S^ν Spaces, from Theory to Practice

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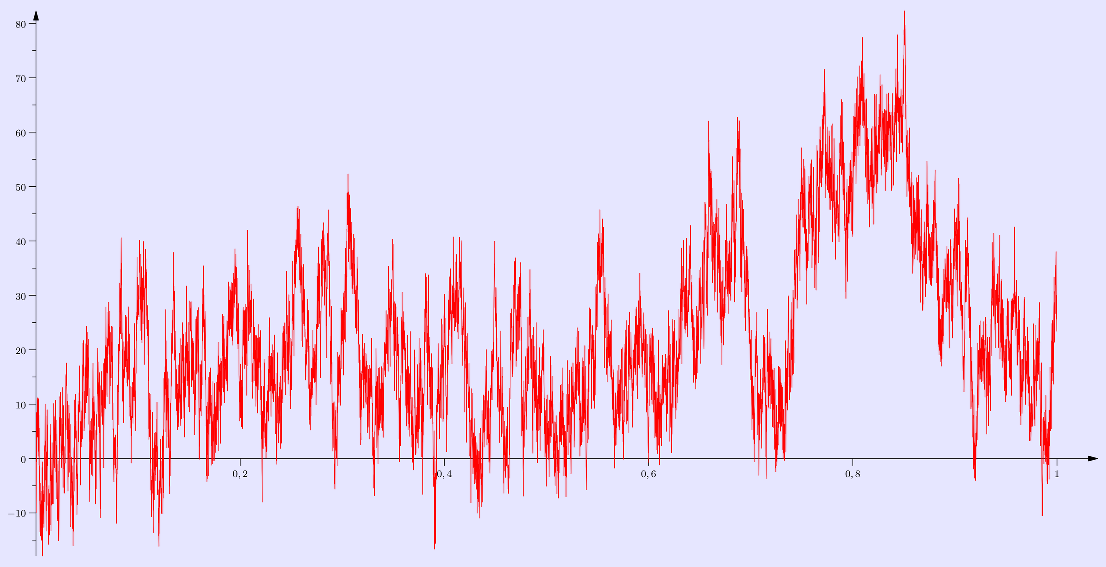
Multifractal Analysis

- Study of very *irregular functions* ;
- We determine the "size" of the set of points x which share the same "irregularity" $h_f(x)$;

$$D_f : h \mapsto \dim_{\mathcal{H}}(\{x : h_f(x) = h\});$$

  it gives a geometrical idea of the *repartition of the irregularity* ;
- In practice, we use a numerically computable function which "approximates" this size ; we use a *Multifractal Formalism*.



For Functions

Definition 1. Let $x \in \mathbb{R}$, $s \in \mathbb{R}_0^+$ and $f \in L^\infty_{loc}$. We denote $f \in C^s(x)$ if there exist a polynomial P of degree stricly smaller than s, a constant $C > 0$ and a neighbourhood Ω of 0 such that

$$|f(x+l) - P(l)| \leq C|l|^s$$

for all $l \in \Omega$.

Definition 2. Let $x \in \mathbb{R}$ and $f \in L^\infty_{loc}$; we denote the *Hölder exponent* of f at a point x by

$$h_f(x) = \sup\{s \in \mathbb{R}_0^+ : f \in C^s(t)\}.$$

For Measures

Definition 3. Let $x \in \mathbb{R}$ and μ a positive Borel measure on $\mathbb{R}$. We denote the *Hölder exponent* of μ at a point x by

$$h_\mu(x) = \liminf_{r \to 0^+} \frac{\log(\mu(B(x,r)))}{\log(r)}.$$

Wavelet

Take a mother wavelet ψ and $\{(\psi_{j,k}) : j \geq 0,\ k \in \{0, \ldots, 2^j - 1\}\}$ an orthonormal basis of $L^2([0;1])$ associated to ψ. We denote by $c_{j,k} = \langle f, \psi_{j,k}\rangle$ the periodized wavelet coefficients of $f \in L^2([0;1])$.

Theorem 4 (Barral, Seuret)**.** *Let μ be a positive Borel measure on $[0;1]$. If f is a function where $c_{j,k} = \mu([k2^{-j}; (k+1)2^{-j}[)$ then $D_f = D_\mu$.*

S^ν Spaces in Theory

Definition 5. We define the *wavelet profil* of a function $f \in L^2([0;1])$ by

$$\nu_f^C(\alpha) = \lim_{\epsilon \to 0^+}\left(\limsup_{j \to +\infty}\left(\frac{\ln(\#E_j(C, \alpha+\epsilon)(f))}{\ln(2^j)}\right)\right)$$

where $E_j(C,\alpha)(f) = \{k : |c_{j,k}| \geq C2^{-\alpha j}\}$.

Proposition 6. *For all $C_1, C_2 > 0$, $\nu_f^{C_1} = \nu_f^{C_2} := \nu_f$.*

Definition 7. Take a function $\nu : \mathbb{R} \to \{-\infty\} \cup [0;1]$ nondecreasing and right-continuous and assume that there exists $\alpha_{min} \geq 0$ such that $\nu(\alpha) = -\infty$ for all $\alpha < \alpha_{min}$ and $\nu(\alpha) \in [0;1]$ for all $\alpha \geq \alpha_{min}$. We define

$$S^\nu = \{f \in L^2([0;1]) : \nu_f(\alpha) \leq \nu(\alpha)\ \forall \alpha \in \mathbb{R}\}.$$

Theorem 8 (Aubry, Bastin, Dispa)**.** *For all $f \in S^\nu$, the function*

$$D_f^\nu(h) = \begin{cases} h \sup_{h' \in]0;h]} \frac{\nu(h')}{h'} & \text{if } h \leq h_{max} := \inf_{h \geq \alpha_{min}} \frac{h}{\nu(h)} \\ 1 & \text{otherwise} \end{cases}$$

is an upper bound of D_f and the set of functions where $D_f^\nu = D_f$ is prevalent in S^ν.

S^ν Spaces in Practice

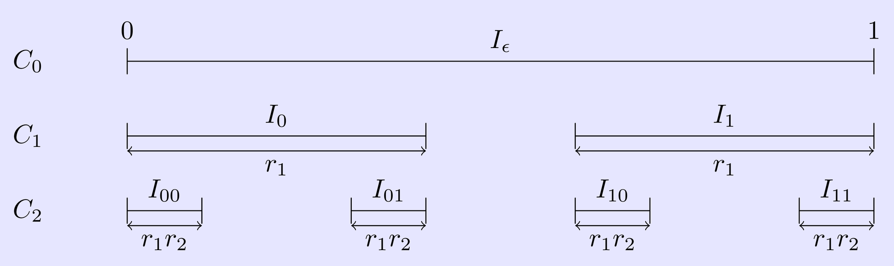
In practice, the constant $C > 0$ of $\nu^C(\alpha)$ is not arbitrary because we have only a finite number of wavelet coefficients :

- If C is too small, the detected value of $\nu_f^C(\alpha)$ will be 1 ;
- If C is too big, the detected value of $\nu_f^C(\alpha)$ will be $-\infty$.

For $\alpha \in \mathbb{R}$, we construct the function $C \mapsto \nu_f^C(\alpha)$.

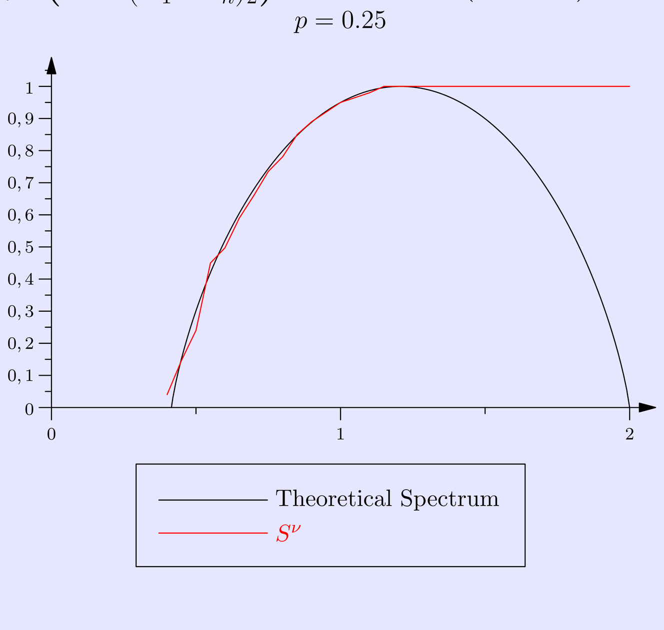
In practice, if $\alpha \geq \alpha_{min}$, this function is decreasing and stabilizes with an approximation of the theoritical value of $\nu_f(\alpha)$.

Spectrum of p-Cantor measure



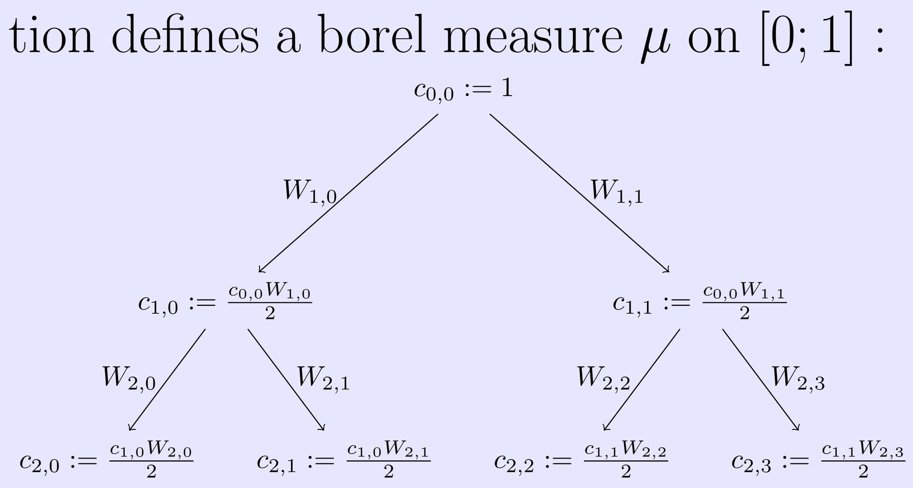
Take $(r_j)_{j \in \mathbb{N}}$ such that $\sup r_j < \frac{1}{2}$ and $\lim_{n \to +\infty} \frac{1}{n}\log(r_1 \ldots r_n)$ exists and is finite. Define $C := C((r_j)_j) = \bigcap_j C_j$. The *$p$-Cantor measure* on C is a measure such that

$$\mu\left(I_{w=(w_1 \ldots w_n)_2}\right) = p^{n-|w|_2}(1-p)^{|w|_2}.$$



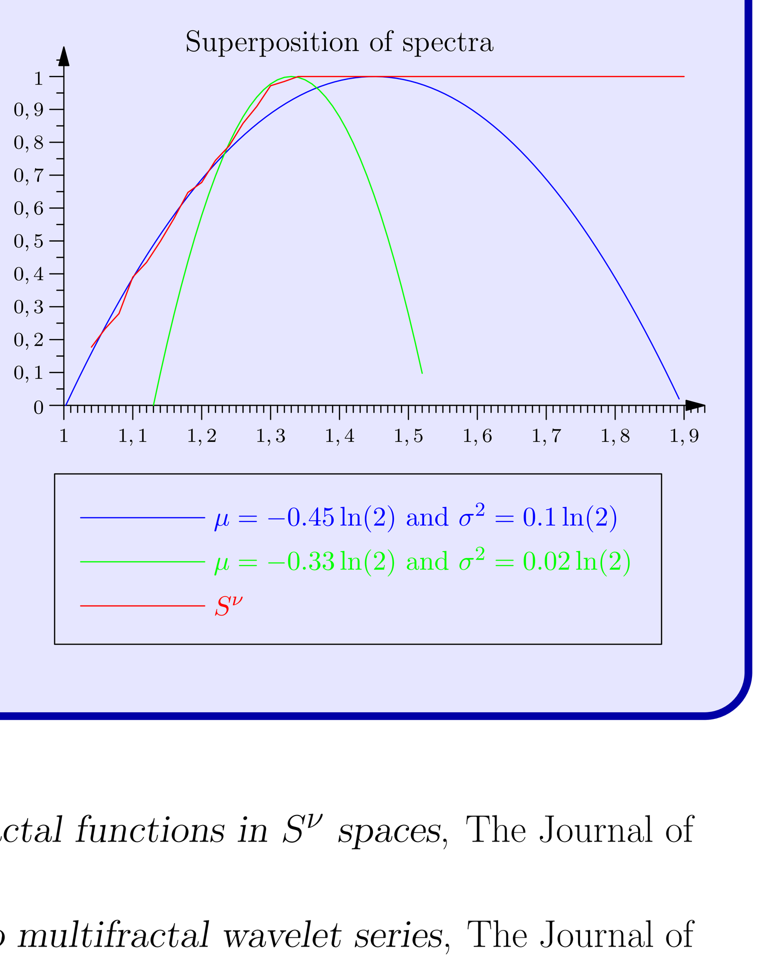
Spectrum of Cascades of Mandelbrot

Take W a positive random variable such that $E[W] = 1$ and $W_{j,k} \sim^{i.i.d} W$. Almost surely, the following construction defines a borel measure μ on $[0;1]$:



with $\mu([k2^{-j}; (k+1)2^{-j}]) = c_{j,k}$.

Take the example where W is a log-normal :



References :

[1] J.-M. Aubry, F. Bastin, S. Dispa *Prevalence of multifractal functions in S^ν spaces*, The Journal of Fourier Analysis and Applications, 2007.

[2] J. Barral and S. Seuret, *From multifractal measures to multifractal wavelet series*, The Journal of Fourier Analysis and Applications, 2002.