Bootstrap Equations for String-Like Amplitude

Kirill Semenov-Tian-Shansky
St. Petersburg State University, Université de Liège au Sart Tilman
E-mail: semenov@pdmi.ras.ru

Abstract

One of the ways to check the consistency of our effective field theory (EFT) approach (see [1]) is to perform the numerical testing of those sum rules for hadron resonance parameters which follow from the system of bootstrap constrains. In this talk we discuss the peculiar features of this procedure for the case of exactly solvable bootstrap model based on Veneziano string amplitude. This allows us to simulate different situations that may encounter in realistic EFT models. We also make a short review of the technique that may be useful for further analysis of various bootstrap systems.

1 Introduction

In papers [2]-[4] an attempt is made to develop an effective field theory formalism suitable for description of hadronic scattering processes at low energies (see also [1] and ref. therein). It was shown that the natural requirements of self-consistency of the perturbation series for scattering amplitudes lead to an infinite system of restrictions (called *bootstrap constraints*) for the physical parameters of a theory. Once solved the bootstrap system would permit to uncover the set of truly independent physical parameters of the effective field theory. Unfortunately, for the present we are unable to point out an explicit solution to the bootstrap system. So, roughly speaking, the only way to check the consistency of our effective theory approach is to perform the numerical testing of the relations (sum rules) for physical parameters of the theory that follow from the bootstrap system. In other words, we need to check if it is possible to approximately saturate those sum rules with the finite number of experimentally known resonances. This kind of verification was done for the cases of πK, πN and KN systems (see ref. in [1]).

In each case it proved to be possible to single out certain groups of sum rules that are very well saturated with known data on the resonance spectrum. At the same time, there were found sum rules which cannot be satisfactorily saturated with the same set of data. Clearly, one of the reasons is that the modern information on hadron spectrum is far from being exhaustive (especially in the region $M > 2$ GeV). That is why only those sum rules that converge sufficiently rapidly could undergo the experimental verification and posses certain predictive power. So it would be extremely instructive to learn how to pick out such sum rules from the infinite system of bootstrap constraints.

The problem is that even the simplest (tree level) bootstrap system for the physical parameters of $2 \to 2$ hadron scattering looks very intricate and awkward. Its analysis is complicated by multiple technical obstacles such as the non-trivial geometry of Mandelstam complex plane, the presence of various symmetries and bulky expressions corresponding to contributions of the high spin resonances. That is why it seems reasonable to simulate different situations that may be encountered in realistic numerical analysis. For this we need to consider a simple model example. A good candidate for such tests is the "toy bootstrap model" (discussed previously in [3]). Surely, the main reason to choose it is that this model is exactly solvable, in a sense that in this case we have in hands the closed analytical expression for the amplitude that satisfies the system of bootstrap constraints.

In this talk we consider the numerical test of sum rules for this bootstrap model and discuss the methods of accelerating their convergency. We will also introduce the matrix methods that (as show our preliminary investigations) may be useful for the further analysis of bootstrap systems.

2 Veneziano String Amplitude: Cauchy Forms

We consider a simple model for the scattering amplitude that is constructed — in accordance with

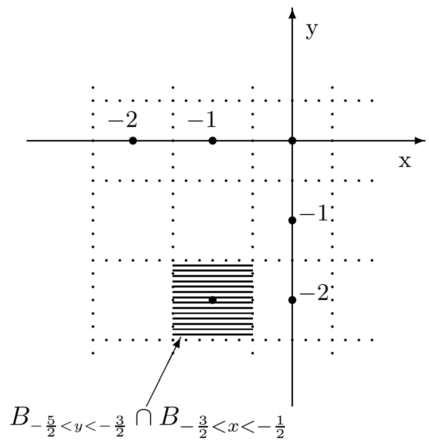

Figure 1: *The plane of intersection of different layers B_x, B_y*

the idea of Veneziano [5] – out of B-function without a tachyon:

$$A(x,y) = (-x-y)B(\tfrac{1}{2}-x, \tfrac{1}{2}-y) = \frac{\Gamma(\frac{1}{2}-x)\Gamma(\frac{1}{2}-y)}{\Gamma(-x-y)}. \tag{1}$$

In this Section we give a short summary of the properties of this function.

The string amplitude has the following specific points (hyperplanes) $(m, n = 0, 1, 2, ...)$:

- Zero hyperplanes: $x + y = n$.
- Pole hyperplanes in x (y fixed, $x + y \neq m$): $x = \frac{1}{2} + n$.
- Pole hyperplanes in y (x fixed, $x + y \neq m$): $y = \frac{1}{2} + n$.
- There are also three series of ambiguity points located at the intersections of the zero hyperplanes with the pole hyperplanes in any variable.

It can be shown that in the system of layers

$$B_{n-\frac{1}{2}<y<n+\frac{1}{2}} \quad n = 0, \pm 1, ...$$

the amplitude $A(x,y)$ is the n-bounded[1] function of one complex variable x and of one real parameter y. In the system of layers

$$B_{n-\frac{1}{2}<x<n+\frac{1}{2}} \quad n = 0, \pm 1, ...$$

$A(x,y)$ has the analogous asymptotic behavior. The layers B_x and B_y intersect as it is shown in Figure 1.

The residues of $A(x,y)$ at poles in variable x are given by the expression

$$r_n(y) \equiv Res\big|_{x=n+\frac{1}{2}} A(x,y) = \frac{1}{n!}(\tfrac{1}{2}+y)\cdots(\tfrac{1}{2}+y+n) \equiv \frac{1}{n!}\,(y+\tfrac{1}{2})_{(n+1)}, \quad n = 0, 1, ..., \tag{2}$$

were $(y+\frac{1}{2})_{(n+1)}$ stands for the so-called Pochhammer symbol (shifted factorial). Residues at poles in y read:

$$\rho_n(x) \equiv Res\big|_{y=n+\frac{1}{2}} A(x,y) = \frac{1}{n!}\,(x+\tfrac{1}{2})_{(n+1)}, \quad n = 0, 1, ... \tag{3}$$

It is often said that the string amplitude is constructed solely from the pole contributions of one channel:

$$A(x,y) = \begin{bmatrix}\text{Smooth}\\ \text{background}\end{bmatrix} + \sum_n \begin{bmatrix}\text{Pole}\\ \text{contributions}\end{bmatrix}.$$

[1] In the sense of contour asymptotics; see [2], [3].

However in [2] it was shown that the meaning of this statement needs to be refined. The technique of Cauchy forms known from the complex analysis allows one to represent $A(x,y)$ in the system of layers B_x, B_y as the uniformly converging series of the following structure:

$$A(x,y) = \begin{bmatrix} \text{Smooth} \\ \text{background} \end{bmatrix} + \sum_n \begin{bmatrix} \text{Pole} \\ \text{contributions} \end{bmatrix} + \begin{bmatrix} \text{Correcting} \\ \text{polynomials} \end{bmatrix}.$$

This is not just a sum of poles in one channel, because the cross-channel poles make explicit contributions to the correcting polynomials as well as to background term. The degree of the correcting polynomial depends on the asymptotic behavior in a given layer.

For example, in the layer $B_{-\frac{1}{2}<y<\frac{1}{2}}$ the asymptotic behavior of $A(x,y)$ corresponds to zero value of the bounding polynomial degree (the amplitude grows slower than a linear function of x but faster than a constant). Hence in the Cauchy expansion we have to take account of the correcting polynomials of $0-$th degree. Thus we obtain the following series:

$$A(x,y) = \alpha_y(y) + \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{(y+\frac{1}{2})_{(n+1)}}{x-n-\frac{1}{2}} + \frac{(y+\frac{1}{2})_{(n+1)}}{n+\frac{1}{2}} \right), \qquad y \in (-\tfrac{1}{2},\ \tfrac{1}{2}). \tag{4}$$

In the layer $B_{-\frac{3}{2}<y<-\frac{1}{2}}$ $A(x,y)$ grows slower than a constant. So, we can write down the Cauchy expansion without correcting polynomials:

$$A(x,y) = \sum_{n=0}^{\infty} \frac{r_n(y)}{(x-n-\frac{1}{2})}\ , \qquad y \in (-\tfrac{3}{2}, -\tfrac{1}{2}). \tag{5}$$

Note, that because the asymptotics becomes "softer" at large negative y, this expansion is also valid in every layer[2] corresponding to $y < -\frac{3}{2}$. The similar Cauchy form can be written for $A(x,y)$ in the layer $B_{-\frac{3}{2}<x<-\frac{1}{2}}$:

$$A(x,y) = \sum_{n=0}^{\infty} \frac{\rho_n(x)}{(y-n-\frac{1}{2})}\ , \qquad x \in (-\tfrac{3}{2}, -\tfrac{1}{2}). \tag{6}$$

3 Bootstrap for the String Amplitude

The full system of bootstrap equations for the string amplitude contains three types of conditions:

- The consistency conditions that arise naturally from the requirement that the Cauchy form in one variable valid in certain layer can be analytically continued into the perpendicular layer. For this reason two Cauchy forms in different variables written in two intersecting layers should coincide with one another in the intersection domain.

  E.g., in the intersection domain $B_{-\frac{5}{2}<y<-\frac{3}{2}} \cap B_{-\frac{3}{2}<x<-\frac{1}{2}}$ (see Fig. 1) we obtain the following condition:

$$\Psi_{-1,-1}(x,y) = \sum_{n=0}^{\infty} \frac{\rho_n(x)}{y-n-\frac{1}{2}} - \sum_{n=0}^{\infty} \frac{r_n(y)}{x-n-\frac{1}{2}} \equiv 0, \quad x \sim -1,\ y \sim -2. \tag{7}$$

  With the help of generating function $\Psi_{-1,-1}(x,y)$ this condition can be rewritten as follows:

$$\frac{\partial^{k+p}}{\partial x^k \partial y^p} \Psi_{-1,-1}(x,y) \Big|_{\substack{x=-1 \\ y=-2}} = 0, \qquad \forall\ k,p = 0,1,\dots\ . \tag{8}$$

- The *collapsing conditions* for excessive degrees of the correcting polynomials.

[2] Except the values corresponding to the coordinates of ambiguity points.

E.g., the Cauchy series (4) written for $A(x,y)$ in the layer $B_{-\frac{1}{2}<y<\frac{1}{2}}$ is also valid in the lower layers with softer asymptotics. However in those layers this expansion should coincide with (5) and we obtain the corresponding collapsing condition:

$$\alpha(y) = -\sum_{n=0}^{\infty} \frac{r_n(y)}{n+\frac{1}{2}}, \quad y < -\tfrac{1}{2}.$$

- The *superconvergency conditions* in the layers with rapidly decreasing asymptotics.

  E.g., in the layer $B_{-\frac{7}{2}<y<-\frac{5}{2}}$ the string amplitude is the (-2)-bounded function of the complex variable x. Thus one can write down the collapsing condition of the previous type for the function $xA(x,y)$ that is -1-bounded in this layer. It reads:

$$\sum_{n=0}^{\infty} r_n(y) = 0, \quad y < -\tfrac{5}{2}.$$

This system is by no doubt badly overdetermined and the problem of pointing out its full subsystem rests a serious challenge. We will return to this question in Sec. 5.

4 Numerical Tests

The numerical testing of sum rules for the resonance parameters is one of the ways to check the consistency of our EFT approach. In this Section we consider various situations that may encounter in realistic EFT models by analyzing the numerical tests of sum rules following from the above-described "toy bootstrap model".

To perform these numerical tests we first need to define a quantity that would allow us to characterize the accuracy of saturation of a given sum rule after taking into account the finite number of items. Because of the symmetry of spectrum, it looks natural to consider the difference of the contributions from the B_x-layer and B_y-layer poles at every step of the computation.

Usually the contributions of the first few poles have a definite (say, positive) sign but, starting from a certain number (N_+), the sign changes. Thus the saturation of a sum rule is provided by the negative contribution of the long tail of distant poles. It compensates gradually the positive contribution of the first few items. As the convergence characteristics we chose the ratio:

$$D(N) \equiv \frac{\Delta S(N)}{S_+} \cdot 100\%,$$

where $\Delta S(N)$ is the discrepancy that remains after one considers N poles in the B_x-layer and N in B_y-layer; S_+ is the sum *of all* positive contributions (those corresponding to the finite number of the first terms)[3].

As an example we perform the numerical testing of the sum rule that follows from the bootstrap condition (8) with different p and k. Certain sum rules of this group are saturated very fast. For example, for $k=0$, $p=2$ first 10 poles saturate the sum rule with 10% discrepancy. The sum rule with $k=0$, $p=1$ saturates even faster: first 10 poles give 3% discrepancy.

It is interesting to note that the scenario of saturation of this sum rule is in accordance with the so-called *local cancellation hypothesis* (see [6]) employed in phenomenological models. It says, that the properties of the amplitude are mostly defined by the resonances with masses close to the energy scale under consideration and, possibly, by nonsingular background. The contributions from all the other resonances "almost cancel among themselves". The fast saturating sum rules provide us with powerful constrains for the parameters of first few poles. The hope is that the reliable sum rules which were found for $\pi N, \pi K$ and KN systems are exactly of this type and, hence, they can be used to study the hadron spectrum.

[3]Certainly it only makes sense for $N > N_+$.

However, we will show that the fast saturation is not always the case and in certain situations the important properties of the amplitude may depend on the large number of distant poles. For this let us consider the sum rule that follows from (8) with $k = 0$, $p = 0$:

$$\Psi_{-1,-1}(x, y)\Big|_{\substack{x=-1\\y=-2}} = 0. \tag{9}$$

The convergence of this sum rule turns out to be very slow. The dependence of the relative discrepancy D on the number of poles taken into account is shown in Fig. 2.

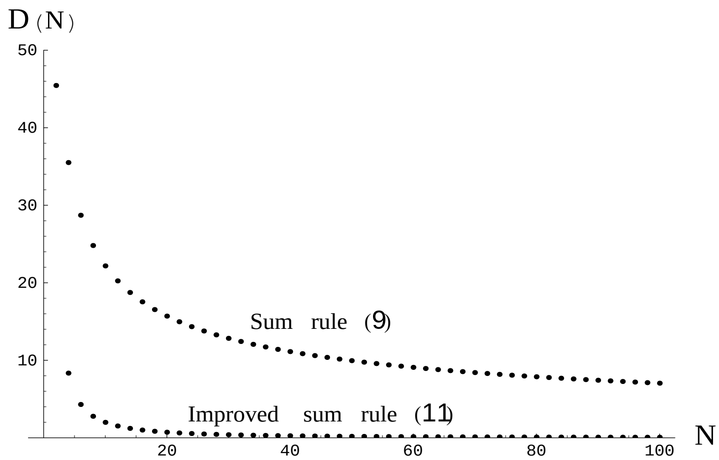

Figure 2: Saturation of sum rule (9) and of improved sum rule (11).

To saturate this sum rule with high accuracy on has to use the detailed information about the resonance spectrum (poles and residues). For example, to saturate it with 99% accuracy (1% discrepancy) one has to take into account more than 5000 poles.

Now we would like to discuss a way that allows one to modify this sum rule so that it will be possible to saturated it with smaller number of poles. The trick we are using is very similar to the conventional method of accelerating the convergency: the partial summation of converging series. Instead of (5) and (6) representing $A(x, y)$ in $B_{\frac{5}{2}<y<\frac{3}{2}}$ and $B_{-\frac{3}{2}<x<-\frac{1}{2}}$ we now make use the Cauchy forms with excessive degree of the correcting polynomial.

The bootstrap conditions in $B_{-\frac{5}{2}<y<-\frac{3}{2}} \cap B_{-\frac{3}{2}<x<-\frac{1}{2}}$ then read:

$$\begin{aligned} &\alpha_x(x) - \alpha_y(y) + \Psi_{0,0}(x, y) = \\ &\alpha_x(x) - \alpha_y(y) + \sum_{n=0}^{\infty}\left(\frac{\rho_n(x)}{y - n - \frac{1}{2}} + \frac{\rho_n(x)}{n + \frac{1}{2}}\right) - \sum_{n=0}^{\infty}\left(\frac{r_n(y)}{x - n - \frac{1}{2}} + \frac{r_n(y)}{n + \frac{1}{2}}\right) \equiv 0, \quad x \sim -1,\ y \sim -2. \end{aligned} \tag{10}$$

We consider the sum rule:

$$\alpha_x(x)|_{x=-1} - \alpha_y(y)|_{y=-2} + \Psi_{0,0}(x, y)\Big|_{\substack{x=-1\\y=-2}} = 0. \tag{11}$$

The smooth parts of the amplitude (background terms) $\alpha_x(x)$ and $\alpha_y(y)$ can be found, for example, from the bootstrap conditions in another layers. From $B_{-\frac{5}{2}<x<-\frac{3}{2}} \cap B_{-\frac{5}{2}<y<-\frac{3}{2}}$ (using (4), (6)) we find:

$$\alpha_y(y) = \left[\sum_{n=0}^{\infty}\frac{\rho_n(x)}{(y - n - \frac{1}{2})} - \sum_{n=0}^{\infty}\left(\frac{r_n(y)}{x - n - \frac{1}{2}} + \frac{r_n(y)}{n + \frac{1}{2}}\right)\right]_{x=-2}. \tag{12}$$

In the similar way from $B_{-\frac{3}{2}<x<-\frac{1}{2}} \cap B_{-\frac{7}{2}<y<-\frac{5}{2}}$:

$$\alpha_x(x) = \left[\sum_{n=0}^{\infty}\frac{r_n(y)}{(x - n - \frac{1}{2})} - \sum_{n=0}^{\infty}\left(\frac{\rho_n(x)}{y - n - \frac{1}{2}} + \frac{\rho_n(x)}{n + \frac{1}{2}}\right)\right]_{y=-3}. \tag{13}$$

The sum rule (11) saturates very fast: taking into account of just 10 first poles one obtains 98% accuracy. The contribution of distant poles (in both variables) is negligible. So, in principle, from this sum rule we also can obtain the constrains for the values of resonance parameters of few first poles.

The studies of possibility to apply the methods of convergency acceleration for the case of realistic EFT models describing hadron scattering to cure certain "bad" sum rules and to derive the fastly saturated sum rules are now in progress.

5 Matrix Methods

The problem of analysis of the system of bootstrap constrains is closely linked to the general theory of analytic continuation. In fact, the bootstrap system expresses the possibility to carry out the analytic continuation of the meromorphic function[4] with given asymptotic regime from one layer to the perpendicular one. Bootstrap constrains result in a set of restriction for the resonance parameters of a theory. One of the most important (and difficult) problems appearing in our EFT approach is formulated as follows: "Should we take into account the bootstrap constraints in all the intersection domains of various layers? Do the bootstrap conditions in the intersection domains of other layers impose any new restrictions on the parameters of the theory in addition to those following from the bootstrap system in a given intersection domain, or they are just a consequence of the bootstrap in one domain?". To study this and the related problems we introduce matrix methods that are of great use in general theory of analytic continuation.

Let us consider the analytic continuation of the analytic (in certain domain D) function of two complex variables. The coefficients a_{ij} of the double Taylor series in the vicinity of (x_1, y_1):

$$F(x,y) = \sum_{i,j} a_{ij}(x_1,y_1)(x-x_1)^i(y-y_1)^j \quad x \sim x_1 \quad y \sim y_1$$

are linked to the coefficients $\tilde{a}_{ij}$ of the expansion around (x_2, y_2) by way of certain matrix operation $\hat{T}(x_2, y_2; x_1, y_1)$:

$$\tilde{a}_{ij}(x_2,y_2) = \sum_{k,l} \hat{T}_{ijkl}(x_2,y_2;x_1,y_1)a_{kl}(x_1,y_1).$$

In fact, the matrix $\hat{T}$ contains the complete information about non-trivial analytic properties of the function F. On the other hand it is this matrix that transforms bootstrap conditions written in the intersection domain of two given layers into the bootstrap system in other domains.

For string amplitude it is possible to present the specific matrix of analytic continuation from, say, (x_1, y_1) to (x_1+1, y_1). This matrix mirrors the recurrent properties of the B-function and transforms the bootstrap system at (x_1, y_1) to another bootstrap system at (x_1+1, y_1):

$$\tilde{a}_{ij}(x_1+1,y_1) = \sum_{k=0}^{i}\sum_{l=0}^{j} \hat{T}_{ijkl}(x_1+1,y_1;x_1,y_1)a_{kl}(x_1,y_1) \ , \tag{14}$$

where

$$\hat{T}_{ijkl}(x_1+1,y_1;x_1,y_1) \equiv \frac{1}{(i-k)!(j-l)!}\left[\frac{\partial^{i+j-k-l}}{\partial x^{i-k}\partial y^{j-l}}\left(\frac{x+y+1}{x+\frac{1}{2}}\right)\right]\Bigg|_{(x_1,y_1)} \ ; \tag{15}$$

a_{ij} and $\tilde{a}_{kl}$ stand for the sets of Taylor expansion coefficients in the vicinity of (x_1, y_1) and (x_1+1, y_1) respectively. The bootstrap system requires that in the domain of intersection of layers the set of Taylor coefficients calculated from the Cauchy expansion in one layer should coincide with that calculated from the Cauchy expansion in the perpendicular layer. So it is exactly the matrix (15) acting as shown in (14) that allows one to pass from the bootstrap system in a given domain to the system in the neighboring domains.

Employing the information on the asymptotic behavior of the string amplitude that is contained in collapsing and superconvergence conditions (see Sec. 3), it turns out possible to show that – due to

[4]This only relates to the lowest order (tree level) constraints. When considering the higher level systems one deals with another kinds of singular points.

recurrent properties of the string amplitude – the bootstrap systems appearing in all the intersection domains of various layers are, in fact, equivalent to one another.

It is possible that the application of methods of the general theory of analytic continuation and of the technique of infinite dimensional matrices for the case of realistic EFT models would also help to single out the independent subsystem of the full system of bootstrap constraints.

6 Conclusions

The exactly solvable “toy bootstrap model” based on Veneziano string amplitude is an outstanding proving ground for the development of methods to handle the bootstrap systems. The hope is that the results we obtained in the framework of this model will deepen our understanding of analytical properties of scattering amplitudes in the EFT approach.

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