# Translation kernels for velocity dependent interactions

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We reconsider first the representation of solutions defined by a condition at the origin and recognize the difficulty of extending the representation outside simple cases. By eliminating the study of solutions defined at the origin in further studies the translation kernels for velocity dependent interactions are constructed only for solutions defined by their behavior at infinity. Two methods are proposed. Their domains of extension are compared and shown to be different.

## 1. INTRODUCTION

The translation kernels were brought to consideration in the early 1950's within the framework of the inverse scattering problem.[1] Their use by Gel'fand—Levitan[2] and Marchenko[3] was decisive in the mathematical solution of the inverse problem for systems without singularities; Ref. 3 contains, in addition to this mathematical solution, an indirect attempt to solve a particular case of a system with singularity, the particular case which appears in the deuteron problem with a tensor for force. Almost at the same time, within his theory of perturbations, Friedrichs[4] introduces the idea of similitude between operators.

Let $A_1$ and $A_2$ be two operators with the same domain. If there exists an operator $U$ with a bounded inverse $U^{-1}$ such that the equation

$$UA_1 = A_2U \tag{1}$$

holds, the operators $A_1$ and $A_2$ will be called similar and $U$ a similitude transformation. The existence of $U^{-1}$ leads to the equation

$$UA_1U^{-1} = A_2. \tag{2}$$

If $A_1$ and $A_2$ operate on a Hilbert space and are both self-adjoint, Eq. (2) leads to the equation

$$UU^{-1}A_1U^*U = A_1 \tag{3}$$

from which it follows that $U$ is a unitary transformation. The point is important when the spectra of $A_1$ and $A_2$ are compared.

The translation operators satisfy Eq. (1); in addition we require they be integral transformations and belong to the category of Volterra operators.

Recently the Clarkson school[5] while developing an idea of Lax[6] constructed a class of nonlinear equations whose solution is connected to that of an inverse scattering problem. By so doing they renewed interest in the inverse problem and it becomes more compelling to specify the class of equations for which a translation kernel exists.

In a first work on the inverse problem,[7] the theory of translation operators was re-examined so as to use them in the inverse problem at fixed energy. Recently, we moved to problems at fixed angular momentum with singularities but we restricted our investigation to differential operators and to systems of coupled differential operators of the form

$$\frac{d^2}{dx^2}\, I + V(x) = \lambda, \tag{4}$$

where $I$ is the unit matrix and $\lambda$ is a constant diagonal matrix.[8-10] The matrix $V(x)$ called potential was separated into

$$V(x) = V_0(x) + V_N(x) \tag{5}$$

with a reference potential $V_0(x)$ and a nuclear potential $V_N(x)$. In most of the cases we took the reference potential to be

$$V_0(x) = \frac{\alpha(\alpha+1)}{x^2} + \frac{\eta}{x}. \tag{6}$$

When $V_0$ is defined by Eq. (6), it contains a centripetal and a Coulomb part, the usual singularities of the nuclear problem.

In each case we examined, the task was to specify the conditions the nuclear potential $V_N(x)$ has to satisfy for a bounded translation kernel to exist. The conditions were, of course, sufficient conditions. However, the method of proof used for their specification was a *constructive* method. The translation kernel was, in principle, constructed. It is a general remark that sufficient conditions obtained by a constructive method are hard to improve. The construction was obtained by transforming partial differential equations of hyperbolic type into Volterra integral equations which incorporate their boundary equations. This latter transformation has, in our opinion, its own importance; therefore another distinct example which uses different boundary conditions is included in this present paper. Extensions of the transformations may be obtained by using the appropriate elementary solutions which are here the Riemann solutions.

To specify the notations and the object of this present paper, we give the following definitions. Let $A_0$ and $A_1$ be two (systems of) differential operators together with

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boundary conditions necessary to specify the solutions their respective equations may possess.

The two types of boundary conditions we studied were conditions of regularity at the origin or conditions of behavior at infinity. The new case discussed in the present paper incorporates conditions at the origin which do not involve regularity. The two first types of conditions are the usual conditions considered in scattering problems. A translation operator $X$ is defined as a bounded operator with an inverse $X^{-1}$,

$$X = 1 + K, \quad X^{-1} = 1 + \tilde{K}.$$

The operator $K$ of the definition is always an integral operator. As a consequence of the definition, $X$ transforms the solutions of the differential operator $A_0$ into the solutions of the operator $A_1$. The operator $A_1$ was written as $A_0 + V_N$; $A_0$ contains the reference potential which may be zero. If the solutions thus transformed are the regular solutions, we call the kernel $K$ a Gel'fand—Levitan kernel: The notation of the Gel'fand—Levitan representation is retained here, but it is extended here to any representation of solutions specified by a condition at the origin. In the case of solutions defined by their behavior at infinity, we will have a Marchenko kernel and a Marchenko representation.

In all the cases we investigated, the conditions were expressed in terms of conditions the nuclear potential $V_N$ should satisfy; whether it was attractive or repulsive was unimportant. Its strength also was irrelevant. The conditions were dependent upon the *absolute* moments of the potential $V_N$ as are the usual conditions for the existence of solutions of the Schrödinger equation. It is important to note that they were, in addition, dependent upon the choice of the reference potential.

Systems of coupled differential operators defined by Eq. (4) belong to a restricted class of operators. However when, for the first time, Levitan[11] defined the concept of translation operation, he stated its use for differential operators of the more general form:

$$a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} - V(x) + \lambda. \tag{7}$$

The extension was obtained, he said through a straightforward extension of the Liouville transformation.[12] As we will see later, he was a little optimistic; the Liouville transformation cannot be used directly except for the $b = 0$ cases, but the idea of Levitan can be pursued and a transformation shown to exist. Already a desire for such an extension appeared in Ref. 7. At that time it was simply recognized that the conditions for existence depend upon the solution of a Cauchy problem but this quest was not pursued. Now many years have passed, progress has been made; not only do we possess, in the Riemann method, a tool for specifying the conditions a nuclear potential has to satisfy, but also we possess the motivation for pursuing the physical application of the extension which once seemed far away and has now become real. While the forms of operators discussed in Ref. 5 are related to the problem of the string amplitude,[13,14] a subject dear to dual theorists,[15] the Sturm—Liouville operators for Eq. (7) contain the differential operators with velocity dependent interactions.[16] The latter have been shown to be equivalent to static interactions with a hard core.[17] After additional work on the inverse problem by Marchenko[18] and Faddeev,[19] Zachariev[20] had just formulated the inverse problem for potentials which depend on the velocity. He has, in addition, proposed a solution for the approximation which involves the replacement of the differential operator by a difference operator.

Attention cannot be restricted however to differential operators; systems of coupled differential operators described by Eq. (8) have to be treated. The extension to such systems is actually realized in the present paper.

The reader is warned that the advice of Levitan of using a Liouville transformation is not followed all the time; we found it more convenient to use a simpler transformation, precisely the one advocated for the construction of equivalent local potentials in Ref. 21, In Sec. 2 the Riemann solutions method is reviewed and a new representation for a solution which is not regular but is defined by a condition at the origin is given for the $l = 0$ case.

In Sec. 3, systems of coupled equations with velocity dependent interactions are discussed. The systems are transformed into simpler ones prior to being subjected to the translation operators techniques. The use of a more direct method is discussed in the conclusion.

## 2. INTRODUCTION OF THE RIEMANN'S SOLUTION

Before proceeding to this introduction, we consider the construction of an irregular solution for a local Schrödinger equation defined by a condition at the origin. For this construction we consider the Volterra equation

$$\xi_l(k,x) = v_l(k,x) + \int_0^x g_l(k;x,y)\, V(y)\, \xi_l(k,y)\, dy \tag{8}$$

with

$$g_l(k;x,y) = [u_l(k,x)\, v_l(k,y) - u_l(k,y) \times v_l(k,x)]/W_r(u_l, v_l). \tag{9}$$

In the Green function $g_l$ of Eq. (9) the Riccati—Bessel $u_l$ and $v_l$ are inserted together with their Wronskian. One has

$$\lim_{x\to 0} v_l(k,x) = \frac{1}{(2l-1)!!\,(kx)^l},$$

$$\lim_{x\to 0} u_l(k,x) = \frac{(kx)^{l+1}}{(2l+1)!!}.$$

Obviously, Eq. (8) is meaningless for $l \neq 0$ except for special classes of interactions.

We will limit ourselves to the $l = 0$ ($s$-wave) case. Then Eq. (8) reads

$$\xi_0(k,x) = \cos kx + \int_0^x \frac{\sin k(x-y)}{k}\, V(y)\, \xi_0(k,y)\, dy. \tag{10}$$

Together with Eq. (10), we consider the possibility of representing $\xi_0(k,x)$ by an integral

$$\xi_0(k,x) = \cos kx + \int_0^x K_*(x,y)\cos ky\, dy. \tag{11}$$

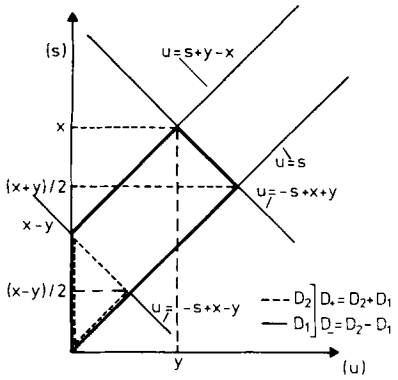


FIG. 1.

In order to exist, the kernel $K_+(x,y)$ has to satisfy the partial differential equation

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right) K_+(x,y)-V(x)K_+(x,y)=0, \tag{12}$$

$$K_+(x,x)=\tfrac{1}{2}\int_0^x V(s)\,ds, \tag{13}$$

$$\frac{\partial}{\partial y}K_+(x,y)\big|_{y=0}=0. \tag{14a}$$

In order to prove the existence of $K_+(x,y)$ one has to transform Eqs. (12), (13), and (14a) into a Volterra equation.

This can be done by using the identity

$$\frac{\sin k(x-s)}{k}\cos ku=\frac{1}{2}\int_{u+s-x}^{u+x-s}\cos kt\,dt. \tag{15}$$

After trivial interchanges of variables valid if $y\geq x/3$ and which are justified by the Tonelli–Fubini's theorems, one obtains

$$\begin{aligned}K_+(x,y)=&\tfrac{1}{2}\Big[\int_0^{(x+y)/2}V(s)\,ds+\int_0^{(x-y)/2}V(s)\,ds\Big]\\&+\tfrac{1}{2}\Big[\int_{(x+y)/2}^{x}V(s)\,ds\int_0^{x-s+y}K_+(s,u)\,du\Big]\\&+\tfrac{1}{2}\Big[\int_{(x-y)/2}^{(x-y)}V(s)\,ds\int_0^{x-s-y}K_+(s,u)\,du\\&+\int_0^{x-y}V(s)\,ds\int_0^{s}K_+(s,u)\,du\\&+\int_0^{(x-y)/2}V(s)\,ds\int_0^{s}K_+(s,u)\,du\\&-\int_{(x+y)/2}^{x}V(s)\,ds\int_0^{s}K_+(s,u)\,du\\&+\int_{x-y}^{(x+y)/2}V(s)\,ds\int_{y+s-x}^{s}K_+(s,u)\,du\Big].\end{aligned} \tag{16}$$

As we said in the introduction we decide to call this representation a Gel'fand–Levitan representation, denoting $K_\pm(x,y)$ the kernels of the two Gel'fand–Levitan representations. Equation (16) of this paper and Eq. (16) of Ref. 8 can be summarized as follows:

$$\begin{aligned}K_\pm(x,y)=&\tfrac{1}{2}\Big[\int_0^{(x+y)/2}V(s)\,ds\pm\int_0^{(x-y)/2}V(s)\,ds\Big]\\&+\tfrac{1}{2}\Big[\int_{(x+y)/2}^{x}V(s)\,ds\int_0^{x-s+y}K_\pm(s,u)\,du\\&\pm\int_{(x-y)/2}^{x-y}V(s)\,ds\int_0^{x-s-y}K_\pm(s,u)\,du\\&+\int_0^{x-y}V(s)\,ds\int_0^{s}K_\pm(s,u)\,du\pm\int_0^{(x-y)/2}V(s)\,ds\\&\times\int_0^{s}K_\pm(s,u)\,du-\int_{(x+y)/2}^{x}V(s)\,ds\int_0^{s}K_\pm(s,u)\,du\\&+\int_{(x-y)}^{(x+y)/2}V(s)\,ds\int_{y+s-x}^{s}K_\pm(s,u)\,du\Big].\end{aligned} \tag{17}$$

$K_-$ must satisfy the same equations (12) and (13) as $K_+$ but the condition (14a) is replaced by

$$K_-(x,y)\big|_{y=0}=0. \tag{14b}$$

At this point it is interesting to visualize the two domains. Let $D_1$ and $D_2$ be the two domains described in Fig. 1,

$$D_\pm=D_1\pm D_2,$$

so Eq. (17) reads

$$\begin{aligned}K_\pm(x,y)=&\tfrac{1}{2}\Big[\int_0^{(x+y)/2}V(s)\,ds\pm\int_0^{(x-y)/2}V(s)\,ds\Big]\\&+\tfrac{1}{2}\Big[\int\int_{D_1}V(s)K(s,u)\,du\,ds\pm\int\int_{D_2}V(s)K(s,u)\,du\,ds\Big].\end{aligned} \tag{18}$$

Retaining the same segments $(u=0, u=s, u+s=x+y, u-s=y-x)$, $(u=0, u=s, s+y=x-y)$ as boundary, $D_1$ and $D_2$ can be defined for $0\leq y\leq x$. With this extension, one can prove that Eq. (18) is quite general and valid for $0\leq y\leq x$.

Using the method of successive approximations, the reader can verify that the two kernels $K_+$ and $K_-$ exist under the conditions that the local potential $V$ possesses absolute moments of order zero and of order one. Interest in the two kernels $K_+$ and $K_-$ has been displayed by Mehta[22] and more recently by Dyson[23] but until now no integral representation has been given for both these kernels. Before proceeding further a discussion of the Gel'fand–Levitan domains is in order. Both are bounded at finite distance and are built up with more than three segments. When one desires to extend the representation to a reference potential $p(x)\neq 0$, the Riemann solution $R$ for the operator

$$\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+p(x)-p(y)$$

enters into the picture. Then the Riemann–Green formula for the domain $D$ and its contour $C_+$ is used,

$$\int_{C_+}\left[R\left(\frac{\partial}{\partial u}K\,ds+\frac{\partial}{\partial s}K\,du\right)\right]-\left[\frac{\partial}{\partial u}R\,ds+\frac{\partial}{\partial s}du\right]K=\iint_D RVK(s,u)\,du\,ds.$$

Two segments $C_+$ are characteristics, a third $u=s$ carries the boundary condition $K(x,x)$. Consequently, except in special circumstances, the reduction

$$\begin{aligned}K_\pm(x,y)=&\tfrac{1}{2}\Big[\int_0^{(x+y)/2}RV\,ds\pm\int_0^{(x-y)/2}RV\,ds\Big]\\&+\tfrac{1}{2}\Big[\int\int_{D_1}RVK\,du\,ds\pm\int\int_{D_2}RVK\,du\,ds\Big]\end{aligned}$$

will not result. Such a special circumstance occurs when the reference potential is the scalar centripetal barrier

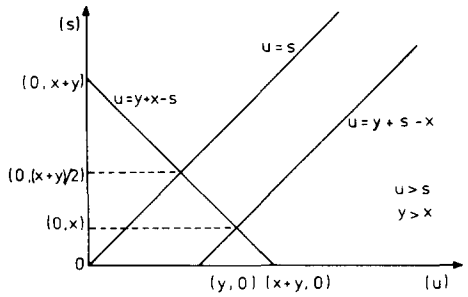


FIG. 2. Marchenko domain.

$$p(x) \equiv -l(l+1)/x^2$$

and when the $K_-$ kernel is considered.

Because of this fundamental deficiency of the two Gel'fand–Levitan representations only the Marchenko representation is considered in the rest of the present paper.

Although Riemann's method can be introduced, as it has been done by Riemann himself, a simpler heuristic presentation is given here. In the following, $D$ denotes Marchenko's domain (Fig. 2).

Let $L$ denote the *scalar* partial differential operator,

$$L = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \{p(x) - p(y)\} \tag{19}$$

and finally let $R(x, y; s, u)$ be Riemann's solution for Eq. (20a), which follows:

$$\begin{aligned} &LR = 0, \\ &\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) R = 0 \quad \text{if } y - x = u - s, \\ &\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) R = 0 \quad \text{if } y + x = u + s, \\ &R(s, u; s, u) = 1. \end{aligned} \tag{20}$$

From the last three Eqs. (20) it follows that

$$R(x, y; s, u) = 1 \quad \text{if } y - x = u - s.$$

One can now prove the two following equalities using Eqs. (19) and (20):

$$L \int\int_D R(x, y; s, u)\, W_1(s, u)\, du\, ds = 2 W_1(x, y), \tag{21}$$

$$L \int_{(x+y)/2}^{\infty} R(x, y; s, s)\, W_2(s)\, ds = 0. \tag{22}$$

The conditions for Eqs. (21) or (22) to be valid are simply the usual conditions for the differentiation under the integral sign.

In addition to Eqs. (21) and (22), provided that the convergence of the integrals is uniform, the following limits exist and have a common value which is zero:

$$\lim_{y \to \infty} \int\int_D R(x, y; s, u)\, W_1(s, u)\, du\, ds, \tag{23}$$

$$\lim_{y \to \infty} \int_{(x+y)/2}^{\infty} R(x, y; s, s)\, W_2(s)\, ds, \tag{24}$$

$$\lim_{y \to x} \int\int_D R(x, y; s, u)\, W_1(s, u)\, du\, ds. \tag{25}$$

In addition to these zero-value limits one has

$$\lim_{y \to x} \int_{(x+y)/2}^{\infty} R(x, y; s, s)\, W_2(s)\, ds = \int_x^{\infty} W_2(s)\, ds. \tag{26}$$

The Marchenko integral representation for the translation kernel is obtained as a consequence of these equations. Let us consider the partial differential equation the kernel has to satisfy:

$$\begin{aligned} &LK(x, y) = V(x)\, K(x, y), \\ &\lim_{y \to \infty} K(x, y) = 0 = \lim_{y \to \infty} \frac{\partial}{\partial y} K(x, y), \\ &K(x, x) = \tfrac{1}{2} \int_x^{\infty} V(s)\, ds. \end{aligned} \tag{27}$$

In view of Eqs. (21)–(26), Eq. (27) is equivalent to the Volterra integral equation

$$\begin{aligned} K(x, y) = {} & \tfrac{1}{2} \int_{(x+y)/2}^{\infty} R(x, y; s, s)\, V(s)\, ds \\ & + \tfrac{1}{2} \int\int_D R(x, y; u, s)\, V(s)\, K(s, u)\, du\, ds. \end{aligned} \tag{28}$$

Since we emphasized the dependence of the conditions on the nuclear potential on the choice of the reference potential, we report the following statement.

If the two potentials $V_1$ and $V_2$ belong to the class of acceptable potentials for the Marchenko representation, the reference potential $V_0$ being chosen, then the potentials $V_1 \pm V_2$ are members of the same class. We write $V_1, V_2 \in C(V_0)$ to denote this property. Furthermore the potential $V_2 \in C(V_1)$, resp. $V_1 \in C(V_2)$, belongs to the class of acceptable potentials, the reference potential being $V_1$, resp. $V_2$. The result can be obtained by estimates on the Riemann's solutions involved; it also results from a simpler argument.[24]

If $V_1$ and $V_2 \in C(V_0)$, then

$$\phi_1(x) = \phi_0(x) + \int_x^{\infty} K_{10}(x, y)\, \phi_0(y)\, dy, \tag{29}$$

$$\phi_0(x) = \phi_2(x) + \int_x^{\infty} K_{02}(x, y)\, \phi_2(y)\, dy. \tag{30}$$

Equation (30) follows the fact that the existence of $K_{20}$ implies that of the inverse kernel $K_{02}$.

We consider now the integral operator

$$\begin{aligned} A(x, y) = {} & K_{10}(x, y) + K_{02}(x, y) \\ & + \int_x^y K_{10}(x, z)\, K_{02}(z, y)\, dz. \end{aligned} \tag{31}$$

From Eq. (31) one gets

$$\begin{aligned} &\int_x^{\infty} A(x, y)\, \phi_2(y)\, dy \\ &\quad = \int_x^{\infty} K_{10}(x, y)\, \phi_2(y)\, dy + \int_x^{\infty} K_{02}(x, y)\, \phi_2(y)\, dy \\ &\qquad + \int_x^{\infty} \int_x^{y} K_{10}(x, z)\, K_{02}(z, y)\, \phi_2(y)\, dz\, dy. \end{aligned} \tag{32}$$

Using Eq. (30) and a permutation of variables, one gets

$$\begin{aligned} \text{rhs} \equiv {} & \int_x^{\infty} K_{10}(x, y)\, \phi_2(y)\, dy + \phi_0(x) - \phi_2(x) \\ & + \int_x^{\infty} dz \int_z^{\infty} K_{10}(x, z)\, K_{02}(z, y)\, \phi_2(y)\, dy \\ \equiv {} & \int_x^{\infty} K_{10}(x, y)\, \phi_2(y)\, dy + \phi_0(x) - \phi_2(x) \\ & + \int_x^{\infty} K_{10}(x, z)[\phi_0(z) - \phi_2(z)]\, dz \equiv \phi_1(x) - \phi_2(x). \end{aligned}$$

The law of composition of Marchenko's kernel is thus established and the statement proved.

The kernel $A(x,y)$ is identical to the kernel $K_{12}(x,y)$ we wanted to construct.

An interesting feature of Eq. (31) follows. Let us assume $V_1 = V_2$, then

$$0 = K_{10}(x,y) + K_{01}(x,y) + \int_x^y K_{10}(x,z)\, K_{01}(z,y)\, dz. \tag{33}$$

Equation (33) is an integral equation for the inverse kernel $K_{01}(x,y)$ of $K_{10}(x,y)$.

## 3. THE TRANSLATION KERNEL FOR VELOCITY DEPENDENT INTERACTIONS

In this section the use of the transformation for Schrödinger equations with velocity dependent interaction (effective mass dependence) is discussed. Two types of methods are used. The first one transforms the velocity dependent potential into an energy dependent operator. The second one transforms the same potential into an angular momentum dependent operator. Advantages of both methods are compared and an extension is proposed for systems of coupled differential equations.

In the $n$-dimensional space, the Schrödinger equation reads

$$-\frac{\hbar^2}{2\mu}\{\nabla[1+p(r)]\nabla - U(r) + E\}\psi(x_1, x_2, \ldots, x_n) = 0. \tag{34}$$

In Eq. (34) the following notation was used,

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right), \quad r^2 = \sum_{i=1}^{n} x_i^2. \tag{35}$$

As in the three-dimensional case, a partial wave decomposition can be used followed by the construction of the reduced radial equation $(u_m(r)/r)$ in the three-dimensional case. For the reduced radial function $u_m(r)$ one has the equation valid for all values of $n$,

$$\left(\frac{d}{dr}\,[1+p(r)]\right)\frac{d}{dr} - [1+p(r)]\frac{\nu(\nu+1)}{r^2} - W + k^2]\, u_m(r) = 0. \tag{36}$$

In Eq. (36) we have denoted by $W(r)$ the potential

$$W(r) = U(r) + \frac{(n-1)}{2}\,\frac{p'}{r},$$

and by the index $\nu$, the number obtained from the integer $m$ by

$$\nu = m + (n-3)/2.$$

In the physical situations the form factor $p(r)$ proportional to the density $\rho(r)$ of the medium, is

$$p(r) \sim \rho(r).$$

This proportionality leads to these two consequences:

$$p'(0) = 0 \tag{37}$$

and

$$\lim_{r\to\infty} r^2 p(r) = 0. \tag{38}$$

In addition to this assumption, we impose a positivity condition

$$1 + p(r) > 0 \quad \forall\, r. \tag{39}$$

If this positivity condition is assumed, no singularity is brought into the operator through the effective mass. Also if Eq. (36) is a matrix equation and $p(r)$ denotes the diagonal matrix

$$p_i(r)\,\delta_{ij} = (p(r))_{ij}, \tag{40}$$

one can define two diagonal matrices,

$$[1+p(r)]^{+1/2} \quad \text{and} \quad [1+p(r)]^{-1/2}. \tag{41}$$

In what follows, solutions may be scalars or matrices. When they will be scalars, the index $m$ is used.

### A. Method number one

Let us define (see Refs. 21 and 25)

$$u_m(r) = [1+p(r)]^{-1/2}\, v_m(r). \tag{42}$$

The factor $[1+p(r)]$ is the Wronskian of two linearly independent solutions of Eq. (36). The transformation defined by Eq. (42) is exactly the transformation from a nonlocal potential to its local equivalent used in Ref. 21.

The new radial equation for $v_m(r)$ follows,

$$\left(\frac{d^2}{dr^2} - \frac{\nu(\nu+1)}{r^2} - W(r,k^2) + k^2\right) v_m(r) = 0.$$

We have therefore defined:

$$W(r,k^2) \equiv W_0(r) + W_1(r) + k^2 W_2(r),$$

$$W_0(r) = -[1+p(r)]^{-1}\left(-\tfrac{1}{2}p'' + \tfrac{1}{4}p'^2[1+p(r)]^{-1} - \frac{(n-1)}{2}\,\frac{p'}{r}\right),$$

$$W_1(r) = [1+p(r)]^{-1/2}\, U(r)[1+p(r)]^{-1/2},$$

$$W_2(r) = p(r)[1+p(r)]^{-1}.$$

When the equation for $v_m(r)$ has been obtained one is in a situation to inquire about the existence of translation operators. For this inquiry one introduces the two operators

$$A_0 \equiv \frac{d^2}{dx^2} - \frac{\nu(\nu+1)}{x^2} + k^2,$$

$$A \equiv \frac{d^2}{dx^2} - \frac{\nu(\nu+1)}{x^2} - W(x,k^2) + k^2,$$

and defines the solutions $v_{0m}(x)$, $v_m(x)$ specified by their behavior at infinity, by

$$A_0 v_{0m}(x) = 0, \quad A v_m(x) = 0.$$

The existence of the integral representation

$$v_m(x) = v_{0m}(x) + \int_x^\infty K(k^2;x,y)\, v_{0m}(y)\, dy,$$

can be discussed. The kernel $K(k^2;x,y)$ satisfies the partial differential equation

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\nu(\nu+1)}{x^2} - W(x,k^2)\right] K(k^2;x,y) = \left[\frac{\partial^2}{\partial y^2} - \frac{\nu(\nu+1)}{y^2}\right] K(k^2,x,y),$$

$$\lim_{y\to\infty} K(k^2,x,y) = \lim_{y\to\infty} \frac{\partial}{\partial y}\, K(k^2,x,y) = 0, \tag{43}$$

$$K(k^2,x,x) = \tfrac{1}{2}\int_x^\infty W(s,k^2)\, ds.$$

With the help of the Riemann solution $R_\nu$,

$$R_\nu \equiv P_\nu(1-2x_1-2x_2+2x_1x_2),$$

for the equation

$$\left[\frac{\partial^2}{\partial x^2}-\frac{\nu(\nu+1)}{x^2}\right] R_\nu=\left[\frac{\partial^2}{\partial y^2}-\frac{\nu(\nu+1)}{y^2}\right] R_\nu,$$

Eq. (43) with its boundary condition can be transformed into a Volterra equation. Using Eq. (28), one has

$$K(k^2;x,y)=\tfrac{1}{2}\int_{(x+y)/2}^{\infty} R_\nu(x,y;s,x)\,W(s,k^2)\,ds + \tfrac{1}{2}\int\int_{D} R_\nu(x,y;s,u)\,W(s,k^2)\,K(k^2;s,u)\,du\,ds. \tag{44}$$

According to previous studies the existence of $K$ follows the possession of moments of order 1 and order $\nu$ by the energy-dependent potential $W(x,k^2)$.

Since

$$W(x,k^2)=W_0(x)+W_1(x)+k^2W_2(x),$$

and $W_0$ and $W_2$ contain the density related form factor $p(r)$ and its derivatives, the condition reduces simply to the possession by the original nuclear potential $U(s)$ of moments of order 1 and order $\nu$.

A natural question arises, that of the analytical dependence of $K$ with respect to the energy $\lambda=k^2$.

Let us define a new function

$$L(\lambda,x,y)\equiv\frac{\partial}{\partial\lambda}K(x,y;k^2).$$

The integral equation for $L$ is

$$L(\lambda,x;y)=\tfrac{1}{2}\int_{(x+y)/2}^{\infty} R_\nu(x,y;s,s)\,p(s)[1+p(s)]^{-1}\,ds + \tfrac{1}{2}\int\int_{D} R_\nu(x,y;s,u)\,p(s)[1+p(s)]^{-1} \times K(\lambda;s,u)\,ds\,du+\tfrac{1}{2}\int\int_{D} R_\nu(x,y;s,u) \times W(s,\lambda)\,L(\lambda;s,u)\,ds\,du. \tag{45}$$

Equation (45) can be solved by the method of successive approximations. The upper bound

$$(x)^\nu\,|K(\lambda;x,y)|\le\tfrac{1}{2}\sigma_\nu\left(\frac{x+y}{2}\right)\exp\sigma_1(x)$$

can be used to obtain an estimate for the zero-order term

$$L_0(\lambda,x,y)=\tfrac{1}{2}\int_{(x+y)/2}^{\infty} ds\,R_\nu(x,y;s,s)\,p(s)[1+p(s)]^{-1} + \tfrac{1}{2}\int\int_{D} R_\nu(x,y;s,u)\,p(s)[1+p(s)]^{-1} \times K(\lambda,s,u)\,du\,ds. \tag{46}$$

By looking to Eqs. (45) and (46) it is obvious that the analyticity of $K$ follows its very existence.

The method just outlined extends itself to systems of coupled differential equations of the form:

$$\frac{d}{dr}[1+p_i(r)]\frac{d}{dr}u_i(r)-[1+p_i(r)]\frac{\nu_i(\nu_i+1)}{r^2}u_i(r) - \frac{(n-1)}{2}\frac{p_i'(r)}{r}u_i(r)-\sum_j U_{ij}(r)\,u_j(r)+k_i^2u_i(r)=0, \tag{47}$$

where we have separated the diagonal operators from the nondiagonal ones. Defining the matrix solution

$$v_i(x)=[1+p_i(x)]^{-1/2}u_i(x),$$

and operating as we did, we obtain

$$\frac{d^2}{dx^2}v_i(x)-\left[\frac{\nu_i(\nu_i+1)}{x^2}+W_i^{(0)}(x)+k_i^2W_i^{(2)}(x)\right]v_i(x) - \sum_j W_{ij}^{(1)}(x)\,v_j(x)+k_i^2v_i(x)=0. \tag{48}$$

Definitions used in Eq. (48) are

$$W_i^{(0)}(x)=[1+p_i(x)]^{-1}\left(-\tfrac{1}{2}p_i''+\tfrac{1}{4}p_i'^2[1+p_i(x)]^{-1} - \frac{(n-1)}{2}\frac{p_i'(x)}{x}\right),$$

$$W_{ij}^{(1)}(x)=[1+p_i(x)]^{-1/2}U_{ij}(x)[1+p_j(x)]^{-1/2},$$

$$W_i^{(2)}(x)=p_i(x)[1+p_i(x)]^{-1}.$$

The matrices

$$W^{(0)}(x)=W_i^{(0)}(x)\,\delta_{ij},$$

$$W^{(1)}(x)=W_{ij}^{(1)}(x),$$

$$W^{(2)}(x)=W_i^{(2)}(x)\,\delta_{ij},$$

$$\Lambda=\nu_i(\nu_i+1)\,\delta_{ij},$$

$$k^2=k_i^2\delta_{ij}$$

are defined and the two operators

$$A_0=\frac{d^2}{dx^2}+k^2-\frac{\Lambda}{x^2},$$

$$A=\frac{d^2}{dx^2}+k^2-\frac{\Lambda}{x^2}+W^{(0)}(x)+k^2W^{(2)}(x)+W^{(1)}(x) \tag{49}$$

are introduced together with the matrix solutions $u_{0\Lambda}, u_\Lambda$.

The translation kernel $K$ which is now a matrix satisfies:

$$\left(\frac{\partial^2}{\partial x^2}+k^2-\frac{\Lambda}{x^2}-W^{(0)}(x)-k^2W^{(2)}(x)-W^{(1)}(x)\right)K(x,y) = \left(\frac{\partial^2}{\partial y^2}\right)K(x,y)+K(x,y)\left(k^2-\frac{\Lambda}{y^2}\right).$$

$$\lim_{y\to\infty}K(x,y)=0=\lim_{y\to\infty}\frac{\partial}{\partial y}K(x,y),$$

$$K(x,x)=\tfrac{1}{2}\int_x^\infty[W^{(0)}(s)+W^{(1)}(s)+k^2W^{(2)}(s)]\,ds. \tag{50}$$

For the discussion of the existence of $K$, the Riemann solutions

$$\Pi\begin{pmatrix}\nu_i,\nu_j\\ x_1,x_2\end{pmatrix}$$

are used.[26] Conclusions are identical to the ones we reported in Ref. 9; they don't need to be repeated in the present paper.

When the matrix kernel $K(k^2;x,y)$ for the operators defined in Eq. (50) has been obtained, one writes the final representation for the matrix solution $u_\Lambda(x)$,

$$u_\Lambda(x)=[1+p(x)]^{-1/2}\{v_{0\Lambda}(x)+\int_x^\infty K(k^2;x,y)\,v_{0\Lambda}(y)\,dy\}. \tag{51}$$

### B. Method number two (the Liouville transformation)

Although less extensive than the first method, this second method has the advantage of not introducing an energy dependence into the transformed interaction and consequently into the transformation kernel. Its lack of extension comes first from the requirement of the same effective mass in all the channels. It comes also from the appearance of potential decreasing like $x^{-3}$ infinity. The second method has nonetheless enough interest to be developed for its own merits. The Liouville transformation takes place in two steps. In the first step, one defines a new radial variable $x$ by

$$x = \int_0^r \frac{dt}{[1+p(t)]^{1/2}} . \tag{52}$$

The definition implies again $1+p(t) > 0$. Then one has a one to one mapping between $r$ and $x$. With this definition and the assumption that $p(t)$ goes to zero when $t$ goes to infinity,

$$\lim_{r\to\infty} \frac{dx}{dr} = \frac{1}{[1+p(r)]^{1/2}} = 1.$$

So instead of Eq. (52) we write definition (53) for $x$ which is its equivalent

$$x = r + \int_0^r \{(1+p(t))^{-1/2} - 1\} dt. \tag{53}$$

From Eq. (53) one sees that $r$ goes to zero with $x$ and $x$ with $r$.

Using Eq. (52) one gets

$$\frac{d}{dr}(1+p)\frac{d}{dr} = \frac{d^2}{dx^2} + \frac{1}{2}\frac{1}{1+p}\dot{p}\frac{d}{dx} . \tag{54}$$

In Eq. (54) $\dot{p}$ denotes $(d/dx)\,p(r(x))$ (not to be confused with $dp/dr$ which we denoted earlier $p'$).

With the help of Eq. (54), Eq. (36) becomes

$$\left(\frac{d^2}{dx^2} + \frac{1}{2}\frac{\dot{p}}{1+p}\frac{d}{dx} - (1+p)\frac{\nu(\nu+1)}{r^2(x)} - W(x) + k^2\right) u_m(x) = 0 \tag{55}$$

with

$$W(x) = U(x) + \frac{(n-1)}{2}\frac{p'}{r} = U(x) + \frac{(n-1)}{2}\frac{\dot{p}}{r}(1+p)^{-1/2} . \tag{56}$$

Again the index $m$ denotes scalar solutions.

The second step of the Liouville transformation is to renormalize the radial wavefunction so as to eliminate the velocity dependence from the equations.

For this purpose one defines

$$u_m(x) = (1+p)^{-1/4} v_m(x) \tag{57}$$

and uses

$$\frac{d}{dx}u = (1+p)^{-1/4}\frac{d}{dx}v - \tfrac{1}{4}\dot{p}(1+p)^{-5/4}v, \tag{58}$$

$$\frac{d^2}{dx^2}u = (1+p)^{-1/4}\frac{d^2}{dx^2}v - \tfrac{1}{2}\dot{p}(1+p)^{-5/4}\frac{d}{dx}v + \tfrac{5}{16}\dot{p}^2(1+p)^{-9/4}v - \tfrac{1}{4}(1+p)^{-5/4}\ddot{p}v. \tag{59}$$

Inserting Eqs. (57)–(59) into Eq. (55) gives

$$\left(\frac{d^2}{dx^2} - [1+p(x)]\frac{\nu(\nu+1)}{r^2(x)} - V + k^2\right) v_m = 0 \tag{60}$$

with

$$V = (1+p)^{1/4}U(x)(1+p)^{-1/4} + \frac{(n-1)}{2}\frac{\dot{p}}{r}(1+p)^{-1/2} - \tfrac{1}{16}[3\dot{p}^2(1+p)^{-1} - 4\ddot{p}](1+p)^{-1}. \tag{61}$$

As $x$ goes to infinity the centripetal barrier goes to

$$\frac{\nu(\nu+1)}{(x+c)^2} ,$$

where $c$ is the constant defined by

$$c = \int_0^\infty \{1 - [1+p(t)]^{-1/2}\} dt.$$

In the same way, by a Taylor expansion, as $x$ goes to zero

$$r(x) \to x\left.\frac{dr}{dx}\right|_{x=0} = x[1+p(0)]^{1/2} .$$

The centripetal barrier at the origin is therefore

$$[1+p(0)]\frac{\nu(\nu+1)}{x^2[1+p(0)]} \equiv \frac{\nu(\nu+1)}{x^2} .$$

Equation (60) is first rewritten as

$$\left[\frac{d^2}{dx^2} - \frac{\nu(\nu+1)}{x^2} - \left(V + [1+p(x)]\frac{\nu(\nu+1)}{r^2(x)} - \frac{\nu(\nu+1)}{x^2}\right) + k^2\right] v_m(x) = 0. \tag{62}$$

To shorten the notations, we define

$$V^{(1)} = [1+p(x)]\frac{\nu(\nu+1)}{r^2(x)} - \frac{\nu(\nu+1)}{x^2} .$$

Notice that the potential $V^{(1)}$ does not possess any singular point at the origin and decreases like $x^{-3}$ at infinity.

Back to Eq. (61) where $V$ is defined, and still assuming the relationship between $p(r)$ and the density, we have

$$\dot{p}(0) = 0.$$

The only singularities at the origin or at infinity of $V + V^{(1)}$ are those possessed by the original potential $U(r)$.

According to the normal procedure the two differential operators

$$A \equiv \frac{d^2}{dx^2} - \frac{\Lambda}{x^2} - (V^{(1)} + V) + k^2,$$

$$A_0 \equiv \frac{d^2}{dx^2} - \frac{\Lambda}{x^2} + k^2,$$

where $V$ may be a $j\times j$ matrix. If present, nonzero threshold energies would require an exponential decrease from $V^{(1)}$ and $V$. The $x^{-3}$ decrease of $V^{(1)}$ obliges $k^2$ to be a scalar. Furthermore, $\sigma_\nu$ found in the bound

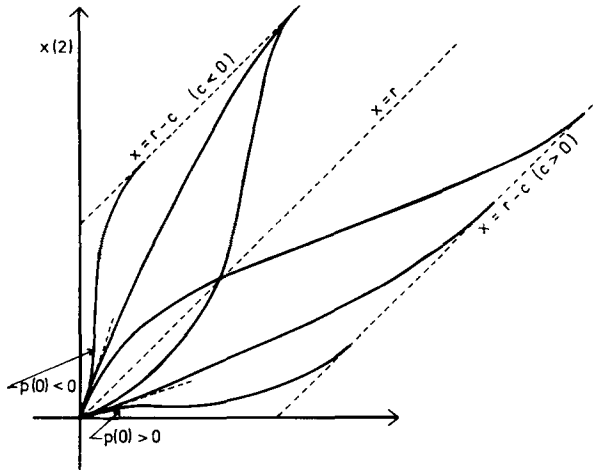


FIG. 3.

for $K$, together with the $x^{-3}$ decrease, restrict the extension to systems with maximum $\nu_j = 3$. Matrix solutions of the differential matrix equations

$$A v_\Lambda(x) = 0, \quad A_0 v_{0\Lambda}(x) = 0,$$

are now related through an integral transformation. We write this transformation as

$$v_\Lambda(x) = v_{0\Lambda}(x) + \int_x^\infty K(x,y)\, v_{0\Lambda}(y)\, dy. \tag{63}$$

The equations for $K(x,y)$ are similar to Eq. (27), within the restricted class of operators specified earlier. $K(x,y)$ exists if the original potential $U$ possesses the appropriate absolute moments.[9] From Eq. (63) one returns to the original variable $x$ by

$$u_\Lambda(r) = [1+p(r)]^{-1/4}[v_{0\Lambda}(x) + \int_x^\infty K(x,y)\, v_{0\Lambda}(y)\, dy] \tag{64}$$

and

$$u_\Lambda(r) = [1+p(r)^{-1/4}\{v_{0\Lambda}[x(r)] + \int_x^\infty K[x(r), y(s)] \times v_{0\Lambda}[y(s)] \times [1+p(s)]^{-1/2}\, ds\}. \tag{65}$$

From Eq. (53) one obtains the asymptotic relation

$$x = r - c + \int_r^\infty \{1 - [1+p(s)]^{-1/2}\}\, ds = r - c + O[\int_r^\infty ds f(s)]. \tag{66}$$

For illustration in Fig. 3, typical curves $x(r)$ are constructed. Returning to the scalar case we have the following asymptotic behaviors:

$$u_m(r) \sim a_m \sin\left(kr - \frac{m\pi}{2} + \delta_m\right), \qquad \phi_m(x) \sim b_m \sin\left(kx - \frac{m\pi}{2} + \eta_m\right). \tag{67}$$

Using Eq. (66), one sees that the limits (67) as $x$ and $r$ go together to infinity imply the simple equation

$$k(r-c) + \eta_m = kr + \delta_m$$

or

$$\eta_m = \delta_m + kc. \tag{68}$$

Equations (66) and (68) are the basis for Calogero choosing a new variable $x = r - c$ and identifying velocity dependent with static hard core interactions.

## 4. CONCLUSION

In the paper we assumed Schrödinger equations with an effective mass and showed the existence of translation operators, in the sense of Marchenko, between the free Schrödinger equation and the Schrödinger equation with an effective mass in addition to the nuclear potential. Two methods were used to achieve this; they had a common characteristic, the full Schrödinger equation was subjected to some transformation prior to being considered for translation kernel purposes. One may wonder whether this preliminary transformation is necessary: The answer is no, but it is convenient as the following will show. The two operators

$$A_0 = a_0(x)\frac{d^2}{dx^2} + b_0(x)\frac{d}{dx} + c_0(x), \tag{69}$$

$$A = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x), \tag{70}$$

may be directly considered. When Eq. (1), $XA_0 = AX$, is developed one obtains the partial differential equation for the kernel

$$K(x,y).$$

They are

$$\frac{\partial^2}{\partial y^2}[K(x,y)\, a_0(y)] - \frac{\partial}{\partial y}[K(x,y)\, b_0(y)] + K(x,y)\, c_0(y) = a_1(x)\frac{\partial^2}{\partial x^2} K(x,y) + b_1(x)\frac{\partial}{\partial x} K(x,y) + c_1(x) K(x,y), \tag{71}$$

with complicated boundary conditions. The restriction for the translation operator to be of the Volterra type leads to the following two constraints on the coefficients of Eqs. (69) and (70):

$$a_1(x) = a_0(x) \tag{72}$$

and

$$b_1(x) - b_0(x) = -K(x,x)\, a_0(x) + a_1(x) K(x,x). \tag{73}$$

Inspection of the system thus obtained, convinced us it was not worth pursuing except when the two conditions

$$a = a_0 \equiv \text{scalar}, \quad b = b_0 = 0$$

are realized.

At the termination of this paper we can assert the validity of the Marchenko representation in a wide variety of physical situations: many-channel scattering in a $n$-dimensional space with centrifugal force or Coulomb force or even both, with or without an effective mass. Along our study we were led to abandon the Gel'fand—Levitan representation. It is not such a damageable result, the connection of the regular physical solution with the physical solution is not maintained when the passage from a one-channel to a many-channel problem is operated. In addition one should say the Marchenko approach is the one which is appealing to the creators of the solution of nonlinear problems via inverse scattering methods.[5]

We must emphasize that the determination of the conditions the nuclear potential should satisfy for the existence of the translation kernel, is only a first step

into the inverse problem. When translation kernels for a class of potentials have been proved to exist, one obtains upper bounds for these translation kernels. Within this class of potentials, it may be possible to construct a fundamental equation between $K(x,y)$ and some spectral matrix $F(x,y)$ as Marchenko or Gel'fand–Levitan did. The upper bounds satisfied by $K(x,y)$ induce upper bounds that the spectral matrix $F(x,y)$ itself verifies. Considering now the fundamental equation as an equation for $K(x,y)$, the necessary bound, which the spectral matrix verifies, becomes the important element in deciding whether or not the fundamental equation possesses a unique solution. This discussion is the essence of the inverse problem.


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# ERRATA

## Erratum: Higher weights of semisimple Lie algebras [J. Math. Phys. 18, 1162 (1977)]


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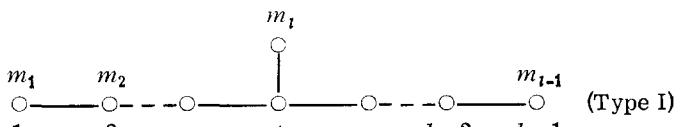
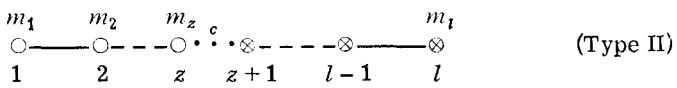
Page 1163, just before Eq. (11): the following are the corrected versions of Types I and II:




(Type I)




(Type II)

Page 1163, at the end of line (– 2) $A_l$ should belong to line (–3).

Page 1165, line 1, left column: The word "will" was blotted out in the printing process.

Page 1167, first line immediately after Eq. (35): the lhs of the equations should read

$$\delta(S_{r-1}^{(i)}) \quad \text{and} \quad \delta(\lambda_{r-1}^{(i)}).$$

Page 1168, Table V: the second line should read

$$+\sum_{k=p}^{l-1}\left(\sum_{i=1}^{p-1}(l-k)\,2im_i+\sum_{i=p}^{i=k}[p^2+(2-p)\,i](l-k)\,m_i\right.$$
$$\left.+\sum_{i=k+1}^{l-1}[p^2+(2-p)\,k](l-i)\,m_i\right.$$

Page 1169, Table VII, rightmost part of lines for $l=7$: The numbers blotted out in the printing process are 16 (first line) and 15 (second line).

Page 1169, Table VII, last column for $l=8$ (last entry under 3): the condition $1 \leq i \leq 16$ should apply to both equations in that last $l=8$ entry.

Page 1170, Table VIII: the large brace should only extend to include the equation $\alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$.

Page 1171, Table IX, left-hand entry 2: the first factor on the rhs of the $\Pi_2(D_l)$ equation should read

$$(m_l + 1).$$

Page 1172, Eq. (46): The lhs partially blotted out in the printing process, should read: $(H_{\alpha_k})^{r,i}_{r,i}$.

Page 1173, Table XI, bottom line of the $E_6$ entry: $C_{21}^2$ and $C_{27}^2$ should read $C_{27}^2$ and $C_{27}^3$, respectively.

Page 1174, Table XII, under 1st labeling:

$$\left(2\,\frac{l-1}{2l}\right) \quad \text{should read} \quad 2\left(\frac{l-1}{2l}\right).$$

Page 1175, the third to last line should not include ⊞.

Page 1176: the rhs of $H_{\alpha_2}$ equation should read

$$\text{diag}(m_2/c,\ m_2/c+1,\ m_2/c+1-2/c,\ \text{etc.}).$$