The translation kernel in the n-dimensional scattering problem

Marcel Coz

Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506

Pierre Rochus[a)]

Physique Nucléaire Théorique, Université de Liège,
Institut de Physique au Sart-Tilman, B—4000 Liège 1, Belgium
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Radial wavefunctions are defined for the n-dimensional scattering problem ($n > 1$) with spherical symmetry by conditions of regularity at the origin or by conditions of behavior at infinity. The existence of translation kernels can therefore be discussed in both instances. The problem of representing regular solutions appears to be essentially different from that of representing irregular solutions. The essential difference originates from the type of domain used in the representation: It is bounded in the first case and unbounded in the second. If one can still compare the ranges of validity of the two types of representation when one is dealing with a scalar situation, upon proceeding to a matrix situation, a comparison is no longer possible.

1. INTRODUCTION

Our first intention was to bring new elements into the discussion of the role of the translation kernel in the potential scattering discussion begun in Refs. 1–3. The first motivation for seeking new elements was noted in Ref. 3: it was the existence of a connection between the partial wave translation kernel and the partial wave Green function. This connection appears when the equations

$$\psi_l(x) = \phi_l(x) + \int K(l,x,y)\phi_l(y)\,dy, \tag{1}$$

$$\psi_l(x) = \phi_l(x) + \int G(l,k;x,y)V_l(y)\phi_l(y)\,dy \tag{2}$$

containing ϕ_l, the radial solution for the "reference" Schrödinger equation, and ψ_l, the radial solution for the complete equation, are considered simultaneously. A second motivation was the technique due to Blazěch.[4] From the partial operator K_l one can construct a global transformation operator κ using any of the two equations

$$\kappa(x,y;\mathbf{n}_1\cdot\mathbf{n}_2) = \sum_l (2l+1)K_l(x,y)P_l[\cos(\mathbf{n}_1\cdot\mathbf{n}_2)] \tag{3a}$$

or

$$\kappa(\mathbf{x},\mathbf{y}) = \sum_l (2l+1)K_l(x,y)P_l[\cos(\mathbf{x}\cdot\mathbf{y})]. \tag{3b}$$

Along these lines in Ref. 2(a) we investigated the conditions for the existence of K_l for the three-dimensional potential scattering problem when the potential possesses spherical symmetry. A first extension of the study was obtained in Ref. 2(b) where the many (finite)-channel scattering problem was solved. A second possible extension may be investigated. It is realized when one wants to consider the n-dimensional problem ($n > 0$). In the study the separation of the $n=1$ case from the $n>1$ cases becomes necessary. While for $n>1$, a radiation condition emerges as a prerequisite,[5] in the $n=1$ case, such a condition cannot be verified. The $n=1$ case was therefore separated and studied in Ref. 3. However, in Refs. 2 and 3, we kept the restrictive condition that the number of channels remain finite. The original purpose of the present paper was therefore to subject the $n>1$ dimensional scattering problem to discussion and to present indications on how the restriction upon the finite number of channels could be removed. However, during the work it became more and more evident that the representation of regular solutions (Gel'fand–Levitan representation) and of irregular solutions defined by an asymptotic condition (Marchenko representation) were not two parallel problems. Consequently while considering possible extensions to the cases considered in Refs. 2(a) and 2(b) the interest shifted to the differences between the two types of representations. Reasons for the differences were brought to light. They were two in number. The first: the Marchenko contour is not closed whereas the Gel'fand–Levitan is closed. This latter is built up of four segments and it is one of these segments which is troublesome. The second: the essential element for the discussion, the Riemann solution itself, has different analytical properties in the two cases. As a by-product of our study, appears the necessity of choosing a complete system of functions which may lead either to a finite set of differential equations or to an infinite set of equations which can be truncated. In Ref. 6 this problem of truncating an infinite number of channels has received some consideration. In the results presented here the condition for the existence of a translation kernel are expressed in terms of requirements imposed on the matrix elements of the potential between two channels.

To avoid any misunderstanding we want to emphasize that we have made no attempt in this paper to apply hyperspherical systems to the many-body problem. (The poor convergence of hyperspherical systems[7] and the lack of "compactness" of the Lippman–Schwinger equation forbid the use of translation operators in the many-body problem.

The present paper is divided into four sections: The Introduction is in Sec. 1, the study of the radial equation for the n-dimensional Schrödinger equation is found in Sec. 2 where the condition for the existence of the

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solutions is stated. In Sec. 3, translation operators are applied to the n-dimensional Schrödinger equation, while in Sec. 4 the existence of translation operators for the matrix n-dimensional radial equations are discussed.

To fix the notations throughout this work the number n denotes the dimension of the space.

As usual with any function of one or many variables, one associates its absolute value by $f \to |f|$. To each matrix A, one will associate the matrix of its absolute value $|A|$ defined by

$$|A|_{ij} = |A_{ij}|,$$

and its Marchenko's norm

$$\|A\| \equiv \sup_i \sum_{j=1} |A_{ij}|.$$

2. THE RADIAL EQUATION

One considers the time independent n-dimensional Schrödinger equation

$$\Delta_n \psi(\mathbf{x}) + [k^2 - V(\mathbf{x})]\psi(\mathbf{x}) = 0. \tag{4}$$

In Eq. (4) one has

$$\Delta_n = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

and $\psi(\mathbf{x})$ denotes the n-dimensional scattering wave. Suppose the interaction potential $V(\mathbf{x})$ possesses spherical symmetry and satisfies the following conditions[7,8]: At the origin and at infinity

$$\int_0^r s\,|V(s)|\,ds < \infty \quad \text{and} \quad \int_r^\infty |V(s)|\,ds < \infty.$$

Then with a sufficiently large $r = |x|$, one can find a solution of Eq. (4) of the form,[10]

$$\psi(\mathbf{x}) = \exp[i(\mathbf{k}\cdot\mathbf{x})] + f(k,\theta)\frac{\exp(ikr)}{r^{(n-1)/2}} + O(r)^{-(n+1)/2}. \tag{5}$$

An alternate way of posing the problem is to seek a solution $u(\mathbf{x}) = \psi(\mathbf{x}) - \exp(i\mathbf{k}\cdot\mathbf{x})$ which satisfies a finiteness condition:

$$|r^{(n-1)/2}u(\mathbf{x})| \le \text{const}, \tag{6a}$$

and verifies uniformly in all directions the Sommerfeld radiation condition[11,5]

$$\lim_{r\to\infty} r^{(n-1)/2}\left(\left|\frac{d}{dr}u(\mathbf{x}) - iku(\mathbf{x})\right|\right) = 0. \tag{6b}$$

This condition is not satisfied by the one-dimensional Schrödinger equation solution.[3] The Sommerfeld condition means, in physical terms, that no energy can be radiated in from infinity.[11] A more satisfying version of conditions 3(a), 3(b) was given by Rellich,[12]

$$\lim_{R\to\infty} \int_{|x|=R} dS \left|\frac{\partial}{\partial r}\psi(\mathbf{x}) - ik\psi(\mathbf{x})\right|^2 = 0. \tag{6c}$$

Condition $\{[(6a)$ and $6(b)] \equiv 6(c)\}$ expresses simply the characterization of the wave function $\psi(\mathbf{x})$ as a vehicle for the energy. It does not tell whether $\psi(\mathbf{x})$ is a solution of a Schrödinger equation for an energy operator H, or whether it is not. The conditions for the existence of H are not hard to satisfy (see Ref. 13); but even if H does not exist, condition (6) may be valid.

In what follows, we use Eq. (5) without assuming the existence of H. Let us set

$$\mathbf{k}\cdot\mathbf{x} = kr\cos\theta.$$

The n-dimensional plane wave expansion follows:

$$\exp(ikr\cos\theta) = \left(\frac{2}{kr}\right)^{(n-2)/2}\Gamma[\tfrac{1}{2}(n-2)] \times \sum_{p=0}^{\infty} i^p\left(p + \frac{n-2}{2}\right)J_{p+(n-2)/2}(kr) \times C_p^{(n-2)/2}(\cos\theta), \tag{7}$$

with the Gegenbauer polynomial $C_p^{(n-2)/2}(\cos\theta)$, see Ref. 14, defined by the generating function

$$\sum_{p=0}^{\infty} r^p C_p^{(n-2)/2}(\cos\theta) = (1 - 2r\cos\theta + r^2)^{-(n-2)/2}.$$

Gegenbauer polynomials are related to Jacobi polynomials by $(\lambda+\frac{1}{2})_p C_p^\lambda(x) = (2\lambda)_p P_p^{\alpha\alpha}(x)$.[15] Therefore, expansions may be found in the literature which employ the Jacobi polynomials.[16,17]

Obviously $C_p^{1/2} = P_p$, see Ref. 14.

Since

$$J_{\nu \atop r\to\infty} \sim \frac{1}{\sqrt{\pi}}\left(\frac{2}{kr}\right)^{1/2}\left(\sin\, kr - (\nu - \tfrac{1}{2})\frac{\pi}{2}\right)(18),$$

one has

$$J_{p+(n-2)/2 \atop r\to\infty} \sim \frac{1}{\pi^{1/2}}\left(\frac{2}{kr}\right)^{1/2}\sin\left[kr - \left(p + \frac{n-3}{2}\right)\frac{\pi}{2}\right].$$

One introduces hyperspherical Bessel functions by

$$j_p(kr) = \tfrac{1}{2}\Gamma[\tfrac{1}{2}(n-2)]\left(\frac{2}{kr}\right)^{(n-2)/2} J_{p+(n-2)/2}(kr). \tag{8}$$

They behave asymptotically as

$$j_p(kr) \sim \tfrac{1}{2}\Gamma[\tfrac{1}{2}(n-2)]\left(\frac{2}{kr}\right)^{(n-2)/2}\left(\frac{1}{\pi^{1/2}}\right)\left(\frac{2}{kr}\right)^{1/2} \times \sin\left[kr - \left(p + \frac{n-3}{2}\right)\frac{\pi}{2}\right].$$

Using Eq. (8) the plane wave expansion of Eq. (7) reads as

$$\exp(ikr\cos\theta) = \sum_p (2p + n - 2)i^p j_p(kr) C_p^{(n-2)/2}(\cos\theta). \tag{9}$$

To obtain the scattering amplitude, the asymptotic form for Eq. (9) is used,

$$(\phi_{\mathrm{PW}})_{r\to\infty} \sim \frac{1}{\pi^{1/2}}\sum_p (2p+n-2)i^p\left(\frac{2}{kr}\right)^{1/2}\left(\frac{2}{kr}\right)^{(n-2)/2} \times \Gamma[\tfrac{1}{2}(n-2)]\frac{i}{2}\left(\exp\left\{-i\left[kr - \left(p+\frac{n-3}{2}\right)\frac{\pi}{2}\right]\right\} - \exp\left\{+i\left[kr - \left(p+\frac{n-3}{2}\right)\frac{\pi}{2}\right]\right\}\right)C_p^{(n-2)/2}(\cos\theta). \tag{10}$$

In Eq. (10) we have used the subscripts PW to denote the words plane wave. From Eq. (10) we get $\psi_{\mathrm{SCATTERED}}$ defined by the equation

$$\psi_{\mathrm{SC}} = \frac{\exp(ikr)}{r^{(n-1)/2}} f(k,\theta);$$

by identification the transition amplitude $f(k,\theta)$ is obtained,

$$f(k,\theta)=i\frac{2^{(n-5)/2}}{\pi^{1/2}}\frac{\Gamma[(n-2)/2]}{k^{(n-1)/2}}\sum_p (2p+n-2) \times \exp[i(n-3)\pi/4](S_p-1)C_p^{(n-2)/2}(\cos\theta). \quad (11)$$

Equation (11) can be assumed therefore from the scattering radiation condition without the adjunction of the assumption that some Hamiltonian exists.

In our work, a Hamiltonian is assumed and Eq. (11) comes from the partial wave decomposition of the scattering solution in a way similar to that followed for the $n=3$ dimensional case.

A time dependent Schrödinger equation is considered,

$$i\hbar\frac{\partial}{\partial t}\phi(\mathbf{x},t)=-\frac{\hbar^2}{2M}\Delta_n\Phi(\mathbf{x},t)+U(\mathbf{x})\Phi(\mathbf{x},t).$$

We write $\phi(\mathbf{x},t)=\psi(\mathbf{x})g(t)$ to obtain the stationary equation,

$$-\frac{\hbar^2}{2M}\Delta_n\psi(\mathbf{x})+U(\mathbf{x})=E\psi(\mathbf{x}),$$

from which the reduced equation results,

$$\Delta_n\psi(\mathbf{x})+k^2\psi(\mathbf{x})-V(\mathbf{x})\psi(\mathbf{x})=0.$$

Let us assume a special form for the interaction $V(\mathbf{x})$,

$$V(x)=V(\|\mathbf{x}\|). \quad (12)$$

We indicate now the behaviors of the radial functions used in the scattering descriptions when r goes to zero.

For the Bessel and Newman functions, as r goes to zero

$$J_{p+1/2}\sim r^{p+1/2} \quad \text{or} \quad J_\nu\sim r^\nu,$$
$$N_{p+1/2}\sim r^{-p-1/2} \quad \text{or} \quad N_\nu\sim r^{-\nu}.$$

When n is even, $\nu=p+(n-2)/2$ is an integer and the Newman function N_ν is defined by $\lim_{\epsilon\to 0}(1/\epsilon)\times[J_{\nu+\epsilon}-(-1)^\nu J_{-\nu-\epsilon}]$.

For hyperspherical Bessel functions the behavior at the origin is

$$j_p\sim\left(\frac{1}{r}\right)^{(n-2)/2}J_{p+(n-2)/2}=r^{-(n-2)/2}r^p r^{n/2}r^{-1}\sim r^p,$$
$$\eta_p\sim\left(\frac{1}{r}\right)^{(n-2)/2}N_{p+(n-2)/2}\sim r^{-(n-2)/2}r^{-p}r^{-n/2}r^{+1}\sim r^{-(p+n-2)},$$

while for the Riccati—Bessel's it is

$$u_p\sim j_p r^{(n-1)/2}\sim r^{p+(n-1)/2},$$
$$v_p=\eta_p r^{(n-1)/2}\sim r^{-p-(n-3)/2}. \quad (13)$$

We had introduced previously the hyperspherical Bessel functions. This is their equation[18,19]:

$$\left[\frac{d^2}{dr^2}+\frac{(n-1)}{r}\frac{d}{dr}+k^2-\frac{p(p+n-2)}{r^2}\right]\begin{matrix}j_p\\ \eta_p\end{matrix}=0. \quad (14)$$

To arrive at the analogous one for the Riccati—Bessel's we use the reducing factor $f(r)=r^{-(n-1)/2}$ and we get the reduced equation

$$\left[\frac{d^2}{dr^2}+k^2-\frac{p(p+n-2)}{r^2}-\frac{1}{4r^2}(n-1)(n-3)\right]\begin{matrix}u_p\\ v_p\end{matrix}=0. \quad (15)$$

It is obvious that the product of the behaviors of u_p, v_p at the origin, namely the product $[p+(n-1)/2]\times[-p-(n-3)/2]$ should equal the coefficient of r^{-2} in Eq. (15).

Together with the Riccati—Bessel functions we should introduce the Riccati—Hankel functions which behave like imaginary exponentials when r goes to infinity.

After these preliminaries we can concern ourselves with the solutions for the radial equation

$$\left[\frac{d^2}{dr^2}+k^2-\frac{(\nu^2-\frac{1}{4})}{r^2}-V(r)\right]\psi(p,k,r)=0 \quad (16)$$

with $\nu=p+(n-2)/2$. In the discussion of Eq. (16) the authoritative treatment of Newton in Ref. 20 is followed. Regular and irregular solutions $\psi(p,k,r)$, $f(p,\pm k,r)$ are respectively defined by conditions at the origin and conditions as r goes to infinity. We write the integral equations which define these solutions,

$$\phi(p,k,r) =u(p,k,r)+\int_0^r g(p,k;r,s)V(s)\phi(p,k,s)\,ds, \quad (17)$$

$$f(p,\pm k,r) =w(p,\pm k,r)-\int_r^\infty g(p,k;r,s)V(s)f(p,\pm k,s)\,ds, \quad (18)$$

where g is the Green function for Eq. (15).

Equation (16) including the factor $(\nu^2-\frac{1}{4})$ allows the use of Levinson—Newton upper bounds[21] which read, with the notation

$$k=x+iy \quad \text{and} \quad \nu=a+ib, \quad a>0,$$
$$|u(p,k,r)|\leqslant C\left[\frac{r}{1+|k|r}\right]^{a+1/2}\exp(|y|r),$$
$$|v(p,k,r)|\leqslant C\left[\frac{1+|k|r}{r}\right]^{a-1/2}\exp(|y|r),$$
$$g(p,k;r,s)=\{u(p,k,r)v(p,k,s)-u(p,k,s)v(p,k,r)\}/W_r(p,k),$$

where $W_r(p,k)$ is the Wronskian of u and v,

$$|g(p,k;r,s)|\leqslant C\frac{1}{|W_r(p,k)|}\left[\frac{r}{1+|k|r}\right]^{a+1/2}\times\left[\frac{1+|k|s}{s}\right]^{a-1/2}\exp(|y|r).$$

With these bounds we can obtain the existence for the regular and irregular solutions provided that

$$\int_0^\infty |V(s)|\,ds<\infty.$$

3. THE TRANSLATION KERNEL (UNCOUPLED EQUATIONS)

Having established the existence of regular and irregular solutions for the reduced radial equation, we turn to the existence of translation kernels.

While the existence of the solutions depends on properties of the Bessel solutions which do not discriminate between values of p, the problem of the existence of the translation kernel is dependent upon the properties of the Legendre functions, with an order p which can be either an integer or half an integer.

The translation kernels are used as in Refs. 1–3 either for the representation of the irregular solutions $f(f, \pm k, r)$,

$$f(p, \pm k, r) \sim \exp\{\pm (ikr)\} \exp i[(n-3)\pi/4] \exp ip\pi/2,$$

or for the representation of the regular solution (solution which behaves at the origin as a Riccati–Bessel).

We introduce first the operator

$$L(x) = \frac{d^2}{dx^2} - \frac{(p(p+n-2))}{x^2} - \frac{(n-1)(n-3)}{4x^2} = \frac{d^2}{dx^2} - \frac{\lambda(\lambda+1)}{x^2}, \tag{19}$$

where we have defined $\lambda = \nu - \frac{1}{2} = p + (n-3)/2$.

The translation kernel $K(x,y)$ is the solution of the Darboux equation[1]

$$L(x)K(x,y) = L(y)K(x+y) + V(x)K(x,y)$$

with appropriate boundary conditions which depend upon whether the regular solution or an irregular solution is represented. These conditions are:

Irregular case	*Regular case*	
(Marchenko)	(Gel'fand–Levitan)	
$\lim_{y\to\infty} K_M(x,y) = \lim_{y\to\infty} \frac{\partial}{\partial y} K_M(x,y) = 0$	$\lim_{y\to 0} K_G(x,y) = 0$	
$K_M(x,x) = \frac{1}{2}\int_x^\infty V(s)\,ds$	$K_G(x,x) = \frac{1}{2}\int_0^x V(s)ds$	(20)
$y \geqslant x$	$x \geqslant y$	

The kernels $K(x,y)$ are the solutions of integral equations which incorporate their boundary conditions.[3]

To find these integral equations one uses Riemann's method. Let $R(x,y;s,u)$ be the solution of the equation

$$L(s)R(x,y;s,u) = L(u)R(x,y;s,u)$$

with the conditions

$$R(x,y;x,y) = 1,$$

$$\frac{\partial R}{\partial s} = \frac{\partial R}{\partial u} \quad \text{when} \quad x+y = u+s,$$

$$\frac{\partial R}{\partial s} = -\frac{\partial R}{\partial u}, \quad \text{when} \quad x-y = u-s.$$

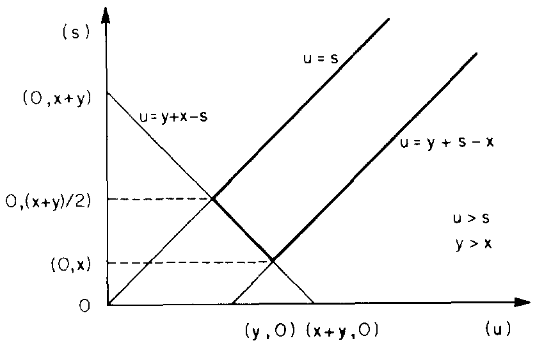

FIG. 1. Marchenko domain.

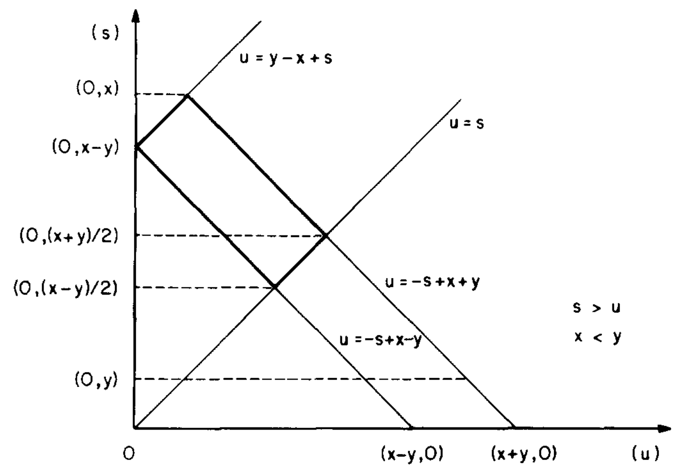

FIG. 2. Gel'fand–Levitan domain.

Let D be any of the two domains D_M, D_G specified in Figs. 1 and 2 and let C_+ be the boundary of the domain. By Green's theorem one obtains

$$\iint R[L(s) - L(u)]K\,du\,ds = \int_{C_+}\left(\frac{\partial}{\partial u}R\,ds + \frac{\partial}{\partial s}R\,du\right)K - \int_{C^+} R\,\frac{\partial}{\partial u}K\,ds + \frac{\partial}{\partial s}K\,du\ . \tag{21}$$

The left-hand side is replaced by $\int\int_D RVK\,du\,ds$ and the integrations are performed on the right-hand side. By so doing, one obtains the integral equations.

Irregular case:

$$K_M(x,y) = \tfrac{1}{2}\int_{(x+y)/2}^\infty ds\,R(x,y;s,s)V(s)\,ds + \tfrac{1}{2}\int\int_{D_M} R(x,y;s,u)V(s)K(s,u)\,du\,ds; \tag{22a}$$

D_M is the Marchenko domain shown in Fig. 1 where we have

$$s \geqslant x; \quad u-s \leqslant y-x; \quad u+s \geqslant y+x; \quad y > x; \quad u \geqslant s.$$

Regular case:

$$K_G(x,y) = \frac{1}{2}\int_{(x-y)/2}^{(x+y)/2} ds\,R(x,y;s,s)V(s)\,ds + \int_{(x-y)/2}^{(x-y)} \frac{d}{ds}[R(x,y;s,-s+x-y)] \times K(s,-s+x-y)\,ds + \frac{1}{2}\iint_{D_G} R(x,y;s,u)V(s)K(s,u)\,du\,ds. \tag{22b}$$

The domain D_G in Eq. (22b) is the Gel'fand–Levitan domain of Fig. 2 where we have

$$s \leqslant x; \quad u-s \geqslant y-x; \quad u+s \leqslant x+y; \quad s \geqslant u;$$
$$s+u \geqslant x-y; \quad x \geqslant y.$$

At this point a word on Eqs. (22a) and (22b) is necessary.

Equation (22a) was obtained under the assumption that the contribution of the contour C_+ is negligible when y goes to infinity. Remembering that the kernel $K_M(x,y)$

is such that

$$\lim_{u\to\infty} K_{\mathrm{M}}(s,u)=\lim_{u\to\infty}\frac{\partial}{\partial u}K_{\mathrm{M}}(s,u)=0,$$

the conditions upon $R(x,y;s,u)$ so that

$$\int_{C_+(u+s\to\infty)}\left(R(x,y;s,u)\frac{d}{du}K(s,u)-\frac{d}{du}R(x,y;s,u)K(s,u)\,du\right)=0$$

are not particularly stringent.

The contour C_+ of Eq. (22b) is closed at finite distance. It is built up of four segments. Two of these segments are characteristic segments. A third one carries the boundary condition

$$\frac{d}{dr}K(r,r)=\tfrac{1}{2}V(r).$$

We are therefore left with a fourth segment whose contribution will in general be different from zero. However in the case of uncoupled equations considered in this section in the presence of the centripetal potential only, the Riemann function is

$$\begin{aligned}R&=\Pi\begin{pmatrix}\lambda,\lambda\\ x_1,x_2\end{pmatrix}\\&=P_\lambda(1-2x_1)-2x_2\int_0^1 P_\lambda(1-2x_1+2x_1t)P_\lambda'(1-2x_2t)\,dt\\&=P_\lambda(1-2x_2)-2x_1\int_0^1 P_\lambda(1-2x_2t)P_\lambda'(1-2x_1+2x_1t)\,dt\\&=P_\lambda(1-2x_1-2x_2+2x_1x_2).\end{aligned}\tag{23}$$

In Eq. (23), P_λ is the Legendre function of order λ $[\lambda=p+(n-3)/2]$, which is an integer or half an integer number according to whether the dimension of the space is odd or even. The Chaundy variables x_1,x_2 are defined[22] as follows:

$$\begin{aligned}x_1&=\frac{(u+s-x-y)(x-y-s+u)}{4xs},\\x_2&=\frac{(x+y-u-s)(x-y+u-s)}{4uy},\\Z&=1-2x_1-2x_2+2x_1x_2.\end{aligned}\tag{24}$$

Along the curve $(x-y=u+s)$ the value of Z is -1. Consequently when λ is an integer (space with odd dimension)

$$R_\lambda(x,y;s,-s+x-y)=P_\lambda(-1)=(-1)^\lambda.$$

The derivative of R_λ with respect to s vanishes and the integral equation has the known reduction

$$\begin{aligned}K_{\mathrm{G}}(x,y)&=\tfrac{1}{2}\int\int_{\partial_{\mathrm{G}}}ds\,R_\lambda(x,y;s,s)V(s)\\&\quad+\tfrac{1}{2}\int\int_{\partial_{\mathrm{G}}}du\,ds\,R_\lambda(x,y;s,u)V(s)K_{\mathrm{G}}(s,u).\end{aligned}\tag{22c}$$

If λ is half an integer, P_λ is not defined for the argument $Z=-1$. The representation of regular solutions for spaces with even dimension will not be discussed here. More trivially, one can verify that applying

$$L(x)-L(y)-V(x)$$

to both sides of Eq. (22c) gives zero only if

$$\left(\frac{\partial}{\partial s}-\frac{\partial}{\partial u}\right)R(x,y;s,u)=0\quad\text{for}\quad u+s=x-y,$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)R(x,y;s,u)=0\quad\text{for}\quad x-y=s+u.$$

Such a circumstance happens when R is the Legendre function $P_\lambda(Z)$ and λ is an integer. One can see it by using characteristic variables

$$\eta=x+y,\quad \eta_0=s+u,\quad \xi=x-y,\quad \xi_0=s-u,$$

and

$$Z=1+\frac{(\xi^2-\xi_0^2)(\eta^2-\eta_0^2)}{(\eta^2-\xi^2)(\eta_0^2-\xi_0^2)},$$

one gets

$$\frac{\partial R_\lambda}{\partial x}+\frac{\partial R_\lambda}{\partial y}=\frac{\partial R_\lambda}{\partial \eta}=\frac{\partial P_\lambda}{\partial \eta}=\frac{\partial P}{\partial Z}\frac{\partial Z}{\partial \eta}.$$

Since

$$\frac{\partial Z}{\partial \eta}=\frac{(\xi^2-\xi_0^2)}{(\xi^2-\xi_0^2)}\frac{2\eta[\eta_0^2-\xi^2]}{(\eta^2-\xi^2)^2},$$

$$\frac{\partial R_\lambda}{\partial \eta}=0\quad\text{for}\quad\begin{cases}\xi=\xi_0 \text{ or } x-y=s-u,\\ \eta_0=\xi \text{ or } s+u=x-y.\end{cases}$$

The result of Ref. 2(a) concerning the extension of the representation from the s-wave to the higher l-waves was dependent upon this circumstance. This important fact was not pointed out at the time. We can consider now the two types of representation.

A. Marchenko representation

In the representation of the irregular solutions the argument Z of the Legendre function is greater than one. Since

$$\begin{aligned}Z&=1-2x_1-2x_2+2x_1x_2\\&=1+\frac{(u+s-x-y)(s-u+y-x)(u+y+s+x)(u+y-s-x)}{8usxy},\end{aligned}\tag{25}$$

and since the four factors in the numerator of Eq. (25) are all positive, we have $Z\geqslant 1$.

On the other hand, in the Marchenko domain we have

$$\begin{aligned}0\leqslant -x_1&=\frac{(u+s-x-y)(y-x+s-u)}{4xs}\\&\leqslant\frac{2(s-x)2(s-x)}{4xs}\leqslant\frac{s}{x}\end{aligned}$$

and

$$0\leqslant x_2=\frac{(u+s-x-y)(y-x+s-u)}{4uy}\leqslant\frac{2uy}{4uy}\leqslant\tfrac{1}{2}.$$

We can write

$$Z=1-2x_1-2x_2(1-x_1)\leqslant 1-2x_1\leqslant 1+2s/x\leqslant 3s/x.$$

Since $Z>1$, one can use the Laplace integral representation[23]

$$P_\lambda(Z)=\frac{1}{\pi}\int_0^\pi[Z+(Z^2-1)^{1/2}\cos\phi]^\lambda\,d\phi\tag{26}$$

for all values of λ.

According for all values of n, the upper bound

$$|P_\lambda(Z)|\leqslant\left(\frac{6s}{x}\right)^\lambda$$

holds and can be used in the study of the representation of the irregular solutions.

From the study of Ref. 2(a), the existence of translation kernels results.

Theorem 1: The translation kernel $K(x,y)$ used in the representation of irregular solutions is uniformly bounded, if the two integrals

$$\sigma_1(x)=\int_x^\infty s\,|\bar{V}(s)|\,ds \quad \text{and} \quad \sigma_\lambda(x)=\int_x^\infty s^\lambda|\bar{V}(s)|\,ds \tag{27}$$

coverge.

In the theorem we have defined $\bar{V}=(6)^\lambda|V(s)|$. To be precise one obtains the bound

$$|K(x,y)|<\tfrac{1}{2}(1/x)^\lambda\sigma_\lambda\left(\frac{x+y}{2}\right)\exp\sigma_1(x). \tag{28}$$

The existence of K is proved for any λ. Therefore, K exists for any n-dimensional space with $1<n<\infty$.

B. Gel'fand-Levitan representation

In the representation of the regular solution, considering Z defined by

$$Z=1-[(x+y-u-s)(u-s+x-y)(u+y+s+x)\times(s+x-u-y)(8usxy)^{-1}],$$

with four positive factors in the numerator, we find $Z\leqslant 1$.

On the other hand, since, in the Gel'fand–Levitan domain we have

$$0\leqslant -x_1=\frac{(x+y-u-s)(x-y+u-s)}{4xs}\leqslant\frac{2x(x-y)}{4xs}\leqslant\frac{2x\cdot 2u}{4xs}\leqslant 1 \tag{29}$$

and

$$0\leqslant x_2=\frac{(x+y-u-s)(x-y+u-s)}{4uy}\leqslant\frac{2y\cdot 2u}{4uy}\leqslant 1, \tag{30}$$

we also can write

$$z=1-2x_2-2x_1(1-x_2)\geqslant 1-2x_2\geqslant -1. \tag{31}$$

The argument Z of the Legendre function being between $+1$ and -1, one must set apart the cases where λ is an integer from the cases where λ is half an integer; the latters will not be considered.

Using the method of Ref. 2(b), when λ is an integer we state the theorem.

Theorem 2: If λ is an integer (n odd) and if the moments σ_0 and σ_1 exist, the kernel $K(x,y)$ used for the representation of the regular solution is bounded. One has

$$|K(x,y)|\leqslant\tfrac{1}{2}\sigma_0\left(\frac{x+y}{2}\right)\exp\sigma_1(x) \tag{32}$$

with

$$\lim_{y\to 0}K(x,y)=0 \quad \text{if } V \text{ is continuous.}$$

The existence of K is proved when the dimension of the space is odd. In addition the integral

$$\int_0^x|K(x,y)|\,dy \tag{33}$$

exists when x goes to infinity.

The present Sec. 3 on uncoupled equations has separated even dimensional from odd dimensional spaces when the potential possesses spherical symmetry. The study already made can be extended without difficulty to the generalized axially symmetric Hamiltonian of Gilbert.[28] For the extension one has simply to consider the Hamiltonian equation

$$\left[\Delta_n+\frac{s}{x_n}\frac{\partial}{\partial x_n}+k^2-V(r)\right]\psi(\mathbf{x})=0, \tag{34}$$

where s is a fixed parameter. The asymptotic behavior of the solution $\psi(\mathbf{x})$ of Eq. (34) is this time

$$\lim_{|r|\to\infty}\psi(\mathbf{x})=\exp(i\mathbf{k}\cdot\mathbf{x})+\frac{f(k,\theta)\exp(ikr)}{r^{1/2(n+s-1)}}+O[r^{-1/2(n+s+1)}]. \tag{35}$$

Equations (34) and (35) together with the method of Sec. 2 lead to the construction of a reduced radial equation,

$$\left(\frac{d^2}{dr^2}+k^2-\frac{p(p+n-2+s)}{r^2}-\frac{1}{4r^2}(n-1+s)(n-3+s)-V(r)\right)\psi(p,k,r)=0. \tag{36}$$

Translation operators can be applied for representing irregular solutions. The Legendre functions to be used in the representation are never polynomials (their order is no longer an integer). The result expressed in Theorem 1 continues to hold provided an appropriate value is given to λ. Needless to say the representation of the regular solutions for this generalized axially symmetric equation is excluded.

4. TRANSLATION KERNEL (MATRIX EQUATIONS)

In Sec. 4, the results of Sec. 3 are extended to matrix differential equations. Solutions, as are custommary in the case, are matrices which are built up using vector solutions. After a discussion of an important Riemann solution in Sec. 4 A, the extension to the n-dimensional problem is attacked in Sec. 4 B. Existence theorems are formulated when a many-channel approach is menaingful.

Before proceeding further, we feel it is useful to discuss the introduction of hyperspherical systems in m-body problems.[29,30] Although the n-dimensional space shares with the m-body ($3m=n+3$) the same number degrees of freedom, the hyperspherical scattering solutions do not possess the exact asymptotic form. Furthermore, the kernel of the Lippman–Schwinger equation associated with the m-body problem is not *compact*.[31] Faddeev equations (or any equivalent) are needed to reach compactness.[32] Whether the method of Zachariev[33] and Raynal[34] removes the "dangerous contributions" is not the point here. The fact is that Refs. 33 and 34 construct nonhomogeneous differential systems, unfitted by nature to linear translation operators. Consequently the m-body problem is not discussed here except when it reduces trivially to a many-channel problem.[35] We will allow, however the number of channels to become infinite.

A. The Riemann solution $C(l,m)$

The solution $C(l,m)$ is constructed because of its use in the following subsection.

We consider the two operators

$$A \equiv \left[\frac{d^2}{dx^2} + k^2 - l^2\right] - V(l, m, x),$$
$$A_0 \equiv \left[\frac{d^2}{dx^2} + k^2 - l^2\right], \tag{37}$$

where l^2 may be a continuous or simply a discrete threshold energy. The partial differential equations for the continuous "matrix" $K(l, m; x, y)$ are

$$\frac{\partial^2}{\partial x^2} K(l, m; x, y) - l^2 K(l, m; x, y) + m^2 K(l, m; x, y)$$
$$= \frac{\partial^2}{\partial y^2} K(l, m; x, y) + \hat{\int} V(l, n, x) K(n, m; x, y)\, dn. \tag{38}$$

In the case of a discrete index the integration is replaced by an ordinary summation.

To Eq. (38) boundary conditions are to be added. We are led to consider the Riemann solutions for the

$$\left[\frac{\partial^2}{\partial x^2} - (l^2 - m^2)\right] C(l, m; x, y) = \frac{\partial^2}{\partial y^2} C(l, m; x, y). \tag{39}$$

These Riemann solutions which we briefly denote $C(l, m)$ are given in Ref. 2(b) where we followed Ref. 36. In characteristic variables they are

$$C(l, m) \equiv J_0[-(4v)^{1/2}] \tag{40}$$

with

$$v = (l^2 - m^2)(\xi - \xi_0)(\eta - \eta_0)$$
$$= (l^2 - m^2)[(x - s)^2 - (y - u)^2].$$

Since the inequality $(x - s)^2 > (y - u)^2$ holds for both the Marchenko and the Gel'fand—Levitan representation, one has

$$C(l, m) = I_0(2\sqrt{v})$$

and therefore

$$|C(l, m)| \leq 1 \quad \text{for} \quad |m| \geq |l|, \tag{41}$$
$$|C(l, m)| \leq \exp 2\sqrt{v} \quad \text{for} \quad |m| \leq |l|. \tag{42}$$

From Eqs. (41) we can obtain the bounds to be used later:

Marchenko representation: Since $s > x$, we have

$$|C(l, m)| < \exp 2|l| s, \quad \forall\, m. \tag{43}$$

Gel'fand—Levitan representation: One can use $u + y \geq x - s > 0$,

$$v < (u + y)^2 (l^2 - m^2),$$
$$|C(l, m)| \leq \exp 2|l| u \exp 2|l| y, \quad \forall\, m. \tag{44}$$

B. The n-dimensional space

One expands the wavefunction in hyperspherical variables,

$$\psi(\mathbf{x}) = \sum_{\nu\alpha} P_{\nu\alpha}(\Omega) R_\nu(r) \frac{1}{r^{(n-1)/2}}. \tag{45}$$

In Eq. (45) we included in the hyperspherical definition the "order" ν and the set α of all the "quantum numbers" necessary for their definition. See Ref. 30(d).

Noting $(\nu, \alpha) \equiv i$ and $(\mu, \beta) \equiv j$, we define the elements of the matrix potential between the i and j "channels" by the integrals

$$V_{ij}(r) = \int P_{\nu\alpha}(\Omega) V(x) P_{\mu\alpha}(\Omega)\, d\Omega. \tag{46}$$

Recalling the definition

$$\lambda_1 = p_i + (n - 3)/2, \tag{47}$$

we introduce *a priori* the system of coupled equations,

$$L_i(x) u_i(x) + V_{ii}(x) u_i(x)$$
$$= \left[\frac{d^2}{dx^2} + k_i^2 - \frac{\lambda_i(\lambda_i + 1)}{x^2} + V_{ii}(x)\right] u_i(x)$$
$$= \sum_{i \neq j} V_{ij}(x) u_j(x). \tag{48}$$

In Eq. (48) one has

$$k_i^2 = (E - E_i) \frac{\hbar^2}{2M}. \tag{49}$$

In Eq. (49), E is the incident energy and E_i is the threshold energy of the ith "channel." The case where all the E_i are set equal to zero (all the k_i^2 are equal to k^2) is discussed first. The extension to the general case which makes use of Sec. 4 A follows.

In the physical applications the system of equations (48) may be infinite. Concerning infinite systems, the question of the existence of translation kernels may be raised. In the following the conditions for the existence of translation kernels are stated first when the order of the system is finite. We indicate hereafter how the conditions should be supplemented when infinite systems are considered.

To obtain our results we use the Riemann solutions $R_{ij}(x, y; s, u)$. They satisfy the equations:

$$L_i(x) R_{ij}(x, y; s, u) = L_j(y) R_{ij}(x, y; s, u), \tag{50}$$

$$R_{ij}(x, y; s, u)$$
$$= \Pi\begin{pmatrix}\lambda_i, \lambda_j \\ x_1, x_2\end{pmatrix}$$
$$= P_{\lambda_i}(1 - 2x_1) - 2x_2 \int_0^1 P_{\lambda_i}(1 - 2x_1 + 2x_1 t)$$
$$\times P'_{\lambda_j}(1 - 2x_2 t)\, dt$$
$$= P_{\lambda_j}(1 - 2x_2) - 2x_1 \int_0^1 P_{\lambda_j}(1 - 2x_2 t)$$
$$\times P'_{\lambda_i}(1 - 2x_1 + 2x_1 t)\, dt, \tag{51}$$

where x_1, x_2 have already been defined in Eqs. (24).

Again we separate the representation of solutions defined by a condition at infinity (Marchenko representation) from the representation of solutions defined by a condition at the origin (Gel'fand—Levitan representation). When the latter representation is considered, along the segment $u + s = x - y$, x_2 equals 1. In the last equation, (51), P_{λ_j} is defined only if λ_j is an integer, that is if n is odd; the representation of regular solutions for even n is again therefore excluded.

(a) *Marchenko representation*: The representation of irregular matrix solutions is discussed for even and odd dimensional spaces. Since we have $0 \leq -x_1 \leq s/x$

and $0 \leq x_2 \leq \frac{1}{2}$, all the arguments of the Legendre functions which appear in the last of the Eqs. (51), are positive. Consequently, we can use the bounds derived from the Laplace integral. This can be done whether the dimension of the space is odd or even. Following the methods used in Ref. 2(b), we obtain

$$|R_{ij}| \leq \left(\frac{7s}{x}\right)^{\lambda_i}. \tag{52}$$

We introduce now the notation

$$\begin{aligned}
&\tilde{V}_{ij}(s) = (7s)^{\lambda_i} V_{ij}(s),\\
&W_{ij}(s) = (7s)^{\lambda_i} V_{ij}(s) \frac{1}{(s)^{\lambda_j}},\\
&\eta\left(\frac{x+y}{2}\right) = \int_{(x+y)/2}^{\infty} \|\tilde{V}(s)\|\, ds,\\
&\xi(x) = \int_x^\infty s\, \|W(s)\|\, dx,\\
&D_{ij}(x) = x^{\lambda_i} \delta_{ij}
\end{aligned} \tag{53}$$

and we can state the existence theorem when the system (48) is finite and all the k_i^2 are set equal.

Theorem 3: If the system considered is finite and if all k_1^2 are equal, and if in addition the two integrals $\sigma_y^{(0)}$ and $\sigma_y^{(1)}$ converge as follows:

$$\begin{aligned}
&\sigma_{ij}^{(0)}(x) = \int_x^\infty t^{\lambda_i} |V_{ij}(t)|\, dt < \infty,\\
&\sigma_{ij}^{(1)}(x) = \int_x^\infty t\, |V_{ij}(t)|\, t^{|\lambda_i - \lambda_j|}\, dt < \infty,
\end{aligned} \tag{54}$$

a translation kernel for the n-dimensional irregular matrix solutions exists. To be precise one has

$$D^{-1}(x) \|K(x,y)\| \leq \tfrac{1}{2} \eta\left(\frac{x+y}{2}\right) \exp\xi(x). \tag{55}$$

By Theorem 3, a restriction has been set on the nondiagonal elements of the matrix potential $V_{ij}(x)$. In addition to this restriction, the theorem requires the existence of absolute moments for the matrix $V_{ij}(x)$. When the order of the matrices introduced in the proof is finite, these matrices have a finite norm but this is not necessarily so when their order is infinite. Then one needs additional requirements. First one must assume

$$\|V\| = \sup_i \sum_{j=1} |V_{ij}| < \infty, \tag{56}$$

so that V possesses a finite norm. Afterwards one should require the existence of the following limits:

$$\lim_{i,j\to\infty} \int_x^\infty t^{|\lambda_i - \lambda_j|+1} |V_{ij}(t)|\, dt < \infty, \tag{57}$$

$$\lim_{i,j\to\infty} \int_x^\infty t^{\lambda_i} |V_{ij}(t)|\, dt < \infty. \tag{58}$$

The exponents (λ_i, λ_j) of Eqs. (57) and (58) represent the angular momenta of the i and j channels. They may remain finite even if the number of channels become infinite.

In the same way as the norm for V was defined in Eq. (55), norms for the infinite matrices V and W, Eq. (53), have to be introduced and should be assumed to exist. Writing

$$\xi(x) = \int_x^\infty s\, \|W(s)\|\, ds < \infty, \tag{59}$$

$$\eta(x) = \int_x^\infty \|V(s)\|\, ds < \infty, \tag{60}$$

and assuming inequalities (57), (58), (59), and (60) are satisfied one takes into account the possibility for i and j to assume infinite values. In this way the upper bound expressed by Eq. (55) remains finite. Now according to Eqs. (48) and (49) threshold energies are included. Assuming the existence of an upper bound K on the set of positive numbers K_i^2 defined by

$$K_i^2 = -k_i^2, \tag{61}$$

and using the Riemann solutions $C(K_i, K_j)$ of Eq. (43), the existence of a translation kernel can be proved in the extended case where the k_i^2 are different. The extension is obtained at the price of requiring an additional exponential decrease of the matrix potential; the measure of which is expressed by

$$4 \sup_{i,j} (|k_i^2 - k_j^2|)^{1/2}. \tag{62}$$

As one realizes, the system of operators introduced in Eq. (37) is a discrete system.

(b) *Gel'fand–Levitan representation*: We recall a first restriction. No consideration is given to even dimensional spaces. The Riemann solutions introduced in the possible integral representation are defined in Eqs. (44) and (51). When $x_2 = 1$ these $R_{ij}(x,y;s,u)$ are no longer constant. The contribution of the segment $u+s = x-y$ to the integral representation has to be included. We are therefore obliged to consider the full integral equations, namely

$$\begin{aligned}
K_{ij}(x,y) = \tfrac{1}{2} &\int_{(x-y)/2}^{(x+y)/2} R_{ij}(x,y;s,s) V_{ij}(s)\\
&+ \tfrac{1}{2} \iint_{0_G} R_{ij}(x,y;s,u) \sum_k V_{ik}(s)\\
&\times K_{kj}(s,u)\, du\, ds\\
&+ \int_{(x-y)/2}^{(x-y)} \frac{d}{ds}[R_{ij}(x,y;s,-s+x-y)]\\
&\times K_{ij}(s,-s+x-y)\, ds.
\end{aligned} \tag{63}$$

In this form Eq. (63) does not seem suitable for the method of successive approximations. A variant is given which is obtained by integrating the last term of Eq. (63) by parts. Denoting this last term by I, one gets

$$\begin{aligned}
I = &R_{ij}(x,y;x-y,0) K_{ij}(x-y,0) - R_{ij}\left(x,y;\frac{x-y}{2}, \frac{x-y}{2}\right)\\
&\times K_{ij}\left(\frac{x-y}{2}, \frac{x-y}{2}\right) - \int_{(x-y)/2}^{(x-y)} R_{ij}(x_1,x_2)\\
&\times \frac{d}{ds} K_{ij}(s,-s+x-y)\, ds.
\end{aligned} \tag{64}$$

Using the boundary conditions $K_{ij}(x,y)$ has to satisfy, Eq. (64) reduces to

$$\begin{aligned}
I = &-\tfrac{1}{2} R_{ij}\left(x,y;\frac{x-y}{2}, \frac{x-y}{2}\right) \int_0^{(x-y)/2} V_{ij}(s)\, ds\\
&- \int_{(x-y)/2}^{(x-y)} R_{ij}(x_1,x_2) \frac{d}{ds} K_{ij}(s,-s+x-y)\, ds.
\end{aligned} \tag{65}$$

Equations (64) and (65) suggest that under proper conditions of convergence the kernel $K_{ij}(x,y)$ may exist in very general circumstances. When all the "channels" are coupled at the same angular momentum λ_i and all have a zero threshold energy the value of the term I is exactly zero and the contribution of the segment $u+s+x-y$, to Eq. (63) disappears. Then under the simple conditions that the matrix potential possesses finite moments of order zero and order one, a translation kernel exists. Due to the difficulty of providing the existence of $K(x,y)$ for the most general form of Eq. (63), no more consideration is given to the Gel'fand–Levitan representation. From now on we will consider only the Marchenko method.[37]

(c) *Physical application*: A physical application concerning the many-channel case for the n-dimensional scattering problem is now discussed. The basic idea is to solve the A-body Hamiltonian

$$H_A\phi_n(\zeta)=\epsilon_n\phi_n(\zeta),$$

before considering the $(A+1)$ Hamiltonian H_{A+1}. Its solutions $\phi_n(\zeta)$ are separated in radial and angular coordinates using hyperspherical variables

$$\phi_n(\zeta)=\sum_{m'} P_{m'}(\Omega)u_{m'n}(\zeta).$$

Channels can be defined by coupling target-generalized angular momenta m' to the incident projectile ones m'' to a total angular momentum m,

$$[P_{m''}(\Omega)\times P_{m'}(\Omega)\,u_{m'n}(\zeta)]_m.$$

It is assumed, as is the case in three-dimensional nuclear problems that the target momenta m' are finite. The projectile angular momenta m'' which can be coupled to some m' to construct m (Clebsch–Gordon generalizations) are finite. The set of λ_i's present in Eq. (48) possesses therefore an upper bound. The left-hand side of expressions Eqs. (57), (58), (59), and (60) can be finite even under the assumption of an infinite number of channels.

If we assume, in addition, that closed channels can be safely neglected if their respective threshold energies are large[6,38] and the constants l and m which appear in the bounds of Sec. 4A for the Riemann solution $C(l,m)$ are themselves bounded.

Theorem 3 can be used to assert the existence of a Marchenko kernel provided the matrix potential has, in addition to the conditions it expresses, adequate exponential decrease. See Eq. (62).

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