}{\partial \widehat{k}}$	$+0.15958$	$+0.13788$	$+0.10164$
	$\frac{\partial \widehat{G}}{\partial \widehat{m}}$	-0.63831	-0.25635	-0.12566
Step 4	α_1	$+0.2425$	$+0.4737$	$+0.6289$
	α_2	-0.9701	-0.8807	-0.7775
Step 5	μ_G	0.7000	0.50061	0.31603
	σ_G	0.65795	0.29108	0.16162
Step 6	β	$+1.0639$	$+1.7199$	$+1.9554$
	p_f	$1.4369e-001$	$4.2727.10^{-2}$	$2.5271.10^{-2}$

Step 1 – Guess design point –

Step 2 – Compute Failure Condition –

Step 3 – Compute the gradient of the Failure Condition –

Step 4 – Compute the orientation of the next design point –

Step 5 – Compute the mean and std of Failure Cond –

Step 6 – Compute the reliability index and probability of failure –

Table 3.11: Detailed results of the FOGSM analysis

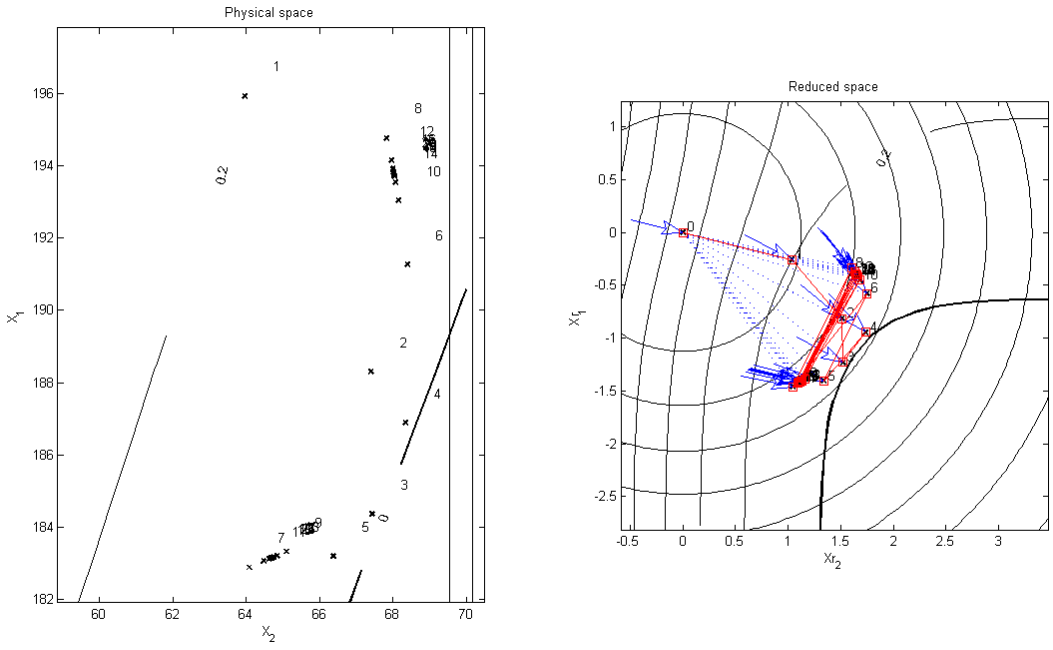

Figure 3.15: Illustration of the divergence of the FOGSM algorithm

$\widehat{G} = 7.1354.10^{-3}$. The corresponding probability of failure is equal to $p_f = 2.4173.10^{-2}$ and has to be compared to the exact probability of failure $p_f = 0.01283$ (Eq. (3.182)). The discrepancies between both results is due to the non convergence of the iterative procedure and to the non linearity of the reduced failure condition[5].

3.3.3 Reliability analysis: illustration of numerical resolution (FOGAM)

The same problem is again considered but this time with the FOGAM method. The considered algorith consists in replacing the actual probability density function by equivalent Gaussian ones (Alg. 14, p. 28). The details for the first step are given in the following:

Step 0.a: The first step consists in providing a first guess for the design point. As in many other method, the mean physical variables are considered:

$$k^{(0)} = 200 \quad ; \quad m^{(0)} = 50 \tag{3.198}$$

Step 0.b: The transformation to the reduced space can't be realized directly because the mean and variances of equivalent gaussian processes have to be determined first. The equivalence consists in imposing the same probability density function and the same cumulative distribution function at the design point, i.e. find the values of $\mu_1^{(0)}$, $\mu_2^{(0)}$, $\sigma_1^{(0)}$ and $\sigma_2^{(0)}$ such that:

$$\begin{aligned}
p_{N\left(\mu_1^{(0)},\sigma_1^{(0)}\right)}\left(k^{(0)}\right) &= p_k\left(k^{(0)}\right) = 0.025 \\
F_{N\left(\mu_1^{(0)},\sigma_1^{(0)}\right)}\left(k^{(0)}\right) &= F_k\left(k^{(0)}\right) = 0.5 \\
p_{N\left(\mu_2^{(0)},\sigma_2^{(0)}\right)}\left(m^{(0)}\right) &= p_m\left(m^{(0)}\right) = 0.025 \\
F_{N\left(\mu_2^{(0)},\sigma_2^{(0)}\right)}\left(m^{(0)}\right) &= F_m\left(m^{(0)}\right) = 0.5
\end{aligned} \tag{3.199}$$

The solution of this system of equations is:

$$\begin{aligned}
\mu_1^{(0)} &= 200 \quad ; \quad \sigma_1^{(0)} = 15.906 \\
\mu_2^{(0)} &= 50 \quad ; \quad \sigma_2^{(0)} = 15.906
\end{aligned} \tag{3.200}$$

The actual non gaussian variables are then replaced by these equivalent ones. This is illustrated at Fig. 3.16. In the top left corner the joint probability function between k and m is schematically represented by concentric rectangles. The joint probability function of the equivalent variables is represented by the dashed lines (concentric ellipses).

Step 0.1: The reduced variables are then defined and computed by:

$$\widehat{k} = \frac{k - \mu_1^{(0)}}{\sigma_1^{(0)}} = 0 \tag{3.201}$$

$$\widehat{m} = \frac{m - \mu_2^{(0)}}{\sigma_2^{(0)}} = 0 \tag{3.202}$$

This point is represented in the lower left corner of figure 3.16 (cross and label "0").

Step 0.2: The failure condition is accordingly transformed. From its representation by level curves in the physical space, its transformation to the reduced space (also represented by level curves in Fig. 3.16) is straightforward. Since the transformation is linear, the linear failure condition remains linear. Its formal expression is:

$$\widehat{G}^{(0)} = 2\sqrt{\frac{15.906\widehat{k} + 200}{15.906\widehat{m} + 50}} - 3.3 \tag{3.203}$$

[5] In this problem, the estimation of the probability of failure, which is finally the aim of the analysis is easier to be performed in the initial space. Indeed the exact probability of failure can be computed from the simple calculation of the surface of the triangle under the failure condition.

Step 1: The reduced design point has already been computed (step 0.1) and,
Step 2: the reduced failure function at this point is equal to:

$$\widehat{G}^{(0)}\left(\widehat{k},\widehat{m}\right)=0.7 \tag{3.204}$$

Step 3: Thanks to the simplicity of the transformation, the gradient of the failure function can be computed easily:

$$\begin{aligned} n_1^{(0)} &= \frac{\partial\widehat{G}^{(0)}}{\partial\widehat{k}}=\sqrt{\frac{15.906\widehat{m}+50}{15.906\widehat{k}+200}}\frac{15.906}{15.906\widehat{m}+50} \\ n_2^{(0)} &= \frac{\partial\widehat{G}^{(0)}}{\partial\widehat{m}}=-15.906\sqrt{\frac{15.906\widehat{m}+50}{15.906\widehat{k}+200}}\frac{15.906\widehat{k}+200}{\left(15.906\widehat{m}+50\right)^2} \end{aligned} \tag{3.205}$$

and its estimation at the design point is:

$$\begin{aligned} n_1^{(0)}\left(\widehat{k},\widehat{m}\right) &= \frac{\partial\widehat{G}^{(0)}}{\partial\widehat{k}}=\frac{15.906}{100}=0.15906 \\ n_2^{(0)}\left(\widehat{k},\widehat{m}\right) &= \frac{\partial\widehat{G}^{(0)}}{\partial\widehat{m}}=--0.63623 \end{aligned} \tag{3.206}$$

Step 4: Exactly as for the AFOSM method the new orientation of the design point is estimated from the knowledge of the gradient:

$$\begin{aligned} \alpha_1^{(0)} &= \frac{n_1^{(0)}}{\left|n^{(0)}\right|}=\frac{0.15906}{0.65581}=0.2425 \\ \alpha_2^{(0)} &= \frac{n_2^{(0)}}{\left|n^{(0)}\right|}=\frac{-0.63623}{0.65581}=-0.9701 \end{aligned}$$

Step 5: The estimated mean and standard deviation of the failure condition are given by:

$$\mu_G = \widehat{G}^{(0)}-\sum_{i=1}^{N}n_i^{(0)}x_i^{(0)}=0.7 \tag{3.207}$$

$$\sigma_G = \sum_{i=1}^{N}\alpha_i^{(0)}n_i^{(0)}=\left|n^{(0)}\right|=0.65581 \tag{3.208}$$

Step 6: These values lead finally to the new reliability index:

$$\beta^{(0)}=\frac{\mu_G}{\sigma_G}=1.0674 \tag{3.209}$$

Step 7: ... and to the estimation of the next design point:

$$\widehat{x}^{(1)}=-\alpha^{(0)}\beta^{(0)}=-\left(0.2425;-0.9701\right)1.0674=\left(-0.2610;1.3517\right) \tag{3.210}$$

Because the reduced space is continuously changing (from an iteration to the next one), the design point has to be computed, for the FOGAM method, in the physical space:

$$\begin{aligned} k^{(0)} &= 15.906\widehat{k}+200=195.88 \\ m^{(0)} &= 15.906\widehat{m}+50=66.471 \end{aligned} \tag{3.211}$$

This point is represented by the label '1' in the physical space as well as in the reduced space. Now the iterations can be repeated, by defining at each iteration a new transformation, i.e. a new reduced space and new reduced failure conditions. Fig. 3.16 illustrate the results obtained for the first few iterations. It can be observed that the failure condition is continuously changing (different level curves) but remains linear. Also the equivalent gaussian variables are also continuously changing. This is indicated by the different joint probability density functions at each iteration (concentric ellipses in the physical space).

Tables 3.12 and 3.13 give thhe summarized and detailed results of the application of this algorithm. They inidcate that the FOGAM is also not suitable for the estimation of the probability of failure. Indeed the summarized results indicate a divergent behaviour of the method. Actually, it could be checked that the location of the design point in the physical space is obtained quite accurately. The lack of convergence comes from the variability, from an iteration to the next one, in the mean and standard deviations of the equivalent variables. Then even if the process is convergent in the physical space, and since the reduced space is continuously changing, the iterative resolution can be unstable in the reduced space. Because the reliability index is defined in the reduce space, this divergent phenomenon leads to the impossibility to estimation correctly the probability of failure.

It.	$\widehat{X}_1^* = \widehat{k}$	$\widehat{X}_2^* = \widehat{m}$	μ_G	σ_G	β	p_f
1	$+0.0000$	$+0.0000$	0.7000	0.6558	1.0674	0.14290
2	-0.2610	$+1.3517$	0.3918	0.2130	1.8388	3.297210^{-2}
3	-1.6270	$+1.4302$	0.2630	0.1465	1.7953	3.630510^{-2}
4	-0.9366	$+1.4977$	6.06610^{-2}	0.09226	0.6575	0.25544
5	-0.6778	$+3.5282$	5.43710^{-2}	0.1103	0.4929	0.31106
6	-0.5022	$+3.3566$	$1.037e+000$	0.3150	3.2928	4.959410^{-4}

Table 3.12: Summarized results of the FOGAM analysis

	Iteration	1	2	3
Step 1	$\widehat{X}_1^* = \widehat{k}$	$+0.0000$	-0.2610	-1.6270
	$\widehat{X}_2^* = \widehat{m}$	$+0.0000$	$+1.3517$	$+1.4302$
	k	$+200.00$	$+195.88$	$+182.07$
	m	$+50.000$	$+66.471$	$+66.947$
Step 2	G	$+0.7.000$	$+0.13331$	-1.706210^{-3}
Step 3	$\frac{\partial \widehat{G}}{\partial \widehat{k}}$	$+0.15906$	$+0.13449$	$+3.842310^{-2}$
	$\frac{\partial \widehat{G}}{\partial \widehat{m}}$	-0.63623	-0.16523	-0.14135
Step 4	α_1	$+0.2425$	$+0.6313$	$+0.2623$
	α_2	$+0.9843$	$+0.9986$	$+0.3832$
Step 5	μ_G	$+0.70000$	$+0.39176$	$+0.26297$
	σ_G	$+0.65581$	$+0.21305$	$+0.14648$
Step 6	β	$+1.0674$	$+1.8388$	$+1.7953$
	p_f	$+0.14290$	$+3.297210^{-2}$	$+3.630510^{-2}$
Step 1 – Guess design point –				
Step 2 – Compute Failure Condition –				
Step 3 – Compute the gradient of the Failure Condition –				
Step 4 – Compute the orientation of the next design point –				
Step 5 – Compute the mean and std of Failure Cond –				
Step 6 – Compute the reliability index and probability of failure –				

Table 3.13: Detailed results of the FOGAM analysis

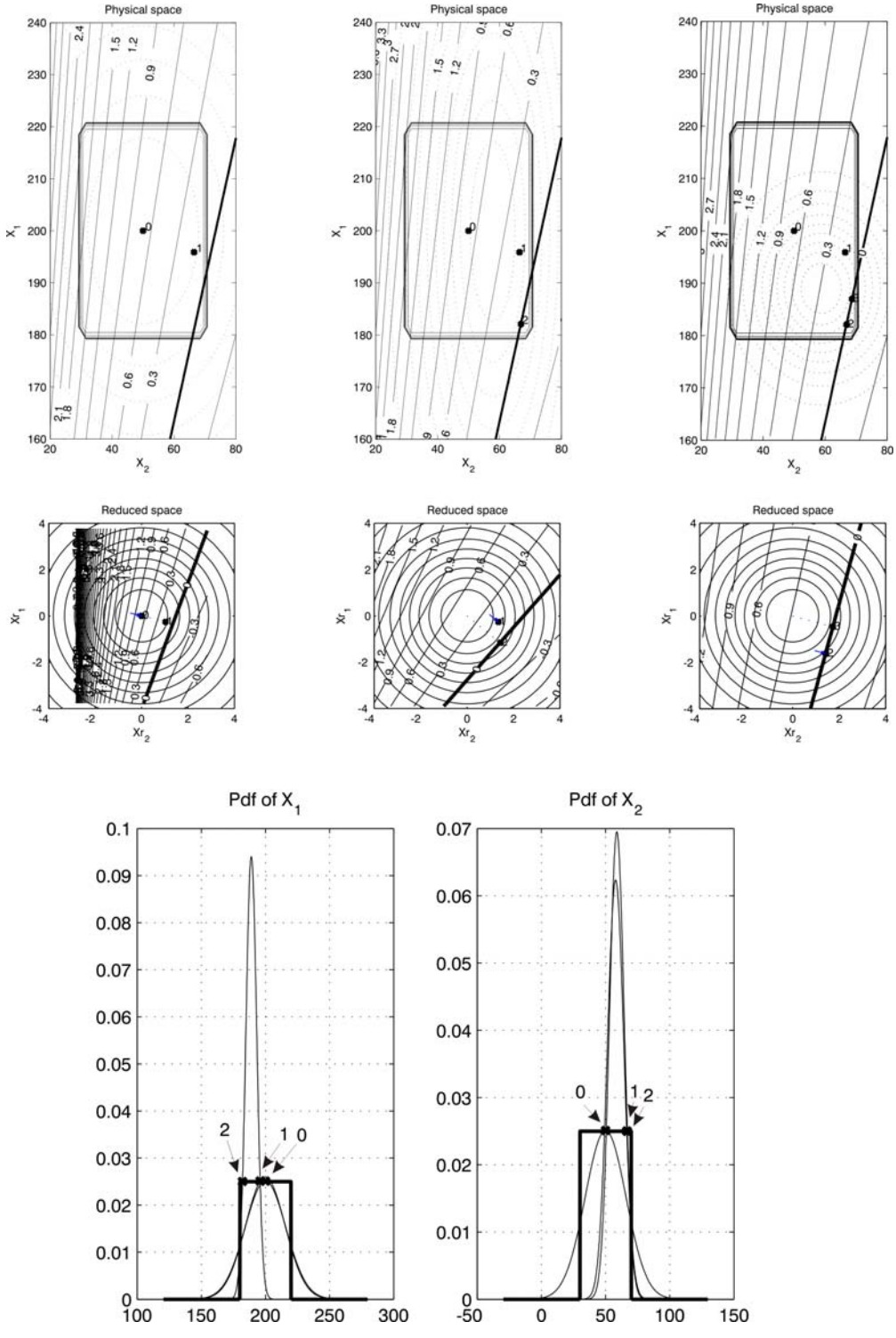

Figure 3.16: Illustration of the FOGAM method

Appendix A

Computer Programs

A.1 AFOSMC

A.1.1 Call to AFOSMC

```
mX = [150 100];
sX = [ 20 30];
r = 0.4;
covX = [sX(1)^2 r*sX(1)*sX(2); r*sX(1)*sX(2) sX(2)^2];

GFct = @FailureCond_Ex1_1;
[Beta, X, XRed, G, GrRed, Alfa, PrFail, MeanG, StdG] = AFOSMC (GFct, mX, covX);
```

A.1.2 AFOSMC subroutine

```
function [Beta,X,XRed,G,GrRed,Alfa,PrFail,MeanG,StdG] = ...
     AFOSMC (GFct, mX, covX,iStart)
% [Beta, X, XRed, G, GrRed, Alfa] = AFOSMC (GFct, mX, covX)
% Returns the results of the AFOSMC (correlated variables)
%
% [Beta, X, XRed, G, GrRed, Alfa] = AFOSM (GFct, mX, covX,iStart) allows
% choosing the starting point (Default:0)
%   iStart = 0: mean variable
%   iStart = 1: like Kusama (beta=3, alfa=(1,1,...,1)/sqrt(N);

if nargin <4; iStart=0;end

Nvar = length(mX);

ep = 1e-6;
NbIter = 5;

Beta= zeros(NbIter,1); X   = zeros(NbIter,Nvar); XRed= zeros(NbIter,Nvar);
G   = zeros(NbIter,1); GrRed=zeros(NbIter,Nvar); Alfa =zeros(NbIter,Nvar);
PrFail=zeros(NbIter,1);MeanG= zeros(NbIter,1);   StdG = zeros(NbIter,1);
grad = zeros(1,Nvar);

switch iStart
    case 0
        beta = 0;
```

```
        alfa = zeros(1,Nvar);
    case 1
        beta = 3;
        alfa = sqrt(ones(1,Nvar)/Nvar);
end

[V, D] = eig(covX); A = (V*sqrt(inv(D)))'; invA=inv(A); % A*covX*A'=I
for icpt = 1:NbIter

    Xs_red = -beta*alfa;
    Xs = Xs_red*invA' + mX;
    GXs = GFct(Xs);

    for i=1:Nvar
        X_red = Xs_red; X_red(i)=X_red(i)+ep;
        X_ = X_red*invA' + mX;
        grad(i) = (GFct(X_)-GXs)/ep;
    end

    SumSq = sqrt(sum(grad.^2));
    alfa = grad / SumSq;

    mZ = GXs - sum(grad.*Xs_red);
    sZ = SumSq;

    beta = mZ/sZ;

    PrFail(icpt) = 1-erfc(-beta/sqrt(2))/2;


    XRed(icpt,:)=Xs_red; X(icpt,:)= Xs; G(icpt)=GXs;
    GrRed(icpt,:)=grad; Alfa(icpt,:)=alfa; Beta(icpt)=beta;
    MeanG(icpt)=mZ; StdG(icpt)=sZ;

end
```

A.2 FOGSM

A.2.1 Call to FOGSM

```
P = [150 30; 100 30];
Typ = {'Ext_Typ.I', 'Ext_Typ.I'};
GFct = @FailureCond_Ex1_1;
[Beta,X,XRed,G,GrRed,Alfa,PrFail,MeanG,StdG] = FOGSM (GFct, P, Typ);
```

A.2.2 FOGSM subroutine

```
function [Beta,X,XRed,G,GrRed,Alfa,PrFail,MeanG,StdG] = FOGSM (GFct, P, Typ)
% [Beta, X, XRed, G, GrRed, Alfa, PrFail, MeanG, StdG] = FOGSM (GFct, P, Typ)
% Returns the results of the FOGSM (uncorrelated variables)
% First Order Gaussian Second Moment
if nargin <4; iStart=0;end
Nvar = length(Typ);
```

```
ep = 1e-6;
NbIter = 6;
Beta= zeros(NbIter,1); X   = zeros(NbIter,Nvar); XRed= zeros(NbIter,Nvar);
G   = zeros(NbIter,1); GrRed=zeros(NbIter,Nvar); Alfa =zeros(NbIter,Nvar);
PrFail=zeros(NbIter,1);MeanG= zeros(NbIter,1);   StdG = zeros(NbIter,1);
grad = zeros(1,Nvar);
switch iStart
    case 0
        beta = 0;
        alfa = zeros(1,Nvar);
    case 1
        beta = 3;
        alfa = sqrt(ones(1,Nvar)/Nvar);
end

for icpt = 1:NbIter

    Xs_red = -beta*alfa;

    Xs=FOGSM_InvTransf(Xs_red,P,Typ);
    GXs = GFct(Xs);

    for i=1:Nvar
        X_red = Xs_red; X_red(i)=X_red(i)+ep;
        X_ = FOGSM_InvTransf(X_red,P,Typ);
        grad(i) = (GFct(X_)-GXs)/ep;
    end

    SumSq = sqrt(sum(grad.^2));
    alfa = grad / SumSq;

    mZ = GXs - sum(grad.*Xs_red);
    sZ = SumSq;

    beta = mZ/sZ;

    PrFail(icpt) = 1-erfc(-beta/sqrt(2))/2;


    XRed(icpt,:)=Xs_red; X(icpt,:)= Xs; G(icpt)=GXs;
    GrRed(icpt,:)=grad; Alfa(icpt,:)=alfa; Beta(icpt)=beta;
    MeanG(icpt)=mZ; StdG(icpt)=sZ;

end
```

A.2.3 FOGSM_InvTransf subroutine

```
function X=FOGSM_InvTransf(XRed,P,Typ)
% X=FOGSM_InvTransf(XRed,P,Typ)
% Variables X are distributed as described by Typ, P and NbP
% XRed is normally distributed
%   SEE ALSO: FOGSM_Transf

for t=1:length(XRed)
```

```
    switch char(Typ(t))
        case 'Ext_Typ.I'
            mu=P(t,1); sigma = P(t,2);
            X(t) = evinv(normcdf(XRed(t),0,1),mu,sigma);
        case 'Uniform'
            a=P(t,1); b = P(t,2);
            X(t) = unifinv(normcdf(XRed(t),0,1),a,b);
    end
end
```

A.3 Monte Carlo (non Gaussian)

A.3.1 Call to MCS_NG

```
P = [150 30; 100 30];
Typ = {'Ext_Typ.I', 'Ext_Typ.I'};
GFct = @FailureCond_Ex1_1;
N = 1e5;
MCS_NG(GFct, P, Typ);
```

A.3.2 MCS_NG subroutine

```
function MCS_NG (G, P, Typ, N, Nbins)
% MCS_NG (G, P, Typ)
%   Compute the probability density function of the scalar result of G
%   G is a function of several random variables described by their own
%   distribution (Typ) and parameters (P)
% MCS_NG (G, P, Typ, N)
%   The size of the sample can be given. Default value: 10000
% MCS_NG (G, P, Typ, N, Nbins)
%   The number of bins can be given. Default value: sqrt(N)/10
if nargin <4; N = 100000; end
if nargin <5; Nbins = floor(sqrt(N)/10); end

Nvar = length(Typ);

if isa(G, 'function_handle')   % Verify that G is a function handle.
    disp (' ')
    disp ('MONTE CARLO method')
    disp ('------------------')

    x=rand(N,Nvar); X=zeros(N,Nvar);
    for t=1:Nvar
        switch char(Typ(t))
            case 'Ext_Typ.I'
                mu=P(t,1); sigma = P(t,2);
                X(:,t)=evinv(x(:,t),mu,sigma);
            case 'Uniform'
                a = P(t,1); b=P(t,2);
                X(:,t)=unifinv(x(:,t),a,b);
            case 'Normal'
                moy = P(t,1); sig=P(t,2);
                X(:,t)=norminv(x(:,t),moy,sig);
```

```
            case 'Beta'
                p1 = P(t,1); p2=P(t,2);
                X(:,t)=betainv(x(:,t),p1,p2);
        end

    end

    g = G(X);

    figure;
    [nn,xx] = histv(g,Nbins);
    subplot(1,2,1); plot(xx,nn); title ('pdf of MCS variable'); grid on
    [n,x] = hist(g,Nbins);
    F_g = cumtrapz(x,n); F_g=F_g/max(F_g);
    subplot(1,2,2); plot(x,F_g); title ('cdf of MCS variable'); grid on

    M = MomentsStatistiques(g,4);
    disp (['  --> Mean    : ' num2str(M(1))])
    disp (['      StandDev: ' num2str(M(2))])
    disp (['      Skewness: ' num2str(M(3))])
    disp (['      kurtosis: ' num2str(M(4))])
else
    disp ('G is not a function handle!')
end
```

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