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PHYSICAL INTERPRETATION AND GENERALIZATION OF MARGUERRE'S SHALLOW SHELL THEORY

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ABSTRACT - Marguerre's shallow shell theory is interpreted by the means of the introduction of a "fictitious initial displacement". A logical interpretation of Marguerre's equilibrium equations follows directly from this point of view. The introduction of fictitious displacements is easy to generalize, and the case of quasi-conical shells is analyzed in detail.

1. INTRODUCTION

Classical shell theories leads to somewhat deficient finite element models due to the fact that rigid body motions are not polynomials in terms of curvilinear coordinates. The only way to overcome this difficulty is to make use of cartesian coordinates, as done by DUPUIS and GOEL [1] .

The use of NOVOZHILOV's shallow shell theory, as demonstrated by IDELSOHN [2] , leads to the further difficulty, that shallow shell solutions do not converge to deep shell solution when the finite element mesh is refined, since Novozhilov's approximation consists of neglecting some terms involving the radii of curvature which do not tend to zero for fine meshes.

Another shallow shell theory has been developed by MARGUERRE [3] in 1938. In this theory, use is made of cartesian coordinates, and the neglected terms do not involve the radii of curvature, but only the slopes of the surface, so that finite element approximation based on this approach do converge to the deep shell solution.

The classical way to introduce Marguerre's theory is to regard it as an approximation of the exact one [2,4] . In the present paper, another interpretation of Marguerre's equation is given, which is to some extent closer to Marguerre's point of view. The main interest of this presentation lies in the fact that it can be generalized to a large class of physical problems, including quasi-conical shells, twisted beams, ... The case of quasi-conical shells is analyzed in detail, and leads to a theory which, with the proper assumptions, degenerates in GECKELER's classical approximation.

2. MARGUERRE'S SHALLOW SHELL THEORY

2.1.

In Marguerre's shell theory, a given point of the shell is represented by its projection on a reference plane and its height h (fig.1). Marguerre's strain expressions are then

$$\bar{\varepsilon}_{xx} = \frac{\partial u}{\partial x} + \frac{\partial h}{\partial x}\frac{\partial w}{\partial x}$$

$$\bar{\varepsilon}_{yy} = \frac{\partial v}{\partial y} + \frac{\partial h}{\partial y}\frac{\partial w}{\partial y}$$

$$\bar{\gamma}_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial h}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial h}{\partial y}\frac{\partial w}{\partial x} \qquad (1)$$

$$\bar{\gamma}_{xz} = \alpha + \frac{\partial w}{\partial x}$$

$$\bar{\gamma}_{yz} = \beta + \frac{\partial w}{\partial y}$$

$$\bar{\chi}_{xx} = \frac{\partial \alpha}{\partial x}$$

$$\bar{\chi}_{yy} = \frac{\partial \beta}{\partial y}$$

$$\bar{\chi}_{xy} = \frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} ,$$

where u,v,w,α,β are the displacements and the rotations defined in the cartesian system of the reference plane. These expressions can be deduced from the exact deformations by a proper simplification rule consisting of neglecting the slopes of the shell in comparison to unity [2] . This point of view seems to be the curtomary one [4] . But Marguerre's presentation was somewhat different. Concerned with stability problems, he considered the height h as some imperfection affecting a plate, and he deduced the preceeding equations in the following way [3] :

$$(1+ \bar{\varepsilon}_{xx})^2 = \left(\frac{\text{new lenght}}{\text{old lenght}}\right)^2 = \frac{(1+ \frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2 + (\frac{\partial h}{\partial x} + \frac{\partial w}{\partial x})^2}{1 + (\frac{\partial h}{\partial x})^2}$$

and neglecting $\bar{\varepsilon}_{xx}^2$, $(\frac{\partial u}{\partial x})^2$, $(\frac{\partial v}{\partial x})^2$ and $(\frac{\partial h}{\partial x})^4$ in comparison with unity, he obtained

$$1 + 2\,\bar{\varepsilon}_{xx} = \frac{1+2\,\frac{\partial u}{\partial x} + (\frac{\partial h}{\partial x})^2 + 2\,\frac{\partial h}{\partial x}\,\frac{\partial w}{\partial x} + (\frac{\partial w}{\partial x})^2}{1 + (\frac{\partial h}{\partial x})^2}$$

$$\simeq \{1 + 2\,\frac{\partial u}{\partial x} + (\frac{\partial h}{\partial x})^2 + 2\,\frac{\partial h}{\partial x}\,\frac{\partial w}{\partial x} + (\frac{\partial w}{\partial x})^2\}\,\{1 - (\frac{\partial h}{\partial x})^2\},$$

i.e.

$$\bar{\varepsilon}_{xx} \simeq \frac{\partial u}{\partial x} + \frac{\partial h}{\partial x}\,\frac{\partial w}{\partial x} + \frac{1}{2}\,(\frac{\partial w}{\partial x})^2 .$$

The last term is of course neglected in the linear theory. The main originality of this approach is, to our opinion, the fact that the height h is considered as some initial displacement.

2.2.

This idea can be generalized in the following way. Let A be a reference structure of simple geometric characteristics, and let B be another structure which does not strongly differ from A. One may imagine that the structure B is obtained from an appropriate relatively short deformation from the structure A, the corresponding fictitious displacement being denoted $\mathring{u}_i$. If the body B is submitted to a subsequent deformation u_i, the total displacement will be

$$\overset{t}{u}_i = \mathring{u}_i + u_i . \qquad (2)$$

The total deformation, from the reference body A to the actual body C may be measured by the "total Green's strain tensor"

$$\overset{t}{g}_{ij} = \frac{1}{2}\,\{D_i\,(\mathring{u}_j + u_j) + D_j\,(\mathring{u}_i + u_i) + D_i\,(\mathring{u}_m + u_m)\,D_j(\mathring{u}_m + u_m)\},$$

and the initial fictitions deformation, by

$$\mathring{g}_{ij} = \frac{1}{2}\,\{D_i\,\mathring{u}_j + D_j\,\mathring{u}_i + D_i\mathring{u}_m\,D_j\,\mathring{u}_m\} .$$

Because B is the true physical body, the effective deformation has to be the deformation between B and C, measured by the following expression

$$g_{ij} = \overset{t}{g}_{ij} - \mathring{g}_{ij} = \frac{1}{2}\,\{D_iu_j + D_ju_i + D_i\,\mathring{u}_m\,D_ju_m + D_j\mathring{u}_m\,D_iu_m + D_i\,u_m\,D_ju_m\} \qquad (3),$$

whose linearized version is

$$\varepsilon_{ij} = \frac{1}{2} \{D_i u_j + D_j u_i + D_i \mathring{u}_m D_i u_m \} \qquad (4)$$

2.3.

Returning to the problem of a shallow shell, one may consider that it is obtained from the reference plate by a purely vertical displacement h, <u>without rotations</u>. As illustrated in fig. 2, an orthogonal section AB becomes a section A'B' which is no longer orthogonal. The initial displacement is then

$$\mathring{u}_1 = 0, \qquad \mathring{u}_2 = 0, \qquad \mathring{u}_3 = h(x,y) \ , \qquad (5)$$

and the introduction of the classical plate hypothesis

$$u_1 = u + \alpha z, \qquad u_2 = v + \beta z, \qquad u_3 = w \ , \qquad (6)$$

leads to equations (1).

2.4.

A correct interpretation of the stress resultants requires some care. Conjugated to Green strains increments, the stresses have to be understood in TREFFTZ's sense [5] . Their interpretation is thus as follows. During the fictitious initial deformation, the material basis vectors $\vec{e}_1$ and $\vec{e}_3$ are transformed in $\vec{g}_1$ and $\vec{g}_3$. It is clear that $\vec{g}_3 = \vec{e}_3$, but $\vec{e}_1$ is stretched and its orientation is modified. A simple inspection of fig. 3 shows that, if ϕ_1 is the angle $\vec{e}_1$ and $\vec{g}_1$,

$$||\vec{g}_1|| = \frac{1}{\cos \phi_1} \qquad (7)$$

A force F dy dz acting on an infinitesimal section has to be decomposed <u>in the basis of the shell</u> , i.e.

$$dydz\ F = dydz\ F_1\ \vec{g}_1 + dydz\ F_3\ \vec{g}_3 \ , \qquad (8)$$

and one then poses

$$N_{11} = F_1 , \qquad Q_1 = F_3 . \tag{9}$$

The effective value of the force in the direction $\vec{g}_1$ is thus

$$\tilde{N}_{11} = F_1 \, ||\vec{g}_1|| = \frac{F_1}{\cos \phi_1} = \frac{N_{11}}{\cos \phi_1} \tag{10}$$

This remark is of primary importance for the understanding of equilibrium equations. Let us consider a shallow shell subject to a vertical load of density p. Equating the virtual work to zero,

$$\begin{aligned} \int_S \{ & N_{xx} \, \delta(\frac{\partial u}{\partial x} + \frac{\partial h}{\partial x}\frac{\partial w}{\partial x}) + N_{xy} \, \delta(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial h}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial h}{\partial y}\frac{\partial w}{\partial x}) \\ & + N_{yy} \, \delta(\frac{\partial v}{\partial y} + \frac{\partial h}{\partial y}\frac{\partial w}{\partial y}) + M_{xx} \, \delta\frac{\partial \alpha}{\partial x} + M_{yy} \, \delta\frac{\partial \beta}{\partial x} + M_{xy} \, \delta(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x}) \\ & + Q_x \, \delta(\alpha + \frac{\partial w}{\partial x}) + Q_y \, \delta(\beta + \frac{\partial w}{\partial y}) - p \, \delta w \} \, dS = 0 , \end{aligned} \tag{11}$$

leads to the following equilibrium equations

$$\delta u \rightarrow \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{12}$$

$$\delta v \rightarrow \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = 0 \tag{13}$$

$$\delta w \rightarrow \frac{\partial}{\partial x}(Q_x + N_{xx}\frac{\partial h}{\partial x} + N_{xy}\frac{\partial h}{\partial y}) + \frac{\partial}{\partial y}(Q_y + N_{xy}\frac{\partial h}{\partial x} + N_{yy}\frac{\partial h}{\partial y}) + p = 0 \tag{14}$$

$$\delta \alpha \rightarrow \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = T_x \tag{15}$$

$$\delta \beta \rightarrow \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = T_y \tag{16}$$

The interpretation of equations (15) and (16) do not raise any problem. For equations (12) and (13), it will be noted that the horizontal projection of $N_{xx}/\cos \phi_1$ is precisely N_{xx} (fig. 4), and this leads to the same equations as in flat shells. Finally, equation (14) is easy to interpret when it is realized that the vertical resultant of Q_x, $N_{xx}/\cos \phi_1$ and $N_{xy}/\cos \phi_2$ is given by (fig. 5)

$$Q_x + N_{xx} \, \mathrm{tg} \, \phi_1 + N_{xy} \, \mathrm{tg} \, \phi_2 = Q_x + N_{xx}\frac{\partial h}{\partial x} + N_{xy}\frac{\partial h}{\partial y} \tag{17}$$

Thus, in disagreement with BISPLINGHOFF [4] , it is seen that Marguerre's equilibrium equations, correctly interpreted, are exact.

2.5.

However, the shallow shell hypothesis is used at two levels: (i) The shell obtained from a deviation of a plate is thiner than the latter. Its thickness is t cos ϕ , (fig. 6) where ϕ is the angle between the reference plane and the shell. The order of magnitude of this error is $(1 - \cos\phi) \approx \frac{\phi^2}{2}$.

(ii) The constitutive equations are set identical to those of the plate. This approximation is of order ϕ [6] . It is thus the largest error . But it should be noted that this error depends only on ϕ , not on the radii of curvature. As a consequence, if one considers a sequence of finite element meshes, the corresponding solutions will converge to the exact one, since $\phi \to 0$.

2.6.

Finally, let us analyse what are the conditions for preserving the rigid body motions in a finite element model. The only modes for which this is not automatic are the rotational ones. They are of the form

$$u_i = e_{ijk} \, \omega_j \, (x_k + \overset{\circ}{u}_k) \, , \tag{18}$$

where e_{ijk} is the alternator symbol. Therefore, the condition is that the initial fictitious displacement should be contained in the finite element model. As an example, is h is a third degree polynomial, the equation (18) has the following explicit form

$$\begin{aligned} u_1 &= \omega_2 \, (h + x_3) - \omega_3 \, x_2 \\ u_2 &= \omega_3 \, x_1 - \omega_1 \, (h + x_3) \\ u_3 &= \omega_1 \, x_2 - \omega_2 \, x_1, \end{aligned} \tag{19}$$

and the rigid body mode will be represented if u and v are at least of degree 3, w at least of degree 1, α and β at least constant. These conditions are in agreement with those obtained by DUPUIS and GOEL [1] for their general model of Kirchhoff-Love's shells in cartesian coordinates.

3. QUASI-CONICAL SHELLS

Axisymmetrical shells may be analysed by a Fourier decomposition scheme, avoiding thereby all the difficulties related to the circumferential curvature $1/R_\theta$. But the meridional curvature $1/R_s$ is the origin of problems similar as in the general case. Therefore, a temptating approach consists in the use of a Marguerre's type theory where the shell will be described by reference to conical frustium, and a "height" defined as the distance between the shell and the cone, measured orthogonally to this one (fig. 7). Following the approach described in (2.2), the height, which depends only on the cone coordinate s, will be considered as an initial displacement of the form

$$\mathring{u}_s = 0 \quad , \quad \mathring{u}_\theta = 0 \, , \quad \mathring{u}_\zeta = h(s) \qquad (20)$$

The following expression will then hold for the strains :

$$\varepsilon_{ij} = \frac{1}{2} \{ u_{i|j} + u_{j|i} + \mathring{u}_{m|i} \; u_{m|j} + \mathring{u}_{m|j} \; u_{m|i} \} \, , \qquad (21)$$

when the bar $|$ denote the covariant derivation, transcribed in the normed basis. Noting that

$$\begin{array}{lll} \mathring{u}_{s|s} = 0 & \mathring{u}_{s|\theta} = 0 & \mathring{u}_{s|\zeta} = 0 \\ \mathring{u}_{\theta|s} = 0 & \mathring{u}_{\theta|\theta} = (1 + \frac{\zeta}{R_\theta})^{-1} \frac{h}{R_\theta} & \mathring{u}_{\theta|\zeta} = 0 \\ \mathring{u}_{\zeta|s} = \frac{dh}{ds} & \mathring{u}_{\zeta|\theta} = 0 & \mathring{u}_{\zeta|\zeta} = 0, \end{array} \qquad (22)$$

one obtains after some classical calculations [6,7]

$$\varepsilon_{ss} = \hat{\varepsilon}_{ss} + \zeta \hat{\chi}_{ss} + \frac{dh}{ds} \frac{\partial w}{\partial s}$$

$$\varepsilon_{\theta\theta} = (1 + \frac{\zeta}{R_\theta})^{-1} (\hat{\varepsilon}_{\theta\theta} + \zeta \hat{\chi}_{\theta\theta}) + (1 + \frac{\zeta}{R_\theta})^{-2} \frac{h}{R_\theta} (\hat{\varepsilon}_{\theta\theta} + \zeta \hat{\chi}_{\theta\theta})$$

$$\gamma_{s\zeta} = \hat{\gamma}_{s\phi} \tag{23}$$

$$\gamma_{\theta\zeta} = (1 + \frac{\zeta}{R_\theta})^{-1}(\hat{\gamma}_{\theta\zeta} + \beta\frac{h}{R_\theta})$$

$$\gamma_{s\theta} = (1 + \frac{\zeta}{R_\theta})^{-1} \{\hat{\gamma}_\theta + \zeta\hat{\delta}_\theta + \frac{dh}{ds}\ \hat{\gamma}_{\theta\zeta} + \frac{h}{R_\theta}(\hat{\gamma}_s + \zeta\hat{\delta}_s)\}$$

$$+ \hat{\gamma}_s + \zeta\hat{\delta}_s - \beta\frac{dh}{ds}$$

$$\varepsilon_{\zeta\zeta} = 0 \quad ,$$

where

$$\hat{\varepsilon}_{ss} = \frac{\partial u}{\partial s}$$

$$\hat{\varepsilon}_{\theta\theta} = \frac{1}{2}\ (\frac{\partial v}{\partial\theta} + u\sin\phi + w\cos\phi)$$

$$\hat{\gamma}_\theta = \frac{1}{r}\ (\frac{\partial u}{\partial\theta} - v\sin\phi)$$

$$\hat{\gamma}_s = \frac{\partial v}{\partial s}$$

$$\hat{\gamma}_{s\zeta} = \alpha + \frac{\partial w}{\partial s}$$

$$\hat{\gamma}_{\theta\zeta} = \beta + \frac{1}{r}\ (\frac{\partial w}{\partial\theta} - v\cos\phi) \tag{24}$$

$$\hat{\chi}_{ss} = \frac{\partial\alpha}{\partial s}$$

$$\hat{\chi}_{\theta\theta} = \frac{1}{r}\ (\frac{\partial\beta}{\partial\theta} + \alpha\sin\phi)$$

$$\hat{\delta}_\theta = \frac{1}{r}\ (\frac{\partial\alpha}{\partial\theta} - \beta\sin\phi)$$

$$\hat{\delta}_s = \frac{\partial\beta}{\partial s}$$

$$R_\theta = r/\cos\phi$$

Assuming that the thickness of the shell verifies

$$(\frac{t}{R_\theta})^2 \ll 1 \ , \tag{25}$$

the expressions $(1 + \frac{\zeta}{R_\theta})^{-k}$ will be correctly approximated by limited Taylor's expansions :

$$(1 + \frac{\zeta}{R_\theta})^{-k} \simeq 1 - k\frac{\zeta}{R_\theta} + O(\eta) ,$$

with

$$\eta = (\frac{t}{R_\theta})^2$$

This leads to the following approximated strains :

$$\varepsilon_{\theta\theta} = \hat{\varepsilon}_{\theta\theta}(1+ \frac{h}{R_\theta}) + \zeta\{\hat{\chi}_{\theta\theta}(1+ \frac{h}{R_\theta}) - \frac{\hat{\varepsilon}_{\theta\theta}}{R_\theta}(1 + 2\frac{h}{R_\theta})\} + O(\eta)$$

$$\gamma_{\theta\zeta} = (\hat{\gamma}_{\theta\zeta} + \beta\frac{h}{R_\theta}) - \frac{\zeta}{R_\theta}(\hat{\gamma}_{\theta\zeta} + \beta\frac{h}{R_\theta}) + O(\eta) \qquad (26)$$

$$\gamma_{s\theta} = \{\hat{\gamma}_\theta + \hat{\gamma}_{\theta\zeta}\frac{dh}{ds} - \beta\frac{dh}{ds} + \hat{\gamma}_s(1+ \frac{h}{R_\theta})\}$$

$$+ \zeta\{\hat{\delta}_\theta + \hat{\delta}_s(1 + \frac{h}{R_\theta}) - \frac{\hat{\gamma}_\theta}{R_\theta} - \frac{1}{R_\theta}\frac{dh}{ds}\hat{\gamma}_{\theta\zeta} - \frac{h}{R_\theta^2}\hat{\gamma}_s\} + O(\eta)$$

A further simplification is possible by making both assumptions

$$[\frac{t}{R_\theta}] << 1 , \qquad [\frac{h}{R_\theta}] << 1 . \qquad (27)$$

But here, a special care is required, because it is not guaranteed that the suppression of some terms will not destroy the representation of rigid body motions. In fact, for such a motion, all strains are equal to zero. By virtue of the independence of the functions constant, ζ , ζ^2,..., any term of Taylor's expansion of the strains will vanish. But our present goal is to omit <u>a part</u> of Taylor's linear terms of the strains, which do not vanish <u>a priori</u>. This point has been developed in detail in [6] . Consider first the expression of $\varepsilon_{\theta\theta}$. The constant term $\hat{\varepsilon}_{\theta\theta}$ necessarily vanishes for a rigid body motion. Consequently, $\hat{\chi}_{\theta\theta}$ will also be zero, and it is permissible to neglect the small term

$$\hat{\chi}_{\theta\theta}\frac{h}{R_\theta} - \hat{\varepsilon}_{\theta\theta}\frac{2h}{R_\theta^2} . \qquad (28)$$

By similar arguments, the whole linear part of $\gamma_{\theta\zeta}$ may be neglected.

But <u>in $\gamma_{s\theta}$, no simplification may be done</u>.

Finally, the strains may be set in the following form :

$$
\begin{aligned}
\varepsilon_{ss} &= \bar{\varepsilon}_{ss} + \zeta\bar{\chi}_{ss} \\
\varepsilon_{\theta\theta} &= \bar{\varepsilon}_{\theta\theta} + \zeta\bar{\chi}_{\theta\theta} \\
\gamma_{s\theta} &= \bar{\gamma}_{s\theta} \\
\gamma_{s\zeta} &= \bar{\gamma}_{s\zeta} \\
\gamma_{\theta\zeta} &= \bar{\gamma}_{\theta\zeta} ,
\end{aligned} \tag{29}
$$

where

$$
\begin{aligned}
\bar{\varepsilon}_{ss} &= \frac{\partial u}{\partial s} + \frac{dh}{ds}\frac{\partial w}{\partial s} \\
\bar{\varepsilon}_{\theta\theta} &= \frac{1}{2}\left(\frac{\partial v}{\partial \theta} + u \sin\phi + w \cos\phi\right) \\
\bar{\gamma}_{s\theta} &= \frac{1}{2}\left\{\frac{\partial u}{\partial \theta} + \frac{dh}{ds}\frac{\partial w}{\partial \theta} - v\left(\sin\phi + \cos\phi\frac{dh}{ds}\right) + \frac{\partial v}{\partial s}(r + h\cos\phi)\right\} \\
\bar{\gamma}_{s\zeta} &= \alpha + \frac{\partial w}{\partial s} \\
\bar{\gamma}_{\theta\zeta} &= \frac{1}{r}\left\{\beta(r+h\cos\phi) + \frac{\partial w}{\partial \theta} - v\cos\phi\right\} \\
\bar{\chi}_{ss} &= \frac{\partial \alpha}{\partial s} \\
\bar{\chi}_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial \beta}{\partial \theta} + \alpha\sin\phi\right) \\
\bar{\chi}_{s\theta} &= \frac{1}{r}\left\{\frac{\partial \alpha}{\partial \theta} - \beta\left(\sin\phi + \cos\phi\frac{dh}{ds}\right) + \frac{\partial \beta}{\partial s}(r+h\cos\phi)\right\} \\
&\quad - \frac{1}{rR_\theta}\left\{\frac{\partial u}{\partial \theta} + \frac{dh}{ds}\frac{\partial w}{\partial \theta} - v\left(\sin\phi + \cos\phi\frac{dh}{ds}\right) + h\cos\phi\frac{\partial v}{\partial s}\right\} .
\end{aligned} \tag{30}
$$

4. <u>EQUILIBRIUM EQUATIONS OF A QUASI-CONICAL SHELL</u>

The equilibrium equations will be obtained by writing the virtual work theorem for an interval $]s_1, s_2[\times]\theta_1, \theta_2[$. The first variation of the strain energy is given by

$$\delta U = \int_{s_1}^{s_2} ds \int_{\theta_1}^{\theta_2} d\theta \int_{-t/2}^{t/2} \{\sigma_{ss}\ \delta\varepsilon_{ss} + \sigma_{\theta\theta}\delta\varepsilon_{\theta\theta} + \tau_{s\theta}\ \delta\gamma_{s\theta} + \tau_{s\zeta}\ \delta\gamma_{s\zeta} + \tau_{\theta\zeta}\ \delta\gamma_{\theta\zeta}\}\ (1+\frac{\zeta}{R_\theta})\ d\zeta\ . \qquad (31)$$

Note that the cone surface element $(1 + \frac{\zeta}{R_\theta})\ d\theta\ ds$ is used, in the same way as Marguerre has written his equations in the reference plane. The (symmetric) generalized forces are thus $(\alpha,\beta = 1,2)$

$$N_{\alpha\beta} = \int_{-t/2}^{t/2} \sigma_{\alpha\beta}\ (1+\frac{\zeta}{R_\theta})\ d\zeta$$

$$M_{\alpha\beta} = \int_{-t/2}^{t/2} \sigma_{\alpha\beta}\ \zeta(1 + \frac{\zeta}{R_\theta})d\zeta \qquad (32)$$

$$Q_\alpha = \int_{-t/2}^{t/2} \tau_{\alpha\zeta}\ (1+\frac{\zeta}{R_\theta})\ d\zeta$$

These definitions allow us to write (31) in the following form

$$\begin{aligned}\delta U = \int_{s_1}^{s_2} ds \int_{\theta_1}^{\theta_2} \Big[& r\ N_{ss}\ \delta\{\frac{\partial u}{\partial s} + \frac{dh}{ds}\frac{\partial w}{\partial s}\} + N_{\theta\theta}\delta\{\frac{\partial v}{\partial\theta} + u\ \sin\phi + w\ \cos\phi\} \\ & + N_{s\theta}\ \delta\{\frac{\partial u}{\partial\theta} + \frac{dh}{ds}\frac{\partial w}{\partial\theta} - v\ (\sin\phi + \cos\phi\ \frac{dh}{ds}) + \frac{\partial v}{\partial s}\ (r + h\ \cos\phi)\} \\ & + r\ M_{ss}\ \delta\{\frac{\partial\alpha}{\partial s}\} + M_{\theta\theta}\ \delta\{\frac{\partial\beta}{\partial\theta} + \alpha\ \sin\phi\ \} \\ & + M_{s\theta}\delta\{\frac{\partial\alpha}{\partial\theta} - \beta(\sin\phi + \cos\phi\ \frac{dh}{ds}) + \frac{\partial\beta}{\partial s}\ (e + h\ \cos\phi\ ) \\ & \qquad - \frac{1}{R_\theta}\ (\frac{\partial u}{\partial\theta} + \frac{dh}{ds}\frac{\partial w}{\partial\theta} - v\ (\sin\phi + \cos\phi\ \frac{dh}{ds}) + h\ \cos\phi\ \frac{\partial v}{\partial s})\ \} \\ & + r\ Q_s\ \ \delta\{\alpha + \frac{\partial w}{\partial s}\} + Q_\theta\ \delta\{\beta\ (r + h\ \cos\phi) + \frac{\partial w}{\partial\theta} - v\ \cos\phi\}\Big]\ d\theta\end{aligned} \qquad (33)$$

Assuming a pressure load, the virtual work is

$$\delta\tau = \int_{s_1}^{s_2} ds \int_{\theta_1}^{\theta_2} pr\ \delta w\ d\theta\ \ , \qquad (34)$$

and one obtains the following equilibrium equations

$$\delta u \rightarrow -\frac{\partial}{\partial s}\ (r\ N_{ss}) + N_{\theta\theta}\ \sin\phi - \frac{\partial N_{s\theta}}{\partial\theta} + \frac{1}{R_\theta}\frac{\partial M_{s\theta}}{\partial\theta} = 0 \qquad (35)$$

$$\delta v \rightarrow -\frac{\partial N_{\theta\theta}}{\partial\theta} - N_{s\theta}\ (\sin\phi + \cos\phi\ \frac{dh}{ds}) - \frac{\partial}{\partial s}\{\ (r + h\ \cos\phi)N_{s\theta}\}$$

$$+ \frac{M_{s\theta}}{R_\theta}\left(\sin\phi + \cos\phi \frac{dh}{ds}\right) + \frac{\partial}{\partial s}\left(\frac{h\cos\phi}{R_\theta} M_{s\theta}\right) - Q_\theta \cos\phi = 0 \tag{36}$$

$$\delta w \rightarrow -\frac{\partial}{\partial s}\left(r\, N_{ss} \frac{dh}{ds}\right) + N_{\theta\theta}\cos\phi - \frac{\partial}{\partial \theta}\left(N_{s\theta}\frac{dh}{ds}\right)$$

$$+ \frac{\partial}{\partial \theta}\left(\frac{M_{s\theta}}{R_\theta}\frac{dh}{ds}\right) - \frac{\partial}{\partial s}(r\, Q_s) - \frac{\partial Q_\theta}{\partial \theta} - pr = 0 \tag{37}$$

$$\delta\alpha \rightarrow -\frac{\partial}{\partial s}(r\, M_{ss}) + M_{\theta\theta}\sin\phi - \frac{\partial M_{s\theta}}{\partial \theta} + r\, Q_s = 0 \tag{38}$$

$$\delta\beta \rightarrow -\frac{\partial M_{\theta\theta}}{\partial \theta} - M_{s\theta}\left(\sin\phi + \cos\phi\frac{dh}{ds}\right) - \frac{\partial}{\partial s}\{(r+h\cos\phi)M_{s\theta}\} + (r+h\cos\phi)Q_\theta = 0 \tag{39}$$

All boundary conditions will be supposed homogeneous. They may be written in the following condensed form :

$$\delta u \rightarrow \Big[r\, N_{ss}\, \delta u \Big]_{s_1}^{s_2} = 0 \quad , \quad \Big[\left(N_{s\theta} - \frac{M_{s\theta}}{R_\theta}\right) \delta u \Big]_{\theta_1}^{\theta_2} = 0 \tag{40}$$

$$\delta v \rightarrow \Big[\left((r+h\cos\phi)N_{s\theta} - \frac{h\cos\phi}{R_\theta} M_{s\theta}\right)\delta v \Big]_{s_1}^{s_2} = 0 \quad , \quad \Big[N_{\theta\theta}\,\delta v \Big]_{\theta_1}^{\theta_2} = 0 \tag{41}$$

$$\delta w \rightarrow \Big[r\left(Q_s + N_{ss}\frac{dh}{ds}\right)\delta w \Big]_{s_1}^{s_2} = 0 \quad , \quad \Big[\left(Q_\theta + \left(N_{s\theta} - \frac{M_{s\theta}}{R\theta}\right)\frac{dh}{ds}\right) \delta w \Big]_{\theta_1}^{\theta_2} = 0 \tag{42}$$

$$\delta\alpha \rightarrow \Big[r\, M_{ss}\, \delta\alpha \Big]_{s_1}^{s_2} = 0 \quad , \quad \Big[M_{s\theta}\; \delta\,\alpha \Big]_{\theta_1}^{\theta_2} = 0 \tag{43}$$

$$\delta\beta \rightarrow \Big[r\, M_{s\theta}\, \delta\beta \Big]_{s_1}^{s_2} = 0 \quad , \quad \Big[M_{\theta\theta}\, \delta\beta \Big]_{\theta_1}^{\theta_2} = 0 \tag{44}$$

It will be noted that the apparition of groups as $N_{s\theta} - \frac{M_{s\theta}}{R_\theta}$ is due to the use of symmetrical resultants, the same phenomenon appearing in deep shells theories. Equations (43) are easy to explain by a direct generalization of the interpretation of the forces in Marguerre's theory. The only puzzling term is $\frac{h\cos\phi}{R_\theta} M_{s\theta}$, in (41). As we shall see, it restores the correct difference between the classical resultants $N_{s\theta}^{(s)}$ and $N_{s\theta}^{(\theta)}$ (fig. 7). In fact, a glance to this figure shows that the equilibrium equation of a little portion of the shell round its normal is given by

$$N_{s\theta}^{(s)} (R_\theta+h)\, ds\, d\theta - N_{s\theta}^{(\theta)} (R_\theta+h)\, ds\, d\theta - N_{s\theta}\, ds\, d\theta = 0 \quad ,$$

and this leads to

$$N_{s\theta}^{(s)} - N_{s\theta}^{(\theta)} = \frac{M_{s\theta}}{R_\theta+h} \quad . \qquad (45)$$

By (41) and (42),

$$N_{s\theta}^{(\theta)} = N_{s\theta} - \frac{M_{s\theta}}{R_\theta}$$

$$N_{s\theta}^{(s)} = N_{s\theta} - \frac{h\cos\phi}{R_\theta(r+h\cos\phi)} M_{s\theta} = N_{s\theta} - \frac{h}{R_\theta(R_\theta+h)} M_{s\theta} \quad ,$$

so that

$$N_{s\theta}^{(\theta)} - N_{s\theta}^{(s)} = \frac{M_{s\theta}}{R_\theta} \left(1 - \frac{h}{R_\theta+h}\right) = \frac{M_{s\theta}}{R_\theta+h} \quad ,$$

which is precisely the correct value.

5. CONSTITUTIVE EQUATION

In contrast with theories such as KALNINS' one [7], the fact that all strains are zero for a rigid body motion allows us to simplify the constitutive equations without risk concerning the representation of these particular modes. The simplest choice is thus a decoupled law of the form ($\alpha, \beta, \gamma, \delta = 1, 2$)

$$N_{\alpha\beta} = t\, C^{(1)}_{\alpha\beta\gamma\delta}\, \bar{\varepsilon}_{\alpha\beta}$$

$$M_{\alpha\beta} = \frac{t^3}{12} C^{(2)}_{\alpha\beta\gamma\delta}\, \bar{\chi}_{\alpha\beta} \qquad (46)$$

$$Q_\alpha = t\, G_{\alpha\beta}\, \bar{\gamma}_{\beta\zeta}$$

Moreover, the approximate representation of stiffened shells by the "smearing" procedure remains possible (In this case, $C^{(1)}_{\alpha\beta\gamma\delta} \neq C^{(2)}_{\alpha\beta\gamma\delta}$).

6. FINITE ELEMENT IMPLEMENTATION

The present formulation seems to be very attractive for finite element applications. Considering a given shell (fig. 9), in each element, use will be made of the cone defined by the extreme nodes, so that conformity will be guaranteed. The rigid body criterium is verified if the degree of displacements is at least equal to the degree of h. When the mesh is refined, the finite element solution converges to the exact deep shell solution, as it is the case with conical elements, but faster than in the latter case.

Finally, note that the correct symmetry conditions on the axis can be derived from the fact that all strains have to be square integrable in the neighborhood of the axis.

7. RELATION WITH GECKELER'S APPROXIMATION [8]

As it is well known, the classical methods to solve axisymmetrical shell problems often consist in a membrane analysis of the shell first, to which a bending correction is applied afterwards at the neighborhood of the points where the first analysis leads to displacement discontinuities, that is, in general, the fixed points and the points of slope or curvature discontinuities of the shell. The correction acts on a short region of the shell whose length is of the order of magnitude of $\sqrt{t.R_\theta}$. Such approximate methods are not totally obsolate after the finite element revolution because their intuitive meaning can be used at the conception level and also for the reason that they offer a simple way of checking numerical results.

Geckeler's method is applicable whenever a shell has an end parallel to the axis and a sufficiently large radius of curvature R_s.
In such a case, the angle between the axis and the shell is small in the region where flexural effects are significant. For quasi-conical shells, a natural procedure consists of using a cylinder as a reference. In the axisymmetrical case, the strains reduce to

$$\bar{\varepsilon}_{ss} = \frac{du}{ds} + \frac{dh}{ds}\frac{dw}{ds} \tag{47}$$

$$\bar{\varepsilon}_{\theta\theta} = \frac{w}{r} \tag{48}$$

$$\bar{\gamma}_{s\zeta} = \alpha + \frac{dw}{ds} \tag{49}$$

$$\bar{\chi}_{ss} = \frac{d\alpha}{ds}\ , \tag{50}$$

and the equilibrium equations become, in the unloaded case,

$$\frac{d}{ds}(r\ N_{ss}) = 0 \tag{51}$$

$$-\frac{d}{ds}(r\ N_{ss}\frac{dh}{ds} + r\ Q_s) + N_{\theta\theta} = 0 \tag{52}$$

$$-\frac{d}{ds}(r\ M_{ss}) + r\ Q_s = 0\ . \tag{53}$$

Making use of the Kirchhoff-Love hypotheses, one sets

$$\alpha = -\frac{dw}{ds}\ . \tag{54}$$

The vertical equilibrium will be unchanged by the flexural effects, so that one may assume

$$N_{ss} = 0\ , \tag{55}$$

and, consequently,

$$N_{\theta\theta} = Et\frac{w}{r}\ , \tag{56}$$

r being the (constant) radius of the reference cylinder. The solution is then easy to obtain by the use of the variables $V = r\ Q_s$ and α. In fact, equation (52) becomes

$$r\frac{d}{ds}(r\ Q_s) = Et\ w \rightarrow \frac{d}{ds}(r\frac{dV}{ds}) + Et\alpha = 0 \tag{57}$$

and, from equation (53) and the constitutive relation

$$M_{ss} = D\frac{d\alpha}{ds}\ , \tag{58}$$

it is easy to obtain

$$\frac{d}{ds}\left(r\frac{d\alpha}{ds}\right) - \frac{V}{D} = 0 \ . \qquad (59)$$

Equations (57) and (59) are precisely Geckeler's ones. Their derivation turn out to be very natural in the context of quasi-conical shells.

8. CONCLUSION

The introduction of a fictitions initial displacement enlightens Marguerre's theory of shells in such a way that its equilibrium equations become perfectly logical. The point of view developed here is generalizable to a large class of problems including the theory of quasi-conical shells which has been treated here and several other physical situations such as weakly pretwisted beams, weakly non prismatic beams, shallow arches etc... The finite element implementation of these theories seems to be very attractive because of the convergence to the exact solution and also for the correct representation of rigid body motions that they allow. The nonlinear case can also be treated following the same approach.

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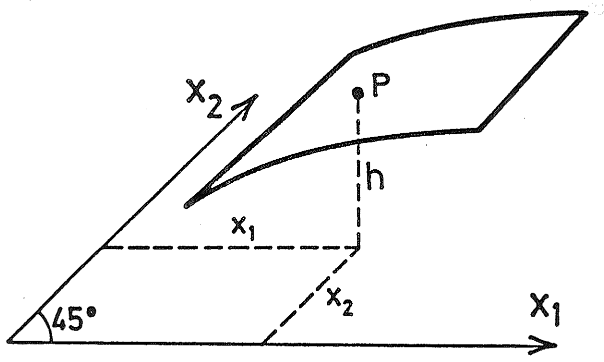


FIG. 1

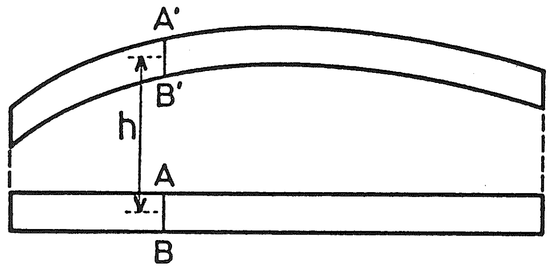


FIG. 2

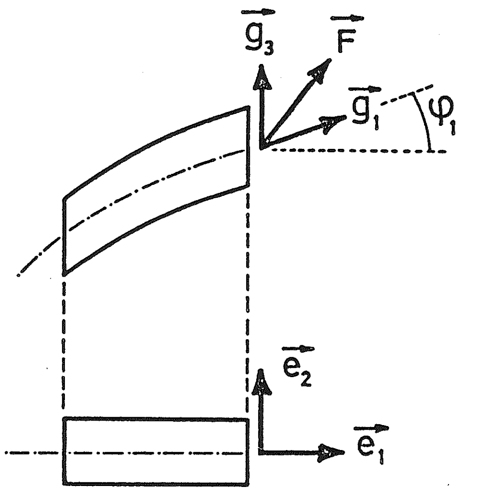


FIG. 3

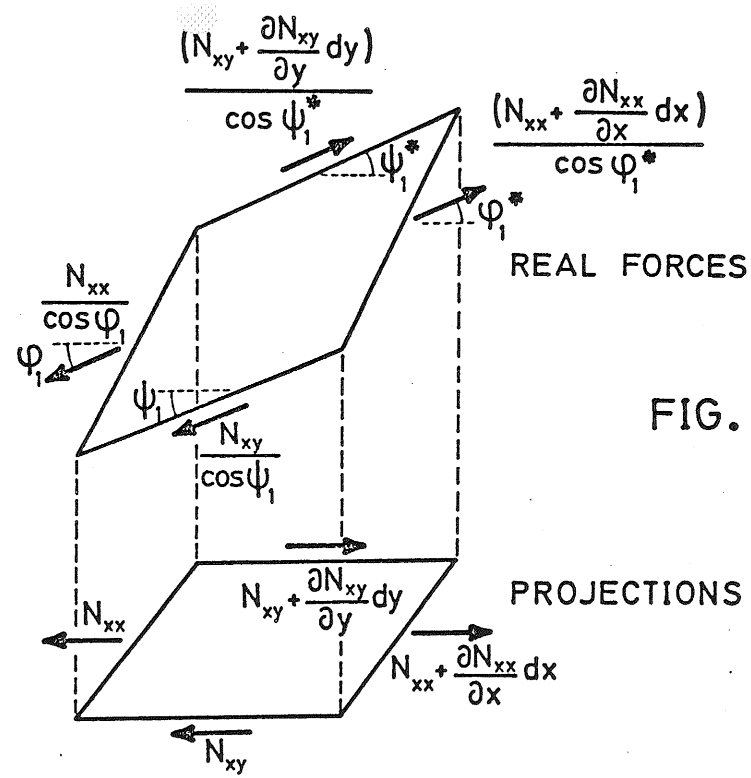
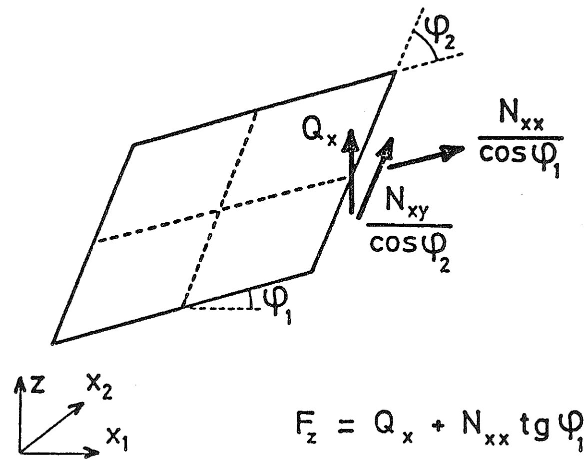


FIG. 4



$$F_z = Q_x + N_{xx}\, tg\, \varphi_1 + N_{xy}\, tg\, \varphi_2$$

FIG. 5

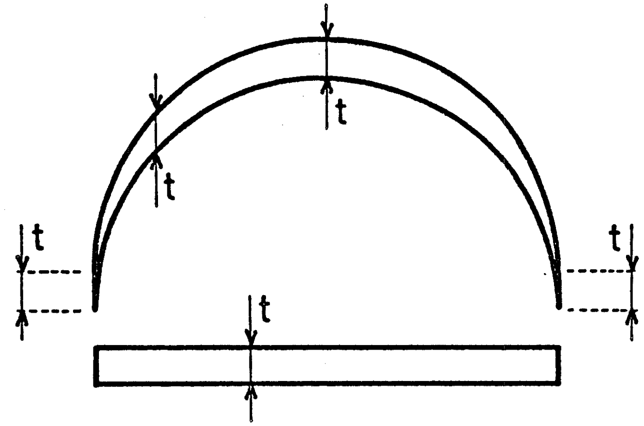


FIG. 6

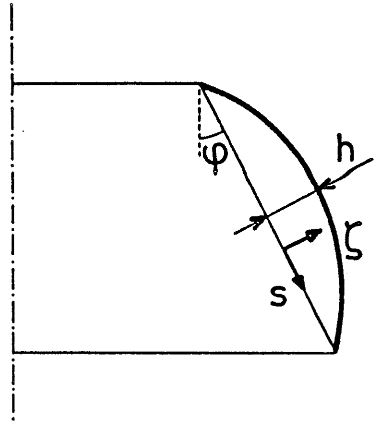


FIG. 7

FIG. 8

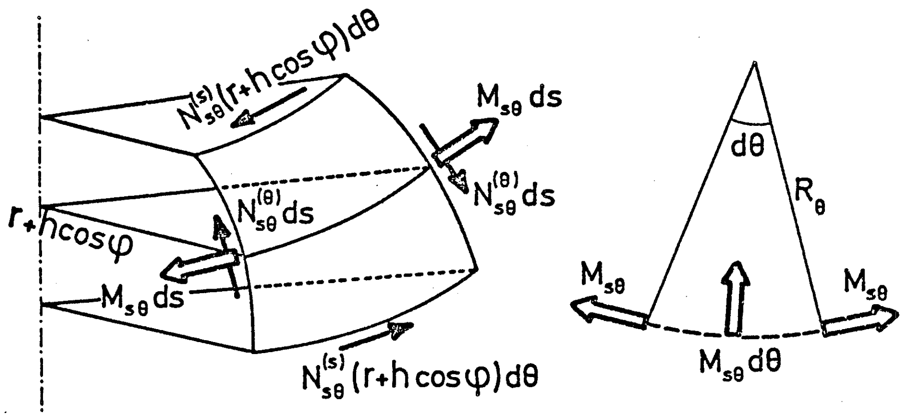


FIG. 9