

Use of the Fourier transform to derive simple expressions for the gravitational lens deflection angle

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ABSTRACT

Knowing that the gravitational lens deflection angle can be expressed as the convolution product between the dimensionless surface mass density $\kappa(\mathbf{x})$ and a simple function of the scaled impact parameter vector \mathbf{x} , we make use of the Fourier transform to derive its analytical expression for the case of mass distributions presenting a homoeoidal symmetry. For this family of models, we obtain the expression of the two components of the deflection angle in the form of integrals performed over the radial coordinate ρ . In the limiting case of axially symmetric lenses, we obviously retrieve the well-known relation $\hat{\alpha}(\mathbf{x}) \propto M(<|\mathbf{x}|)\mathbf{x}/|\mathbf{x}|^2$. Furthermore, we derive explicit solutions for the deflection angle characterized by dimensionless surface mass density profiles such as $\kappa \propto (\rho_c^2 + \rho^2)^{-\nu}$; corresponding to the non-singular isothermal ellipsoid model for the particular case $\nu = 1/2$. Let us insist that all these results are obtained without using the complex formalism introduced by Bourassa and Kantowski. Further straightforward applications of this Fourier approach are suggested in the conclusions of this work.

Key words: gravitational lensing: strong – methods: analytical.

1 INTRODUCTION

Gravitational lens effects consist in the bending of light rays from a background source, under the influence of a foreground mass distribution along the line of sight (e.g. stars, galaxies, clusters of galaxies, etc.). The study of this phenomenon requires realistic lens models which fully account for the observed lensed image configurations, lensed image amplifications and/or time delays between lensed images. From observation, we know that an appreciable fraction of galaxies shows elliptical isophotes which suggest an elliptical mass distribution. First, axially symmetric lenses perturbed by an external large-scale gravitational field, the external shear, were studied in detail by Chang & Refsdal (1984), Kochanek (1991), Wambsganss & Paczyński (1994) and An & Evans (2006), etc. Those types of models have the advantage to be mathematically simple to use. For instance, for the case of a nearly perfect alignment among a point-like source, a power-law mass distribution perturbed by an external shear and an observer, all lens model parameters can be directly derived from the first-order astrometric equations (Wertz, Pelgrims & Surdej 2012). Another approach consists in considering lenses characterized by an elliptical symmetry, i.e. with isodensity contours represented by concentric ellipses (so-called homoeoidal

symmetry). Although the parameters of the elliptical isophotes may vary with the major axis, the isophotes may as well turn out to be ‘twisted’. An elliptical lens is considered to be an adequate model to represent observed gravitational lens systems (Schramm 1990). We note that elliptical potentials may also be used as lens models and these are mathematically easier to handle than elliptical mass distributions. But for large ellipticities (e.g. $\epsilon \gtrsim 0.5$), the corresponding mass distributions turn out to be unphysical (Kassiola & Kovner 1993).

Whether considering the determination of the image positions, image amplification ratios and/or time delays between observed lensed images, the expression of the deflection angle (or deflection potential) needs to be calculated for any given model. According to the complexity of the expression for the dimensionless surface mass density $\kappa(\mathbf{x})$, the determination of the deflection angle may turn out to be very complicated. For the case of elliptical mass distributions, Bourassa, Kantowski & Norton (1973) first derived the expression of the deflection angle using a complex formalism (Bourassa & Kantowski 1975; corrected by Bray 1984) which is rather difficult to apply. According to this formalism, the lens equation is given by

$$y_c = x_c + I_c^*(x_c), \quad (1)$$

with the complex deflection angle (the scattering function)

$$I_c(x_c) = -2 \operatorname{sign}(x_c) \cos(\beta) \int_0^\rho \frac{\rho' \kappa(\rho')}{\sqrt{x_c^2 - \rho'^2 \sin^2(\beta)}} d\rho', \quad (2)$$

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where $y_c = y_1 + \iota y_2$ represents the complex source position, $x_c = x_1 + \iota x_2$ the complex image positions, ρ , the radial coordinate and $1/\cos(\beta) = (1 - \epsilon)$ with ϵ the ellipticity of the mass distribution.

It turns out that separating the two components of the deflection angle (equation (2)) is extremely difficult (Schneider, Ehlers & Falco 1992). An alternative equivalent formulation has been derived by Schramm (1990) from results of the classical three-dimensional potential and ellipse theories. However, analytical solutions occur only for very special or simple surface mass density expressions. Let us however note that for numerical evaluation, their formulation represents a very efficient tool.

In this paper, we propose to use the well-known fact that the deflection angle can be expressed as a convolution product (e.g. Bartelmann 2003), in order to analytically derive, with the help of Fourier analysis, the expression of the two components of the deflection angle (Section 2) without the need of invoking complex expressions.

We first apply this new approach to the case of elliptical lenses and derive the explicit expression of the deflection components for mass distributions obeying the law $\kappa \propto (\rho_c^2 + \rho^2)^{-\nu}$, in particular for the cases of the non-singular isothermal ellipsoid (NSIE; $\nu = 1/2$) and axially symmetric lenses ($f = 1$, see Section 3). Some general conclusions form the last section.

2 DEFLECTION ANGLE AND FOURIER ANALYSIS

2.1 Basic equations and definitions

Since we adopt the thin lens approximation, a general mass distribution $\rho(\xi_x, \xi_y, \xi_z)$ is represented by its projected surface mass density $\Sigma(\xi) = \int_{\mathbb{R}} \rho(\xi, \xi_z) d\xi_z$, where $\xi = (\xi_x, \xi_y)$ corresponds to the impact parameter vector defined in the lens plane, perpendicular to the line of sight (ξ_z direction). Moreover, for a given light ray characterized by a normalized impact parameter $x = \xi/\xi_0$, where ξ_0 represents a scaled factor which is dependent on the lens model,¹ the scaled deflection angle is defined by (Schneider et al. 1992)

$$\hat{\alpha}(x) = -\frac{1}{\pi} \iint_{\mathbb{R}^2} \kappa(x') \frac{x - x'}{|x - x'|^2} dx', \quad (3)$$

where $\kappa(x) = \Sigma(\xi_0 x)/\Sigma_{\text{cri}}$ represents the dimensionless surface mass density (also called convergence) with Σ_{cri} , the critical surface mass density defined by

$$\Sigma_{\text{cri}} = \frac{c^2 D_{\text{OS}}}{4 G D_{\text{LS}} D_{\text{OL}}}. \quad (4)$$

For the case of simple lens models (e.g. axially symmetric lens models), the expression of $\hat{\alpha}(x)$ may be derived straightforwardly from equation (3). However, it turns out to be generally more complicated.

2.2 Deflection angle as a convolution product

Considering $\kappa(x)$ which either has a sufficiently rapid decay at infinity or is locally integrable, the expression of the deflection

angle, equation (3), can be expressed as a convolution product (e.g. Bartelmann 2003):

$$\hat{\alpha}(x) = -\frac{1}{\pi} \kappa(x) \otimes \frac{x}{|x|^2}. \quad (5)$$

Making use of the *Fourier convolution theorem* which states that the Fourier transform of a convolution product is equal to the pointwise product of the Fourier transforms of the functions to be convolved (Bracewell 1999), the expression of the deflection angle can be expressed as

$$\hat{\alpha}(x) = -2 \mathcal{F}^+ \left[\mathcal{F}^- [\kappa(x)] \mathcal{F}^- \left[\frac{x}{|x|^2} \right] \right], \quad (6)$$

where

$$\mathcal{F}^\mp[f] = \frac{1}{2\pi} \iint_{\mathbb{R}^2} f(x) e^{\mp i x \cdot z} dx. \quad (7)$$

In equation (6), the Fourier transform $\mathcal{F}^- [x/|x|^2]$ is independent of the lens model and only needs to be calculated once. Its expression is simply given by

$$\mathcal{F}^- \left[\frac{x}{|x|^2} \right] = -\frac{i z}{|z|^2}, \quad (8)$$

where $z = (z_1, z_2)$ represents the conjugate variables of (x_1, x_2) in the Fourier space. As a consequence, equation (6) becomes

$$\hat{\alpha}(x) = 2 i \mathcal{F}^+ \left[\mathcal{F}^- [\kappa(x)] \frac{z}{|z|^2} \right], \quad (9)$$

which remains valid for any expression of the dimensionless surface mass density $\kappa(x)$.

3 HOMOEIDAL SYMMETRIC LENSES

For the case of homoeoidal symmetric mass distributions, the first step now consists in explicitly expressing the Fourier transform of the dimensionless surface mass density $\kappa(x)$. Since the shape of the isodensity contours consists of concentric ellipses, κ can be expressed in terms of the coordinates (ρ, ϕ) defined such as

$$\begin{cases} x_1 = \rho \cos(\phi) \\ x_2 = \frac{\rho}{f} \sin(\phi), \end{cases} \quad (10)$$

where $0 < f \leq 1$ represents the axis ratio of the ellipses and is related to the ellipticity ϵ by

$$f = 1 - \epsilon. \quad (11)$$

Under such conditions, we have $\kappa(x) = \kappa(\rho)$. Therefore, the expression of the Fourier transform of $\kappa(\rho)$ becomes

$$\begin{aligned} \mathcal{F}^- [\kappa(\rho)] &= \frac{1}{2\pi f} \int_0^{+\infty} \rho' \kappa(\rho') \left\{ \int_0^{2\pi} e^{-i \rho' z \cos(\phi' - \theta)/f} d\phi' \right\} d\rho', \\ &= \frac{1}{f} \int_0^{+\infty} \rho' \kappa(\rho') J_0 \left(\frac{\rho' z}{f} \right) d\rho', \end{aligned} \quad (12)$$

where $J_0(\rho' z/f)$ represents the zero-order Bessel function of the first kind and (z, θ) , the conjugate coordinates of (ρ, ϕ) in the Fourier space defined such as

$$\begin{cases} z_1 = \frac{z}{f} \cos(\theta) \\ z_2 = z \sin(\theta). \end{cases} \quad (13)$$

¹ In the case of axially symmetric lens models, the scaled factor equals the *Einstein radius* ξ_E . Furthermore, the value of the *Einstein ring angular radius* is given by $\theta_E = \xi_E/D_{\text{OL}}$.

By substituting equation (12) into equation (9), the i th component of the deflection angle reduces to

$$\hat{\alpha}_i = -\frac{1}{\pi f} \iint_I \left\{ \int_0^{+\infty} \rho' \kappa(\rho') J_0 \left(\frac{\rho' z}{f} \right) d\rho' \right\} \times \frac{F_i(\theta) \sin \left(\frac{\rho z}{f} \cos(\theta - \phi) \right)}{\cos^2(\theta) + f^2 \sin^2(\theta)} dz d\theta, \quad (14)$$

where $I = [0, +\infty[\times [0, 2]$, $i \in \{1, 2\}$ and $F_i(\theta)$ is simply defined by

$$F_i(\theta) = \begin{cases} \cos(\theta) & \text{if } i = 1 \\ f \sin(\theta) & \text{if } i = 2. \end{cases} \quad (15)$$

In the remainder part of this demonstration, we will focus our calculation on the 1st component. Let us apply the Fubini's theorem² in order to first calculate the z -integral. It turns out that, for $\rho' \geq \rho \cos(\theta - \phi)$, the z -integral vanishes. Then, for $0 < \rho' < \rho \cos(\theta - \phi)$, one finds (Gradshteyn & Ryzhik 2007, 6.671, p. 718)

$$\int_0^{+\infty} J_0 \left(\frac{\rho' z}{f} \right) \sin \left(\frac{\rho z}{f} \cos(\theta - \phi) \right) dz = \frac{f}{\sqrt{\rho^2 \cos^2(\theta - \phi) - \rho'^2}}. \quad (16)$$

After substituting equation (16) into equation (14), the latter equation can be reduced to

$$\hat{\alpha}_1 = -\frac{1}{\pi} \int_0^2 \frac{\cos(\theta)}{\cos^2(\theta) + f^2 \sin^2(\theta)} \times \left\{ \int_0^{\rho \cos(\theta - \phi)} \frac{\rho' \kappa(\rho') d\rho'}{\sqrt{\rho^2 \cos^2(\theta - \phi) - \rho'^2}} \right\} d\theta. \quad (17)$$

After applying the Fubini's theorem once again, substituting $\theta - \phi$ by Θ and calculating the adequate upper and lower limits, equation (17) reduces to

$$\hat{\alpha}_1 = -\frac{2}{\pi} \int_0^\rho \rho' \kappa(\rho') I(\rho') d\rho', \quad (18)$$

where the integral $I(\rho')$ is given by

$$I(\rho') = \int_{-l}^l \frac{\cos(\Theta + \phi)}{(\cos^2(\Theta + \phi) + f^2 \sin^2(\Theta + \phi)) \sqrt{\rho^2 \cos^2(\Theta) - \rho'^2}} d\Theta, \quad (19)$$

with $l = \arccos(\rho'/\rho)$. The resolution of the Θ -integral is not trivial. This integral can be expressed as the difference between two integrals by developing $\cos(\Theta + \phi) = \cos(\Theta)\cos(\phi) - \sin(\Theta)\sin(\phi)$. We note that for the 2nd component of the deflection angle, the term to be developed is $\sin(\Theta + \phi) = \sin(\Theta)\cos(\phi) + \cos(\Theta)\sin(\phi)$ which leads to the sum of the two same integrals over Θ but multiplied by $\sin(\phi)$ (resp. $\cos(\phi)$) instead of $\cos(\phi)$ (resp. $\sin(\phi)$). After

some mathematical developments, the Θ -integral takes the form

$$I(\rho') = \frac{\pi\sqrt{2}}{2} \text{sign}(\cos(\phi)) \left(\frac{1}{\sqrt{\lambda^2 \rho'^2 + \omega_1^2}} + \frac{1}{\sqrt{\lambda^2 \rho'^2 + \omega_2^2}} \right), \quad (20)$$

where the expressions of λ , ω_1 and ω_2 are given by

$$\lambda = \sqrt{2}\sqrt{1 - f^2}, \quad (21)$$

$$\omega_1 = \sqrt{2} \rho (f \cos(\phi) + \iota \sin(\phi)), \quad (22)$$

and

$$\omega_2 = \sqrt{2} \rho (f \cos(\phi) - \iota \sin(\phi)). \quad (23)$$

Here, we note that because $\omega_2 = \omega_1^*$, the term in brackets is real, and therefore we have $I(\rho') \in \mathbb{R}$, for any values of $0 < f \leq 1$, $\phi \in [0, 2]$ and $\rho \in [0, +\infty[$. As a consequence, even if equation (20) contains imaginary terms, the whole expression is a real valued function of the real variable $\rho' \in [0, \rho]$. The 2nd component can be similarly derived. In fact, the Θ -integral reduces, in this case, to

$$I(\rho') = \frac{\iota\pi\sqrt{2}}{2} \text{sign}(\cos(\phi)) \left(\frac{1}{\sqrt{\lambda^2 \rho'^2 + \omega_1^2}} - \frac{1}{\sqrt{\lambda^2 \rho'^2 + \omega_2^2}} \right). \quad (24)$$

The term in brackets is a purely imaginary quantity. But since the whole expression is multiplied by the imaginary unit ι , the former is a real valued function of the real variable ρ' . Finally, the expressions of the two components of the deflection angle reduce to

$$\hat{\alpha}_1(\rho, \phi) = -\sqrt{2} \text{sign}(\cos(\phi)) \times \left\{ \int_0^\rho \frac{\rho' \kappa(\rho') d\rho'}{\sqrt{\lambda^2 \rho'^2 + \omega_1^2}} + \int_0^\rho \frac{\rho' \kappa(\rho') d\rho'}{\sqrt{\lambda^2 \rho'^2 + \omega_2^2}} \right\}, \quad (25)$$

and

$$\hat{\alpha}_2(\rho, \phi) = -\iota \sqrt{2} \text{sign}(\cos(\phi)) \times \left\{ \int_0^\rho \frac{\rho' \kappa(\rho') d\rho'}{\sqrt{\lambda^2 \rho'^2 + \omega_1^2}} - \int_0^\rho \frac{\rho' \kappa(\rho') d\rho'}{\sqrt{\lambda^2 \rho'^2 + \omega_2^2}} \right\}. \quad (26)$$

Furthermore, in order to demonstrate that $\hat{\alpha}_1 = \text{Re}(I_c^*(x_c))$ and $\hat{\alpha}_2 = \text{Im}(I_c^*(x_c))$ with $I_c(x_c)$ defined by equation (2), we simply calculate $\hat{\alpha}_1 + \iota \hat{\alpha}_2$ using equations (10), (25) and (26)

$$\begin{aligned} \hat{\alpha}_1 + \iota \hat{\alpha}_2 &= -2\sqrt{2} \text{sign}(\cos(\phi)) \int_0^\rho \frac{\rho' \kappa(\rho') d\rho'}{\sqrt{\lambda^2 \rho'^2 + \omega_2^2}}, \\ &= -2 \text{sign}(x_c) \frac{1}{f} \int_0^\rho \frac{\rho' \kappa(\rho') d\rho'}{\sqrt{-(1 - \frac{1}{f^2})\rho'^2 + x_c^{*2}}}, \\ &= -2 \text{sign}(x_c) \cos(\beta) \int_0^\rho \frac{\rho' \kappa(\rho') d\rho'}{\sqrt{x_c^{*2} - \rho'^2 \sin^2(\beta)}}, \\ &= I_c^*(x_c), \end{aligned} \quad (27)$$

where $\cos(\beta) = 1/f$. Finally, we have succeeded in demonstrating that $\hat{\alpha}_1$ and $\hat{\alpha}_2$ constitute the real and imaginary parts of the complex deflection angle defined by Bourassa & Kantowski (1975).

² The Fubini's theorem sets conditions which allows the order of integration to be changed in iterated integrals.

For practical purposes, equations (25) and (26) can be used to derive the analytical expression of the two deflection angle components.

Let us first consider the family of models characterized by the dimensionless surface mass density having the form

$$\kappa(\rho) = \frac{\kappa_0}{(\rho_c^2 + \rho^2)^v}, \quad (28)$$

with κ_0 being a constant, ρ_c the core scale and v a positive real number. The latter dimensionless surface mass density includes a large class of elliptical lens models. The expressions of the components of the deflection angle deduced from equations (25) and (26) reduce to

$$\hat{\alpha}_1(\rho, \phi) = \kappa_0 \lambda^{2v-2} \sqrt{2} (\Omega(\rho, \phi, \omega_1) + \Omega(\rho, \phi, \omega_2)), \quad (29)$$

and

$$\hat{\alpha}_2(\rho, \phi) = \iota \kappa_0 \lambda^{2v-2} \sqrt{2} (\Omega(\rho, \phi, \omega_1) - \Omega(\rho, \phi, \omega_2)), \quad (30)$$

where $\Omega(\rho, \phi, \omega_i)$ is defined by

$$\begin{aligned} \Omega(\rho, \phi, \omega_i) &= \frac{\omega_i}{(\lambda^2 \rho_c^2 - \omega_i^2)^v} \\ &\times \left(-\sqrt{1 + \frac{\lambda^2 \rho^2}{\omega_i^2}} {}_2F_1 \left[\frac{1}{2}, \nu, \frac{3}{2}; -\frac{\lambda^2 \rho^2 + \omega_i^2}{\lambda^2 \rho_c^2 - \omega_i^2} \right] \right. \\ &\quad \left. + {}_2F_1 \left[\frac{1}{2}, \nu, \frac{3}{2}; -\frac{\omega_i^2}{\lambda^2 \rho_c^2 - \omega_i^2} \right] \right), \quad (31) \end{aligned}$$

with ${}_2F_1(a, b, c, z)$ representing the Gauss hypergeometric function and ω_i is defined by equations (22) and (23). We note that since $\omega_2 = \omega_1^*$, we have $\Omega(\rho, \phi, \omega_2) = \Omega^*(\rho, \phi, \omega_1)$.

Let us now consider the so-called NSIE lens model, in which dimensionless surface mass density is obtained from (28) with $\kappa_0 = \sqrt{f}/2$ and $v = 1/2$:

$$\kappa(\rho) = \frac{\sqrt{f}}{2 \sqrt{\rho_c^2 + \rho^2}}, \quad (32)$$

where ρ_c still represents the dimensionless core radius. With the help of equations (29) and (30), we directly deduce the analytical expressions for the two components of the deflection angle, without having to use the complex representation of the lens theory altogether with the results of Bourassa & Kantowski (1975) and Bray (1984). Therefore, since we have

$$\operatorname{arcsinh}(x) = \log \left(x + \sqrt{x^2 + 1} \right) \quad (33)$$

and

$$\arctan(q) = -\frac{\iota}{2} \log \left(\frac{1 + \iota q}{1 - \iota q} \right), \quad (34)$$

the components of the deflection angle can be expressed as

$$\hat{\alpha}_1(\rho, \phi) = \frac{\sqrt{f}}{f'} \operatorname{sign}(\cos(\phi)) \log \left[\frac{A}{B} \right], \quad (35)$$

and

$$\hat{\alpha}_2(\rho, \phi) = \frac{\sqrt{f}}{2f'} \operatorname{sign}(\cos(\phi)) \arctan \left(\frac{C}{D} \right), \quad (36)$$

where $f' = \sqrt{1 - f^2}$ and

$$A = \left(1 + \frac{\rho_c^2}{\rho^2} \right) - \left(f \frac{\rho_c}{\rho} - f' \cos(\phi) \operatorname{sign}(\cos(\phi)) \right)^2, \quad (37)$$

$$B = -f^2 \frac{\rho_c^2}{\rho^2} + \left(\sqrt{1 + \frac{\rho_c^2}{\rho^2}} + f' \cos(\phi) \operatorname{sign}(\cos(\phi)) \right)^2, \quad (38)$$

$$\begin{aligned} C &= f \rho^2 + f'^2 \rho_c \sqrt{\rho^2 + \rho_c^2} \\ &\quad + f' \rho \left(\rho_c + f \sqrt{\rho^2 + \rho_c^2} \cos(\phi) \operatorname{sign}(\cos(\phi)) \right), \quad (39) \end{aligned}$$

and

$$\begin{aligned} D &= \rho \sin(\phi) \left[-f'^2 \rho \cos(\phi) \right. \\ &\quad \left. + f' \left(f \rho_c - \sqrt{\rho^2 + \rho_c^2} \right) \operatorname{sign}(\cos(\phi)) \right]. \quad (40) \end{aligned}$$

We note that equations (35) and (36) are not identical to the expressions derived by Kormann, Schneider & Bartelmann (1994, equations 62a–e) but remain rigorously equivalent. However, our derived expressions of the deflection angle do not involve complex quantities, unlike those derived by Kormann et al. (1994). Once more, the reader may thus perceive the real interest of using the Fourier approach developed in this work. Of course, for the case of $\rho_c = 0$, the latter equations reduce to those for the singular isothermal ellipsoid (SIE) case. In fact, for the singular case $\rho_c = 0$, equations (29) and (30) take the simple form

$$\begin{aligned} \hat{\alpha}_1(\rho, \phi) &= -\sqrt{2} \kappa_0 \frac{\rho^{2-v}}{(2-v)} \\ &\times \left(\frac{1}{\omega_1} {}_2F_1 \left[\frac{1}{2}, 1 - \frac{\nu}{2}, 2 - \frac{\nu}{2}; -\frac{\lambda^2}{\omega_1^2} \rho^2 \right] \right. \\ &\quad \left. + \frac{1}{\omega_2} {}_2F_1 \left[\frac{1}{2}, 1 - \frac{\nu}{2}, 2 - \frac{\nu}{2}; -\frac{\lambda^2}{\omega_2^2} \rho^2 \right] \right), \quad (41) \end{aligned}$$

and

$$\begin{aligned} \hat{\alpha}_2(\rho, \phi) &= -\iota \sqrt{2} \kappa_0 \frac{\rho^{2-v}}{(2-v)} \\ &\times \left(\frac{1}{\omega_1} {}_2F_1 \left[\frac{1}{2}, 1 - \frac{\nu}{2}, 2 - \frac{\nu}{2}; -\frac{\lambda^2}{\omega_1^2} \rho^2 \right] \right. \\ &\quad \left. - \frac{1}{\omega_2} {}_2F_1 \left[\frac{1}{2}, 1 - \frac{\nu}{2}, 2 - \frac{\nu}{2}; -\frac{\lambda^2}{\omega_2^2} \rho^2 \right] \right). \quad (42) \end{aligned}$$

We note that the latter equations only remain valid for $v \in \mathbb{R}^+ \setminus \mathbb{Z} \cup \{1\}$. For the case $v = 1$ and $\kappa_0 = \sqrt{f}/2$, i.e. the SIE model, the two components of the deflection angle take the form

$$\begin{aligned} \hat{\alpha}_1(\rho, \phi) &= -\frac{\sqrt{f}}{2f'} \left(\operatorname{arcsinh} \left(\frac{f'}{f \cos(\phi) + \iota \sin(\phi)} \right) \right. \\ &\quad \left. + \operatorname{arcsinh} \left(\frac{f'}{f \cos(\phi) - \iota \sin(\phi)} \right) \right) \quad (43) \end{aligned}$$

and

$$\begin{aligned} \hat{\alpha}_2(\rho, \phi) &= -\frac{\iota \sqrt{f}}{2f'} \left(\operatorname{arcsinh} \left(\frac{f'}{f \cos(\phi) + \iota \sin(\phi)} \right) \right. \\ &\quad \left. - \operatorname{arcsinh} \left(\frac{f'}{f \cos(\phi) - \iota \sin(\phi)} \right) \right). \quad (44) \end{aligned}$$

Using the fact that $\sinh x = -i \sin(ix)$ and the addition properties of the logarithmic functions, we straightforwardly retrieve the expected expression for the deflection angle introduced by Kormann et al. (1994)

$$\hat{\alpha}(\mathbf{x}) = -\frac{\sqrt{f}}{f'} \left[\operatorname{arcsinh} \left(\frac{f'}{f} \cos(\varphi) \right) \mathbf{e}_1 + \operatorname{arcsin} \left(f' \sin(\varphi) \right) \mathbf{e}_2 \right], \quad (45)$$

where φ represents the angular coordinate defined in the polar coordinate system and \mathbf{e}_1 (resp. \mathbf{e}_2) represents the unit vector along the direction x_1 (resp. x_2).

Finally, for the case $f = 1$, i.e. for axially symmetric lenses, we note that $\lambda = 0$. Therefore, in equations (25) and (26), we deduce that $\sqrt{\lambda^2 \rho'^2 + \omega_i^2} = \sqrt{2} \rho (\cos(\phi) \pm i \sin(\phi)) \operatorname{sign}(\cos(\phi))$ is no longer a function of ρ' . As a result, the expressions for the two components of the deflection angle take the expected form

$$\hat{\alpha}_1(\rho) = -\frac{\cos(\phi)}{\pi\rho} M(\leq \rho) \quad (46)$$

and

$$\hat{\alpha}_2(\pi\rho) = -\frac{\sin(\phi)}{\pi\rho} M(\leq \rho), \quad (47)$$

where $M(\leq \rho) = 2\pi \int_0^\rho \rho' \kappa(\rho') d\rho'$. From these two last equations, we may retrieve the components of the deflection angle for any axially symmetric lenses, e.g. the singular isothermal sphere or spherical NFW lensing models.

4 CONCLUSIONS

Although it is a well-known fact that the deflection angle can be expressed as a convolution product between the dimensionless surface mass density and the simple kernel $\mathbf{x}/|\mathbf{x}|^2$, its analytical derivation using the Fourier analysis in a systematic way proves to be a very efficient and appropriate alternative method. For this purpose, we have presented the derivation of the expression for the two components of the deflection angle using the Fourier formalism. From basic theorems of the Fourier and integrals analysis, we have derived the expression of the two components of the deflection angle for the case of homoeoidal symmetric lenses. This result is consistent with the one already derived by Bourassa & Kantowski (1975) but, in our case, we have obtained the expression of the two components separately instead of a unique equation expressed in the complex formalism. As a consequence, equations (25) and (26) turn out to be

very simple and useful in order to calculate the components of the deflection angle. Indeed, for the case of the NSIE, we have derived more simple expressions for the two components of the deflection angle than those previously derived by Kormann et al. (1994). More generally speaking, it should also be straightforward to derive the expression of the deflection angle for the case of a pixelated lens mass distribution such that each pixel in the lens plane represents a constant area with a fixed surface mass density. Following such an analytical approach, it should be easy and convenient to model any unknown gravitational lens mass distribution. By means of such an expression for the deflection angle, it should also be straightforward to analytically determine the values of the expected time delays between pairs of lensed images.

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