A GENERAL THEORY OF DUAL ERROR BOUNDS BY FINITE ELEMENTS

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Summary - This paper presents a general error bound theory for linear elliptic problems, which constitutes an extension of particular results obtained earlier by B. Fraeijs de Veubeke.

1. INTRODUCTION

The dual analysis technique, which was mainly developed by Fraeijs de Veubeke and his colleagues /1 to 9/, consists in a double computation of the same linear elliptic problem by two different ways. The first or primal one is based on displacement models while the second or dual one makes use of equilibrium models. Under certain conditions, such a dual analysis leads to an useful energetic measure of the discretization errors of both approaches.

Up to now, the applicability of this dual error bound technique seemed to be restricted to two particular cases corresponding, if we take stress analysis as an example, to zero applied loads or zero imposed displacements.

In this paper, an analysis is proposed, which proves the possibility to generalize dual analysis concepts to all cases of boundary conditions, by a slightly different evaluation of the error bounds. The theory is developed in a general abstract frame and includes as particular cases the two classical results due to Fraeijs de Veubeke /3,6/. The last section illustrates these concepts in the case of linear elasticity.
2. VARIATIONAL PROBLEMS

Let $E$ be a Hilbert space and $E_0$ a closed subspace of $E$. A bilinear form $a(p, q)$ is given on $E$, which verifies the following conditions:

\[
\begin{align*}
|a(p, q)| & \leq C \|p\|_E \|q\|_E \quad \text{(boundedness)} \\
a(p, q) &= a(q, p) \quad \text{(symmetry)} \\
a(p, p) &\geq \beta \|p\|^2_E, \quad \beta > 0, \quad \forall \, p \in E_0 \quad \text{(E - ellipticity)}
\end{align*}
\]

As usual, symbol $E'$ will denote the dual space of $E$.

In what follows, we will be concerned by variational problems which have the following general form:

\[\text{V.E. - Let be given } q_0 \in E \text{ and } f \in E'. \text{ Find } q \in E_0 \text{ such that } a(q_0 + q, q) = f(q) \quad \forall \, q \in E_0.\]

The existence and unicity of the solution of such a problem is guaranteed by Lax-Milgram's theorem /10,13/. Moreover, taking $q = \hat{q}$ in (4) leads to

\[a(\hat{q}, \hat{q}) = f(\hat{q}) - a(q_0, \hat{q}),\]

which implies by the properties of the bilinear form

\[\|q\|_E \leq \frac{A}{B} (\|f\|_{E'} + C \|q_0\|_E),\]

an inequality which proves that this problem is a well-posed one.

We emphasize that most interest is in general devoted to the complete solution

\[\hat{f} = q_0 + \hat{q}\]

In this view, note that there are a lot of different $q_0$'s which lead to the same complete solution. Consider for instance $q_{o1} \neq q_{o2}$. With $q_{o1}$, the variational equation will be

\[a(q_{o1} + \hat{q}, q) = f(q) \quad \forall \, q \in E_0.\]

With $q_{o2}$, it will be

\[a(q_{o2} + \hat{q}, q) = f(q) \quad \forall \, q \in E_0.\]

Taking the difference between these two equations, one obtains

\[a( (q_{o1} + \hat{q}) - (q_{o2} + \hat{q}), q) = 0 \quad \forall \, q \in E_0.\]

Under the condition

\[ (q_{o1} + \hat{q}) - (q_{o2} + \hat{q}) \in E_0, \]

this element itself may be taken as a particular $q$. This implies by (3)

\[0 = a( (q_{o1} + \hat{q}) - (q_{o2} + \hat{q}), (q_{o1} + \hat{q}) - (q_{o2} + \hat{q}) )\]
\[ \beta \| (q_{o1} + \hat{\alpha}_1) - (q_{o2} + \hat{\alpha}_2) \|_E^2, \]

that is to say
\[ q_{o1} + \hat{\alpha}_1 = q_{o2} + \hat{\alpha}_2. \]

As condition (7) is equivalent to
\[ q_{o1} - q_{o2} \in E_0, \]
we shall say that two elements \( q_{o1} \) and \( q_{o2} \) of \( E \) are \( E_0 \)-equivalent if they differ only by an element of \( E_0 \). Thus, two variational problems involving the same bilinear form, the same \( f \) and two \( E_0 \)-equivalent \( q_o \)'s lead to the same complete solution.

Now, a very simple lemma which will be useful in the following. Suppose that \( f(E_0) \neq \{ 0 \} \) unless \( f = 0 \). Then, among all \( q_o \) which are \( E_0 \)-equivalent to a given \( q_o \), it is possible to select (at least) one particular \( \tilde{q}_o \) verifying \( f(\tilde{q}_o) = 0 \).

Remark first that if \( f(E_0) = \{ 0 \} \), \( f \) is equivalent to zero in \( E_0 \), and may be replaced by zero without change of the solution. Our restriction is thus natural.

If \( f = 0 \), the lemma is of trivial nature. In other cases, consider \( p \in E_0 \) such that \( f(p) \neq 0 \). Taking
\[ \tilde{q}_o = q_o - \frac{f(q_o)}{f(p)} p \]  \hspace{1cm} (9)
leads the announced result.

As is well known, every symmetric variational problem of the form \( \min_{q \in E} \) is equivalent to the minimization of the functional
\[ \mathcal{R}(q_o + q) = \frac{1}{2} a(q_o + q, q_o + q) - f(q_o + q) \]
among all \( q \in E_0 \) or, in other words, to the following minimization problem:
\[ \text{M.P.} - \text{Find} \ \hat{q} \in q_o + E_0 \ \text{which minimizes in this linear manifold the functional} \]
\[ \mathcal{R}(r) = \frac{1}{2} a(r, r) - f(r). \]  \hspace{1cm} (10)

3. **RAYLEIGH-RITZ APPROXIMATIONS**

The basis of Rayleigh-Ritz approximations (including conforming finite element techniques) is to replace the space \( E \) by a finite dimensional subspace \( E_h \). Subspace \( E_0 \) is then replaced by \( E_{oh} = E_h \cap E_0 \).

It is necessary to make the three following restrictions:
\[ R_1 : E_{oh} \neq \{ 0 \} \]
\[ R_2 : f(E_{oh}) \neq \{ 0 \} \]
$R_3$: The given $q_o$ is $E_o$-equivalent to some $\bar{q}_o \in E_o$.

Conditions $R_1$ and $R_2$, which only exclude "too little" subspaces leading to meaningless solutions, are fulfilled in all reasonable idealizations. In contrast, condition $R_3$ is a real restriction. Its purpose is to preserve the inclusion

$$\bar{q}_o + E_{oh} \subset \bar{q}_o + E_o.$$

Rayleigh-Ritz' technique consists to minimize $\mathcal{J}(r)$ on $\bar{q}_o + E_{oh}$. If the three preceding conditions are verified, this approximate problem will be called in the following a regular internal approximation. For simplicity, overbars on $q_o$ will be omitted.

The approximate solution $\hat{f}_h = q_o + \hat{a}_h$ verifies

$$a(q_o + \hat{a}_h, q_h) = f(q_h)$$

whatever $q_h \in E_o$. Since $q_o + E_{oh} \subset q_o + E_o$, one has the inequality

$$\mathcal{J}(\hat{f}_h) = \min_{r_h \in q_o + E_{oh}} \mathcal{J}(r_h) \geq \min_{r \in q_o + E_o} \mathcal{J}(r) = \mathcal{J}(\hat{f})$$

(12)

Now, the quantity

$$d(\hat{f}_h, \hat{f}) = \left( a(\hat{f}_h - \hat{f}, \hat{f}_h - \hat{f}) \right)^{1/2}$$

will be called the energetic distance between $\hat{f}$ and $\hat{f}_h$. The word distance is justified by the fact that, since $\hat{f} - \hat{f}_h \in E_o$,

$$d^2(\hat{f}, \hat{f}_h) \geq \beta \| \hat{f} - \hat{f}_h \|_E^2.$$

Calculating this distance, one has

$$d^2(\hat{f}, \hat{f}_h) = a((q_o + \hat{a}_h) - (q_o + \hat{a}), (q_o + \hat{a}_h) - (q_o + \hat{a}))$$

$$= a(q_o + \hat{a}_h, q_o + \hat{a}_h) + a(q_o + \hat{a}, q_o + \hat{a})$$

$$- 2a(q_o + \hat{a}, q_o + \hat{a}_h).$$

Noting that

$$2a(q_o + \hat{a}, q_o + \hat{a}_h) = 2a(q_o + \hat{a}, q_o + \hat{a}) + 2a(q_o + \hat{a}, \hat{a}_h - \hat{a})$$

$$= 2a(q_o + \hat{a}, q_o + \hat{a}) + 2\mathcal{F}(\hat{a}_h - \hat{a}),$$

where account has been taken of relation (4) and that $\hat{a}_h - \hat{a} \in E_o$,

one obtains

$$d^2(\hat{f}, \hat{f}_h) = a(q_o + \hat{a}_h, q_o + \hat{a}_h) - 2\mathcal{F}(q_o + \hat{a}_h) + 2\mathcal{F}(q_o + \hat{a})$$

$$- a(q_o + \hat{a}, q_o + \hat{a}),$$

i.e. the following fundamental result:
\[ d^2(\mathbf{f}, \mathbf{f}_n) = 2 \left( \mathcal{K}_n(\mathbf{f}_n) - \mathcal{K}(\mathbf{f}) \right). \]  

(13)

Note finally that the three restrictions \( R_1 \) to \( R_3 \) permit to suppose, as in the undiscretized problem, that \( f(q_0) = 0 \), \( q_0 \) being in \( E_{oh}^* \).

4. **GENERAL DUAL ANALYSIS SITUATIONS**

The general dual analysis frame may be described as follows. Let \( V \) and \( H \) be two Hilbert spaces, equipped with scalar products \( (u,v)_V \) and \( (\mathbf{e}, \mathbf{\eta})_H \). Let \( \mathcal{D} \) be a linear operator from \( V \) to \( H \), which is bounded,

\[ \|\mathcal{D}\|_H \leq M_1 \|\mathcal{D}\|_V \]  

(14)

Let now \( V_0 \) be a closed subspace of \( V \), in which the following ellipticity condition is verified

\[ (\mathcal{D}u, \mathcal{D}u)_H \geq \alpha \|u\|_V^2 \quad \forall \ u \in V_0, \ \alpha > 0 \]  

(15)

This inequality implies that the image \( \mathcal{D}V_0 \) of \( V_0 \) in \( H \) is a closed subspace of \( H \).

Finally, let \( C \) be a linear operator from \( H \) to \( H \) which exhibits the following properties:

\[
\begin{cases}
\|C\mathbf{e}\|_H \leq M_2 \|\mathbf{e}\|_H \quad \text{(boundedness)} \\
(\mathbf{e}, \mathbf{\eta})_H = (\mathbf{e}, C\mathbf{\eta})_H \quad \text{(hermiticity)} \\
(\mathbf{e}, \mathbf{\eta})_H \geq \gamma \|\mathbf{e}\|_H^2, \gamma > 0 \quad \text{(ellipticity)}
\end{cases}
\]  

(16)-(18)

These conditions imply the existence of an inverse operator \( C^{-1} \) which is also bounded, hermitic and elliptic on \( H \). In fact, for a given \( \mathbf{\sigma} \in H \), the equation

\[ C\mathbf{e} = \mathbf{\sigma} \]  

(19)

corresponds to seeking an element \( \mathbf{e} \in H \) such that

\[ (C\mathbf{e} - \mathbf{\sigma}, \mathbf{\eta})_H = 0 \quad \forall \ \mathbf{\eta} \in H \]

or equivalently,

\[ (C\mathbf{e}, \mathbf{\eta})_H = (\mathbf{\sigma}, \mathbf{\eta})_H \quad \forall \ \mathbf{\eta} \in H \]

By virtue of Lax-Milgram's theorem, this problem admits an unique solution verifying

\[ \|\mathbf{e}\|_H \leq \frac{1}{\gamma} \|\mathbf{\sigma}\|_H, \]

that is to say

\[ \|C^{-1}\mathbf{\sigma}\|_H \leq \frac{1}{\gamma} \|\mathbf{\sigma}\|_H. \]  

(20)

Now, taking \( \mathbf{e} = C^{-1}\mathbf{\sigma} \) in (16) leads

\[ \|C^{-1}\mathbf{\sigma}\|_H \geq \frac{1}{M_2} \|\mathbf{\sigma}\|_H \]
and this implies by (18) the ellipticity relation
\[ (\sigma, c^{-1}\sigma)_H = (c^{-1}\sigma, c^{-1}\sigma)_H \geq \frac{\gamma}{\lambda^2} \| e^{-1} \sigma \|_H^2 \geq \frac{\gamma}{\lambda^2} \| e^{-1} \sigma \|_H^2 \]
\[ (\sigma, c^{-1}\sigma)_H = (c^{-1}\sigma, \tau)_H \]

(21)

Now, by setting \( \epsilon = c^{-1}\sigma \) and \( \eta = c^{-1}\tau \) in (17), one obtains
\[ (\sigma, c^{-1}\tau)_H = (c^{-1}\sigma, \tau)_H \]

that is, the hermiticity relation for \( c^{-1}\).

5. THE PRIMAL PROBLEM

The primal problem consists as follows
\[ P.P. \quad -u_0 \in V \text{ and } f \in V' \text{ being given, find } \hat{u} \in V_o \text{ such that} \]
\[ (C(\partial u + \partial \hat{u}), \partial \hat{u})_H = f(u) \]

(22)

It is easy to see that this problem is a particular form of problem \( P.P. \) described in section 2. This results from the inequalities
\[ \begin{align*}
& \| (C \partial u, \partial v)_H \| \leq M_2 \| \partial u \|_H \| \partial v \|_H \leq M_2 M_1^2 \| u \|_V \| v \|_V \\
& (C \partial u, \partial \hat{u})_H \geq \frac{\gamma}{\lambda} \| \partial u \|_H^2 > \frac{\alpha}{\gamma} \| u \|_V^2 \quad \forall u \in V_o
\end{align*} \]

The primal problem is also equivalent to the minimization of the primal functional
\[ \mathcal{K}_1(w) = \frac{1}{2} (C \partial w, \partial w) - f(w) \]
on the linear manifold \( u_0 + V_o \). Consequently, any regular internal approximation of this problem will lead to
\[ \mathcal{K}_1^2(\theta, \theta) = (C(\partial \theta_h - \partial \theta), \partial \theta_h - \partial \theta) = 2 (\mathcal{K}_1(\theta) - \mathcal{K}_1(\theta)) \]

(23)

6. THE DUAL PROBLEM

Let us introduce the subspace \( S_o \), defined as the orthogonal subspace of \( \partial V_o \) in \( H \),
\[ S_o = \{ \sigma \in H \mid (\sigma, \partial v)_H = \sigma, \partial v \in V_o \} \]

(24)

This is naturally a closed subspace. Owing to the fact that \( \partial V_o \) is also closed, one concludes to the interrelations
\[ S_o^\perp = \partial V_o, \quad \partial V_o^\perp = S_o. \]

(25)

For any arbitrary functional \( f \in V' \), it is possible to find an element \( \sigma_o \in H \) such that
\[ (\sigma_o, \partial v)_H = f(v) \quad \forall v \in V_o. \]

(26)

In fact, any
\[ \sigma_o = C(\partial u_o + \partial \alpha) + \tau , \]
where \((u_o + \alpha)\) is the solution of the primal problem and \(\tau \in S_o\) visibly satisfies to condition (28).

Now, the dual problem is as follows

\textbf{D.P. - A functional} \(f \in V\) and an element \(u_o \in V\) being given, let \(\sigma_o\) be such that
\[ (\sigma_o, \partial v)_H = f(v) \quad \forall \ v \in V_o. \]

Find \(\sigma \in S_o\) verifying
\[ (c^{-1}(\sigma_o + \sigma), \sigma)_H = (\partial u_o, \sigma)_H \quad (29) \]
whatever \(\sigma \in S_o\).

This problem is also a particular form of problem \(V.P.\) described in section 2, since
\[ \begin{align*}
\| (c^{-1}(\sigma, \tau)_H \| 
\leq \frac{1}{2} \| \sigma \|_H \| \tau \|_H \\
( c^{-1}(\sigma, \sigma)_H \geq \frac{2}{m^2_2} \| \sigma \|_H^2 \quad \forall \sigma \in H \supset S_o. 
\end{align*} \]

The fact that the solution \(\hat{\sigma}\) verifies
\[ (c^{-1}(\sigma_o + \hat{\sigma}) - \partial u_o, \sigma)_H = 0 \quad \forall \sigma \in S_o \]
implies
\[ c^{-1}(\sigma_o + \hat{\sigma}) - \partial u_o \in S_o \perp = \partial V_o, \]
that is to say, there exists \(\partial v \in V_o\) such that
\[ c^{-1}(\sigma_o + \hat{\sigma}) = \partial u_o + \partial v, \]
or
\[ \sigma_o + \hat{\sigma} = C(\partial u_o + \partial v). \]
It follows from the definition of \(\sigma_o\) and \(\hat{\sigma}\) that
\[ f(u) = (\sigma_o + \hat{\sigma}, \partial u)_H = (C(\partial u_o + \partial v), \partial u)_H \quad \forall \ u \in V_o \]
and this proves that \(v\) is precisely the solution \(\hat{\alpha}\) of the primal problem. In other words, the primal and the dual problem lead to two different forms of the same solution.

The dual problem is equivalent to the minimization of the dual functional
\[ \delta^t_2(\rho) = \frac{1}{2} (c^{-1}(\rho, \rho)_H - (\partial u_o, \rho)_H \quad (30) \]
on the linear manifold \(\sigma_o + S_o\). Consequently, any regular internal approximation of this problem will lead to the following error bound:
\[ d_2^2(\hat{\varrho}_h, \hat{\varrho}) = (c^{-1}(\hat{\varrho}_h - \hat{\varrho}), \hat{\varrho}_h - \hat{\varrho})_H = 2(\mathcal{H}_2(\hat{\varrho}_h) - \mathcal{H}_2(\hat{\varrho})) \tag{31} \]

7. **DUAL ERROR BOUNDS**

It is possible to uniformize the two distances \( d_1 \) and \( d_2 \) by noting that

\[ d_1^2(\vartheta_h, \vartheta) = (c(\vartheta_h - \vartheta), \vartheta_h - \vartheta)_H = (c^{-1}c(\vartheta_h - \vartheta), c(\vartheta_h - \vartheta))_H = d_2^2(c\vartheta_h, c\vartheta). \tag{32} \]

Now, the fact that \( \hat{\varrho} = c\vartheta \) implies

\[ 0 = d_2^2(\hat{\varrho}, c\vartheta) = (c^{-1}(\hat{\varrho} - c\vartheta), \hat{\varrho} - c\vartheta)_H = (c^{-1}\hat{\varrho}, \hat{\varrho})_H + (c\vartheta, \vartheta)_H - 2(c\vartheta, \hat{\varrho})_H \]

The last term may be transformed as follows:

\[ -2(c\vartheta, \hat{\varrho})_H = -2(\sigma_o + \hat{\varrho}, \vartheta_o + \vartheta)_H = -2(\sigma_o + \hat{\varrho}, \vartheta_o)_H - 2(\sigma_o + \hat{\varrho}, \vartheta)_H - f(\vartheta) = -2(\sigma_o + \hat{\varrho}, \vartheta_o)_H - f(u_o + \vartheta), \]

where account has been taken to (28) and the fact that one may always assume that \( f(u_o) = 0 \). As a result, one obtains

\[ \mathcal{H}_1(\vartheta) + \mathcal{H}_2(\hat{\varrho}) = 0 \tag{33} \]

Let now \( \vartheta_h \) and \( \hat{\varrho}_h \) be regular internal approximations of \( \vartheta \) and \( \hat{\varrho} \).

The fact that \( \hat{\varrho} = c\vartheta \) implies

\[ d_2^2(\hat{\varrho}_h, c\vartheta_h) = (c^{-1}(\hat{\varrho}_h - c\vartheta_h), \hat{\varrho}_h - c\vartheta_h)_H = (c^{-1}(\hat{\varrho}_h - \hat{\varrho} + c\vartheta - c\vartheta_h), \hat{\varrho}_h - \hat{\varrho} + c\vartheta - c\vartheta_h)_H = d_2^2(\hat{\varrho}_h, \hat{\varrho}) + d_2^2(c\vartheta_h, c\vartheta_h) + 2(\hat{\varrho}_h - \hat{\varrho}, \vartheta - \vartheta_h)_H \]

Given that

\[ \hat{\varrho}_h - \hat{\varrho} = \sigma_h - \sigma \in S_o \]

\[ \vartheta_h - \vartheta = \vartheta_h - \vartheta \in \partial V_0 = S_o \perp, \]

the last term vanishes, so that

\[ d_2^2(\hat{\varrho}_h, c\vartheta_h) = d_2^2(\hat{\varrho}_h, \hat{\varrho}) + d_2^2(\vartheta_h, \vartheta_h) = 2(\mathcal{H}_2(\hat{\varrho}_h) - \mathcal{H}_2(\hat{\varrho})) + 2(\mathcal{H}_1(\vartheta_h) - \mathcal{H}_1(\hat{\varrho})) \]

and from (33), we find the following expression for the distance
\[ d_2^2(\hat{\rho}_h, c \phi_h) = 2 (\mathcal{K}_1(\phi_h) + \mathcal{K}_2(\hat{\rho}_h)) \]  

(34)

Of course,

\[ d_2^2(\phi, \phi_h) \leq d_1^2(\phi, \phi_h) + d_2^2(\hat{\rho}, \hat{\rho}_h) = 2(\mathcal{K}_1(\phi_h) + \mathcal{K}_2(\hat{\rho}_h)) \]  

(35)

and similarly,

\[ d_2^2(\hat{\rho}, \hat{\rho}_h) \leq 2(\mathcal{K}_1(\phi_h) + \mathcal{K}_2(\hat{\rho}_h)) \]  

(36)

Results (34) to (36) form the basis of a general dual error bound analysis. For the same physical situation, perform two computations, the first one being a regular internal approximation of the primal problem, and the second one, a regular internal approximation of the dual problem. Then, inequalities (35) and (36) provide error bounds for both discretization errors. This result, obtained here in the general inhomogeneous case, is of great interest at a practical point of view, from it permits a secure evaluation of the actual error, independently of any convergence result. In order to qualify a given analysis, the above error bounds may be compared to the computed energetic norms, i.e., \((c^{-1}\hat{\rho}_h, \hat{\rho}_h)_H\) and \((c \phi_h, \phi_h)_H\).

Two particular results of this type were obtained by Prasijjs de Veubeke /1,2,3,6/ and widely used by his colleagues /4,5,7,8/. They correspond to the following situations

a) \( u_0 = 0 \). In this case, \( \phi = \hat{\phi}, \phi_h = \phi_h \) and, by (4),

\[ \mathcal{K}_1(\phi_h) = \frac{1}{2} (c \phi_h, \phi_h)_H - f(\phi_h) = - \frac{1}{2}(c \phi_h, \phi_h)_H \]

and similarly,

\[ \mathcal{K}_1(\phi) = - \frac{1}{2} (c \phi, \phi)_H, \]

while

\[ \mathcal{K}_2(\hat{\rho}_h) = \frac{1}{2} (c^{-1}\hat{\rho}_h, \hat{\rho}_h)_H, \quad \mathcal{K}_2(\hat{\rho}) = \frac{1}{2} (c^{-1}\hat{\rho}, \hat{\rho})_H, \]

so that

\[ d_2^2(\hat{\rho}_h, c \phi_h) = (c^{-1}\hat{\rho}_h, \hat{\rho}_h)_H - (c \phi_h, \phi_h)_H \]  

(37)

From equations (25), (31) and (33), one can deduce

\[ (c \phi_h, \phi_h)_H \leq (c \phi, \phi)_H = (c^{-1}\hat{\rho}, \hat{\rho})_H \quad (c^{-1}\hat{\rho}_h, \hat{\rho}_h)_H \]  

(38)

b) \( \sigma_0 = 0 \). Here, the situation is reversed. One has

\[ \mathcal{K}_2(\phi_h) = - \frac{1}{2} (c^{-1}\phi_h, \phi_h)_H, \quad \mathcal{K}_2(\hat{\rho}) = - \frac{1}{2} (c^{-1}\hat{\rho}, \hat{\rho})_H \]

and

\[ \mathcal{K}_1(\phi) = \frac{1}{2} (c \phi, \phi)_H, \quad \mathcal{K}_1(\phi_h) = \frac{1}{2} (c \phi_h, \phi_h)_H. \]
so that
\[ d^2_h \left( \hat{\rho}_h, \circ \partial \hat{\phi}_h \right) = \langle c \circ \hat{\phi}_h, \circ \hat{\phi}_h \rangle = \langle c^{-1} \hat{\rho}_h, \hat{\rho}_h \rangle \]  
(39)

and
\[ \langle c^{-1} \hat{\rho}_h, \hat{\rho}_h \rangle \leq \langle c^{-1} \hat{\rho}_h, \hat{\rho}_h \rangle = \langle c \circ \hat{\phi}_h, \circ \hat{\phi}_h \rangle \leq \langle c \circ \hat{\phi}_h, \circ \hat{\phi}_h \rangle \]
(40)

Relations (38) and (40) are known by structural analysts as bounds on the direct influence coefficients.

8. APPLICATION TO STRESS ANALYSIS

Let \( \Omega \) be an open set of \( \mathbb{R}^3 \), and define
\[ \mathcal{V} = \left( H^1(\Omega) \right)^3 = \left\{ u = (u_1, u_2, u_3) \mid u_1 \in L^2(\Omega), D_1 u_2 \in L^2(\Omega) \right\}, \]
(41)
equipped with the scalar product
\[ (u, v)_{\mathcal{V}} = \int_\Omega (u_1 v_1 + D_1 u_1 D_1 v_1) \, dx \]
(42)
where repeated indices imply summation over the three values 1, 2, 3. The boundary \( \Gamma \) of \( \Omega \) consists of two parts \( \Gamma_1 \) and \( \Gamma_2 \), in such a manner that
\[ \begin{cases} \Gamma & \subset \Gamma_1 \cup \Gamma_2 \\ \int_{\Gamma_1} ds & \neq 0, \\ \int_{\Gamma_2} ds & = 0. \end{cases} \]
(43)

Let \( \mathcal{V}_0 \) be the subspace of displacement whose traces on \( \Gamma_1 \) vanish
\[ \mathcal{V}_0 = \left\{ u \in \mathcal{V} \mid u \mid_{\Gamma_1} = 0 \right\} \]
(44)

The operator \( \partial \) is defined as
\[ \partial_{ij} u = \frac{1}{2} (D_{i} u_{j} + D_{j} u_{i}) \]
(45)
and transforms \( \mathcal{V} \) in the space given by
\[ \mathcal{H} = \left\{ \epsilon_{ij} \mid \epsilon_{ij} = \epsilon_{ji}, \epsilon_{ij} \in L^2(\Omega) \right\} \]
(46)

From Korn's inequality /11,13/, any \( u \in \mathcal{V}_0 \) verifies
\[ \int_\Omega \partial_{ij} u \partial_{ij} v \, dx \geq \alpha \| u \|_{\mathcal{V}}^2, \quad \alpha > 0 \]
(47)

Finally, let the operator \( C \) be defined by its 21 independent components (which may depend upon \( x \))
\[ C_{ijkl}(x) = C_{jikl}(x) = C_{ijlk}(x) = C_{klij}(x) \in L^\infty(\Omega), \]
(48)
with an uniform positive definiteness condition on \( \Omega \):\[ C_{ijkl}(x) \epsilon_{ij} \epsilon_{kl} \geq \gamma \epsilon_{ij} \epsilon_{ij}, \quad \gamma > 0 \]
(49)
The classical formulation of elasticity problems consists to solve (in the distributional sense) the following equations:
\[
\begin{align*}
\begin{cases}
D^{j}_{i} (c_{ijkl} \partial_{kl} u) + f_{i} = 0 & \text{in } \Omega, \quad f_{i} \in L^{2}(\Omega) \\
n^{j}_{i} (c_{ijkl} \partial_{kl} u) = t_{i} & \text{on } \Gamma_{2}, \quad t_{i} \in H^{-\frac{1}{2}}(\Gamma_{2}) \\
u_{i} = \bar{u}_{i} & \text{on } \Gamma_{1}, \quad \bar{u}_{i} \in H^{\frac{1}{2}}(\Gamma_{1})
\end{cases}
\end{align*}
\]

(For the definition of $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$, see e.g. /10, 11/)

The primal formulation consists to solve this system in the virtual work sense. Let first $u^{0} \in V$ be such that

\[
u^{0}_{i} = \bar{u}_{i} \quad \text{on } \Gamma_{1}.
\]

As pointed in section 2, one may assume that

\[
\int_{\Omega} f_{i} u^{0}_{i} \, dx + \int_{\Gamma_{2}} t_{i} u^{0}_{i} \, dS = 0
\]

In the primal formulation, the functional to minimize is

\[
\mathcal{H}_{1}(u) = \frac{1}{2} \int_{\Omega} c_{ijkl} \partial_{ij} u \partial_{kl} u \, dx - \int_{\Omega} f_{i} u_{i} \, dx - \int_{\Gamma_{2}} t_{i} u_{i} \, dS
\]

This minimization has to be performed in the manifold $u^{0} + V_{0}$.

It is the well-known total energy principle.

In the dual formulation, our first task is to seek a particular field $\sigma^{0} \in H$ such that

\[
\int_{\Omega} \sigma^{0}_{ij} \partial_{ij} u \, dx = \int_{\Omega} f_{i} u_{i} \, dx + \int_{\Gamma_{2}} t_{i} u_{i} \, dS \quad \forall u \in V_{0}
\]

i.e. verifying the equilibrium conditions in the virtual work sense.

Noting that the space $(\mathcal{D}(\Omega))^{3}$ of smooth displacements with compact support within $\Omega$ is contained in $V_{0}$, one can see that (54) implies

\[
D^{j}_{i} \sigma^{0}_{ij} + f_{i} = 0 \quad \text{in } \Omega,
\]

in the distributitional sense. Therefore, the vectors

\[\sigma_{ij} = (\sigma_{1j}, \sigma_{2j}, \sigma_{3j})\]

are elements of the space

\[
H^{\text{div}}(\Omega) = \{ p = (p_{1}, p_{2}, p_{3}) \mid p_{i} \in L^{2}(\Omega), \ \text{div} \ p \in L^{2}(\Omega) \}
\]

On this space, it is possible /11/ to define boundary traces

\[
n^{j}_{i} \sigma_{jj} \in H^{-\frac{1}{2}}(\Gamma)
\]

which, by virtue of (54) and (55), will here verify

\[
n^{j}_{i} \sigma_{jj} = t_{i} \quad \text{on } \Gamma_{2},
\]

as can be verified by a formal integration by parts.

Subspace $S_{0}$ is defined as

\[
S_{0} = \{ \sigma \in H \mid \int_{\Omega} \sigma_{ij} \partial_{ij} u \, dx = 0 \quad \forall u \in V_{0} \}
\]
By a similar argument as for $\sigma^o$, each $\sigma \in S_o$ verifies
\[
\begin{align*}
\sigma_i &= (\sigma_{1i}, \sigma_{2i}, \sigma_{3i})_{H_{\text{div}}(\Omega)} \\
D_i \sigma_{ji} &= 0 \text{ in } \Omega \\
n_j \sigma_{ji} &= 0 \text{ on } \Gamma_2
\end{align*}
\]
(58)

The dual problem consists thus in the minimization in $\sigma^o + S_o$ of the dual functional
\[
\mathcal{J}_2(\sigma) = \frac{1}{2} \int_\Omega C^{-1}_{ijkl} \sigma_{ij} \sigma_{kl} \, dx - \int_{\Gamma_2} \sigma_{ij} \partial_i u^o \, dx
\]
This expression may be reduced to a more familiar one by transforming the last term in the following way. First, an integration by parts gives
\[
\int_\Omega \sigma_{ij} \partial_i u^o \, dx = \frac{1}{2} \int_\Omega (\sigma_{ij} (D_i u_j^o + D_j u_i^o)) \, dx = \int_{\Gamma_1} \sigma_{ij} n_j u^o_i \, ds - \int_{\Gamma_2} (D_j \sigma_{ji}) u^o_i \, dx
\]
Recalling that $\sigma_{ij} \in \sigma^o + S_o$ and taking account to relations (55) to (58), one obtains
\[
\int_{\Gamma_1} t_i u^o_i \, ds + \int_{\Gamma_2} n_j \sigma_{ji} u^o_i \, ds + \int_{\Gamma_1} f_i u^o_i \, dx = \int_{\Gamma_2} n_j \sigma_{ji} u^o_i \, ds
\]
the last expression resulting from assumption (54). Finally,
\[
\mathcal{J}_2(\sigma) = \frac{1}{2} \int_\Omega C^{-1}_{ijkl} \sigma_{ij} \sigma_{kl} \, dx - \int_{\Gamma_1} n_j \sigma_{ji} u^o_i \, ds \quad .
\]
(59)

The reader will recognize the complementary energy principle.

We now turn our attention on finite element approximations of these two problems. The case of the primal problem (displacement model) is rather simple. Condition $V_h \subset V$ is the well-known conformity condition. Restriction $R_3$ of section 3 means that the Dirichlet conditions on $\Gamma_1$ have to be taken in account exactly in the finite element model.

The dual problem requires somewhat more care. Condition $H_h \subset H$ requires only
\[
\sigma_{ij} \in L^2(\Omega),
\]
a condition which is fulfilled by all local polynomial approximations on a bounded set. The necessity of ensuring interelement transmissions appears from the definitions of $\sigma^o$ and $S_o$. If $\mathcal{K}_h$ is the set of finite elements $K$, one may write
\[
\int_{\Omega} \sigma_{ij}^o \partial_{ij} u \, dx = \varepsilon \int_{k \in \Omega_h} \sigma_{ij}^o \partial_{ij} u \, dx = \\
\varepsilon \int_{k \in \Omega_h} n_j \sigma_{ji}^o u_i \, ds - \varepsilon \int_{k \in \Omega_h} (D_j \sigma_{ji}^o) u_i \, dx
\]

the integration by parts being justified by the fact that \( \sigma^o \) is smooth in each element. The conditions are now

\[
\begin{align*}
\{ & D_j \sigma_{ji}^o + f_i = 0 \quad \text{in each } k \in \Omega_h \\
& n_j \sigma_{ji}^o = \text{equilibrated at interelement boundaries} \\
& n_j \sigma_{ji}^o = t_i \quad \text{on } \Gamma_2
\end{align*}
\]

(60)

By a similar way, one finds that a stress field \( \sigma \) is contained in \( S_0 \) if the preceding relations are verified with \( f_i = 0 \) and \( t_i = 0 \).

These are the classical relations for equilibrium elements. Restriction \( R_2 \) has here the meaning that equations (60) can be solved exactly by the finite element model.

Under these conditions, if \( \hat{u}^h \) and \( \hat{\sigma}^h \) are the finite element solutions of a same elastic problem whose exact solution is given by \( \hat{u} \) and \( \hat{\sigma} \), the dual error bounds are

\[
a) \int_{\Omega} c_{ijkl}^o (\hat{\sigma}_{ij}^h - \hat{\sigma}_{ij}^p) \partial_{pq} \hat{u}^h (\hat{\sigma}_{kl}^h - \hat{\sigma}_{kl}^p) \partial_{rs} \hat{u}^h \, dx = 2(\mathcal{A}_1(\hat{u}^h) + \mathcal{A}_2(\hat{\sigma}^h)) \quad (61)
\]

\[
b) \int_{\Omega} c_{ijkl}^o (\hat{\sigma}_{ij}^h - \hat{\sigma}_{ij}^p) (\hat{\sigma}_{kl}^h - \hat{\sigma}_{kl}^p) \, dx \leq 2(\mathcal{A}_1(\hat{u}^h) + \mathcal{A}_2(\hat{\sigma}^h)) \quad (62)
\]

\[
c) \int_{\Omega} c_{ijkl} (\partial_{ij} \hat{u}^h - \partial_{ij} \hat{u}) (\partial_{kl} \hat{u}^h - \partial_{kl} \hat{u}) \, dx = 2(\mathcal{A}_1(\hat{u}^h) + \mathcal{A}_2(\hat{\sigma}^h)) \quad (63)
\]

9. CONCLUSION

A general dual error bound theory has been presented, which works in all inhomogeneous problems and includes earlier results due to Fraeis de Veubeke.

Presented in a general abstract frame, this theory is applicable to many physical situations including, among others, elasticity, thermal conduction problems, diffusion, electrostatics, incompressible lubrication, etc. The application to elasticity has been treated as an illustration.
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