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Atomistic to continuum limits for computational materials science. (English summary)

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From the text: "The present article is an overview of some mathematical results, which provide elements of a rigorous basis for some multiscale computations in materials science. The emphasis is laid upon atomistic to continuum limits for crystalline materials. Various mathematical approaches are addressed. The setting is stationary. The relation to existing techniques used in the engineering literature is investigated.

"Despite the huge effort devoted to the subject, a lot remains to be done, both on the theoretical and on the numerical sides. Most continuum models are far from being completely analyzed. The relation of continuum theories with atomistic ones is not yet fully understood for physically relevant models. This is a motivation for further work, both on the mathematical analysis of these problems, and on their numerical analysis."

Reviewed by *P. Rochus* (Liège)

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.