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Extension of the Marchenko Equation to Non-Hermitian Differential Systems

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The results of Marchenko and Ljance are unified and extended. We prove the existence of a fundamental equation for an $l \neq 0$ non-Hermitian system and identify which the scattering data are. However we have included neither threshold energies nor Coulomb interactions. Conventional methods are used to determine which necessary conditions must be imposed upon the scattering data so that they can be used in the inverse problem for definite angular momentum. Indications are given on how to overcome our restrictions for the proof of the existence of the fundamental equation and the analysis of its properties.

1. INTRODUCTION

The present paper contains no new tool. The methods used were proposed by such theorists as Newton [1a–c), Marchenko [2a], and Ljance [3a]. The interest of the paper lies in its extension of these methods: it concerns non-Hermitian differential systems so it applies to dissipative systems. However in contrast to Refs. [2a, 3a] it allows for $l \neq 0$ angular momentum coupling. As such it extends Eq. (4.13) of Ref. [1a] and has the same generality as Eq. (9.32) of Ref. [1b]. Except for its noninclusion of Coulomb forces and of channel threshold energies, the present paper, which extends Marchenko's formalism, deals with a many-(finite-) channel effective Hamiltonian. Cox [1d] studying the Gel'Fand–Levitan formalism includes the channel threshold energies.

We have proved the existence of translation kernels in a very large set of circumstances (roughly speaking, all the circumstances that one can encounter when dealing with nuclear scattering [4a–g]. The conditions for the existence of a translation kernel

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were expressed in terms of inequalities that integrals of the potential $U(x)$ must satisfy. Their general form (notation σ) is

$$\sigma(x) = \int_x^\infty |f(s)|\, \| U(s)\|\, ds < \infty,$$

$$\| U(s)\| = \sup_i \sum_j |U_{ij}(s)|;$$

the $|f(s)|$ and $|U_{ij}(s)|$ are the absolute values of $f(s)$ and $U_{ij}(s)$. The expression $\sigma(x)$ contains the case where $f(s)$ reduces to $|s|^\alpha$. We will refer to these inequalities as "moments" of the "nuclear potential." These conditions changed when the "reference potential" itself was changed. At no time however, while in the process of establishing our results, have we made any use of the Hermitian property of the matrix nuclear potential. Since translation kernels exist whenever the conditions of moments appropriate to the choice of a reference potential are satisfied, and this, whether or not the nuclear potential is Hermitian, one can wonder whether it is not possible to go further inside the inverse problem.

As is well-known, the final goal of the inverse problem is the reconstruction of the differential Schrödinger equation from a reference equation. Following Marchenko, the steps for this reconstruction are as follows:

(a) to prove the existence of a fundamental equation;

(b) to recognize and enumerate the scattering data;

(c) to formulate necessary and sufficient conditions to be imposed on the set of data candidates to be the "scattering data" for a Schrödinger equation;

(d) finally, to construct the nuclear potential. To these steps, Marchenko in Refs. [2b. c] added the important philosophical discussion of the stability of the reconstruction.

The first step therefore is to ask whether a fundamental equation exists. A first result of this paper is to prove such an existence, in a case which extends Refs. [2a, 3a] and to suggest that the Marchenko method of special transformations may be avoided in the solution of physical inverse problems. In the proof we follow Marchenko, that is, we do not postulate the existence of a Parseval identity. Although threshold energies and Coulomb forces have not been considered, their incorporation in the first step of the work does not generate essential difficulties as will be shown. Unhappily this is not true for the subsequent steps.

During Sections 2, 3, and 4 of this work it becomes apparent that an essential difference exists between Hermitian and non-Hermitian systems; while for the first one the existence of translation kernels and behavior of T-matrices for $k = 0$ guarantee a Marchenko equation, for the second, one must make sure that the point spectrum and the set of spectral singularities remain discrete.

This difference leads us in the second step (Section 5) to separate Hermitian from non-Hermitian systems. We found that it was important to specify why the methods which worked in Hermitian cases were of no use in non-Hermitian cases. A separate paper, on the subject, was therefore written; it is summarized here where we will simply insist on the results already obtained in Ref. [2a]: while for Hermitian systems the uniqueness of the solution of the fundamental equation depends on properties possessed by the homogeneous equation associated with the continuum part of the spectrum, no such dependence exists in a non-Hermitian system. The incentive to analyse the non-Hermitian problem develops when we become aware of Ref. [3a] and its development in Ref. [3b].

Since the solution for the $l = 0$ complex but scalar case was given by Ljance, it was appropriate to look for an extension to a system and if possible to an $l \neq 0$ system. In the same way, it was appropriate to solve the inverse problem at fixed energy as in Ref. [5] using the tools provided by Ljance.

Our development is restricted to the discussion of the fundamental equation related to non-Hermitian systems and stops at the point when one needs to determine whether a set of elements has the properties required in order to constitute a set of authentic scattering data.

To proceed with the study, the potential is separated into a reference potential plus a nuclear potential; it is then assumed that the nuclear potential allows the existence of a translation kernel Refs. [4a–g]. A limitation of our work should however be indicated; we restrict ourselves to solutions specified by a condition at infinity (Jost's solutions). Reasons for this limitation are also found in the aforementioned set of references.

We can now delineate the general plan of our paper. In Section 2 notations are recalled and an important function denoted $\Delta(k)$ is defined. In Section 3, the zeros of $\Delta(k)$, possible eigenvalues for the differential system, are discussed. Section 4 establishes first the existence of a fundamental equation. Afterwards remarks are formulated to orient the remainder of the work. In Section 5 the uniqueness of the solution for the fundamental equation and the complete continuity of its kernel are proved.

2. Notation

To fix the notations, we consider the non-Hermitian system of operators

$$H - \Lambda^2 = -\left[\frac{d^2}{dx^2} - \frac{L(L+1)}{x^2} - U(x)\right] - \Lambda^2, \tag{1}$$

where Λ^2 is a scalar matrix with $\Lambda^2_{ij} = k^2 \delta_{ij}$. In Eq. (1) we note

$$U_{ij}(x) = V_{ij}(x) + iW_{ij}(x),$$
$$(L)_{ij} = l_i \delta_{ij}.$$

The system defined by (1) is not Hermitian, i.e., it does not possess the Hermitian symmetry

$$U^*_{ji}(x) \neq U_{ij}(x),$$

where the asterisk denotes the complex conjugation. The following notations are used:

$$(U^*)_{ij} = U^*_{ij},$$

$$(U^T)_{ij} = U_{ji},$$

$$U^+ = U^{T*} \quad \text{or} \quad (U^+)_{ij} = U^*_{ij}.$$

Throughout the paper it will be assumed that the matrix $(-1)^L$ is scalar, i.e., parity conservation is assumed. Together with the system of operators defined by (1), we consider the following adjoint system:

$$\bar{H} - \Lambda^2 = -\left[\frac{d^2}{dx^2} - \frac{L(L+1)}{x^2} - U^+\right] - \Lambda^2. \tag{2}$$

We will also note U^+ as $\bar{U}$.

A reference system of operators is introduced by

$$\frac{d^2}{dx^2} - \frac{L(L+1)}{x^2} + \Lambda^2$$

for (1) and (2). The nuclear potentials U and U^+ satisfy the conditions for the existence of translation kernels. These conditions are rederived in Section 5 when it is necessary to establish a bound for the kernel of the fundamental equation. Let l be the largest angular momentum present in (1) or (2); the necessary conditions include the moments of order 0 and 1 of the interaction $U(s)$. They are

$$\int_x^\infty s^\alpha \| U(s)\| \, ds < \infty, \qquad \alpha = 0, 1. \tag{3a}$$

If one wants the translation kernel to be absolutely integrable, one needs to require (3a) for $\alpha = 0,..., l, l+1$. In the non-Hermitian case we replace (3a) by the Naimark condition of Ref. [3c]:

$$\int_0^\infty \exp(\epsilon s) \| U(s)\| \, ds < \infty. \tag{3b}$$

The Naimark condition assures the analyticity of the S-matrix in the strip $|\operatorname{Im} k| \leqslant \epsilon$; it guarantees also that the point spectrum and the spectral singularities remain discrete and do not accumulate on the real axis. Condition (3b) is not a necessary condition for this discreteness as shown by the example where the interaction contains an imaginary part which is a "small" perturbation over a real part verifying (3a).

A third condition similar to (3a) and denoted (3c) is also introduced:

$$\int_0^\infty s^\alpha \| U(s)\| \, ds < \infty, \qquad \alpha = 1, 2. \tag{3c}$$

To complete our notations we use "barred" and "unbarred" elements: unbarred elements refer to (1), barred to (2). Unless it is explicitly stated the solutions of the equations, we consider are $n \times n$ matrices.

The solutions ψ and $\bar{\psi}$ of the equations $(H - \Lambda^2)\psi = 0$ (1) and $(\bar{H} - \Lambda^2)\bar{\psi} = 0$ (2), their limits, and the S-matrix depend on the wave number k; they can therefore be studied within the framework of the theory of complex variables and their analyticity can be discussed as depending on k.

Now we consider the special solutions of the Eqs. (1) and (2) which are defined by their behavior at infinity: they are denoted F and $\bar{F}$ and called the Jost solutions. Their Fourier-like integral representation is what we call the Marchenko representation. Since we assume the existence of a translation kernel, for the Marchenko representation, we write directly:

$$F_+(k, x) = H_1(kx) + \int_x^\infty K(x, y)\, H_1(ky)\, dy, \tag{4a}$$

$$\bar{F}_+(k, x) = H_1(kx) + \int_x^\infty \bar{K}(x, y)\, H_1(ky)\, dy; \tag{4b}$$

$H_1(kx)$ denotes accordingly the diagonal matrix:

$$[H_1(\pm kx)]_{ij} = h_1^{l_i}(\pm kx)\, \delta_{ij} \underset{x\to\infty}{\simeq} e^{i(\pm kx - l_i\pi/2)}\delta_{ij}\,. \tag{5}$$

We will introduce later $F_-(k, x)$ related to $H_2(kx)$ by an equation similar to (4a)

$$H_2(kx) \simeq e^{-i(kx - l_i\pi/2)}\delta_{ij}\,.$$

Due to condition (3b), $F_+(k, x)$ is well-defined for $\operatorname{Im} k > -\epsilon/2$. Due to the existence of K, when x goes to infinity for (Im $k > -\epsilon/2$, we can write

$$[\bar{F}_\pm(k, x))]_{ij} \sim [F_\pm(k, x)]_{ij} \sim e^{\pm i(kx - l_i\pi/2)}\delta_{ij}\,.$$

When x goes to zero, the following limit Refs. [1b, c] is introduced

$$H_1(kx) \underset{x\to 0}{\simeq} (2L-1)!!\,(kx)^{-L} + O(k^{-L+2}x^{-L+2}). \tag{6}$$

If $U = U^+$, we have

$$F_+(k, x) = \bar{F}_+(k, x)$$

and

$$K(x, y) = \bar{K}(x, y).$$

If $U = U^T$ we have instead

$$[\bar{F}_\pm(-k^*, x)]^* = (-)^L F_\pm(k, x) \qquad \text{for} \quad \operatorname{Im} k \geqslant -\epsilon/2.$$

As in the scalar case, solutions which are regular at the origin may be defined. However, in contrast to the scalar case it is difficult to define a regular solution by a boundary condition. The integral equation for the matrix case which would have been the direct generalization of the integral equation for the scalar case diverges at $r = 0$ unless negative moments of the off-diagonal part of the interaction are assumed. Newton [1a, c] has shown that if one adds to the inhomogeneity of the previous integral equation an appropriate and constant term, dependent on the off-diagonal part of the interaction, and which is a right multiple of the regular solution of (1) without interaction, the integral is made to converge. The convergence is obtained via a counterterm. The result is a quite complicated equation which is not given in the present paper since we do not use it; the interested reader may refer to Ref. [1a] or [1c] where an explicit form for the equation is found in the neutron–proton case. The solution $G(k, x)$ for this corrected equation will be called, from now on, the regular solution of Eq. (1). It can be shown [1a, c] that $G(k, x)$ is an analytic matrix function of k for $k \neq 0$, under the simple condition (3c); in addition we have the equality:

$$\bar{G}(k^2, x) = [G(k^{2*}, x)]^*. \tag{1}$$

In general, we consider $U \neq U^+$ and $\bar{F}_+(k, x) \neq F_+(k, x)$. This is the reason for introducing the bar symbol.

A lemma is now established which extends the Wronskian theorem to non-Hermitian systems. We consider the two adjoint systems of differential equations:

$$\left[\frac{d^2}{dx^2} + k^2 - \frac{L(L+1)}{x^2} - U\right] u(k^2, x) = 0, \tag{8a}$$

$$\left[\frac{d^2}{dx^2} + k^{2*} - \frac{L(L+1)}{x^2} - U^+\right] v(k^{2*}, x) = 0. \tag{8b}$$

Taking the adjoint of Eq. (8b) gives

$$v^+(k^{2*}, x)\left[\frac{d^2}{dx^2} + k^2 - \frac{L(L+1)}{x^2} - U\right] = 0. \tag{8c}$$

Equation (8c) is postmultiplied by u, while Eq. (8a) is premultiplied by v^+. Subtracting one result from the other yields

$$\begin{gathered} v^+(k^{2*})\,\frac{d^2}{dx^2}\,u - \frac{d^2}{dx^2}\,v^+(k^{2*})\,u = 0, \\ v^+(k^{*2})\,u' - v^{+\prime}(k^{*2})\,u = \text{const}, \end{gathered} \tag{9}$$

Wronskian $[v^+, u] = Wr[v^+, u] = \text{const}$. We state the following lemma.

LEMMA. *Let $u(k^2, x)$ and $v(k^{2*}, x)$ be solutions of Eqs.* (8a) *and* (8b); *the Wronskian* Wr$[v^*, u]$ *is independent of the x-variable.*

As a consequence of the lemma, some Wronskians are computed since they will be used later. We denote by $e^{-i(kx-L\pi/2)}$ the diagonal matrix whose elements are $e^{-i(kx-l_i\pi/2)}\,\delta_{ij}$. Due to Eqs. (3a) and (3c) and due to the existence of K and Eqs. (4a) and (4b), Eqs. (1) and (2) have a fundamental system of solutions for any nonzero value of k in the strip $|\operatorname{Im} k| < \epsilon/2$. These independent solutions have the following asymptotic behavior:

$$F_{\pm}(k, x) \underset{x\to\infty}{\simeq} e^{\pm i(kx-L\pi/2)},$$

$$\bar{F}_{\pm}(k, x) \underset{x\to\infty}{\simeq} e^{\pm i(kx-L\pi/2)}.$$

In the strip $|\operatorname{Im} k| < \epsilon/2$ we have

$$F_{\pm}(-k, x) = (-)^L F_{\mp}(k, x), \tag{10a}$$

$$\bar{F}_{\pm}(-k, x) = (-)^L \bar{F}_{\mp}(k, x). \tag{10b}$$

With (10a) one completes (7) by

$$[\bar{F}_{+}(-k^*, x)]^* = (-1)^L F_{+}(k, x) = F_{-}(-k, x).$$

Consequently, the Wronskians between solutions with the same + or − subscript are

$$\operatorname{Wr}[\bar{F}^{+}(\pm k^*, x), F(\pm k, x)] = \pm 2ik, \tag{10c}$$

$$\operatorname{Wr}[\bar{F}^{+}(\mp k^*, x), F(\pm k, x)] = 0. \tag{10d}$$

We also have

$$\operatorname{Wr}[\bar{F}_{\pm}^{+}(k^*, x), F_{\pm}(k, x)] = \pm 2ik, \tag{10e}$$

$$\operatorname{Wr}[\bar{F}_{\pm}^{+}(k^*, x), F_{\mp}(k, x)] = 0. \tag{10f}$$

Until the end of this section we assume $k \neq 0$ and introduce the solution $G(k, x)$ regular at the origin.

The irregular solutions $F_{+}(k, x)$ and $F_{+}(-k, x)$ of Eq. (4a) constitute a fundamental system for Eq. (1). The same is true for the two irregular solutions $F_{+}(k, x)$ and $F_{-}(k, x)$. One can therefore find two constant matrices A and B (A' and B') and express the regular solution G in the strip $|\operatorname{Im} k| \leqslant \epsilon/2$, $k \neq 0$, by

$$G(k, x) = [F_{+}(k, x)A + F_{-}(k, x)B], \tag{11}$$

$$G(k, x) = [F_{+}(k, x)A' + F_{+}(-k, x)B']. \tag{12}$$

Use of the system of Eqs. (10) gives

$$\operatorname{Wr}[\bar{F}_{+}^{+}(k^*, x), G(k, x)] = 2ikA, \tag{13}$$

$$\operatorname{Wr}[\bar{F}_{-}^{+}(k^*, x), G(k, x)] = -2ikB. \tag{14}$$

We are led to define

$$\bar{F}_{\pm}^{+}(k^*) = \operatorname{Wr}[\bar{F}_{\pm}^{+}(k^*, x), G(k, x)]\, k^L, \tag{15a}$$

for all values of x including $x = 0$.

With the help of Eq. (15), we obtain A and B:

$$A = -\frac{i}{2}\frac{1}{k}\bar{F}_{+}^{+}(k^*)\, k^{-L},$$

$$B = +\frac{i}{2}\frac{1}{k}\bar{F}_{-}^{+}(k^*)\, k^{-L}. \tag{16}$$

The regular solution $G(k, x)$ is therefore expressed by the equation:

$$G(k, x) = \frac{i}{2}[F_{-}(k, x)\bar{F}_{-}^{+}(k^*) - F_{+}(k, x)\bar{F}_{+}^{+}(k^*)]\frac{1}{k^{L+1}}. \tag{17a}$$

Since Eq. (17a) contains F_{+} and F_{-}, its domain of validity is restricted to the strip $|\operatorname{Im} k| \leqslant \epsilon/2$. Just as we write Eq. (17), we can express $\bar{G}(k, x)$:

$$\bar{G}(k, x) = \frac{i}{2}[\bar{F}_{-}(k, x) F_{-}^{+}(k^*) - \bar{F}_{+}(k, x) F_{+}^{+}(k^*)]\frac{1}{k^{L+1}}. \tag{17b}$$

In Eq. (17b) the following definition has been used:

$$F_{\pm}^{+}(k^*) = \operatorname{Wr}[F_{\pm}^{+}(k^*, x), \bar{G}(k, x)]\, k^L. \tag{15b}$$

The Wronskian

$$\operatorname{Wr}[\bar{G}^{+}(k^*, x), G(k, x)] = 0$$

is used to connect the two equations (17a, b). It gives for $|\operatorname{Im} k| < \epsilon/2$:

$$F_{-}(k)\bar{F}_{-}^{+}(k^*) = F_{+}(k)\bar{F}_{+}^{+}(k^*).$$

We rewrite Eq. (17a) as

$$G(k, x) = \frac{-1}{2i}[F_{-}(k, x) - F_{+}(k, x)\, S(k)]\,\bar{F}_{-}^{+}(k^*)\frac{1}{k^{L+1}}, \tag{18}$$

where we have defined

$$S(k) = F_{+}^{-1}(k)\, F_{-}(k) = \bar{F}_{+}^{+}(k^*)\,\bar{F}_{-}^{+-1}(k^*). \tag{19}$$

We might also have obtained

$$G(k, x) = \frac{-1}{2i}[F_{+}(-k, x) - F_{+}(k, x)\, S'(k)]\,\bar{F}_{+}^{+}(-k^*)\frac{1}{k^{L+1}} \tag{20}$$

using the factorizable matrix $S'(k)$ associated with $S(k)$; the matrix $S'(k)$ is defined as follows:

$$S'(k) = F_{+}^{-1}(k)\, F_{+}(-k) = \bar{F}_{+}^{+}(k^*)\, \bar{F}_{+}^{+}(-k^*)^{-1}; \tag{21}$$

since $F_{+}(-k, x) = (-L)^L F_{-}(k, x)$, one has $(-1)^L S'(k) = S(k)$.

Equation (20) shows that the zeros of det $F_{+}(k)$ occurring for Im $k \geqslant 0$ play an important role in any forthcoming discussion.

3. Zeros of $\Delta(k) = \det F(k)$

In this section all the F's are F_{+}'s so we omit the subscript $+$. In addition to remaining in a physical situation we restrict ourselves to matrix nuclear potentials

$$U = V + iW$$

with V and W real and symmetric matrices. Such symmetry belongs to the optical model potential and to its generalized forms as derived by Feshbach [6] or by Fonda and Newton [7a, b]. Due to this symmetry, F and $\bar{F}$ are no longer independent matrices but are related by the following equality:

$$\bar{F}(\mp k^*, x)]^* = F(\pm k, x)(-)^L \qquad \text{for} \quad \text{Im}\, k \geqslant \epsilon/2, \tag{22a}$$

and

$$\bar{G}(k^2, x) = [G(k^{2*}, x)]^*. \tag{22b}$$

If $U = U^T$, then due to Eqs. (22a) and (22b), Ref. [8],

$$\bar{F}_{+}^{+}(\mp k^*) = (-1)^L F_{+}^{T}(\pm k) = F_{-}^{T}(\mp k) \qquad \text{and} \qquad S = S^T. \tag{23}$$

Thus a symmetric potential gives a symmetric S matrix. Let $\phi(x)$ be a normalizable *vector* solution of Eq. (1); we get

$$\left(\frac{d^2}{dx^2} + k^2\right) \phi(x) - \left(\frac{L(L+1)}{x^2} + V + iW\right) \phi(x) = 0. \tag{24}$$

The formal adjoint of Eq. (24) is

$$\left(\frac{d^2}{dx^2} + k^{2*}\right) \phi^+(x) - \phi^+(x) \left(\frac{L(L+1)}{x^2} + V - iW\right) = 0. \tag{25}$$

Since ϕ is a normalized vector, we have

$$\int \phi^+(x)\, \phi(x)\, dx < \infty.$$

Therefore we obtain

$$\text{Im}\, k^2 \int \phi^+(x)\, \phi(x)\, dx = \int \phi^+(x)\, W(x)\, \phi(x)\, dx.$$

Let us assume that for any vector solution of Eq. (24)

$$\int \phi^+(x)\, W(x)\, \phi(x)\, dx \leqslant 0. \tag{26}$$

Then Im $k^2 \leqslant 0$. The inequality (26) applied to scattering states measures the loss in the current density. It guarantees that the potential is absorptive. Absorptive potentials are the only ones obtained by the reduction of a Hermitian Hamiltonian to one of its subspaces [6, 7]; they appear also in any closed physical system [8]. A sufficient condition to have inequality (26) is that the skew-Hermitian part of H be semi-definite negative. Condition (26) is considerably stronger for local interactions than for nonlocal ones.

Normalizable states occur for the eigenvalues $E_n = R_n + i\Gamma_n$ ($\Gamma_n \leqslant 0$); then $k_n = \alpha_n + i\beta_n$ ($\beta_n \geqslant 0$), that is, $\pi/2 < \arg k_n < \pi$, and $F(k_n)^{-1}$ has a pole. Then $\Delta(k_n)$ vanishes. To discuss the zeros of $\Delta(k_n)$, we define a matrix solution $H(k, x)$ of Eq. (1), after Newton [1c] by the equation

$$H(k, x) = H_1(kx)\, \Delta(k) - \left(H_1(kx) \int_0^x u(ks) + u(kx) \int_x^\infty H_1(ks)\right) \frac{1}{k}\, U(s)\, H(k, s)\, ds. \tag{27}$$

The matrix $u(kx)$ which appears in (27) is the Ricatti–Bessel matrix; it behaves as $(kx)^{L+1}/(2L+1)!!$ when x goes to zero and as $\sin(kx - L\pi/2)$ when x goes to infinity. As x approaches infinity one obtains

$$H(k, x) \simeq H_1(kx) A(k).$$

As a consequence the following yields $H(k, x) = F(k, x) A(k)$, where the matrix $A(k)$ is defined by the equation

$$A(k) = \Delta(k)\, I - \int_0^\infty u(ks)\, \frac{1}{k}\, U(s)\, F(k, s)\, ds\, A(k), \tag{28}$$

and $F(k, x)$ is defined by Eq. (4a). To obtain a more transparent expression for $H(k, x)$, the integral equation for $F(k, x)$ is considered:

$$F(k, x) = H_1(k, x) + \int_x^\infty [H_1(k, x)\, u(ks) - H_1(ks)\, u(kx)]\, \frac{1}{k}\, U(s)\, F(k, s)\, ds. \tag{29}$$

By Eq. (15b) one gets

$$F(k) = -k^L\, \text{Wr}[\bar{G}^+(k^*, x), F(k, x)].$$

Since $U = U^T$, we also have, using (22b),

$$F(k) = -k^L \operatorname{Wr}[G^T(k, x), F(k, x)].$$

Therefore defining

$$-J(k) = \lim_{x\to 0} \operatorname{Wr}[G^T(k, x), H_1(kx)]$$

and using Eq. (27), we write

$$k^{-L}F(k) = J(k)\left[1 + \int_0^\infty u(ks)\frac{1}{k}\, U(s)\, F(k, s)\, ds\right]. \tag{30a}$$

We have assumed in (30)

$$0 = \lim_{x\to 0} \operatorname{Wr}[G^T(k, x), u(kx)] \int_x^\infty H_1(ks)\frac{1}{k}\, U(s)\, F(k, s)\, ds. \tag{30b}$$

According to [1c] this conjecture is verified when the nondiagonal elements of the potential satisfy

$$\int_0 |V_{ij}(x)|\, x^{-m}\, dx < \infty, \qquad m = |l_i - l_j|.$$

We show that the conjecture has a larger domain of validity by particularizing a theorem of [2a, p. 167] concerning the coupling of an $l = 0$ with an $l = 2$ wave. The particularization of the theorem reads: if the interaction is constant for $x \neq 0$; there exist two independent solutions S_1 and S_2 whose behavior at the origin is as follows:

$$S_1 = \begin{pmatrix} x + O(x^2), & O(x^4) \\ O(x^2), & x^3 + O(x^4) \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 1 + O(x), & O(1/x) \\ O(x), & 1/x^2 + O(1/x) \end{pmatrix}.$$

The symbol $O(\cdots)$ denotes a quantity whose ratio to that contained in the bracket is finite on passing to the limit.

With $G^T \simeq S_1^T$ and $F \simeq S_2$ one obtains in (30b) a matrix which vanishes as $O(x)$ when x goes to zero.

With Eq. (30), we write Eq. (28):

$$A(k) = \Delta(k)I - [J^{-1}(k)\, k^{-L}F(k) - 1]A(k)$$

or

$$A(k) = \Delta(k)\, F^{-1}(k)\, k^L J(k).$$

The matrix $F^{-1}(k)$ is constructed in the usual way by using the determinant of F and the transposed matrix of the cofactors $\chi(k)$:

$$F^{-1}(k) = \frac{1}{\Delta(k)}\, \chi(k).$$

This yields the final definition:

$$H(k, x) = F(k, x)\chi(k)\, k^L J(k). \tag{31}$$

The exact form of (31) is less important than the existence of a matrix $A(k)$ such that [see also Eqs. (44a) and (45b)]

$$H(k, x) = F(k, x)A(k).$$

With Eq. (31) we consider the case where Γ_n is a nonzero negative number. The k_n itself is strictly complex, not merely imaginary. Therefore the elements of $H(k_n, x)$ decrease exponentially when x goes to infinity as $\exp(-\beta_n x)$ does:

$$k_n = \alpha_n + i\beta_n .$$

On the other hand, since $\Delta(k_n) = 0$, there exists a vector solution

$$H(k_n, x)\eta,$$

where η is some specific vector. This η-vector solution represents therefore a normalizable state which is regular at the origin.

In other words if $\Delta(k_n) = 0$, $\operatorname{Im} k_n \geqslant 0$, we have proved the existence of a singular matrix $A(k)$ or equivalently that of $p < n$ linearly independent vectors. The rank of $A(k)$ is p and the dimension of its kernel is $n - p$. Now the left product of $A(k)$ by $F(k, x)$ reproduces the physical solution.

For the same discussion of the zeros of $\Delta(k)$, instead of (27), we may consider the Wronskian

$$-k^{+L} \operatorname{Wr}[G^T(k, x), F(k, x)] = F(k). \tag{32}$$

If $\Delta(k)$ has a zero of order m for $k = k_j$, there exists a vector $a(k_j)$ such that

$$\lim_{k\to k_j} \operatorname{Wr}[G^T(k, x), F(k, x)]\, a(k_j) = (k - k_j)^p\, E(k_j), \qquad 1 \leqslant p \leqslant m \leqslant n, \tag{33}$$

where $E(k_j)$ is a complex number. Following [1b, Eq. (9.15)] we write

$$G(k_j, x)\, b_0 = F(k_j, x)a(k_j). \tag{34}$$

The l.h.s. of (34) represents a vector which vanishes at $x = 0$, while its r.h.s. decreases exponentially. It represents therefore a normalizable state giving us the result we wanted.

Bound states may also exist for $\Gamma_n = 0$ for $k_n = i\beta_n$ purely imaginary if $\Delta(i\beta_n)$ vanishes. Now if $\Delta(k)$ vanishes for $k_n > 0$, the state is not normalizable. The vanishing $\Delta(k)$ for $k = 0$ represents or does not represent a normalizable state: a special discussion is needed [1a].

In the case of absorptive potentials $\Gamma_n \leqslant 0$ and $\beta_n \geqslant 0$, k_n corresponding to a normalizable state belongs to the third quadrant.

The possibility of $\Delta(k) = 0$ for real k_n is not excluded by the discussion. If $\Delta(k)$ vanishes for real k one has a spectral singularity. Since $\Delta(k_n) = 0$ implies $\Delta(-k_n) = 0$ for real k_n, spectral singularities are excluded for Hermitian interactions.

If the interaction is absorptive, a general argument of [3b] shows that a spectral singularity occurs only for $k_n < 0$. If the potential is simply complex, it may occur at any real k.

Of course there is no way of proving that the poles of $F(k)^{-1}$ are simple. However, it can be proved that the number and the order of these poles is finite.

Since the solution for the scalar case has been given by Naimark [3b], we extend his solution to the matrix case. Results obtained in [1c] are used. The idea of the proof is that $\Delta(k)^{-1}$ [resp. $F(k)^{-1}$] is a meromorphic function [resp. matrix] in the upper half plane whose limit at infinity is one [resp. unity]. In a first step one shows with [1c] that

$$\lim_{|k|\to\infty} \Delta(k) = 1 + o\left(\frac{1}{k}\right). \tag{35}$$

Equation (35) excludes the possibility of an accumulation point at infinity for the zeros of $\Delta(k)$. In a second step one discusses the zeros of $\Delta(k)$ on the real axis $k^2 > 0$. When one deals with a non-Hermitian potential a condition stronger than (3a) is necessary. Here is the first essential difference between Hermitian and non-Hermitian systems. Let us assume condition (3b), although a somewhat weaker condition (3d) has been shown to be sufficient.

$$\int_0^\infty \exp[\epsilon x] \, \| U(x) \| \, dx < \infty, \tag{3b}$$

$$\int_0^\infty \exp[\epsilon x^{1/2}] \, \| U(x) \| \, dx < \infty. \tag{3d}$$

If there exists some $\epsilon > 0$ such that (3b) is satisfied, the function $\Delta(k)$ which goes to one at infinity is a holomorphic function of k in $\operatorname{Im} k > -\epsilon/2$. The set of its zeros in $\operatorname{Im} k \geqslant 0$ is therefore finite in number and in multiplicity.

Since in the Hermitian case the equality

$$\Delta(k) = \Delta^*(-k^*)$$

excludes the zeros from the real axis $k \neq 0$, a strong condition, as (3b) or even (3d) is not necessary to obtain the finiteness of the set of zeros.

With (3b) real zeros of $\Delta(k)$ may exist but they will be discrete. A further result which uses (3c) can be obtained if the potential is absorptive, then real zeros are restricted to the $k \leqslant 0$ part of the real axis; obviously if it is emissive, they are restricted to $k \geqslant 0$. They belong to the continuum spectrum and have received the name spectral singularities (elements of the continuum spectrum where the Fredholm determinant vanishes).

When scalar potentials are concerned, the analysis of these zeros is simple. When they happen for $k < 0$, they describe the absorption by a black sphere. The wave-function vanishes at the origin and has the asymptotic behavior of a purely incoming wave [10]. Their presence in the resolution of identity limits the Parseval identity to the subset of $L_2(0, \infty)$ made of elements with an exponential decrease. Happily this subset is dense in $L_2(0, \infty)$.

There is no presently known condition on the functional form of a complex potential which prevents the existence of spectral singularities. Condition (3a) guarantees the existence of a translation kernel and condition (3c) guarantees in the Hermitian case that the regular solution introduced previously is an analytic function of k. The same two conditions guarantee also, in the Hermitian case, that the discrete spectrum remains bounded and finite.

In the non-Hermitian case (3a) and (3c) still guarantee the analyticity of the regular solution but a stronger condition must be used to keep the set of spectral singularities and the point spectrum discrete as does Naimark's condition (3b).

Section 3 may be extended to situations which incorporate threshold energies; a first price to pay is to require more from the interactions, a second price is to allow bound states in the continuum.

4. The Fundamental Equation

In the present section, we follow Marchenko's method, that is, we do not postulate the Parseval identity. Another attitude can be accepted: it consists in proving that the class of equations under scrutiny possesses Parseval's identity. When the Parseval identity has been obtained, one deduces the fundamental equation. This attitude is that of Refs. [1a, *d*].

Following Ref. [2a] we rewrite Eq. (18) for $|\operatorname{Im} k| \leqslant \epsilon/2$, $k \neq 0$, using Eq. (23) valid for a symmetric matrix potential:

$$(-2i) G(k, x)\, k^L [F_+^T(k)]^{-1} k = [F_-(k, x) - F_+(k, x) S(k)]. \tag{36}$$

Now we transform (36) by the introduction of translation kernels. The right-hand side becomes

$$\text{r.h.s.} = H_2(kx) + \int_0^\infty K(x, y)\, H_2(ky)\, dy - H_1(kx)\, S(k)$$
$$- \int_0^\infty K(x, y)\, H_1(ky)\, S(k).$$

The matrix $1 - S(k) = T(k)$ is inserted for a second modification. One gets

$$\text{r.h.s.} = [H_2(kx) - H_1(kx)] + H_1(kx)\, T(k)$$
$$+ \int_x^\infty K(x, z)\, dz [H_2(kz) - H_1(kz)] + \int_x^\infty K(x, z)\, H_1(kz)\, T(k)\, dz.$$

Finally with $-2iJ(kx) = [H_2(kx) - H_1(kx)]$ we have

$$\text{r.h.s.} = -2iJ(kx) + H_1(kx)\,T(k) - 2i\int_x^\infty K(x,z)\,J(kz)\,dz + \int_x^\infty K(x,z)\,H_1(kz)\,dz\,T(k).$$

So we get the equation

$$-2i[G(k,x)\,k^L F_+{}^T(k)^{-1}\,k - J(kx)] = H_1(kx)\,T(k) - 2i\int_x^\infty K(x,z)\,J(kz)\,dz + \int_x^\infty K(x,z)\,H_1(kz)\,dz\,T(k). \quad (37)$$

The identity

$$I[\delta(x-y) - \delta(x+y)(-1)^L] = \frac{1}{i\pi}\int_{-\infty}^{\infty}[J(kx)\,H_1(ky)]\,dk$$

which contains the unit matrix I is used to transform Eq. (37). Both sides of (37) are postmultiplied by

$$H_1(ky)\,dk;$$

and a k-integration is performed using the upper half plane.

For the sake of simplicity, spectral singularities are excluded; then the contour is made of the segments $[-L, -\delta]$ and $[+\delta, L]$. We use the symbol integral from $-L$ to L to denote

$$\int_{-L}^{+L} = \lim_{\delta\to 0}\int_{-L}^{-\delta} + \int_{\delta}^{L} + \int_{\Gamma},$$

where the half circle of radius δ centered at $k = 0$ is called Γ.

Now we define F-continuous, in short F_c, by

$$2\pi F_c(x,y) = \lim_{L\to\infty}\left[\int_{-L}^{-\delta} + \int_{\delta}^{-L}\right] H_1(kx)\,T(k)\,H_1(ky)\,dk; \quad (38)$$

in an obvious way $F_\Gamma(x, y)$ is also defined. Integration is performed using the identity for the δ-functions and remembering that $K(x, y) = 0$ if $x > y$. With (38) one gets

$$\lim_{\substack{L\to\infty\\ \delta\to 0}}\left[\int_{-L}^{L} H_1(kx)\,T(k)\,H_1(ky)\,dk - 2i\int_x^\infty dz\,K(x,z)\int_{-L}^{L} dk\,J(kz)\,H_1(ky) + \int_x^\infty dz\,K(x,z)\int_{-L}^{L} H_1(kz)\,T(k)\,H_1(ky)\,dk\right]$$
$$= 2\pi\Big\{F_c(x,y) + F_\Gamma(x,y) + K(x,y) - K(x,-y)(-)^L + \int_x^\infty dz\,K(x,z)[F_c(x,y) + F_\Gamma(x,y)]\Big\}. \quad (39)$$

It will be proved later that F_c is a continuous matrix of its arguments for $y \geq x > 0$. This very continuity leads to restrictions on the T-matrix which can no longer be required simply to possess a Fourier transform as in Ref. [2a].

To be specific, using estimate (3.9) of Ref. [1b], one has

$$|\,2\pi F_c(x,y)|_{ij} \leq c\int_{-\infty}^{\infty} e^{-\operatorname{Im} kx}\left[\frac{|\,k\,|\,x+1}{|\,k\,|\,x}\right]^{l_i} |\,T_{ij}\,|\,e^{-\operatorname{Im} k\pi}\left[\frac{|\,k\,|\,y+1}{|\,k\,|\,y}\right]^{l_j} dk.$$

This restricts the class of available T-matrices. (See Ref. [1c, Eq. (15.118)], for low-energy behavior of T-matrices); therefore high moments of the interaction are required for $F_c(x, y)$, to exist. We consider now the transformation of the l.h.s. for Eq. (39):

$$\lim_{\substack{L\to\infty\\ \delta\to 0}} -2i\left[\int_{-L}^{L} G(k,x)\,k^L(F_+{}^T(k)^{-1} - 1)\,H_1(ky)\,k\,dk - \int_{-L}^{+L}[J(kx) - G(k,x)\,k^{L+1}]\,H_1(ky)\,dk\right].$$

The method used consists in using the properties of analytic functions of complex variables. It happens that the l.h.s. of Eq. (37) represents the "physical" rather than the "regular" solution. In the case of nonzero threshold energies, the integrants contain the projection operator onto the open-channel system. As shown in Ref. [1d] the end result is an analytic matrix; the methods described hereafter will therefore apply with only minor difficulties. When no threshold is included, the situation is easier: the value of (39) can be obtained by the evaluation of the integral along C together with that of the residues of the integrant for $\operatorname{Im} k \geq 0$. Since

$$\lim_{|k|\to\infty} |\,G(k,x)\,k^L[F_+{}^T(k)^{-1} - 1]\,H_1(ky)\,k\,| \to 0, \quad (40a)$$

$$\lim_{|k|\to\infty} |\,J(kx) - G(k,x)\,k^{L+1}\,| \to 0. \quad (40b)$$

The integral along the circle $|\,k\,| = \infty$ vanishes and we are simply left with the contributions of the poles. The latter come from the zeros of

$$[\det F_+{}^T(k)]^{-1} = \Delta_+(k)^{-1} \equiv (\Delta)^{-1},$$

at $k = k_j = a_j + ib_j$.

We write, therefore,

$$\text{l.h.s.} = 2i\pi\sum \text{Res.}[kG(k,x)\,k^L F_+{}^T(k)^{-1}\,H_1(ky)](-2i).$$

In contrast to the case of Ref. [2a] the poles of $F_+{}^T(k)$ are not necessarily simple. We write $F_+{}^T(k)^{-1}$ as $(\Delta)^{-1}\chi_+{}^T(k)$ by introducing the transposed matrix of cofactors $\chi_+{}^T(k)$ of $F_+{}^T(k)$. Let $k_j = a_j + b_j$ be the value of k for which Δ vanishes. Assume that k_j is a m_j-order pole of $F_+{}^T(k)^{-1}$; the residue for $k = k_j$ is obtained by the formula:

$$\text{Residue} = \frac{1}{(m_j-1)!}\left(\frac{d}{dk}\right)^{(m_j-1)}[G(k,x)\,k^L(\Delta)^{-1}\,\chi_+{}^T(k)\,kH_1(ky)(k-k_j)^{m_j}]_{k=k_j}. \quad (41)$$

Since Eq. (36) represents the physical solution in the strip $|\operatorname{Im} k| \leqslant \epsilon/2$, one cannot as shown in [1a, b], except in special cases, use Eqs. (17) and (19) to affirm that in the limit $k = k_j$ the bracket in (41) reduces to

$$-2iF_+(k, x)\, S(k)\, H_1(ky)(k - k_j)^{m_j}.$$

What must be done is to find an expression for

$$G(k, x)\, F_+^T(k)^{-1}$$

valid in the half plane $\operatorname{Im} k \geqslant 0$.

To obtain such a representation one must use two solutions of Eq. (1) linearly independent and analytic in the upper half plane of k.

One such representation uses an irregular solution $I(k, x)$ which is analytic in the k-plane, as the regular solution $G(k, x)$ is Ref. [1a], which satisfies the Wronskian

$$\operatorname{Wr}[\bar{I}^+(k^*, x), G(k, x)] = 1$$

and is defined as a linear combination of two independent solutions constructed at some finite $x_0 \neq 0$.

We write $G(k, x)$ as a linear combination of $F_+(k, x)$ and $I(k, x)$

$$G(k, x) = F_+(k, x)A'' + I(k, x)B'',$$

and use

$$\bar{F}_+^+(k^*) = \operatorname{Wr}[\bar{F}_+^+(k^*, x), G(k, x)]\, k^L,$$
$$\mathscr{F}_+^+(k) = \operatorname{Wr}[\bar{F}_+^+(k, x), I(k, x)].$$

Thus we obtain

$$A'' = \mathscr{F}_+^{-1}(k),$$
$$B'' = \mathscr{F}_+^+(k^*)^{-1}\, \bar{F}_+^T(k^*)\, k^{-L}.$$

A representation for $G(k, x)$ is therefore

$$G(k, x) = F_+(k, x)\, \mathscr{F}_+^{-1}(k) + I(k, x)\, \mathscr{F}^+(k^*)^{-1} \times \bar{F}_+^+(k^*)\, k^{-L}. \tag{42a}$$

Consequently we have

$$G(k, x)\, k^L F_+^T(k)^{-1}$$
$$= \{F_+(k, x)\, \mathscr{F}_+^{-1}(k) + I(k, x)\, \mathscr{F}^+(k^*)^{-1}\, \bar{F}_+^+(k^*)\, k^{-L}\}\, k^L F_+^T(k)^{-1}.$$

In the bracket of Eq. (41) we replace

$$G(k, x)\, k^L F_+^T(k)^{-1}\, k H_1(ky)(k - k_j)^{m_j}$$

by (42b)

$$F_+(k, x)\, \mathscr{F}_+^{-1}(k)\, k^L F_+^T(k)^{-1}\, k H_1(ky)(k - k_j)^{m_j}.$$

In other words we make the substitution

$$G(k, x)\, k^L F_+^T(k)^{-1} \rightarrow F_+(k, x) C(k). \tag{42c}$$

The matrix $C(k)$ is defined by identification (42b) and (42c). An m-order differential operator $\mathscr{D}_{m_j}(k)$ expressed by Eq. (43),

$$\mathscr{D}_{m_j}(k) = \left(\frac{d}{dk}\right)^{(m_j-1)} \left[\frac{1}{(m_j - 1)!}(k - k_j)^{m_j}\right], \tag{43}$$

is introduced. With Eqs. (41), (42a), and (43) the l.h.s. becomes

$$\text{l.h.s.} = -2\pi i \sum_j \mathscr{D}_{m_j}(k)[F_+(k, x)\, C(k)\, H_1(ky)]_{k=k_j}. \tag{44}$$

Now we use the Marchenko integral representation of Eq. (4a) to replace $F_+(k, x)$ in Eq. (44). So we get after dividing by 2π:

$$-i \sum_j [\mathscr{D}_{m_j}(k)[H_1(kx)\, C(k)\, H_1(ky)]_{k=k_j}$$
$$-i \int_x^\infty dz\, K(x, z) \sum_j [\mathscr{D}_{m_j}(k)[H_1(kz)\, C(k)\, H_1(ky)]_{k=k_j}. \tag{45}$$

A new kernel $F(x, y)$ is defined by adding the residues and the Γ-contour contributions

$$F_c(x, y) + F_\Gamma(x, y) + \sum_j \mathscr{D}_{m_j}(k)\, i[H_1(kx)\, C(k)\, H_1(k, y)]_{k=k_j} = F(x, y). \tag{46}$$

Since for $y < x$, $K(x, y) = 0$, we obtain the following fundamental matrix equation which is valid for $0 < x < y$:

$$0 = F(x, y) + K(x, y) + \int_x^\infty K(x, z)\, F(z, y)\, dz. \tag{47}$$

If a finite number of spectral singularities is present the method remains valid; the contour integration of (38) must be modified [3a]. After proving the existence of a fundamental equation in their respective cases, Marchenko and Ljance derived Parseval's identities from this very existence. Marchenko added the converse result that Parseval's identities allow the derivation of his fundamental equation. When one follows Marchenko's arguments, one sees that if the kernel $\bar{K}^+(x, y)$ relative to Eq. (2) is bounded, the existence of Marchenko's fundamental equations are reciprocal. The reciprocity was already used in Ref. [1b] for obtaining Gel'Fand–Levitan fundamental equations: Newton proved in a first step that the interactions, of the class he was concerned with, possessed Parseval's identity. In a second step which was the consequence of the first, a fundamental equation was established between any two equations of the class.

When Coulomb interactions were present the proof of a fundamental equation results from the knowledge that the Parseval identity exists.

At this stage of our inquiry Eq. (47) can be regarded as an equation for $F(x, y)$ the spectral matrix. For fixed $K(x, y)$, Eq. (47) is a Volterra equation. In Appendix B, using our assumptions on the interaction U, we find an upper bound for $F(x, y)$. In the inverse problem, the situation is reversed; the spectral matrix $F(x, y)$ is now known and the translation kernel $K(x, y)$ becomes the unknown element. For fixed x, Eq. (47) is a Fredholm equation: its study requires the knowledge of the properties of $F(x, y)$.

Equation (47) has the same form as the equation obtained in Ref. [2a]; it is valid for Hermitian as well as for non-Hermitian systems. It contains the coupling of angular momenta different from zero. One assumption used in its derivation was that the translation kernel $K(x, y)$ exists. This assumption contains an explicit upper bound for $K(x, y)$ which will allow the analysis of $F(x, y)$. We present now three remarks as orientations for the continuation of the paper.

Remark 1. The construction of $F(x, y)$, Eq. (46), is first recalled: $F(x, y)$ is symmetric, i.e., $F(x, y) = F^T(y, x)$; its general form is a weighted product of irregular solutions of Eq. (1), x being the variable, with irregular solutions of the same equation, y being the variable. Letting V_0 be the interaction, the matrix $F(x, y)$ obeys the partial differential equation:

$$\frac{\partial^2}{\partial x^2} F(x, y) - \frac{\partial^2}{\partial y^2} F(x, y) = V_0(x)\, F(x, y) - F(x, y)\, V_0(y).$$

In our case $V_0(x) = +L(L+1)\, x^{-2}$. Since the remark has a heuristic character, we assume therefore that all derivations and integrations under the integral sign are allowed. The operators $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$ applied to Eq. (47) give

$$\frac{\partial^2}{\partial x^2} K(x, y) + \frac{\partial^2}{\partial x^2} F(x, y) - \frac{d}{dx} [K(x, x)\, F(x, y)]$$
$$- \frac{\partial}{\partial x} [K(x, s)]_{s=x} F(x, y) + \int_x^\infty \frac{\partial^2}{\partial x^2} K(x, z)\, F(z, y)\, dx = 0; \tag{48}$$

$$\frac{\partial^2 K(x, y)}{\partial y^2} + \frac{\partial^2 F(x, y)}{\partial y^2} - K(x, x) \frac{\partial}{\partial x} F(x, y) + \frac{\partial}{\partial s} [K(x, s)]_{s=x} F(x, y)$$
$$+ \int_x^\infty \frac{\partial^2}{\partial z^2} [K(x, z)]\, F(z, y)\, dz - \int_x^\infty K(x, z)\, V_0(z)\, F(z, y)\, dz$$
$$+ \int_x^\infty K(x, z)\, F(z, y)\, V_0(y)\, dz = 0. \tag{49}$$

In deriving Eq. (49) use has been made of the equation satisfied by the kernel

$F(z, y)$ and also of the technique of integration by parts. The operators $A_0(y)$ and $A(x)$ are introduced:

$$A_0(y) = \frac{\partial^2}{\partial y^2} - V_0(y) + k^2, \tag{50a}$$

$$A(x) = \frac{\partial^2}{\partial x^2} - V_0(x) + k^2 + 2 \frac{d}{dx} [K(x, x)]; \tag{50b}$$

$A(x)$ is applied to the left-hand side of Eq. (57), while $A_0(y)$ is applied to the right-hand side of Eq. (47). Subtraction gives

$$H(x, y) + \int_x^\infty H(x, z)\, F(z, y)\, dy = 0, \qquad \lim_{y\to\infty} H(x, y) = 0. \tag{51}$$

In Eq. (51), $H(x, z)$ denotes

$$H(x, z) = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2}\right) K(x, z) - V_0(x)\, K(x, z)$$
$$+ K(x, z)\, V_0(z) + 2 \frac{d}{dx} [K(x, x)]\, K(x, z). \tag{52}$$

If Eq. (47) has a unique solution, $H(x, y)$ must vanish. In other words $K(x, y)$ is the solution of a non linear partial differential equation. The necessity for the vanishing of $H(x, y)$ results from the fact that Eq. (51) is the homogeneous equation associated with Eq. (47).

We conclude the first remark. If the equation, candidate for the title of fundamental equation, has a unique solution with derivatives up to the second order, then its solution is such that

$$\lim_{y\to\infty} K(x, y) = 0,$$
$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - V_0(x) + 2 \frac{d}{dx} K(x, x)\right] K(x, y) + K(x, y)\, V_0(y) = 0. \tag{53}$$

It results from Eq. (53) that the uniqueness of the solution of Eq. (47) is a necessary element for its use in the solution of the inverse problem. It is not however a sufficient condition [2a].

Remark 2. Let us assume that the translation kernel has a bound of the following type

$$\| K(x, y) \| \leqslant A_\epsilon(x) g(y), \qquad 0 < \epsilon \leqslant x < y \tag{54}$$

with

$$\int_x^\infty | f(t) \, dt < \infty,$$

$$\int_x^\infty | g(t) \, dt < \infty.$$

The following theorem can be proved:

THEOREM 4.1. *If the kernel $F(x, y)$ is constructed from data leading to a translation kernel having the separable bound (53), then the fundamental equation as the equation for the unknown matrix $K(x, y)$ has a unique solution.*

To prove the theorem we consider the homogeneous equation:

$$\phi_x(t) + \int_x^\infty \phi_x(z) \, F(z, t) \, dz = 0 \tag{55a}$$

and prove that its only bounded solution is the trivial solution. To Eq. (55a) for $\phi_x(t)$, we associate Eq. (55b) for $\psi_x(t)$ which follows:

$$\phi_x(t) = \psi_x(t) + \int_x^t \psi_x(s) \, K(s, t) \, ds, \qquad s > x, \tag{55b}$$

where K is the translation kernel associated with F. If the only bounded solution for Eq. (55b) is $\psi_x(t) \equiv 0$, then the solution $\phi_x(t)$ for Eq. (55a) vanishes identically.

Following Ref. [3a] we introduce Eq. (55b) into Eq. (55a). Interchanging the order of integration we get

$$\psi_x(t) + \int_x^t \psi_x(s) \, K(s, t) \, ds + \int_x^\infty \psi_x(z) \, F(x, t) \, dz$$
$$+ \int_x^\infty ds \, \psi_x(s) \int_s^\infty K(s, z) \, F(z, t) \, dz = 0. \tag{56}$$

From the fundamental equation we get

$$\int_s^\infty K(s, z) \, F(z, t) \, dz + K(s, t) + F(s, t) = 0. \tag{57}$$

Equation (57) is introduced into Eq. (56) and identical terms on both sides of the equal sign are suppressed. These techniques yield

$$\psi_x(t) = \int_t^\infty ds \, \psi_x(s) \, K(s, t). \tag{58}$$

We prove that the only bounded solution of Eq. (58) is the trivial solution.

By transforming Eqs. (55) into Eq. (58), immense progress has been made. No longer are we discussing a Fredholm equation, instead we are dealing with a Volterra equation [11]. Following Goursat [11] we symbolize Eq. (58) as

$$\psi = \psi K.$$

By iterations of the kernel K which is bounded by a rapidly decreasing function, one gets

$$\psi = \psi K = \psi K^2 = \cdots = \psi K^n.$$

If ψ is bounded, it is easy, using the bound for K, to prove that $| \psi | = \lim_{n \to \infty} | \psi K^n | = 0$. Therefore, $| \psi |$ and $\psi_x(t)$ vanish and the theorem is proved.

Remark 3. We now develop a remark concerning the construction of the solution $K(x, y)$ of Eq. (40). To this end we assume that the symmetric kernel $F(x, y)$ can be approximated by a finite-rank kernel:

$$F(x, y) = \sum_{\alpha\beta} \phi^\alpha(x) \, C^{\alpha\beta} \phi^\beta(y), \tag{59}$$

where the ϕ^α's form a complete set of orthonormal matrix functions. As proved in Rietz and Nagy [12] this assumption means that the kernel $F(x, y)$ is completely continuous in L^2 or is compact. With this assumption, the solution $K(x, y)$ of Eq. (47) can be written:

$$K(x, y) = \sum_{\alpha\beta} \phi^\alpha(x) \, D^{\alpha\beta}(x) \, \phi^\beta(y). \tag{60}$$

The following definition is introduced:

$$A^{\alpha\beta}(x) = \int_x^\infty \phi^\alpha(z) \, \phi^\beta(z) \, dz, \tag{61}$$

with $A^{\alpha\beta}(0) = \delta^{\alpha\beta}$ and $A^{\alpha\beta}(\infty) = 0$. Equations (59)–(62) are introduced in Eq. (47). This yields

$$\sum_\alpha \phi^\alpha(x)[C^{\alpha\delta} + D^{\alpha\delta}(x)] + \sum_{\alpha,\beta,\gamma} \phi^\alpha(x) \, D^{\alpha\beta}(x) \, A^{\beta\gamma}(x) \, C^{\gamma\delta} = 0.$$

Defining $M^{\beta\delta}(x) = \sum_\gamma A^{\beta\gamma}(x) \, C^{\gamma\delta}$, we obtain

$$\sum_{\alpha,\beta} \phi^\alpha(x) \, D^{\alpha\beta}(x)[\delta^{\beta\delta} + M^{\beta\delta}] = - \sum_\alpha \phi^\alpha(x) \, C^{\alpha\delta}.$$

Since we assume the uniqueness of the solution of Eq. (47), we define the matrix

$$(1 + M)^{-1} = I \,.$$

and we obtain

$$\sum_{\alpha} \phi^\alpha(x)\, D^{\alpha\beta}(x) = -\sum_{\alpha,\delta} \phi^\alpha(x)\, C^{\alpha\delta} I^{\delta\beta},$$

$$K(x, y) = -\sum_{\alpha,\beta,\delta} \phi^\alpha(x)\, C^{\alpha\delta} I^{\delta\beta}(x)\, \phi^\beta(y)$$

$$= -\sum_{\alpha,\beta} \phi^\alpha(x)[CI]^{\alpha\beta}\, \phi^\beta(y). \tag{63a}$$

When $F(x, y)$ has been constructed as in Eq. (46) it is symmetric in x and y and the indices of the $C_{\alpha\beta}$ matrix are related to its k dependence: the matrix is diagonal in k. We will assume that $F(x, y)$ can be represented by a discrete sum and write $C^{\alpha\beta} \equiv C^\alpha\, \delta_{\alpha\beta}$; by redefining the ϕ's accordingly, Eq. (63a) becomes

$$K(x, x) = -\sum_{\alpha\beta} \phi^\alpha(x)\, C^\alpha I^{\alpha\beta} \phi^\beta(y). \tag{63b}$$

In the scalar case ($n = 1$) the $\phi^\beta(y)$'s as well as the C^α's are scalar quantities: the order of the terms in (63b) can be permuted; thus we get

$$K(x, x) = -\sum_{\alpha\beta} I^{\alpha\beta} \phi^\alpha(x)\, \phi^\beta(x)\, C^\alpha$$

$$= \sum_{\alpha\beta} [(1 + M)^{-1}]^{\alpha\beta} \left[\frac{d}{dx}(1 + M)\right]^{\beta\alpha}; \tag{64}$$

$$K(x, x) = \operatorname{Tr}\left[(1 + M)^{-1} \frac{d}{dx}(1 + M)\right]$$

$$= \frac{d}{dx} \ln \det(1 + M), \tag{65}$$

where Tr stands for trace and where we have used the well-known property $d/dx\, [\ln(\det U)]\, d/dx = \operatorname{Tr}[U^{-1}U']$. Equation (65) extends a result of Faddeev [13] to scalar fundamental equations admitting a finite-rank approximation.

Let us denote by K^{21} the translation operator from an operator with a potential V^1 to an operator with a potential V^2

$$2\frac{d}{dx} K^{21}(x, x) = V^1(x) - V^2(x). \tag{66}$$

Equations (65) and (66) give

$$V^1(x) - V^2(x) = 2\frac{d^2}{dx^2} \ln \det[1 + M^{21}].$$

In the same way one has

$$V^3(x) - V^1(x) = 2\frac{d^2}{dx^2} \ln \det[1 + M^{31}],$$

$$V^3(x) - V^2(x) = 2\frac{d^2}{dx^2} \ln \det[1 + M^{32}].$$

Use of the property of the logarithmic function then gives

$$\det[1 + M^{31}]/\det[1 + M^{32}] = \det[1 + M^{21}]\exp[C_1 + C_2 x];$$

Use of $A(\infty) = 0$ implies $C_1 = C_2 = 0$. From Remarks 2 and 3, estimates for $F(x, y)$ are essential for the constructive solution of the nuclear potential, a construction which is the aim of the inverse problem.

5. The Kernel of the Fundamental Equation

In Section 4 the existence of a fundamental equation was proved for any $0 < x < y$. Existence of translation kernels, behavior of T-matrices for $k = 0$ and discreteness of the set of spectral elements were used in the proof. In order to possess these properties, non-Hermitian systems need strong condition is not needed by Hermitian systems. Using the continuity argument we extend its validity for any y ($0 < x \leqslant y$). In addition we will show that its kernel called the spectral matrix is such that $F(x, x)$ is continuous even when x goes to zero. The results are obtained from the estimates for the spectral matrix and its (x and y) partial derivatives: these estimates are derived from estimates on the translation kernel. In the search for the estimates for $K(x, y)$ a clear separation is again found between Hermitian and non-Hermitian cases. While the bound concerning the former tells which conditions the interaction must satisfy for the theory to develop, the non-Hermitian bound is a consequence of a prior condition on the interaction.

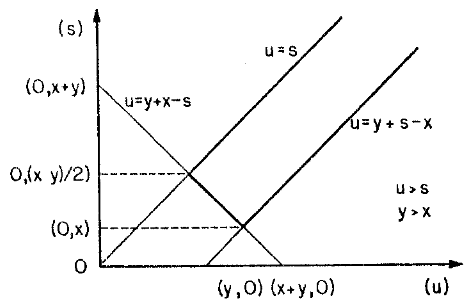

Fig. 1. Marchenko's domain.

The integral equation for $K_{ij}(x, y)$ is now recalled:

$$K_{ij}(x, y) = \frac{1}{2} \int_{(x+y)/2}^{\infty} R_{ij}(x, y; s, s)\, U_{ij}(s)\, ds$$

$$+ \frac{1}{2} \iint_{\mathscr{D}} R_{ij}(x, y; u, s) \sum_m U_{im}(s)\, K_{mj}(s, u)\, du\, ds. \qquad (67)$$

In Eq. (67), $\mathscr{D}$ is the Marchenko domain: it is represented in Fig. 1.

The R_{ij} of Eq. (67) is the Riemann solution for the equation

$$L_i(x)\, R_{ij}(x, y) = L_j(y)\, R_{ij}(x, y),$$

with

$$L_i(x) = \frac{\partial^2}{\partial x^2} - \frac{l_i(l_i + 1)}{x^2}.$$

The expression of the Riemann solution is found in Ref. [14]. Appendix A proves, following [4g], that

$$| R_{ij} | \leqslant \left(A \frac{ys}{xu}\right)^{l_i}.$$

Since y/x and $(y/x)(s/u)$ are greater than 1, we use with $l = l_{\max}$ the bound

$$| R_{ij} | \leqslant \left(A \frac{y}{x} \frac{s}{u}\right)^{l}.$$

We define a modified translation kernel

$$K(x, y) \leqslant \left(A \frac{y}{x}\right)^{l} \tilde{K}(x, y)$$

and a modified Riemann solution

$$R_{ij} = \left(A \frac{y}{x} \frac{s}{u}\right)^{l} \tilde{R}_{ij}.$$

Use of the method of successive approximations and of Marchenko's norm for a matrix A defined as [2a],

$$\| A \| = \sup_i \sum_j | A_{ij} |,$$

yields

$$\| \tilde{K}(x, y) \| \leqslant \sigma_0 \left(\frac{x + y}{2}\right) \exp \sigma_1(x),$$

where

$$\sigma_i(x) = \int_x^{\infty} \| U(s) \|\, s^i\, ds.$$

It results that

$$\| K(x, y) \| \leqslant \left(A \frac{y}{x}\right)^{l} \sigma_0 \left(\frac{x + y}{2}\right) \exp \sigma_1(x). \qquad (68)$$

For $K(x, y)$ to be absolutely integrable an l-moment is required. Furthermore, $K(x, y)$ is continuous for $y \geqslant x$ and $K(x, x)$ is continuous for $x \geqslant 0$.

We show now that the derivatives $\partial/\partial x\, K(x, y)$ and $\partial/\partial y\, K(x, y)$ are bounded. For this purpose, we use characteristic variables:

$$\eta = \frac{y + x}{2}, \qquad \xi = \frac{y - x}{2}, \qquad \eta_0 = \frac{u + s}{2}, \qquad \xi_0 = \frac{u - s}{2}.$$

From the equation for $K_{ij}(x, y)$,

$$K_{ij}(\eta, \xi) = \frac{1}{2} \int_{\eta}^{\infty} d\eta_0\, R_{ij}(\eta, \xi; \eta_0, 0)\, U_{ij}(\eta_0)$$

$$+ \sum_l \int_{\eta}^{\infty} d\eta_0 \int_0^{\xi} d\xi_0\, R_{ij}(\eta, \xi; \eta_0, \xi_0)\, U_{il}(\eta_0 - \xi_0)\, K_{lj}(\eta_0, \xi_0),$$

the equation for $\partial/\partial\eta\, K_{ij}$ is derived:

$$\frac{\partial}{\partial \eta} K_{ij}(\eta, \xi) = -\frac{1}{2} R_{ij}(\eta, \xi, \eta, 0)\, U_{ij}(\eta) + \frac{1}{2} \int_{\eta}^{\infty} d\eta_0 \frac{\partial}{\partial \eta} R_{ij}(\eta, \xi, \eta_0, 0)\, U_{ij}(\eta_0)$$

$$- \sum_l \int_0^{\xi} d\xi_0\, R_{ij}(\eta, \xi; \eta, \xi_0)\, U_{il}(\eta - \xi_0)\, K_{lj}(\eta, \xi_0)$$

$$+ \sum_l \int_{\eta}^{\infty} d\eta_0 \int_0^{\xi} d\xi_0 \frac{\partial}{\partial \eta} R_{ij}(\eta, \xi; \eta_0, \xi_0)\, U_{il}(\eta_0 - \xi_0)\, K_{lj}(\eta_0 \xi_0). \qquad (69)$$

Bound (68) for K_{ij} together with the bounds for R_{ij} and $\partial/\partial n\, R_{ij}$ found in Appendix A are used. A bound for the n-derivative of K, valid for $x > 0$, results. The extension of the domain of validity of the fundamental equation to any $y \geqslant x > 0$ results directly from the bounds for $\partial/\partial\xi\, K$. The continuity of $F(x, x)$ when x goes to zero results from the continuity of $K(x, x)$.

If we are dealing with a non-Hermitian system and if we assume the Naimark condition,

$$\int_0^{\infty} \exp[\epsilon s]\, \| U(s) \|\, ds < \infty; \qquad (70)$$

it is necessary to specify which bound the translation kernel receives as a consequence. For this purpose, we define another modified kernel $\mathscr{K}(x, y)$ as follows:

$$\left(\frac{Ay}{x}\right)^l \mathscr{K}(x, y) = K(x, y) \exp[\epsilon(x + y)/2].$$

Its equation deduced from Eq. (67) reads:

$$\mathscr{K}(x, y) = \frac{1}{2} \exp[\epsilon(x + y)] \int_{(x+y)/2}^{\infty} \tilde{R}(x, y; s, s)\, U(s)\, ds + \frac{1}{2} \exp[\epsilon(x + y)]$$

$$\times \iint_{\mathscr{D}} \tilde{R}(x, y; u, s)\, U(s) \exp[-\epsilon(u + s)] \left(\frac{u}{s}\right)^l \mathscr{K}(s, u)\, du\, ds. \quad (71)$$

By the method of successive approximations and with the definitions

$$s_0(x) = \int_x^{\infty} \exp[\epsilon s] \, \| U(s) \| \, ds,$$

$$\sigma_i(x) = \int_x^{\infty} s^i \, \| U(s) \| \, ds,$$

we get

$$\| \mathscr{K}(x, y) \| \leqslant \frac{1}{2} s_0 \left(\frac{x + y}{2}\right) \exp \sigma_1(x).$$

Returning to the original kernel $K(x, y)$, we have

$$\| K(x, y) \| \leqslant \left(A \frac{y}{x}\right)^l \exp\left[-\frac{\epsilon(x + y)}{2}\right] s_0 \left(\frac{x + y}{2}\right) \exp \sigma_1(x).$$

With (70) the existence of s_l and of σ_1 for all $x > x_0$ is assured. Therefore we have

$$\| K(x, y) \| \leqslant (A/x_0)^l \exp[-\epsilon(x + y)/2] f(x_0). \quad (72)$$

According to the preceding discussion the analysis of the Hermitian case must be separated from that of the non-Hermitian case.

II.1. *The Hermitian Case*

Two scalar differential operators where the superscript l denotes the angular momentum

$$D_+{}^l = x^l \frac{d}{dx} \frac{1}{x^l},$$

$$D_-{}^l = \frac{1}{x^l} \frac{d}{dx} x^l, \quad (73)$$

are introduced. When they are applied to the Riccati–Hankel functions $h_1{}^l(kx)$, Eq. (5), they give

$$D_+{}^l h_1^{l-1}(kx) = -k h_1{}^l(kx),$$

$$D_-{}^l h_1{}^l(kx) = k h_1^{l-1}(kx).$$

We consider now the integral transformation

$$T_1[\phi] = \int_{-\infty}^{\infty} h_1{}^l(k\xi)\, \phi(\xi)\, d\xi$$

and assume that $\phi(\xi)$ is well behaved at infinity and at the origin:

$$\phi(\xi) \in L^{2\prime}(-\infty, \infty);$$

$L^{2\prime}(-\infty, \infty)$ is the dense subset of $L^2(-\infty, \infty)$ made of elements vanishing as x^{l+1} at the origin and having derivatives up to the order l which belong to $L^2(-\infty, \infty)$. With our assumption we obtain

$$T_1[\phi] = \int_{-\infty}^{\infty} h_1^{l-1}(k\xi) \frac{1}{k} D_-{}^l[\phi(\xi)]\, d\xi,$$

after one integration by parts.

Defining now

$$\mathscr{D}_-{}^l = D_-{}^1 \times D_-{}^2 \times \cdots \times D_-{}^l,$$

we have

$$\mathscr{D}_-{}^l h_1{}^l(kx) = k^l \exp ikx,$$

$$T_1[\phi] = \int_{-\infty}^{\infty} \exp ik\xi \frac{1}{(k)^l} \mathscr{D}_-{}^l[\phi(\xi)]\, d\xi. \quad (74)$$

The notations $\vec{\mathscr{D}}_-$ and $\overleftarrow{\mathscr{D}}_-$ are used to denote operations from the left or from the right.

In a straightforward fashion diagonal differential operators are introduced

$$[\mathscr{D}_-]_{ij} = \delta_{ij} D_-^{l_i}. \quad (75)$$

When $\phi(x)$ is a matrix, the two operators

$$\phi \overleftarrow{\mathscr{D}}_- \quad \text{and} \quad \vec{\mathscr{D}}_- \phi$$

are obviously different.

$$\psi \overleftarrow{\mathscr{D}}_- = [\vec{\mathscr{D}}_- \psi^T]^T.$$

If each element of ϕ belongs to $L^{2'}(-\infty, \infty)$, one gets

$$\int_{-\infty}^{\infty} d\xi\ \phi(\xi)\ H_1(k\xi) = \int_{-\infty}^{\infty} d\xi\ \phi(\xi)\ \overleftarrow{\mathscr{D}}_{-} k^{-L} \exp ik\xi \tag{76}$$

and similarly

$$\int_{-\infty}^{\infty} d\xi\ H_1(k\xi)\ \phi(\xi) = \int_{-\infty}^{\infty} d\xi \exp ik\xi\ k^{-L} \overrightarrow{\mathscr{D}}_{-} \phi(\xi). \tag{77}$$

With (73)–(77) we transform $y(t)$ which we define as

$$y(t) = \int_{-\infty}^{\infty} \phi(x)\ F_C(x, t)\ dx, \tag{78}$$

$$y(t) = \int_{-\infty}^{\infty} \phi(x)\ dx \frac{1}{2\pi} \int_{-\infty}^{\infty} H_1(kx)[1 - S(k)]\ H_1(kt)\ dk. \tag{79}$$

We note $y(t) = T_2[\phi]$.

$$y(t) = T_2[\phi] = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} dk\ \phi(\xi)\ H_1(k\xi)\ T(k)\ H_1(kt^4.$$

The operator $\overleftarrow{\mathscr{D}}_{-}$ is applied to both sides of the equation:

$$[y(t)]\ \overleftarrow{\mathscr{D}}_{-} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} dk\ \phi(\xi)\ H_1(k\xi)\ T(k)(k)^L \exp ikt.$$

The r.h.s. of $[y(t)]\ \overleftarrow{\mathscr{D}}_{-}$ is integrated by parts:

$$[y(t)]\ \overleftarrow{\mathscr{D}}_{-} = [\phi(-t)\ \overleftarrow{\mathscr{D}}_{-} - \frac{1}{2\pi} \int_{-\infty}^{\infty} dx [\phi(x)]\ \overleftarrow{\mathscr{D}}_{-}$$
$$\times \int_{-\infty}^{\infty} dk \exp ikx \left(\frac{1}{k}\right)^L S(k)\ k^L \exp ikt. \tag{81}$$

Let us introduce $\tilde{\phi}$ by

$$\tilde{\phi}(-k) = \int_{-\infty}^{\infty} [\phi(\xi)]\ \overleftarrow{\mathscr{D}}_{-} k^{-L} \exp ik\xi\ d\xi = \int_{-\infty}^{\infty} \phi(\xi)\ H_1{}^l(k\xi)\ d\xi.$$

We can write

$$[y(t)]\ \overleftarrow{\mathscr{D}}_{-} = [\phi(-t)]\ \overleftarrow{\mathscr{D}}_{-} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\phi}(-k)\ S(k)\ k^L \exp ikt\ dk.$$

If $\phi(\xi)$ belongs to $L^2(-\infty, \infty)$ and if in addition there exists an $\epsilon > 0$ such that $\phi(\xi)$ and the necessary derivatives of $\phi(\xi)$ vanish for $\xi \leqslant \epsilon$, then

$$[y(t)]\ \overleftarrow{\mathscr{D}}_{-} = \left[\int_{\epsilon}^{\infty} \phi(\xi)\ F_C(\xi, t)\ d\xi\right] \overleftarrow{\mathscr{D}}_{-} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(-k)[S(k)]\ dk, \tag{82}$$

where

$$\hat{\phi}(-k) = k^2 \tilde{\phi}(-k).$$

With Eq. (82) we can show that the homogeneous equations related to Eq. (47) can be reduced to the homogeneous equations related to the ordinary $l = 0$ Marchenko equation [15]. In addition, one shows that the constant matrices which for $l \neq 0$ intervene in Eq. (47) are still normalization matrices: they are singular Hermitian positive semidefinite matrices. The Marchenko results obtained for $l = 0$ can therefore be naturally extended to the $l \neq 0$ Hermitian case by requiring a proper behavior of the T-matrices. So we get:

LEMMA 1. *Any solution of the equation* $\epsilon \geqslant 0$

$$\lambda \phi(t) + \int_{\epsilon}^{\infty} \phi(\xi)\ F_C(\xi, t)\ d\xi = 0 \tag{83}$$

$|\lambda| = 1$, *which belongs to* $L^1(\epsilon, \infty)$, *belongs also to* $L^2(\epsilon, \infty)$.

LEMMA 2. *The eigenvalues* λ *of Eq.* (83) *when* $\epsilon > 0$ *are all smaller than unity*

$$|\lambda| < 1.$$

LEMMA 3. (*It cannot be applied to non-Hermitian interactions since it is based on the assumption* $U = U^+$ *which implies that the* $M_n{}^2$ *are hermitian semidefinite matrices*).
For $\epsilon \geqslant 0$ *the solution of the equation*

$$\phi(t) + \int_{\epsilon}^{\infty} \phi(\xi)\ F(\xi, t)\ d\xi = 0 \tag{84}$$

is also a solution of Eq. (83), $\lambda = 1$ *and in addition satisfies the equation*

$$\tilde{\phi}(ik_n) = \tilde{\phi}(-ik_n)\ M_n,$$

where $M_n{}^2 \equiv C(ik_n)$.

Now we can consider the $\epsilon = 0$ case. To do so we define $F(x, y)$ from the scattering data, that is, we write

$$F(x, y) = F_c(x, y) + F_d(x, y).$$

(When this last definition is meaningful we say that $F_c(x, y)$ has a double Hankel transform.)

Correlatively we have

$$F_d(x, y) = \sum_n H_1(ik_n x)\, M_n{}^2 H_1(ik_n y)$$

with

$$M_n{}^2 = C(ik_n).$$

Let us assume that the matrix $S(k)$ has a double Hankel transform so that the spectral kernel for $F_c(x, y)$ is meaningful.

THEOREM 5.1. *The equations* ($\epsilon = 0$)

$$\text{I.} \qquad x(t) + \int_0^\infty x(\xi)\, F(\xi, t)\, d\xi = 0, \qquad 0 \leqslant t \leqslant \infty,$$

$$\text{II.} \quad \pm y(t) + \int_0^\infty y(\xi)\, F_c(-\xi, -t)\, d\xi = 0, \qquad 0 \leqslant t \leqslant \infty,$$

have no nontrivial solution in $\hat{L}_n(0, \infty)$ *(see Appendix C for definition of* $\hat{L}$*). In addition, the number of linearly independent solutions of the equation*

$$\text{III.} \quad z(t) = \int_0^\infty z(\xi)\, F_c(\xi, t)\, d\xi, \qquad 0 < t < \infty,$$

is equal to the sum of the rank of the matrices M.

The proof of the statement concerning I uses Lemma 3 and cannot therefore be applied to non-Hermitian interactions. In addition it can be shown [15] that the statements concerning I, II, and III is equivalent to the statements that the unitary matrix $S(k)$ has a double Hankel transform and that the set of discrete elements obeys Levinson's formula concerning the normalizable states.

II.2. *Non-Hermitian Case*

Its solution requires that the spectral singularities and the discrete spectrum remain finite in $\operatorname{Im} k \geqslant 0$.

To guarantee this finite character we assume (70). We have now, within this class of short-range interactions, the existence of a fundamental equation and the bounds (72) and (B5). With these bounds we are in a situation to approach the inverse problem but first we need to specify what we mean when we say that the spectral kernel is constructed from the scattering data.

Back to Eq. (46) we see that if a differential equation (1) satisfies (3b) a spectral matrix $F(x, y)$ exists. Its construction uses the S-matrix and a set of constant matrices obtained when the $\mathscr{D}_m(k)$ are applied to the matrices $C(k)$ for $k = k_j$. When an S-matrix and the appropriate number of constant matrices have been obtained from a differential equation (1a) obeying (3b) and an $F(x, y)$ constructed as in (46), the $F(x, y)$ matrix will be said to be constructed from scattering data.

THEOREM 5.2. *If the kernel* $F(x, y)$ *is constructed from the differential system, that is, from the scattering data, the fundamental equation has a unique solution.*

We now have (72), that is,

$$\| K(x, y)\| \leqslant (A/x_0)^l \exp[-\epsilon(x + y)/2] f(x_0).$$

The conditions of factorization expressed in Theorem 4.1 are realized. Theorem 4.1 can be used and consequently Theorem 5.2 is proved.

From the fundamental equation (47), one gets

$$\frac{\partial}{\partial x} F(x, y) + \frac{\partial}{\partial x} K(x, y) + \int_x^\infty \frac{\partial}{\partial x} K(x, z)\, F(z, y)\, dz - K(x, x)\, F(x, y) = 0. \quad (85)$$

Equation (85) provides an upper bound for $\partial F/\partial x\,(x, y)$ for $y \geqslant x \geqslant x_0 > 0$ which depends on the bound for $\partial/\partial x\, K(x, y)$. From the bound (72), one checks that

$$\lim_{x\to\infty} F(x, y) = 0.$$

Using conventional methods it is not hard to prove the two following inequalities:

$$\int_{x_0}^\infty \| F_c(s, y)\|\, ds < \infty,$$

$$\int_{x_0}^\infty s\, \| F_c(s, y)\|^2\, ds < \infty,$$

for all $x > x_0 > 0$; F_c is simultaneously integrable and square integrable.

We move now to the second theorem for the constructive solution of the inverse problem. If the potential realizes the conditions expressed in (70), we can write (Appendix B)

$$\| F(x, y)\| \leqslant \Gamma_1(x_0) \exp[-\epsilon(x + y)]. \quad (86a)$$

$$\| K(x, y)\| \leqslant \Gamma(x_0) \exp[-\epsilon(x + y)]. \quad (86b)$$

The x and y derivatives of K and F will exhibit for $x \geqslant x_0$ the same exponential decrease:

$$\exp[-\epsilon(x + y)].$$

When the potential obeys condition (70), the spectral matrix necessarily obeys (86a). In the inverse problem the spectral matrix is given: if one decides to remain within the class of short-range interactions (70), one must impose (86a) on the spectral matrix. In other words the ingredients for the construction of a spectral matrix must carry an exponential decrease into the construction.

The rows or the columns of the matrix $F(x, y)$ are elements of $L_n^1(x_0, \infty)$ (See Appendix C for notation). We consider the linear operator **F** in $L_n^1(x_0, \infty)$ defined by

$$\mathbf{F}[\xi] = \int_{x_0}^{\infty} \xi(x) F(x, y)\, dx = \eta(y).$$

LEMMA. *F is a normed transformation of $L_n^1(x_0, \infty)$ into $L_n^1(x_0, \infty)$.*

Proof. The norm in $L_n^1(x_0, \infty)$ is defined by

$$\| \xi \| = \int_{x_0} \sum_{i=1}^{n} | \xi_i(x)|\, dx.$$

The norm of η is therefore

$$\| \eta \| = \int_{x_0}^{\infty} \sum_{i=1}^{n} | \eta_i(y)|\, dy$$
$$\leqslant \frac{2n}{\epsilon} \Gamma_1(x_0)\, e^{-\epsilon x_0} \| \xi \|.$$

From this we conclude that the norm of **F** is

$$\| F \| \leqslant \frac{2n}{\epsilon} \Gamma_1(x_0)\, e^{-\epsilon x_0}.$$

The same method with the appropriate norms shows that **F** is a normed transformation of $L_n^\infty(x_0, \infty)$ as well as $L_n^2(x_0, \infty)$ into themselves.

THEOREM 5.3. **F** *is a completely continuous operator in $L_n^1(x_0, \infty)$, $L_n^2(x_0, \infty)$ and $L_n^\infty(x_0, \infty)$.*

Since the method in the three cases is basically the same, we restrict ourselves to the $L_n^1(x_0, \infty)$ case.

Proof. Consider $\eta(y) = \mathbf{F}(\xi)$ with $\xi \in L_n^1(x_0, \infty)$ and $\| \xi \| \leqslant 1$.
The norm of η satisfies

$$\| \eta \| < \| F \| \| \xi \| < \| F \|.$$

Thus for any $\delta > 0$, there exists a positive h such that

$$\int_{x_0}^{\infty} | \eta(t + h) - \eta(t)|\, dt \leqslant \int_{x_0}^{\infty} dt \int_{x_0}^{\infty} dx \,| \xi(x)| \,| F(x, t + h) - F(x, t)|$$
$$= \int_{x_0}^{\infty} | \xi(x)|\, dx \int_{x_0}^{\infty} dt \,| F(x, t + h) - F(x, t)|$$
$$= \int_{x_0}^{\infty} | \xi(x)|\, dx \int_{x_0}^{\infty} dt \left| \frac{\partial}{\partial t} F(x, t)\, h \right| \leqslant \delta. \tag{87}$$

Also, for any $\delta > 0$, there exists another positive h such that

$$\int_{x_0}^{\infty} | \eta(t + h) - \eta(t)|\, dt \leqslant \| \xi \| \,| h |\, \Gamma_3(x_0) \frac{\epsilon}{2} n \exp\left(-\frac{x_0}{\epsilon}\right)$$
$$\leqslant \delta. \tag{88}$$

Finally, for any $\delta > 0$, there exists a number X such that for any $N > X$

$$\int_N^{\infty} | \eta(t)|\, dt \leqslant \int_N^{\infty} dt \int_\epsilon^{\infty} | \xi(x)| \,| F(x, t)|\, dx$$
$$< \int_\epsilon^{\infty} | \xi(x)| \int_N^{\infty} | F(x, t)|\, dt$$
$$< \frac{2n}{\epsilon} \Gamma(N)\, e^{-\epsilon N} \| \xi \| \leqslant \delta; \tag{89}$$

in (89) use has been made of the property that $\Gamma(N)$ is a decreasing function of N_0.
It results from (87)–(89) that the set of vector functions $\eta(y)$ is compact in $L_n^1(x_0, \infty)$. The operator **F** is therefore completely continuous in $L_n^1(x_0, \infty)$. The transformation generated by F is a completely continuous transformation of L^1, L^2, and L^∞ into themselves. It can therefore be approximated by a finite-rank transformation. Use can be made of remark 3 of section 4 to construct the unique solution $K(x, y)$ of the fundamental equation:

COROLLARY. *The equation*

$$a(x) + \lambda\xi(x) + \mathbf{F}[\xi] = 0,$$

where $a(x) \in L_n^1(x_0, \infty)$, has a unique solution $\xi(x)$ if the homogeneous equation

$$\lambda\xi(x) + \mathbf{F}[\xi] = 0$$

has no other solution in $L_n^1(x_0, \infty)$ than the trivial solution.

6. CONCLUSION

When one deals with non-Hermitian interactions, *one does not know if Eq.* (90) *has no* $\lambda = 1$ *nontrivial solution. Indeed this cannot occur when the set of elements are "authentic" scattering elements according to* (5.2). *However if the candidate to be the spectral matrix is exponentially bounded and if Eq.* (90) *has no* $\lambda = 1$ *nontrivial solution, then Eq.* (47) *has a unique solution $K(x, y)$ which in turn is itself exponentially bounded.*

We can now summarize our findings. Let us assume that we have a set of scattering data such that $F(x, y)$ is exponentially bounded *and such that the related Eq.* (90) *has no* $\lambda = 1$ *nontrivial solution. Then one can construct* $K(x, y)$ *and* $V(x)$ *by*

$$V(x) = -\tfrac{1}{2}K(x, x).$$

However we have no guarantee that the scattering data obtained by solving the Schrödinger equation formed with $V(x)$ are identical to the scattering data used to construct $F(x, y)$.

If we assume in addition that the scattering data

$$\begin{bmatrix} \text{an } S\text{-matrix defined for real } k \\ \begin{cases} S(k)\, S(-k) = 1 \\ S(k) = 1 + o(1/k) \qquad k \to \infty \end{cases} \\ \text{a set of normalizable state energies with a given multiplicity} \\ \text{a set of constant matrices} \end{bmatrix}$$

are such that they constitute the data for a unique factorization of the S-matrix, then we have the guarantee we were looking for. As we have shown the fundamental equation as an equation for $F(x, y)$ has a unique solution. The unique factorization of the S-matrix guarantees that the unique F has a unique decomposition

$$F = F_C + F_D .$$

The trail to follow in order to solve the inverse problem completely is well defined: Given a set of "scattering data," verify that they lead to a unique factorization of the S-matrix. References [16a–16c, 17] having already explored the subject, the complete solution to the inverse problem should therefore follow their lead.

Appendix A[1]

We are concerned with the Riemann solutions $R_{ij}(\eta, \xi; \eta_0, \xi_0)$ and their derivatives with respect to η and ξ. They are given by the integral equation

$$R_{ij}(\eta, \xi; \eta_0, \xi_0) = 1 + \int_{\xi_0}^{\xi} d\xi_1 \int_{\eta}^{\eta_0} d\eta_1 \left[\frac{l_i(l_i+1)}{(\eta_1 - \xi_1)^2} - \frac{l_j(l_j+1)}{(\eta_1 + \xi_1)^2}\right] \times R_{ij}(\eta_1, \xi_1; \eta_0, \xi_0); \tag{A1}$$

Chaundy's variables x_1 and x_2 are used [14]: they are

$$x_1 = \frac{(\eta - \eta_0)(\xi - \xi_0)}{(\eta - \xi)(\eta_0 - \xi_0)}; \qquad x_2 = \frac{(\eta_0 - \eta)(\xi - \xi_0)}{(\eta + \xi)(\eta_0 + \xi_0)}. \tag{A2}$$

[1] See Ref. [4g] for details.

In Marchenko's domain one can prove that $x_1 < 0$ and that $0 < x_2 < \frac{1}{2}$; therefore

$$(1 - 2x_1)\frac{(\eta - \xi)(\eta_0 + \xi_0)}{(\eta + \xi)(\eta_0 - \xi_0)} \leqslant 2. \tag{A3}$$

To obtain an appropriate bound for R_{ij}, its expression, as given by Chaundy [14], is used

$$R_{ij} = P_{l_i}(1 - 2x_2) - 2x_1 \int_0^1 P_{l_j}(1 - 2x_2 t)\, P'_{l_i}(1 - 2x_1x_2 + 2x_1 t)\, dt,$$

where P_{l_i} is the Legendre polynomial of degree l_i. Consequently

$$\begin{aligned} | R_{ij} | &\leqslant P_{l_i}(1 - 2x_1) \leqslant \left[\frac{4(\eta + \xi)(\eta_0 - \xi_0)}{(\eta - \xi)(\eta_0 + \xi_0)}\right]^{l_i}; \\ \frac{\partial}{\partial\eta} R_{ij} &= \int_{\xi_0}^{\xi} \left[\frac{l_i(l_i+1)}{(\eta - \xi_1)^2} - \frac{l_j(l_j+1)}{(\eta + \xi_1)^2}\right] d\xi_1\, R_{ij}(\xi_1, \eta; \xi_0, \eta_0). \end{aligned} \tag{A4}$$

One notices that $(\eta + \xi_1)/(n - \xi_1)$ is an increasing function of ξ_1 and obtain

$$\left|\frac{\partial}{\partial\eta} R^0_{ij}\right| \leqslant \left[\frac{4(\eta + \xi)(\eta_0 - \xi_0)}{(\eta - \xi)(\eta_0 + \xi_0)}\right]^{l_i} 2l_j(l_j + 1)\frac{1}{(\eta - \xi)}. \tag{A5}$$

We will therefore use the following bounds:

$$| R | \leqslant \left(A\frac{y}{x}\frac{s}{u}\right)^l,$$

$$\left|\frac{\partial}{\partial\xi} R\right| \leqslant \left(A\frac{y}{x}\frac{s}{u}\right)^l \frac{2l(l+1)}{x},$$

$$\left|\frac{\partial}{\partial\eta} R\right| \leqslant \left(A\frac{y}{x}\frac{s}{u}\right)^l \frac{2l(l+1)}{x},$$

where we have returned to the physical variables and set $l = l_{\max}$, $A = 4$.

Appendix B

The fundamental equation is solved as if it were an equation for the spectral matrix $F(x, y)$ with the translation kernel as a free term.

In the Hermitian case we set

$$K(x, y) = \left(\frac{y}{x}\right)^l \tilde{K}(x, y),$$

$$F(x, y) = \left(\frac{y}{x}\right)^l \tilde{F}(x, y),$$

$$l = l_{\max},$$

in Eq. (47); consequently we get

$$\|\tilde{F}(x, y)\| \leqslant \|\tilde{K}(x, y)\| + \int_x^\infty \|\tilde{K}(x, t)\| \, \|\tilde{F}(t, y)\| \, dt. \tag{B1}$$

Using the moments σ_0 and σ_1 of the interaction $U(s)$ we have

$$\|\tilde{K}(x, y)\| \leqslant \frac{1}{2} \sigma_0 \left(\frac{x+y}{2}\right) \exp \sigma_1(x)$$

$$\leqslant \frac{1}{2} \sigma_0(x) \exp \sigma_1(x). \tag{B2}$$

Iterations of (B1) using (B2) yield

$$\|\tilde{F}(x, y)\|_0 < \tfrac{1}{2}\sigma_0(x) \exp \sigma_1(x),$$

$$\|F(x, y)\|_1 \leqslant \tfrac{1}{2}\sigma_0(x) \exp \sigma_1(x) \int_x^\infty \tfrac{1}{2}\sigma_0(t) \exp \sigma_1(t) \, dt \tag{B3}$$

with

$$\eta(x) = \tfrac{1}{2}\sigma_0(x) \exp \sigma_1(x),$$

$$\|\tilde{F}(x, y)\| \leqslant \eta(x) \exp \int_x^\infty \eta(t) \, dt. \tag{B4}$$

From (B4) the bound

$$\|F(x, y)\| \leqslant \left(\frac{y}{x}\right)^l \eta(x) \exp \int_x^\infty \eta(t) \, dt$$

follows, as well as the continuity of $F(x, x)$ for all x including $x = 0$. If the non-Hermitian case bound (72) is used,

$$\|F(x, y)\|_0 \leqslant G(x_0)\exp[-\epsilon(x+y)] = |K(x, y)|$$

with

$$G(x_0) = f(x_0) \left(\frac{A}{x_0}\right)^l,$$

$$\|F(x, y)\| \leqslant G^2(x_0) \int_x^\infty \exp[-\epsilon(x+t)] \exp[-\epsilon(t+y)] \, dt$$

$$\leqslant G^2(x_0) \exp[-\epsilon(x+y)] \frac{\exp[-2\epsilon x]}{2}.$$

By mathematical induction one gets

$$\|F(x, y)\| < \exp[-\epsilon(x+y)]\exp[G(x_0) \exp - \epsilon x_0]$$

$$\leqslant \exp[-\epsilon(x+y)] \, \Gamma(x_0) \tag{B5}$$

with $\Gamma(x_0) = \exp[G(x_0) \exp - \epsilon x_0]$. It is worthwhile to note that if K is constructed from Eq. (67), F can be obtained by solving Eq. (47). In other words, F is the unique bounded solution of Eq. (47) when K is given.

Appendix C

Let us consider a linear space of vector functions of n components (n finite):

$$f(t) = (f_1(t), f_2(t), \ldots, f_n(t)).$$

By $L_n^p(\epsilon, \infty)$, we denote the set of n-dimensional vectors $f(t)$ which satisfy

$$\|f\|_p = \left\{\int_\epsilon^\infty \sum_{i=1}^n |f_i(x)|^p \, dx\right\}^{1/p} < \infty \tag{C1}$$

for p finite or

$$\|f\|_\infty = \sup_{1 \leqslant i \leqslant n; \epsilon \leqslant x < \infty} \text{L.U.B.} \; |f_i(x)| \tag{C2}$$

for $p = \infty$ (where L.U.B. stands for least upper bound). In this paper, we consider only $p = 1, 2, \infty$. The norms defined in Eqs. (C1) and (C2) satisfy the usual properties of a norm [18]

(a) $\|f\| \geqslant 0;$ (C3)

$\|f\| = 0$ if and only if $f = 0$ (p.p) (p.p. means almost everywhere)

(b) $\|cf\| = |c| \, \|f\|,$ (C4)

(c) $\|f + g\| \leqslant \|f\| + \|g\|,$ (C5)

(d) $\|f\| = \|f^*\| = \| \, |f| \, \|,$ (C6)

(e) if $|f_i| < |g_i|$ $(i = 1, n)$, then $\|f\| \leqslant \|g\|$ (C7)

(f) $\sum_i f_i g_i^* \in L_1^1$ if $f \in L_n^p$ and $g \in L_n^q$

with $[1/p + 1/q] = 1$. We wrote L_1^1 since we were dealing with a one-dimensional object. In addition we have the Hölder inequality:

$$\left|\sum_i \int_\epsilon^\infty f_i g_i^* \, dx\right| \leqslant \|f\|_p \, \|g\|_q ; \tag{C8}$$

(g) Schwarz's inequality: if f, $g \in L_n^2$; then we have $\sum_i f_i g_i^* \in L_1^1$ and

$$\left| \sum_i \int_\epsilon^\infty f_i g_i^* \, dx \right| \leqslant \| f \|_2 \cdot \| g \|_2 ; \tag{C9}$$

(h) if $f \in L_n^1 \cap L_n^\infty$, then $F \in L_n^2$.

Indeed, since $f \in L_n^\infty$, we have $|f_i| \leqslant C$ (p.p.) and

$$\sum_i |f_i|^2 < C \sum_i |f_i| \in L^1.$$

We consider also the space $\hat{L}_n(\epsilon, \infty)$ of all the functions which are expressible as the sum of two functions, one belonging to $L_n^1(\epsilon, \infty)$ and the other to both $L_n^2(\epsilon, \infty)$ and $L_n^\infty(\epsilon, \infty)$.

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