Implementation of the Multifractal Formalism on S^ν Spaces

Thomas Kleyntssens
tkleyntssens@ulg.ac.be

Joint work with Samuel Nicolay

Université de Liège

BMS

Introduction

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- We determine the "size" of the set of points which share the same "irregularity" ;
- In practice, we use a numerically computable function which "approximates" this size.

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   - Example : Thermodynamic Multifractal Formalism
3. **S^{ν} Spaces**
   - Definitions
   - Properties
   - Multifractal Formalism on S^{ν} Spaces
4. **Implementation of the Multifractal Formalism on S^{ν} Spaces**
   - In practice
   - Examples : Monofractal Functions
   - Examples : Multifractal Functions
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Definition (Hölder continuous)

Let $t \in \mathbb{R}$, $s \in \mathbb{R}_0^+$ and $f \in L^\infty_{loc}$. We denote $f \in C^s(t)$ if there exist a polynomial P of degree stricly smaller than s, a constant $C > 0$ and a neighbourhood Ω of 0 such that

$$|f(t+l) - P(l)| \leq C|l|^s$$

for all $l \in \Omega$, . If $s \in]0;1]$, the condition is equivalent to

$$|f(t+l) - f(t)| \leq C|l|^s.$$

Definition (Holder exponent)

Let $t \in \mathbb{R}$ and $f \in L^\infty_{loc}$; we denote the **Hölder exponent** of f at point t by

$$h_f(t) = \sup\{s \in \mathbb{R}_0^+ : f \in C^s(t)\}$$

Proposition

- If $f \in C^s(t)$ then f is continuous at point t.
- If $f \in C^s(t)$ with $s > 1$ then f is differentiable at point t and $f'(t) = P'(0)$.
- If f is differentiable at point t then $f \in C^1(t)$.

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Conclusion

- If $h_f(t) \in]0;1[$ then f is continous but not differentiable at point t.
- So f is very irregular in the neighbourhood of t and the Hölder exponent gives an information on the irregularity of the function.

Definition

The **spectrum of singularities** of a function f is the function

$$D_f(h) = \dim_{\mathcal{H}}(\{t : h_f(t) = h\})$$

where $\dim_{\mathcal{H}}$ is the **Hausdorff's dimension**.

The function f is **monofractal** if there exists only one finite h such that $D_f(h) \neq -\infty$.

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The goal of a **multifractal formalism** is to find function spaces and a numerically computable function such that

- this function is an **upper bound** of the spectrum of singularities of the functions that belong to these spaces ;
- there exist functions where this function **is equal** to their spectrum ;
- there exist « **a lot of** » functions where the equality is verified.

For example, the notion « a lot of » can be the notion of **prevalent**.

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Take a mother wavelet ψ with enough zero moments and $\{2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$ an orthonormal basis of $L^2(\mathbb{R})$.

Any function $f \in L^2(\mathbb{R})$ can be written

$$f(x) = \sum_{j,k\in\mathbb{Z}} c_{j,k}\psi(2^j x - k).$$

Denote $\tau_f(q) = \liminf_{j\to\infty} \frac{\log\left(2^{-j}\sum_k |c_{j,k}|^q\right)}{log(2^{-j})}$, q_f the unique solution of $\tau_f(q) = 0$ and

$$D_f^{WM}(h) = \inf_{q\geq q_f} \{hq - \tau_f(q)\}.$$

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The main inconveniences of this method are that

- D^{WM} is always a concave function ;
- D^{WM} is increasing with slope $\geq q_f$.

The second inconvenience can be taken care of using **Wavelet Leaders** instead of wavelet coefficients (Lashermes, 2005 [11]).

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Definition

We define the **wavelet profil** of a function f by

$$\nu_f(\alpha) = \lim_{\epsilon \to 0^+} \left(\limsup_{j \to +\infty} \left(\frac{\ln(\# E_j(1, \alpha + \epsilon)(f))}{\ln(2^j)} \right) \right)$$

where $E_j(C, \alpha)(f) = \{k : |c_{j,k}| \geq C 2^{-\alpha j}\}$ and $(c_{j,k})_{j \in \mathbb{N}, k \in \{0, \ldots, 2^j - 1\}}$ are periodized wavelet coefficients of f.

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The function $\nu_f : \mathbb{R} \to \{-\infty\} \cup [0; 1]$ is nondecreasing and right-continuous and there exists $\alpha_{min} > 0$ such that $\nu_f(\alpha) = -\infty$ for all $\alpha < \alpha_{min}$ and $\nu_f(\alpha) \in [0; 1]$ for all $\alpha \geq \alpha_{min}$.

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Take a function $\nu : \mathbb{R} \to \{-\infty\} \cup [0;1]$ nondecreasing and right-continuous and suppose there exists $\alpha_{min} > 0$ such that $\nu(\alpha) = -\infty$ for all $\alpha < \alpha_{min}$ and $\nu(\alpha) \in [0;1]$ for all $\alpha \geq \alpha_{min}$.

Definition (Aubry, Bastin, Dispa, Jaffard, 2006 [4])

We define $S^\nu = \{f \in L^2([0;1]) : \nu_f(\alpha) \leq \nu(\alpha)\ \forall \alpha \in \mathbb{R}\}$.

The S^ν space is independent of the wavelet basis (Jaffard, 2004 [8]).

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If we denote

$$\nu_f^C(\alpha) = \lim_{\epsilon \to 0^+} \left(\limsup_{j \to +\infty} \left(\frac{\ln(\# E_j(C, \alpha + \epsilon)(f))}{\ln(2^j)} \right) \right)$$

then $\nu_f^C(\alpha) = \nu_f(\alpha)$ for all $\alpha \in \mathbb{R}$ and $C > 0$.

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Fix $\alpha \in \mathbb{R}$ and $C > 0$. Let $\epsilon > 0$. Take $\epsilon' < \epsilon$ and denote $\epsilon'' = \epsilon - \epsilon' > 0$.
Let $J > 0$ such that for all $j \geq J$, $C 2^{-\epsilon'' j} < 1$.
So, if $|c_{j,k}| \geq 2^{-(\alpha+\epsilon')j}$ then $|c_{j,k}| \geq 2^{-(\alpha+\epsilon')j} C 2^{-\epsilon'' j} = C 2^{-(\alpha+\epsilon)j}$.
Thus, $\forall j > J$, we have $\# E_j(C, \alpha+\epsilon)(f) \geq \# E_j(1, \alpha+\epsilon')(f)$. So,

$$\limsup_{j \to +\infty} \left(\frac{\ln(\# E_j(C, \alpha+\epsilon)(f))}{\ln(2^j)} \right) \geq \limsup_{j \to +\infty} \left(\frac{\ln(\# E_j(1, \alpha+\epsilon')(f))}{\ln(2^j)} \right).$$

So that $\nu_f^C(\alpha) \geq \nu_f(\alpha)$.

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Fix $\alpha \in \mathbb{R}$ and $C > 0$. Let $\epsilon > 0$. Take $\epsilon' < \epsilon$.
Let $J > 0$ such that for all $j \geq J$, $2^{-\epsilon' j} < C$.
So, if $|c_{j,k}| \geq C2^{-(\alpha+\epsilon)j}$ then $|c_{j,k}| \geq 2^{-(\alpha+\epsilon+\epsilon')j}$.
Thus, $\forall j > J$, we have $\#E_j(C,\alpha+\epsilon)(f) \leq \#E_j(1,\alpha+\epsilon+\epsilon')$.
Hence,

$$\limsup_{j\to+\infty}\left(\frac{\ln(\#E_j(C,\alpha+\epsilon)(f))}{\ln(2^j)}\right) \leq \limsup_{j\to+\infty}\left(\frac{\ln(\#E_j(1,\alpha+\epsilon+\epsilon')(f))}{\ln(2^j)}\right).$$

So that $\nu_f^C(\alpha) \leq \nu_f(\alpha)$.

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Theorem (Aubry, Bastin, Dispa, 2007 [2])

The set

$$\{f \in S^\nu : \nu_f(\alpha) = \nu(\alpha)\ \forall \alpha \in \mathbb{R}\}$$

is prevalent in S^ν.

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Denote

$$h_{max} = \inf_{h \geq \alpha_{min}} \frac{h}{\nu(h)}$$

and

$$D_f^\nu(h) = \begin{cases} h \sup_{h' \in]0;h]} \frac{\nu(h')}{h'} & \text{if } h \leq h_{max} \\ 1 & \text{otherwise} \end{cases}$$

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Theorem (Aubry, Bastin, Dispa, 2007 [2])

- For all $f \in S^\nu$, $D_f(h) \leq D_f^\nu(h)$ for all $h \in \mathbb{R}$.
- The set

$$\left\{ f \in S^\nu : D_f(h) = \begin{cases} D_f^\nu(h) & \text{if } h \leq h_{max} \\ -\infty & \text{otherwise} \end{cases} \right\}$$

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- We must calculate an approximation of

$$\nu_f(\alpha) = \lim_{\epsilon \to 0^+} \left(\limsup_{j \to +\infty} \left(\frac{\ln(\# E_j(1, \alpha + \epsilon)(f))}{\ln(2^j)} \right) \right)$$

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where $E_j(C, \alpha)(f) = \{k : |c_{j,k}| \geq C 2^{-\alpha j}\}$.
This means that

$$\# E_j(C, \alpha)(f) \sim 2^{-\nu_f^{C}(\alpha) j}$$

for j "large enough". So, we can approximate $\nu_f^C(\alpha)$ by the slope of

$$j \in \mathbb{N} \mapsto \frac{\ln(\# E_j(C, \alpha)(f))}{\ln(2)}$$

for j "large enough".

For a fixed α, the main problem is to determine a good constant C because we have only a finite number of wavelet coefficients :

- If C is too small, the detected value of $\nu_f^C(\alpha)$ will be 1 ;
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We construct the function

$$C \mapsto \nu_f^C(\alpha).$$

In theory, the constant is arbitrary, so, in practice, this function must stabilize if $\alpha \geq \alpha_{min}$.

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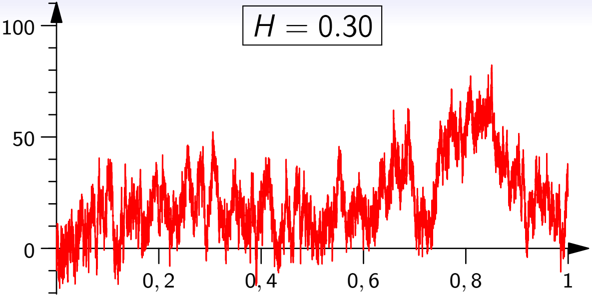
Fractional Brownian Motion



Theorem

Consider a Fractional Brownian motion with a parameter $H \in]0;1[$ defined on a probability space $(\Omega, \mathcal{A}, \mu)$. For almost every $\omega \in \Omega$, the Fractional Brownian Walk associated to ω is mono-Hölder with an exponent H.

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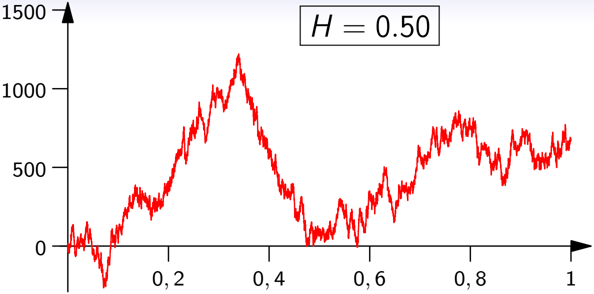

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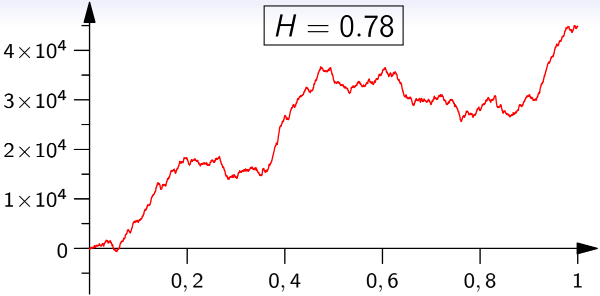

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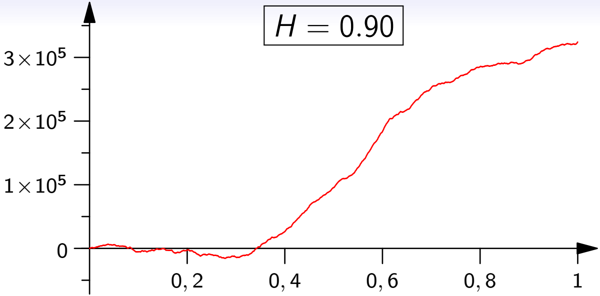

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Method, with the S^ν spaces, to determine the exponent H of a Fractionnal Brownian Walk :
Fix $\alpha \in]0;1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

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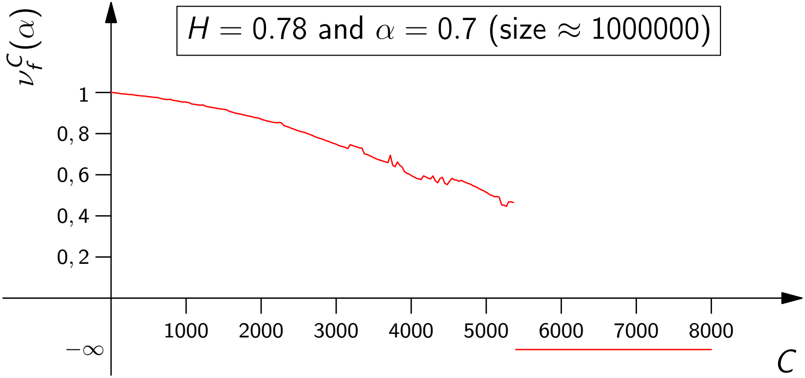

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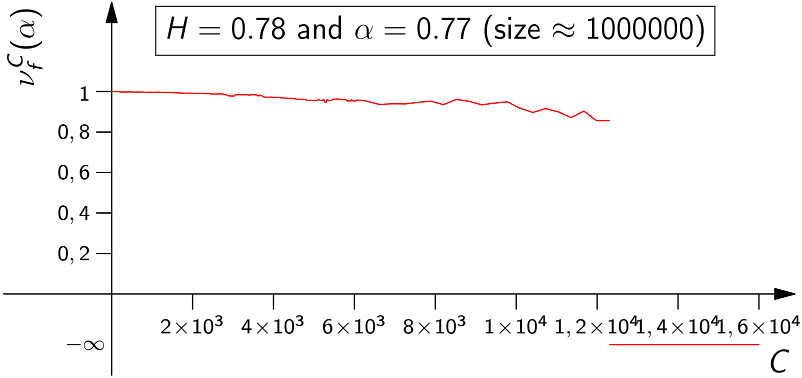

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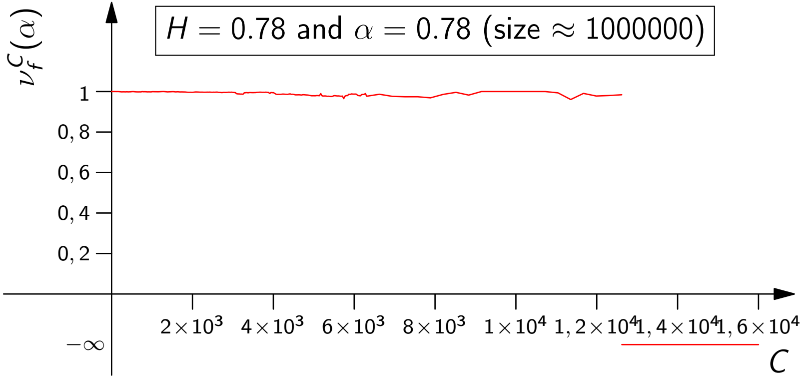

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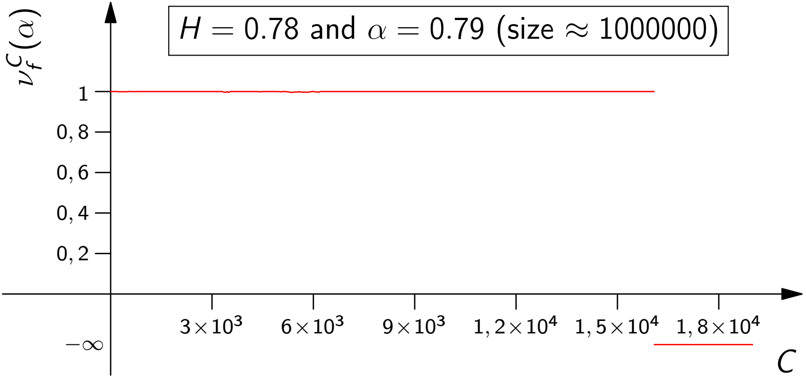


Fractional Brownian Motion

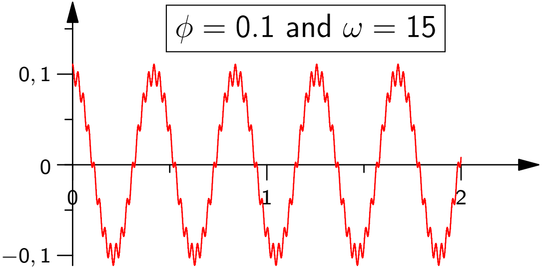
Test on 1000 Fractional Brownian Walks of a size $\approx$ 1000000 with an exponent H.

Error mean	S^ν Method	WLM
$H \leq 0.5$	0.013439	0.0297254
$H \geq 0.5$	0.00904878	0.00687781
Some H	0.0112439	0.0183016

Weierstrass Function

Take $\phi \in]0;1[$, $\omega > 0$ such that $\phi\omega > 1$. We define

$$f(x) = \sum_{n=0}^{+\infty} \phi^n \cos(\omega^n x).$$



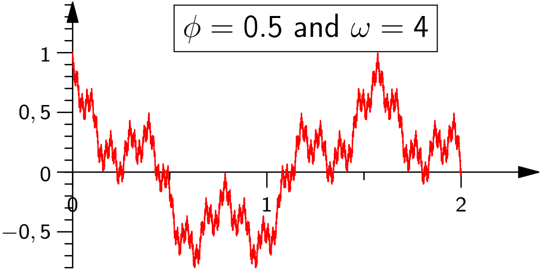
Theorem

The Weistrass Function with parameters ϕ, ω is mono-Hölder with an exponent $\frac{-\log(\phi)}{\log(\omega)}$.

Weierstrass Function

Take $\phi \in]0;1[$, $\omega > 0$ such that $\phi\omega > 1$. We define

$$f(x) = \sum_{n=0}^{+\infty} \phi^n \cos(\omega^n x).$$



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Weierstrass Function

Test on 100 Weierstrass Functions with a size $\approx$ 1000000.

	S^ν Method	WLM
Error mean	0.0265912	0.027018

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Lebesgue-Davenport Function

Definition

Let $t \in [0;1[$. We can write $t = (0.t_1 t_2 \dots t_n \dots)_2$. We define the **Lebesgue-Davenport function** evaluated in t by

$$f(t) = (x(t), y(t))$$

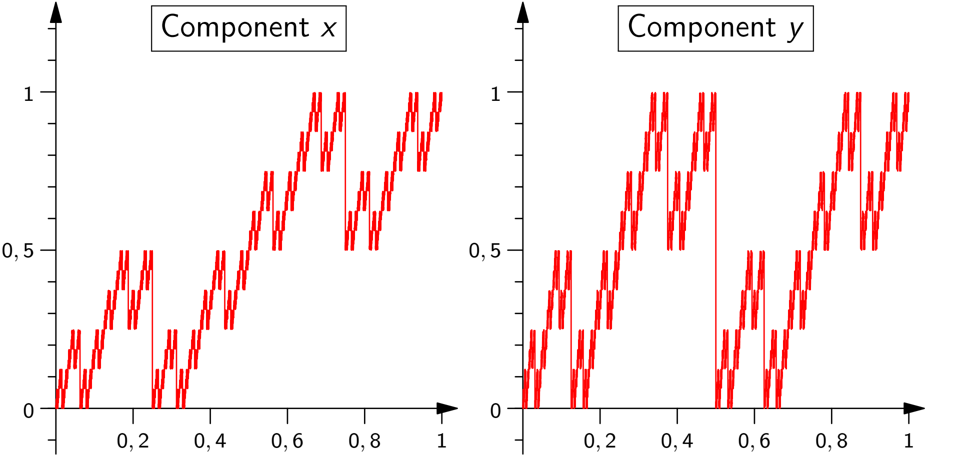
where $x(t) = (0.t_1 t_3 \dots)_2$ and $y(t) = (0.t_2 t_4 \dots)_2$. We extend this function in $t = 1$ by picking $f(1) = (1,1)$.

Theorem (Jaffard, 2004 [9])

The spectrum of singularities of the Lebesgue-Davenport function is given by

$$D_f(h) = \begin{cases} 2h & \text{if } h \in [0, 0.5] \\ -\infty & \text{otherwise} \end{cases}$$

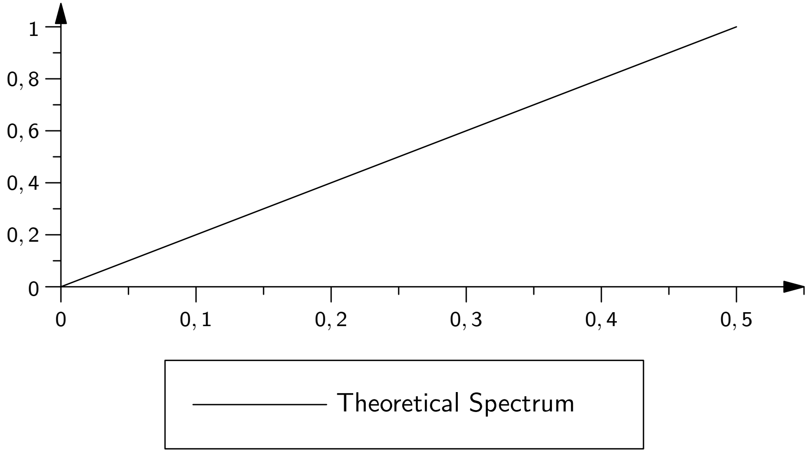
Lebesgue-Davenport Function



Lebesgue-Davenport Function

Jaffard, Nicolay, 2009 [10]

Size $\approx$ 8000000



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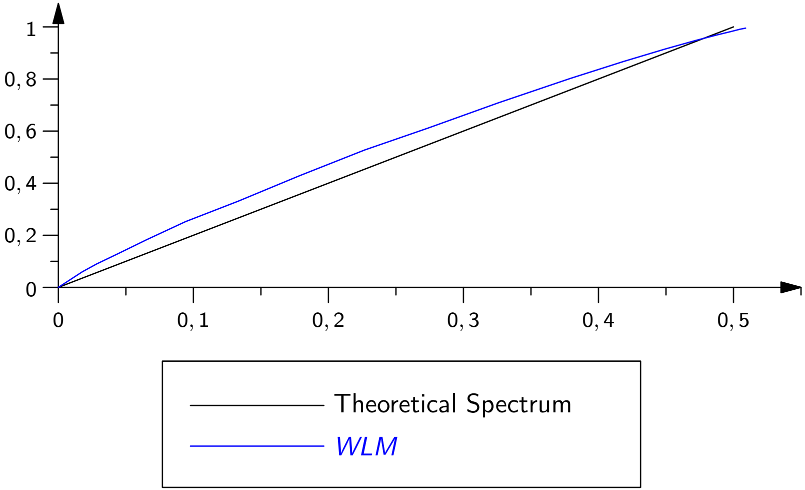

Lebesgue-Davenport Function

Fix $\alpha \in]0;1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

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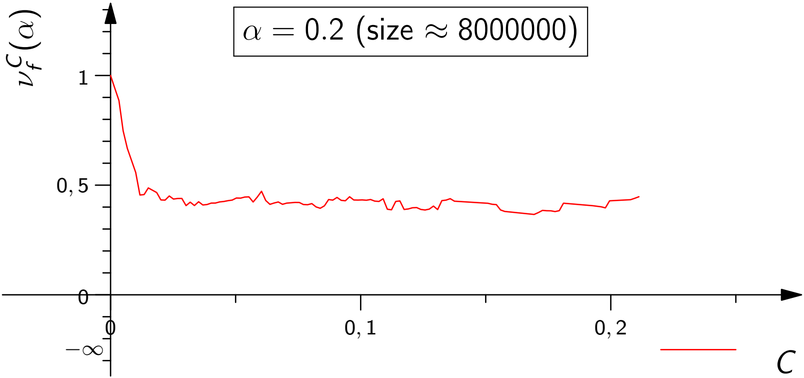

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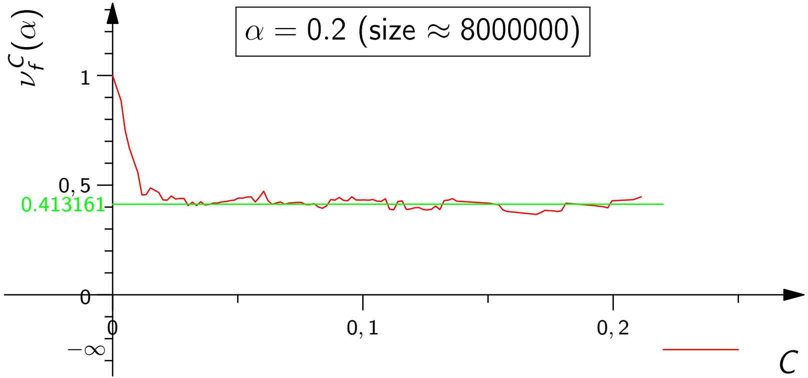

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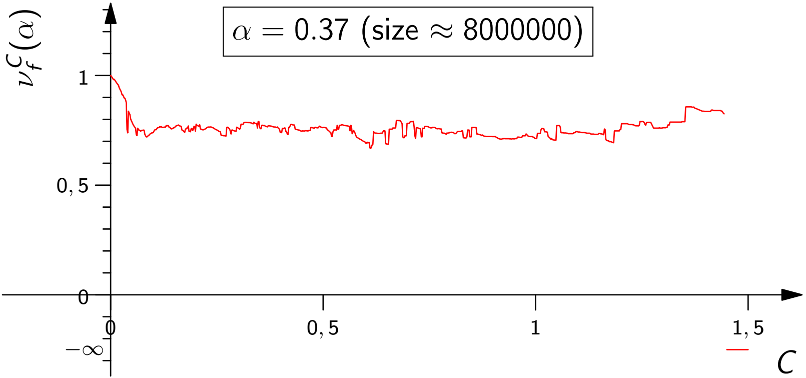

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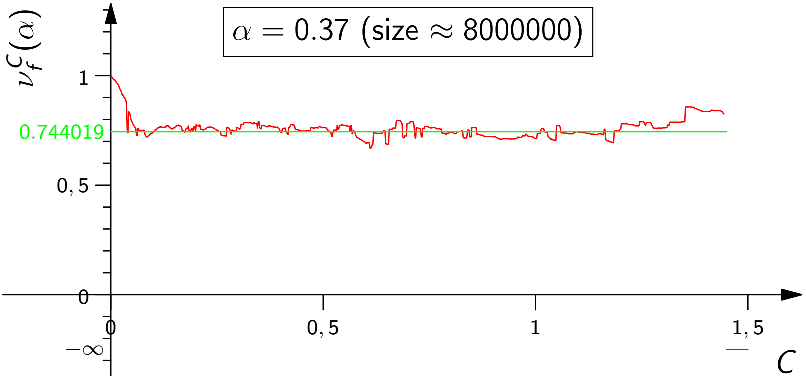

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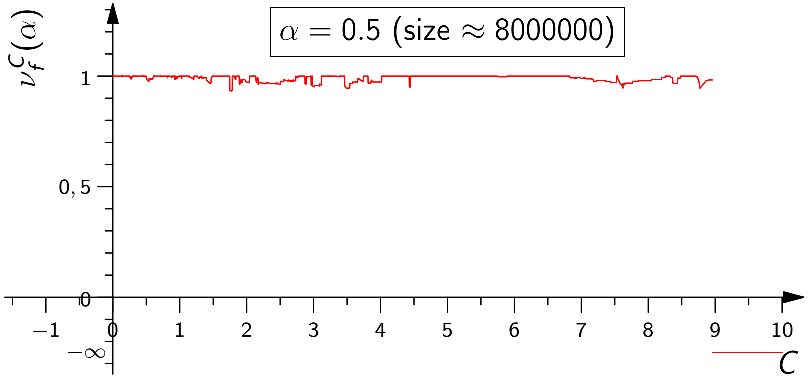

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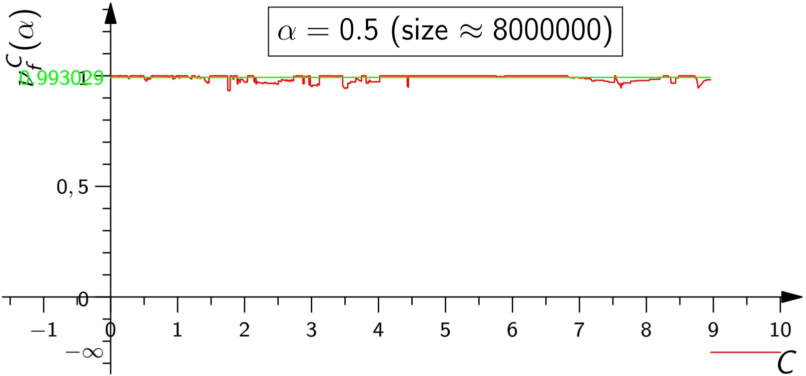

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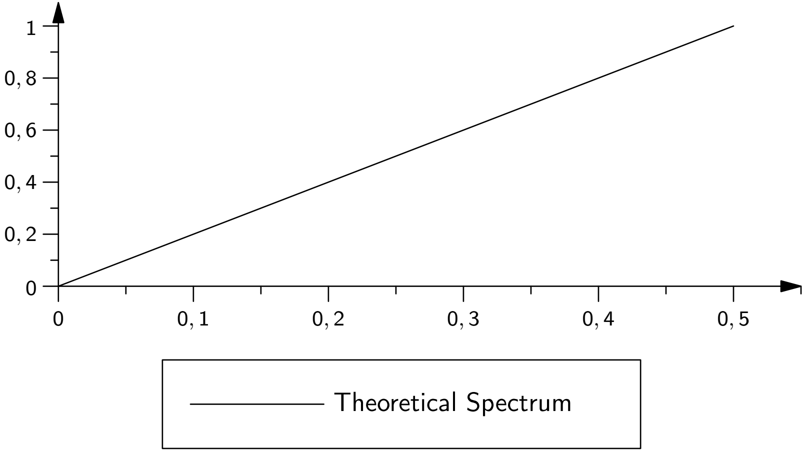

Lebesgue-Davenport Function

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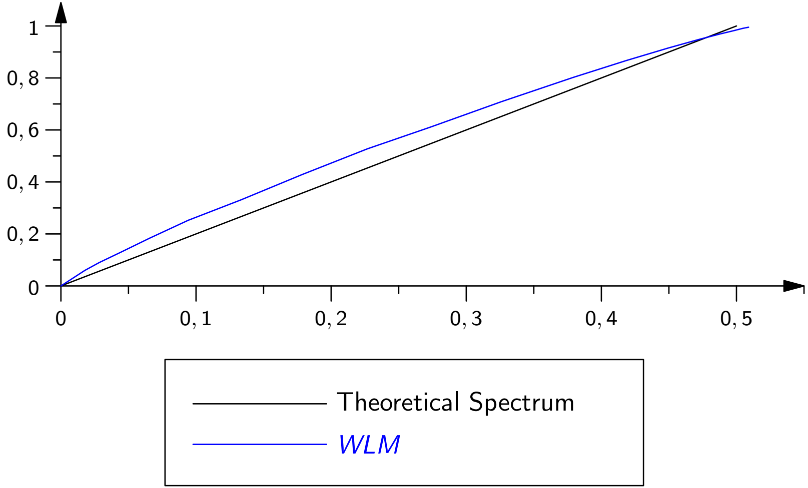

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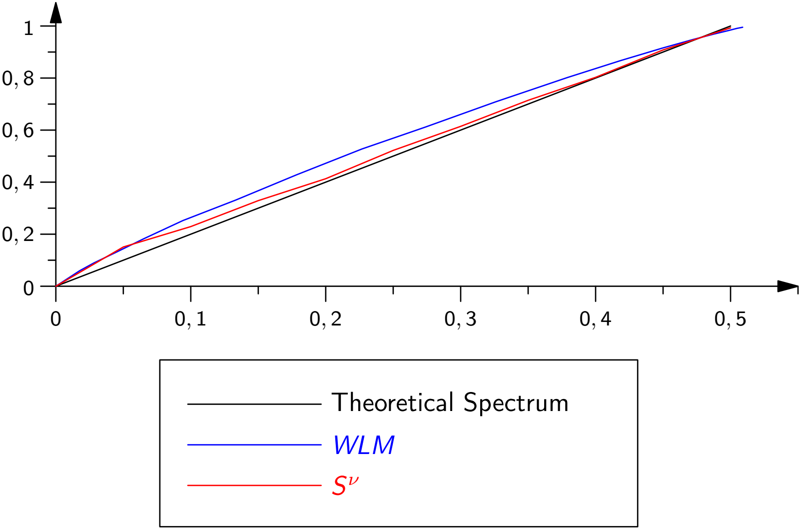

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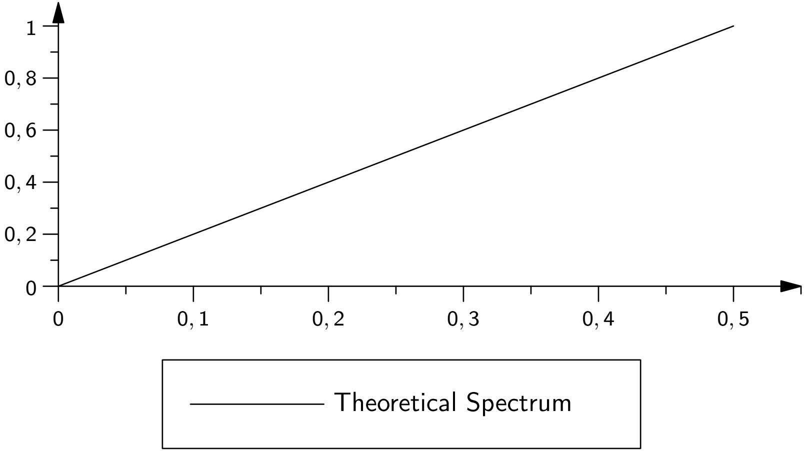

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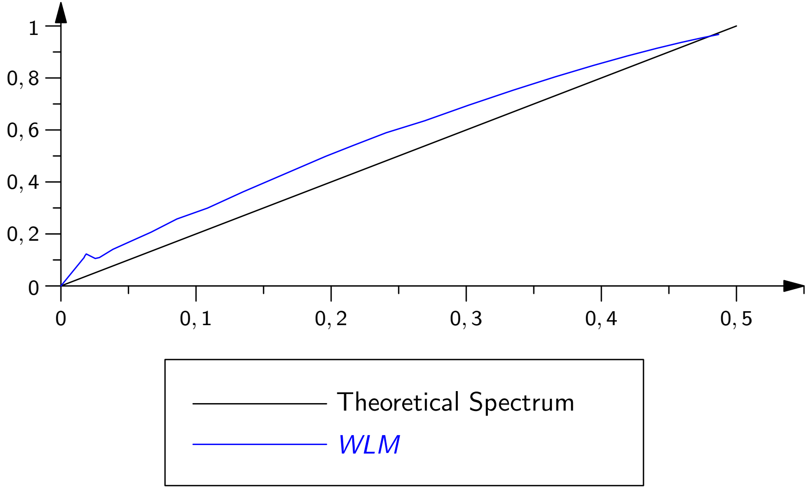

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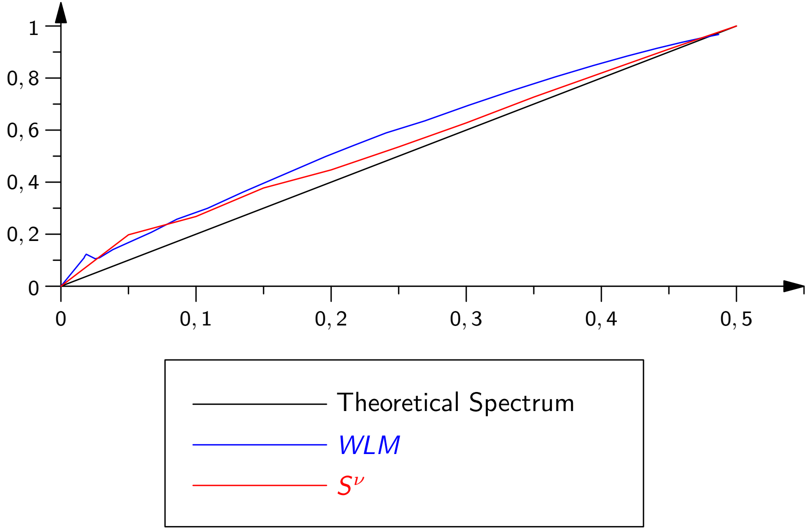


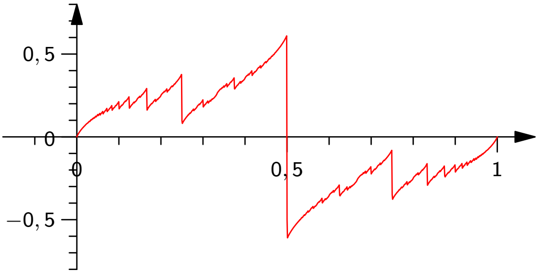
Riemann Function

Theorem (Arneodo, Jaffard [1])

Define the function

$$f(x) = \sum_{n=1}^{+\infty} \frac{[nx]}{n^2}.$$

The spectrum of singularities of this function is $D(h) = h$ for $h \in [0, 1]$.



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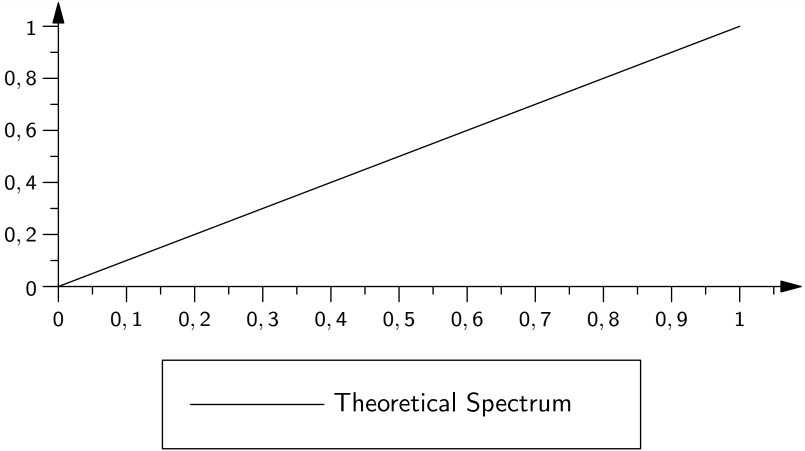

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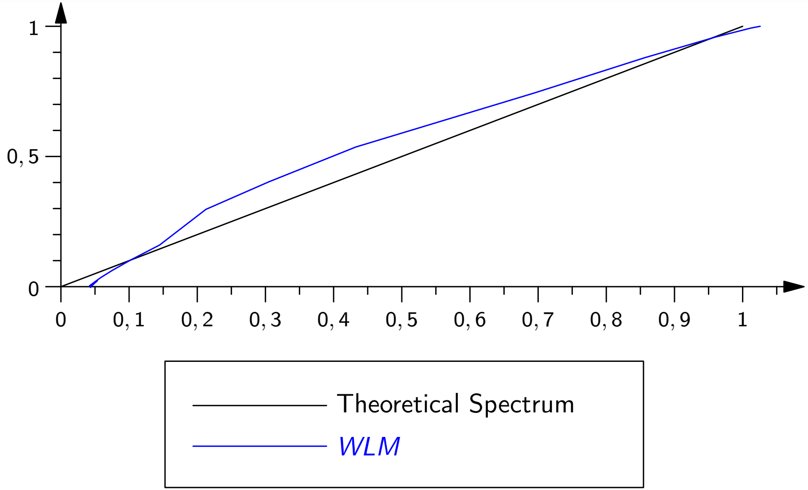

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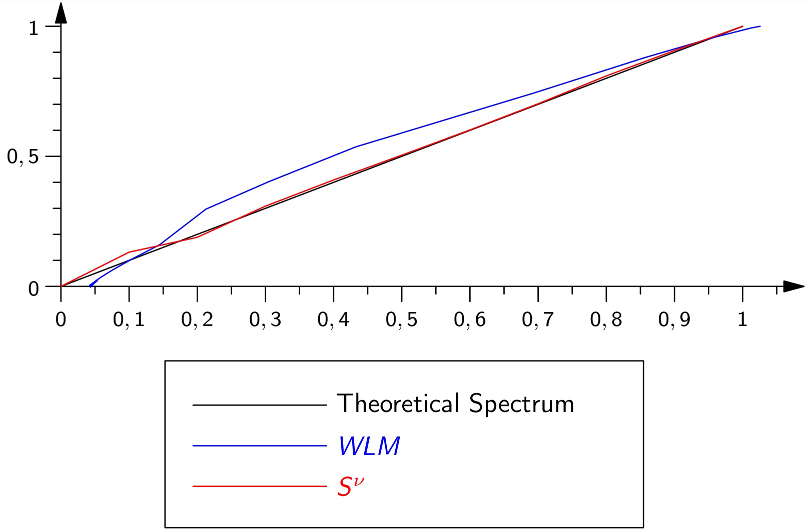

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- it should also be tested on real-life signals (turbulence, and so on) ;
- ...

Thank you for your attention !

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