Mise en oeuvre du formalisme multifractal sur les espaces S^ν

Implementation of the Multifractal Formalism on S^ν Spaces

Thomas Kleyntssens
tkleyntssens@ulg.ac.be

Joint work with Samuel Nicolay

Université de Liège

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Introduction

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- ▶ We determine the "size" of the set of points which share the same "irregularity" ;

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- Studying very irregular functions ;

- We determine the "size" of the set of points which share the same "irregularity" ;

- In practice, we use a numerically computable function which "approximates" this size.

Plan

Plan

Definition (Hölder continuous)

Let $t \in \mathbb{R}$, $s \in \mathbb{R}_0^+$ and $f \in L_{loc}^\infty$. We denote $f \in C^s(t)$ if there exist a polynomial P of degree strictly smaller than s, a constant $C > 0$ and a neighbourhood Ω of 0 such that

$$|f(t + l) - P(l)| \leq C|l|^s$$

for all $l \in \Omega$, . If $s \in]0; 1]$, the condition is equivalent to

$$|f(t + l) - f(t)| \leq C|l|^s.$$

Definition (Holder exponent)

Let $t \in \mathbb{R}$ and $f \in L_{loc}^\infty$; we denote the **Hölder exponent** of f at point t by

$$h_f(t) = \sup\{s \in \mathbb{R}_0^+ : f \in C^s(t)\}$$

Proposition

- If $f \in C^s(t)$ then f is continuous at point t.

- If $f \in C^s(t)$ with $s > 1$ then f is differentiable at point t and $f'(t) = P'(0)$.

- If f is differentiable at point t then $f \in C^1(t)$.

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Conclusion

▶ If $h_f(t) \in]0; 1[$ then f is continous but not differentiable at point t.

▶ So f is very irregular in the neighbourhood of t and the Hölder exponent gives an information on the irregularity of the function.

▶ More $h_f(t)$ is small, more the function is irregular in the neighbourhood of t.

Definition

The **spectrum of singularities** of a function f is the function

$$D_f(h) = \dim_{\mathcal{H}}(\{t : h_f(t) = h\})$$

where $\dim_{\mathcal{H}}$ is the **Hausdorff's dimension**.

The function f is **monofractal** if there exists only one finite h such that $D_f(h) \neq -\infty$.

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The goal of a **multifractal formalism** is to find function spaces and a numerically computable function such that

- this function is an **upper bound** of the spectrum of singularities of the functions that belong to these spaces;

- there exist functions where this function **is equal** to their spectrum;

- there exist « **a lot of** » functions where the equality is verified.

For example, the notion « a lot of » can be the notion of **prevalent**.

Definition

Let X be a complete metric vector space. A set A is called **shy** if it exists a Borel set $B \supseteq A$ and a Borel measure μ such that

- It exists a compact $K \subset X$ such that $0 < \mu(K) < +\infty$;

- For all $x \in X$, $\mu(B + x) = 0$.

A set is **prevalent** if its complement is shy.

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A set is **prevalent** if its complement is shy.

Proposition (Hunt, Sauer, Yorke, 1992 [8])

- In finite dimension, a set is prevalent if and only if it has full Lebesgue measure.

- A prevalent set is dense.

- If A is prevalent then $x + A$ and λA is prevalent for all $x \in X$ and $\lambda \neq 0$.

- A countable intersection of prevalent sets is prevalent.

Plan

Take a mother wavelet ψ with enough zero moments and $\{2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$ an orthonormal basis of $L^2(\mathbb{R})$.

Any function $f \in L^2(\mathbb{R})$ can be written

$$f(x) = \sum_{j,k \in \mathbb{Z}} c_{j,k}\psi(2^j x - k).$$

Denote $\tau_f(q) = \liminf\limits_{j \to \infty} \dfrac{\log\left(2^{-j} \sum_k \left|c_{j,k}\right|^q\right)}{log(2^{-j})}$, q_f the unique solution

of $\tau_f(q) = 0$ and

$$D_f^{WM}(h) = \inf_{q \geq q_f} \{hq - \tau_f(q)\}.$$

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Theorem (Jaffard, Fraysse, 1997 [7, 9])

Let τ a concave function.

- If $q \geq q_f$, then $f \in B_{q,\infty}^{\frac{\tau(q)+1}{q}} \Rightarrow D_f(h) \leq D_f^{WM}(h)$ for all $h \in \mathbb{R}$.

- The knowledge that $\tau_f(q) \geq \tau(q)$ for all $q \geq q_f$ implies that
  $f \in B^\tau := \bigcap_{q \geq q_f} B_{q,\infty}^{\frac{\tau(q)+1}{q}}$ and the set

  $$\left\{ f \in B^\tau : D_f(h) = \left\{ \begin{array}{ll} \inf_{q \geq q_f}\{hq - \tau(q)\} & \text{if this inf is } \leq 1 \\ -\infty & \text{else} \end{array} \right. \right\}$$

  is prevalent in B^τ.

Denote $\tau_f(q) = \liminf\limits_{j \to \infty} \dfrac{\log\left(2^{-j} \sum_k \left|c_{j,k}\right|^q\right)}{\log(2^{-j})}$, q_f the unique solution of $\tau_f(q) = 0$ and

$$D_f^{WM}(h) = \inf_{q \geq q_f} \{hq - \tau_f(q)\}.$$

The main inconveniences of this method are that

- D^{WM} is always a concave function ;

- D^{WM} is increasing with slope $\geq q_f$.

The second inconvenience can be taken care of using **Wavelet Leaders** instead of wavelet coefficients (Lashermes, 2005 [13]).

Plan

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Definition

We define the **wavelet profil** of a function f by

$$\nu_f(\alpha) = \lim_{\epsilon \to 0^+} \left(\limsup_{j \to +\infty} \left(\frac{\ln(\# E_j(1, \alpha + \epsilon)(f))}{\ln(2^j)} \right) \right)$$

where $E_j(C, \alpha)(f) = \{k : |c_{j,k}| \geq C2^{-\alpha j}\}$ and $(c_{j,k})_{j \in \mathbb{N}, k \in \{0,\ldots,2^j-1\}}$ are periodized wavelet coefficients of f.

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The function $\nu_f : \mathbb{R} \to \{-\infty\} \cup [0; 1]$ is nondecreasing and right-continuous and there exists $\alpha_{min} \geq 0$ such that $\nu_f(\alpha) = -\infty$ for all $\alpha < \alpha_{min}$ and $\nu_f(\alpha) \in [0; 1]$ for all $\alpha \geq \alpha_{min}$.

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Take a function $\nu : \mathbb{R} \to \{-\infty\} \cup [0; 1]$ nondecreasing and right-continuous and suppose there exists $\alpha_{min} \geq 0$ such that $\nu(\alpha) = -\infty$ for all $\alpha < \alpha_{min}$ and $\nu(\alpha) \in [0; 1]$ for all $\alpha \geq \alpha_{min}$.

Definition

We define $S^\nu = \{f \in L^2([0; 1]) : \nu_f(\alpha) \leq \nu(\alpha) \,\forall \alpha \in \mathbb{R}\}$.

The S^ν space is independent of the wavelet basis (Jaffard, 2004 [10]).

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Equivalent Definitions (Aubry, Bastin, Dispa, Jaffard, 2006 [5])

- We define $S^\nu = \{f \in L^2([0; 1]) : \nu_f(\alpha) \leq \nu(\alpha) \ \forall \alpha \in \mathbb{R}\}$.

- Its equivalent to define

$$S^\nu = \{f \in L^2([0; 1]) : \forall \alpha \in \mathbb{R}, \ \forall \epsilon > 0, \ \forall C > 0,$$
$$\exists J > 0 \text{ such that } \forall j \geq J, \ \#E_j(C, \alpha)(f) \leq 2^{(\nu(\alpha)+\epsilon)j}\}$$

where $E_j(C, \alpha)(f) = \{k : |c_{j,k}| \geq C2^{-\alpha j}\}$ and $(c_{j,k})_{j\in\mathbb{N}, k\in\{0,\dots,2^j-1\}}$ are periodized wavelet coefficients of f.

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The **wavelet profil** of a function f is

$$\nu_f(\alpha) = \lim_{\epsilon \to 0^+} \left(\limsup_{j \to +\infty} \left(\frac{\ln(\#E_j(1, \alpha + \epsilon)(f))}{\ln(2^j)} \right) \right)$$

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Proposition

If we denote

$$\nu_f^C(\alpha) = \lim_{\epsilon \to 0^+} \left(\limsup_{j \to +\infty} \left(\frac{\ln(\#E_j(C, \alpha + \epsilon)(f))}{\ln(2^j)} \right) \right)$$

then $\nu_f^C(\alpha) = \nu_f(\alpha)$ for all $\alpha \in \mathbb{R}$ and $C > 0$.

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then $\nu_f^C(\alpha) = \nu_f(\alpha)$ for all $\alpha \in \mathbb{R}$ and $C > 0$.

Fix $\alpha \in \mathbb{R}$ and $C > 0$. Let $\epsilon > 0$. Take $\epsilon' < \epsilon$ and denote $\epsilon'' = \epsilon - \epsilon' > 0$.

Let $J > 0$ such that for all $j \geq J$, $C2^{-\epsilon'' j} < 1$.

So, if $|c_{j,k}| \geq 2^{-(\alpha+\epsilon')j}$ then $|c_{j,k}| \geq 2^{-(\alpha+\epsilon')j} C 2^{-\epsilon'' j} = C2^{-(\alpha+\epsilon)j}$.

Thus, $\forall j > J$, we have $\# E_j(C, \alpha + \epsilon)(f) \geq \# E_j(1, \alpha + \epsilon')(f)$. So,

$$\limsup_{j \to +\infty} \left(\frac{\ln(\# E_j(C, \alpha + \epsilon)(f))}{\ln(2^j)} \right) \geq \limsup_{j \to +\infty} \left(\frac{\ln(\# E_j(1, \alpha + \epsilon')(f))}{\ln(2^j)} \right).$$

So that $\nu_f^C(\alpha) \geq \nu_f(\alpha)$.

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then $\nu_f^C(\alpha) = \nu_f(\alpha)$ for all $\alpha \in \mathbb{R}$ and $C > 0$.

Fix $\alpha \in \mathbb{R}$ and $C > 0$. Let $\epsilon > 0$. Take $\epsilon' < \epsilon$.

Let $J > 0$ such that for all $j \geq J$, $2^{-\epsilon' j} < C$.

So, if $|c_{j,k}| \geq C2^{-(\alpha+\epsilon)j}$ then $|c_{j,k}| \geq 2^{-(\alpha+\epsilon+\epsilon')j}$.

Thus, $\forall j > J$, we have $\#E_j(C, \alpha + \epsilon)(f) \leq \#E_j(1, \alpha + \epsilon + \epsilon')$.

Hence,

$$\limsup_{j \to +\infty} \left(\frac{\ln(\#E_j(C, \alpha + \epsilon)(f))}{\ln(2^j)} \right) \leq \limsup_{j \to +\infty} \left(\frac{\ln(\#E_j(1, \alpha + \epsilon + \epsilon')(f))}{\ln(2^j)} \right).$$

So that $\nu_f^C(\alpha) \leq \nu_f(\alpha)$.

Theorem (Aubry, Bastin, Dispa, Jaffard, 2007 [4])

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Theorem (Aubry, Bastin, Dispa, 2007 [3])

The set

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is prevalent in S^ν.

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Theorem (Aubry, Bastin, Dispa, Jaffard, 2006 [5])

Define

$$\tau(q) = \inf_{h \le h_{max}} (hp - \nu(h)).$$

We have $S^\nu = B^\tau$ if and only if ν is concave.

Plan

Denote

$$h_{max} = \inf_{h \geq \alpha_{min}} \frac{h}{\nu(h)}$$

and

$$D_f^\nu(h) = \begin{cases} h \sup_{h' \in]0;h]} \frac{\nu(h')}{h'} & \text{if } h \leq h_{max} \\ 1 & \text{otherwise} \end{cases}$$

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Theorem (Aubry, Bastin, Dispa, 2007 [3])

- For all $f \in S^\nu$, $D_f(h) \leq D_f^\nu(h)$ for all $h \in \mathbb{R}$.

- The set

$$\left\{ f \in S^\nu : D_f(h) = \left\{ \begin{array}{ll} D_f^\nu(h) & \text{if } h \leq h_{max} \\ -\infty & \text{otherwise} \end{array} \right. \right\}$$

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▶ We must calculate an approximation of

$$\nu_f(\alpha) = \lim_{\epsilon \to 0^+} \left(\limsup_{j \to +\infty} \left(\frac{\ln(\#E_j(1, \alpha + \epsilon)(f))}{\ln(2^j)} \right) \right)$$

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where $E_j(C, \alpha)(f) = \{k : |c_{j,k}| \geq C2^{-\alpha j}\}$.
This means that
$$\# E_j(C, \alpha)(f) \sim 2^{-\nu_f^C(\alpha)j}$$

for j "large enough". So, we can approximate $\nu_f^C(\alpha)$ by the slope of

$$j \in \mathbb{N} \mapsto \frac{\ln(\# E_j(C, \alpha)(f))}{\ln(2)}$$

for j "large enough".

For a fixed α, the main problem is to determine a good constant C because we have only a finite number of wavelet coefficients :

▶ If C is too small, the detected value of $\nu_f^C(\alpha)$ will be 1 ;

▶ If C is too big, the detected value of $\nu_f^C(\alpha)$ will be $-\infty$.

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We construct the function

$$C \mapsto \nu_f^C(\alpha).$$

In theory, the constant is arbitrary, so, in practice, this function must stabilize if $\alpha \geq \alpha_{min}$.

Plan

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Fractional Brownian Motion

$H = 0.30$

Theorem

Consider a Fractional Brownian motion with a parameter $H \in]0; 1[$ defined on a probability space $(\Omega, \mathcal{A}, \mu)$. For almost every $\omega \in \Omega$, the **Fractional Brownian Walk** associated to ω is mono-Hölder with an exponent H.

Fractional Brownian Motion

$H = 0.50$

Theorem

Consider a Fractional Brownian motion with a parameter $H \in]0; 1[$ defined on a probability space $(\Omega, \mathcal{A}, \mu)$. For almost every $\omega \in \Omega$, the **Fractional Brownian Walk** associated to ω is mono-Hölder with an exponent H.

Fractional Brownian Motion

$H = 0.78$

Theorem

Consider a Fractional Brownian motion with a parameter $H \in]0; 1[$ defined on a probability space $(\Omega, \mathcal{A}, \mu)$. For almost every $\omega \in \Omega$, the **Fractional Brownian Walk** associated to ω is mono-Hölder with an exponent H.

Fractional Brownian Motion

$$H = 0.90$$

Theorem

Consider a Fractional Brownian motion with a parameter $H \in]0; 1[$ defined on a probability space $(\Omega, \mathcal{A}, \mu)$. For almost every $\omega \in \Omega$, the **Fractional Brownian Walk** associated to ω is mono-Hölder with an exponent H.

Fractional Brownian Motion

Method, with the S^ν spaces, to determine the exponent H of a Fractionnal Brownian Walk :
Fix $\alpha \in]0; 1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

Fractional Brownian Motion

Method, with the S^ν spaces, to determine the exponent H of a Fractionnal Brownian Walk :

Fix $\alpha \in]0; 1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

$H = 0.78$, $\alpha = 0.7$ and $C = 2000$

Fractional Brownian Motion

Method, with the S^ν spaces, to determine the exponent H of a Fractionnal Brownian Walk :

Fix $\alpha \in]0; 1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

$H = 0.78$, $\alpha = 0.7$ and $C = 2000$
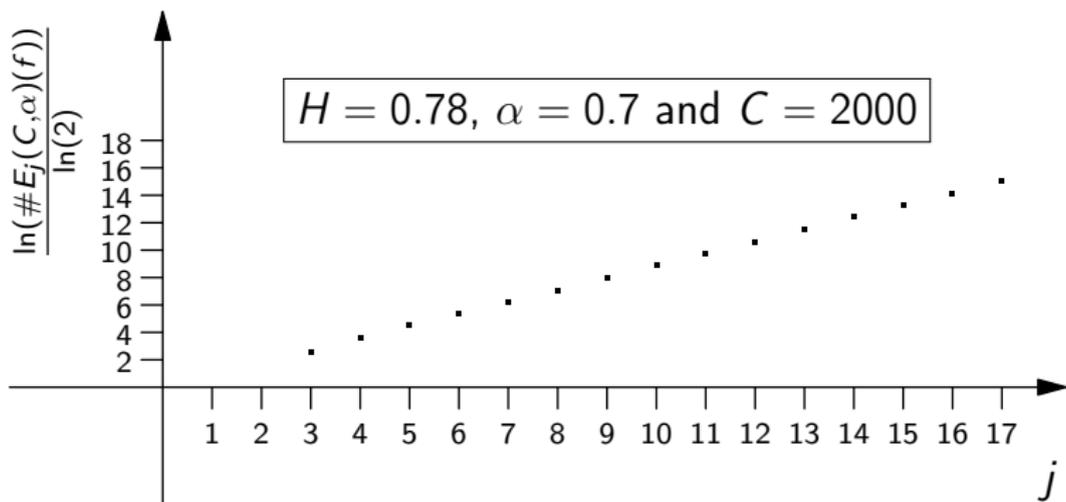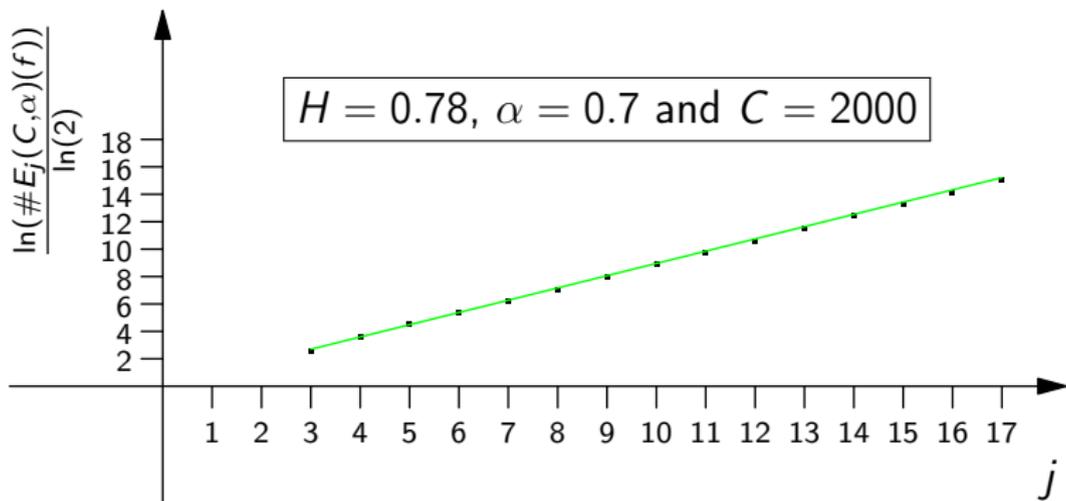
Slope $= \nu_f^C(\alpha) = 0.894441$

Fractional Brownian Motion

Method, with the S^ν spaces, to determine the exponent H of a Fractionnal Brownian Walk :

Fix $\alpha \in]0; 1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

$H = 0.78$, $\alpha = 0.7$ and $C = 5000$

Fractional Brownian Motion

Method, with the S^ν spaces, to determine the exponent H of a Fractionnal Brownian Walk :

Fix $\alpha \in]0; 1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

$H = 0.78$, $\alpha = 0.7$ and $C = 5000$
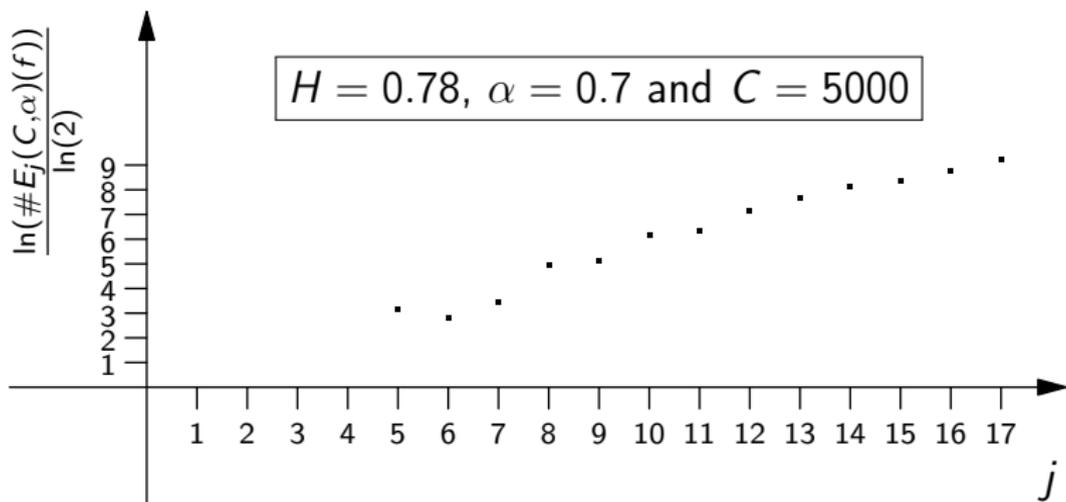
Slope $= \nu_f^C(\alpha) = 0.450242$

Fractional Brownian Motion

Method, with the S^ν spaces, to determine the exponent H of a Fractionnal Brownian Walk :

Fix $\alpha \in]0; 1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

$H = 0.78$ and $\alpha = 0.7$ (size ≈ 1000000)

Fractional Brownian Motion

Method, with the S^ν spaces, to determine the exponent H of a Fractionnal Brownian Walk :

Fix $\alpha \in]0; 1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

$H = 0.78$ and $\alpha = 0.77$ (size ≈ 1000000)

Fractional Brownian Motion

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$H = 0.78$ and $\alpha = 0.78$ (size ≈ 1000000)

Fractional Brownian Motion

Method, with the S^ν spaces, to determine the exponent H of a Fractionnal Brownian Walk :

Fix $\alpha \in]0;1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

$H = 0.78$ and $\alpha = 0.79$ (size ≈ 1000000)

Fractional Brownian Motion

Test on 1000 Fractional Brownian Walks of a size ≈ 1000000 with an exponent H.

Error mean	S^ν Method	WLM
$H \leq 0.5$	0.013439	0.0297254
$H \geq 0.5$	0.00904878	0.00687781
Some H	0.0112439	0.0183016

Weierstrass Function
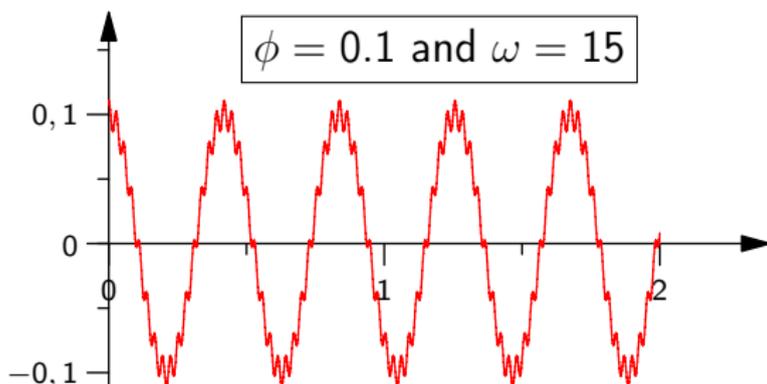
Take $\phi \in]0;1[$, $\omega > 0$ such that $\phi\omega > 1$. We define

$$f(x) = \sum_{n=0}^{+\infty} \phi^n \cos(\omega^n x).$$

$\phi = 0.1$ and $\omega = 15$

Theorem

The Weistrass Function with parameters ϕ, ω is mono-Hölder with an exponent $\frac{-\log(\phi)}{\log(\omega)}$.

Weierstrass Function

Take $\phi \in]0; 1[$, $\omega > 0$ such that $\phi\omega > 1$. We define

$$f(x) = \sum_{n=0}^{+\infty} \phi^n \cos(\omega^n x).$$

$\phi = 0.5$ and $\omega = 4$
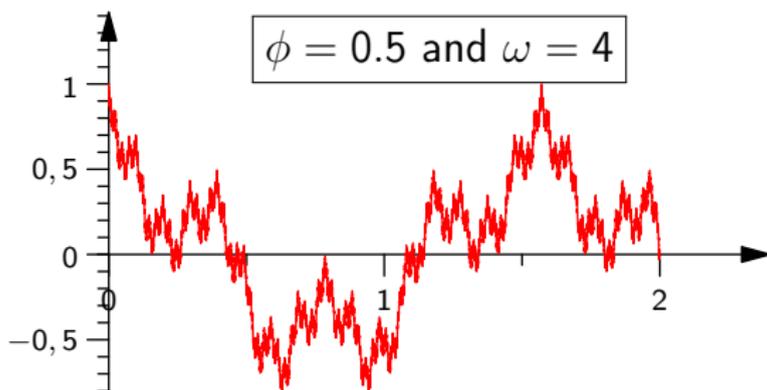
Theorem

The Weistrass Function with parameters ϕ, ω is mono-Hölder with an exponent $\frac{-\log(\phi)}{\log(\omega)}$.

Weierstrass Function

Test on 100 Weierstrass Functions with a size ≈ 1000000.

	S^ν Method	WLM
Error mean	0.0265912	0.027018

Plan

Lebesgue-Davenport Function

Definition

Let $t \in [0; 1[$. We can write $t = (0.t_1 t_2 \ldots t_n \ldots)_2$. We define the **Lebesgue-Davenport function** evaluated in t by

$$f(t) = (x(t), y(t))$$

where $x(t) = (0.t_1 t_3 \ldots)_2$ and $y(t) = (0.t_2 t_4 \ldots)_2$. We extend this function in $t = 1$ by picking $f(1) = (1, 1)$.

Theorem (Jaffard, 2004 [11])

The spectrum of singularities of the Lebesgue-Davenport function is given by

$$D_f(h) = \begin{cases} 2h & \text{if } h \in [0, 0.5] \\ -\infty & \text{otherwise} \end{cases}$$

Lebesgue–Davenport Function

Lebesgue-Davenport Function

Jaffard, Nicolay, 2009 [12]

Size ≈ 8000000

——— Theoretical Spectrum

Lebesgue-Davenport Function

Jaffard, Nicolay, 2009 [12]

Size ≈ 8000000

	Theoretical Spectrum
	WLM

Lebesgue-Davenport Function

Fix $\alpha \in]0; 1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

Lebesgue-Davenport Function

Fix $\alpha \in]0; 1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

$\alpha = 0.2$ (size ≈ 8000000)

Lebesgue–Davenport Function

Fix $\alpha \in\ ]0; 1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

Lebesgue-Davenport Function

Fix $\alpha \in]0; 1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

$\alpha = 0.37$ (size ≈ 8000000)

Lebesgue-Davenport Function

Fix $\alpha \in]0; 1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

Lebesgue-Davenport Function

Fix $\alpha \in\ ]0; 1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

$\alpha = 0.5$ (size ≈ 8000000)

Lebesgue-Davenport Function

Fix $\alpha \in]0; 1[$ and examine the function $C > 0 \mapsto \nu_f^C(\alpha)$.

Lebesgue-Davenport Function

Size ≈ 8000000

Theoretical Spectrum

Lebesgue-Davenport Function

Size ≈ 8000000

Lebesgue-Davenport Function
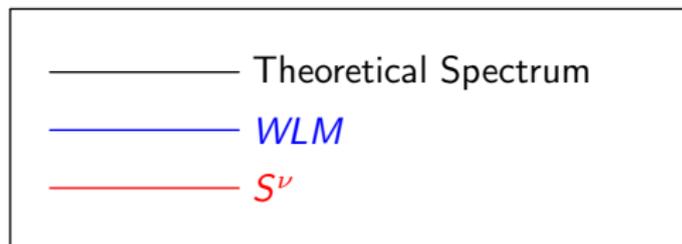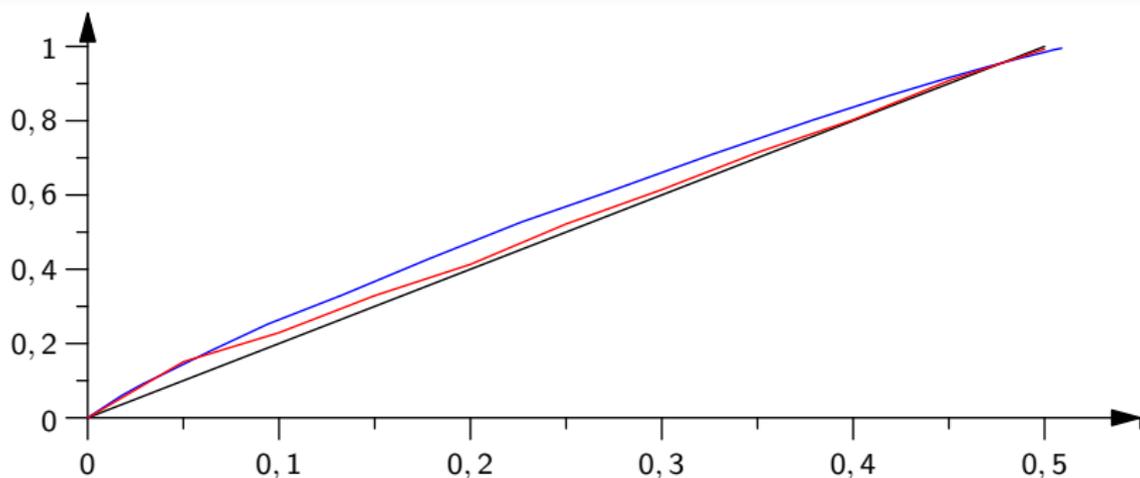
Size ≈ 8000000

Legend:
- Theoretical Spectrum (black)
- *WLM* (blue)
- S^ν (red)

Lebesgue-Davenport Function

Size ≈ 2000000

Theoretical Spectrum

Lebesgue-Davenport Function

Size ≈ 2000000

Theoretical Spectrum

WLM
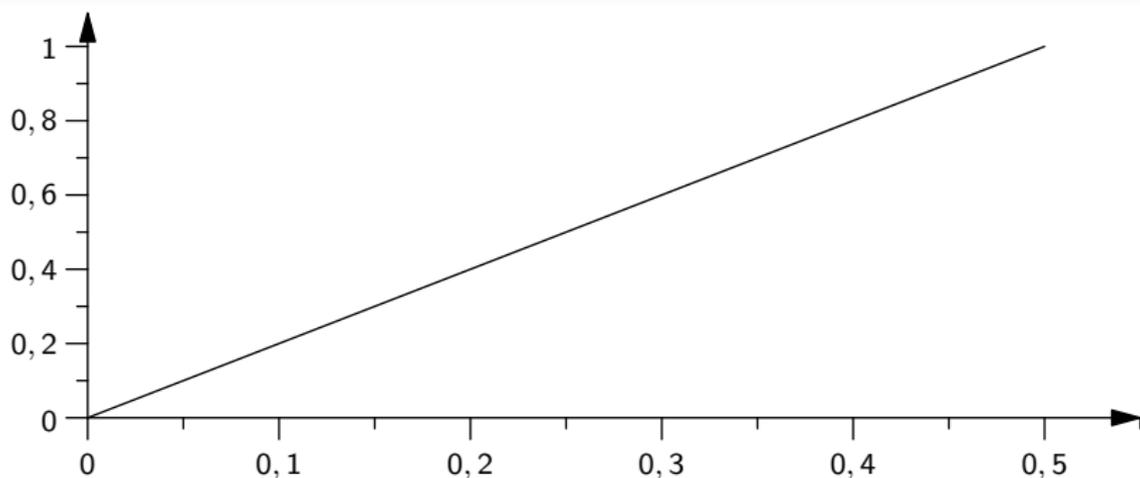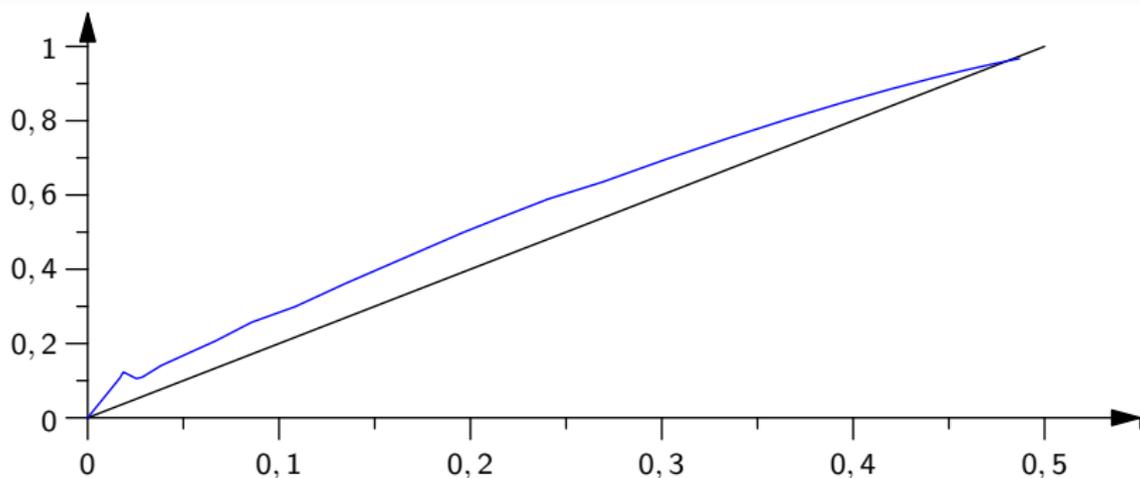
Lebesgue-Davenport Function
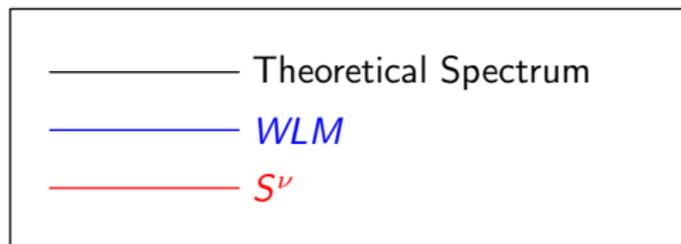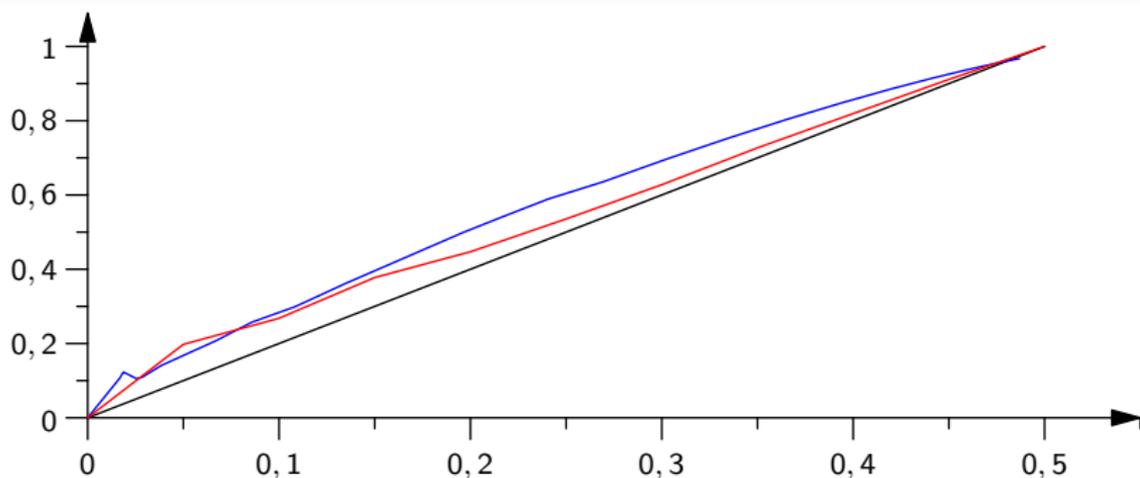
Size ≈ 2000000

Riemann Function

Theorem

Define the function

$$f(x) = \sum_{n=1}^{+\infty} \frac{[nx]}{n^2}.$$

The spectrum of singularities of this function is $D(h) = h$ for $h \in [0, 1]$.

Riemann Function
Size ≈ 2000000

———— Theoretical Spectrum

Riemann Function
Size ≈ 2000000

Riemann Function
Size ≈ 2000000

Theoretical Spectrum

WLM

S^{ν}

Plan

Deterministic Cascades

Deterministic Cascades

$p = 0.25$

Theoretical Spectrum
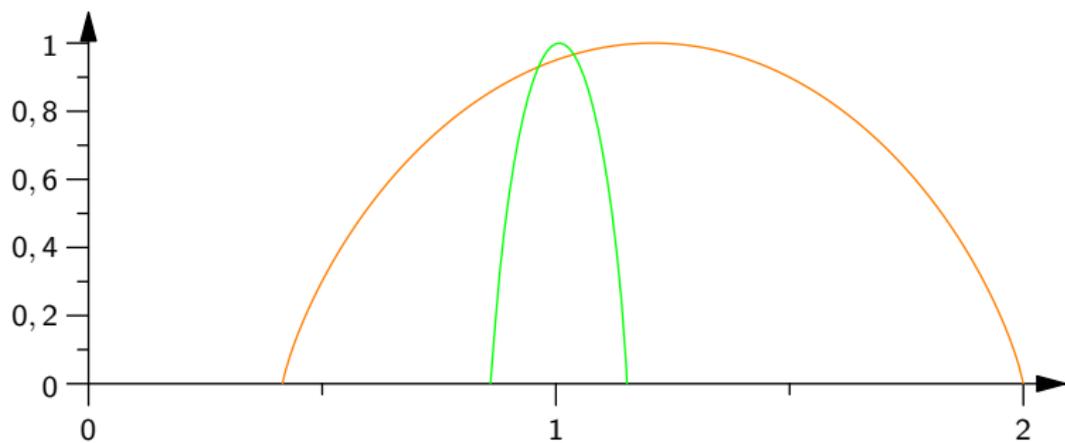
Deterministic Cascades

$p = 0.25$

Theoretical Spectrum
S^ν

Deterministic Cascades

Deterministic Cascades

Theoretical Spectra of Singularities

Deterministic Cascades

Theoretical Spectra of Singularities

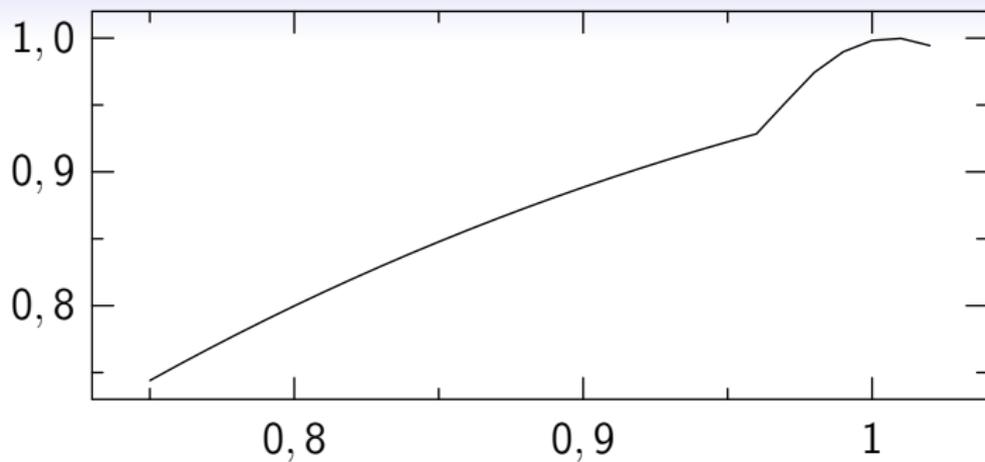|	
———	$p_1 = 0.25$
———	$p_2 = 0.45$
———	Superposition

Deterministic Cascades

——————— Theoritical Spectrum

Deterministic Cascades

Theoritical Spectrum

WLM

Deterministic Cascades

Theoritical Spectrum
WLM
S^ν

Deterministic Cascades

Theoretical Spectra of Singularities

——	$p = 0.2$
——	$p = 0.32$
——	$p = 0.45$

Deterministic Cascades

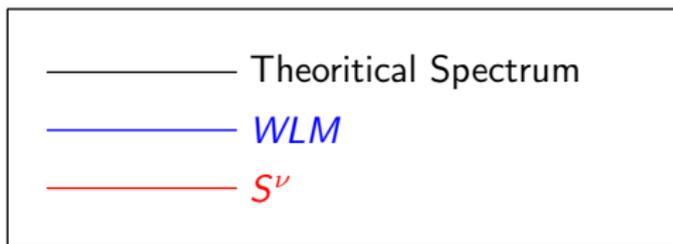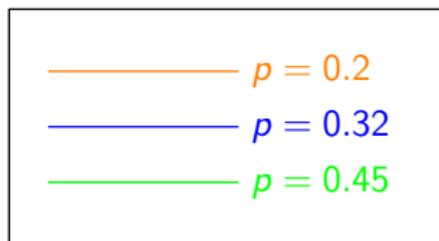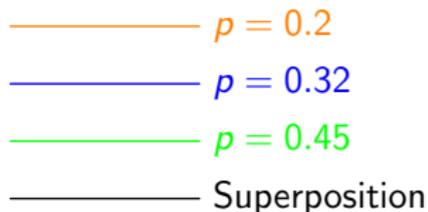## Theoretical Spectra of Singularities

Deterministic Cascades

———— Theoritical Spectrum

Deterministic Cascades
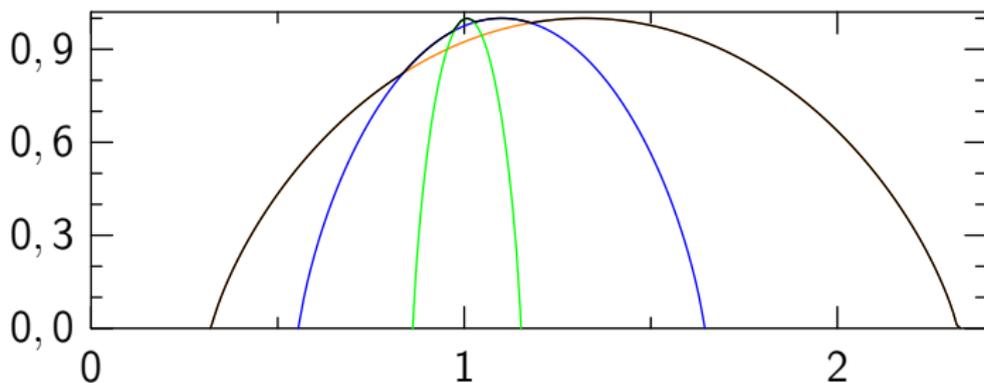

Theoritical Spectrum

S^ν

Plan

Conclusion

▶ For monofractal functions, the S^ν method is as accurate as the previous method ;

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Conclusion

- For monofractal functions, the S^ν method is as accurate as the previous method ;

- For multifractal functions, the S^ν method detectes a better spectrum of singularities than the previous method ;

- This method detectes non-concave spectra of singularities.

In the future,

- it should also be tested on real-life signals (turbulence, and so on) ;

- ...

Thank you for your attention!

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