Higher symmetries of the conformal Laplacian

F. Radoux, ULg (IAP DYGEST)

Joint work with J.-P. Michel (ULg, IAP DYGEST) and J. Silhan (Masaryk University in Brno)

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Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

Introduction

- On $(\mathbb{R}^2, g_0)$, we consider the Schrödinger equation

$$\Delta\phi = E\phi,$$

where

$$\Delta = \partial_x^2 + \partial_y^2, \quad E \in \mathbb{R}.$$

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Second order conformal symmetries of Δ_Y
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Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
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$$\Delta = \partial_x^2 + \partial_y^2, \quad E \in \mathbb{R}.$$

- Coordinates (u, v) separate this equation $\Longleftrightarrow \exists$ solution of the form $f(u)g(v)$
- Coordinates (u, v) orthogonal $\Longleftrightarrow g_0(\partial_u, \partial_v) = 0$

- There exist 4 families of orthogonal separating coordinates systems :

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

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4. Elliptic coordinates (α, β) :

$$\begin{cases} x = \sqrt{d}\cos(\alpha)\cosh(\beta) \\ y = \sqrt{d}\sin(\alpha)\sinh(\beta) \end{cases}$$

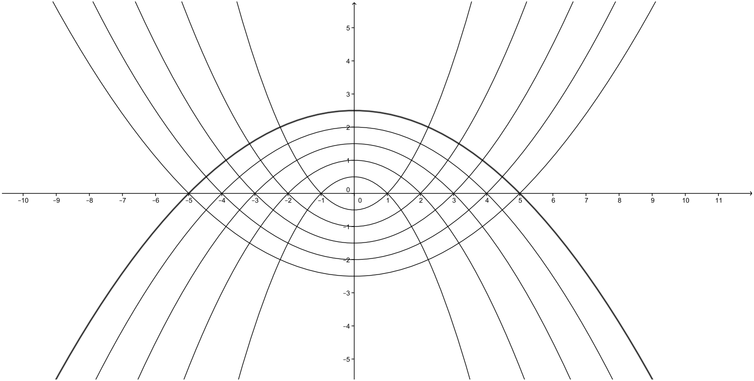

Figure: Coordinates lines corresponding to the parabolic coordinates system

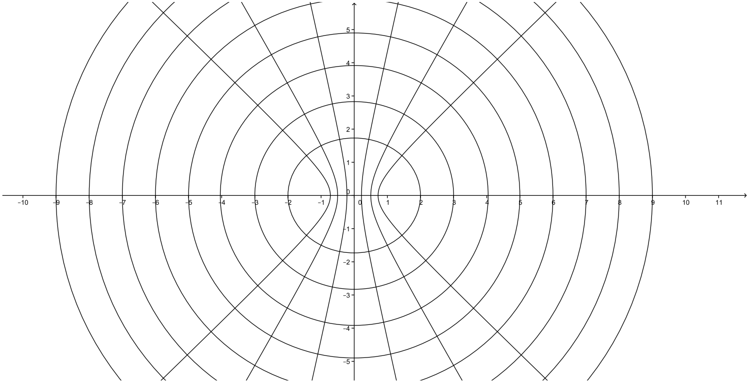

Figure: Coordinates lines corresponding to the elliptic coordinates system

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

- Separating coordinates systems allow to simplify the resolution of the Schrödinger equation :

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- Example : in cartesian coordinates (x, y), $f(x)g(y)$ is a solution of $\Delta\phi = E\phi$ iff

$$\begin{cases} \partial_x^2 f - E_1 f = 0 \\ \partial_y^2 g - (E - E_1)g = 0 \end{cases}$$

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

- Bijective correspondence

$$\{\text{Separating coordinates systems}\}$$

$$\longleftrightarrow$$

$$\{\text{Second order symmetries of } \Delta : \text{second order differential operators } D \text{ such that } [\Delta, D] = 0\}$$

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Coordinates system	Symmetry
(x, y)	∂_x^2
(r, θ)	L_θ^2
(ξ, η)	$\frac{1}{2}(\partial_x L_\theta + L_\theta \partial_x)$
(α, β)	$L_\theta^2 + d\partial_x^2$

with $L_\theta = x\partial_y - y\partial_x$

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries

Examples
DiPirro system
Conformal Stäckel metrics in dimension 3

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- Link between the symmetry and the coordinates system : if the second-order part of D reads as

$$\begin{pmatrix} \partial_x & \partial_y \end{pmatrix} A \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix},$$

the eigenvectors of A are tangent to the coordinates lines.

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- Example : second-order part of L_θ^2 :

$$\begin{pmatrix} \partial_x & \partial_y \end{pmatrix} \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix},$$

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eigenvectors of A in this case :

$$\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} -y \\ x \end{pmatrix}$$

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
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- On a n-dimensional pseudo-Riemannian manifold (M, g),

$$\Delta_Y := \nabla_i \mathrm{g}^{ij} \nabla_j - \frac{n-2}{4(n-1)} \mathrm{Sc},$$

where Sc is the scalar curvature of g.

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

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- Symmetry of Δ_Y : $D \in \mathcal{D}(M)$ such that $[\Delta_Y, D] = 0$
- Conformal symmetry of Δ_Y : $D_1 \in \mathcal{D}(M)$ such that $\exists D_2 \in \mathcal{D}(M)$ such that $\Delta_Y \circ D_1 = D_2 \circ \Delta_Y$

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

- (M, g) conformally flat : for each $x \in M$, there exist a neighborhood U of x and a function f on U such that $e^{2f} g$ is flat on U

  Conformal symmetries of Δ_Y known (M. Eastwood, J.-P. Michel)

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

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  Conformal symmetries of Δ_Y known (M. Eastwood, J.-P. Michel)
- (M, g) Einstein : $\mathrm{Ric} = \frac{1}{n}\mathrm{Sc}\ \mathrm{g}$
  Existence of a second order symmetry (B. Carter)

1 Second order conformal symmetries of Δ_Y
- Conformal Killing tensors
- Structure of the conformal symmetries

2 Examples
- DiPirro system
- Conformal Stäckel metrics in dimension 3

3 Application to the R-separation

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- $\sigma(\Delta_Y) = H = \mathrm{g}^{ij} p_i p_j$, then $\{H, \sigma(D)\} \in (H)$, i.e. $\sigma(D)$ is a conformal Killing tensor

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

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- The existence of a (conformal) Killing tensor is necessary to have the existence of a (conformal) symmetry of Δ_Y
- Is this condition sufficient ?

- If K is a (conformal) Killing tensor of degree 2, there exists a (conformal) symmetry of Δ_Y with K as principal symbol iff $\mathrm{Obs}(K)^\flat$ is an exact one-form, where

$$\mathrm{Obs} = \frac{2(n-2)}{3(n+1)} p_i \partial_{p_j} \partial_{p_l} \left(C^k{}_{jl}{}^i \nabla_k - 3 A_{jl}{}^i \right)$$

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- C : Weyl tensor :

$$C_{abcd} = R_{abcd} - \frac{2}{n-2} (\mathrm{g}_{a[c} \mathrm{Ric}_{d]b} - \mathrm{g}_{b[c} \mathrm{Ric}_{d]a}) + \frac{2}{(n-1)(n-2)} \mathrm{Sc}\, \mathrm{g}_{a[c} \mathrm{g}_{d]b}$$

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- A : Cotton-York tensor :

$$A_{ijk} = \nabla_k \mathrm{Ric}_{ij} - \nabla_j \mathrm{Ric}_{ik} + \frac{1}{2(n-1)} \left(\nabla_j \mathrm{Sc}\ g_{ik} - \nabla_k \mathrm{Sc}\ g_{ij} \right)$$

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

- If $\mathrm{Obs}(K)^\flat = 2df$, the (conformal) symmetries of Δ_Y whose the principal symbol is given by K are of the form

$$\mathcal{Q}(K) - f + L_X + c,$$

where X is a (conformal) Killing vector field, where $c \in \mathbb{R}$ and where $\mathcal{Q}$ denotes the natural and conformally invariant quantization (see works by P. Mathonet, F. Radoux, A. Cap, J. Šilhan).

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  Hamiltonian $H = \mathrm{g}^{ij} p_i p_j$:

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  Killing tensor K :

$$\frac{c(x_3)a(x_1,x_2)p_1^2+c(x_3)b(x_1,x_2)p_2^2-\gamma(x_1,x_2)p_3^2}{\gamma(x_1,x_2)+c(x_3)},$$

  $a, b, \gamma \in C^\infty(\mathbb{R}^2)$, $c \in C^\infty(\mathbb{R})$.

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  $a, b, \gamma \in C^\infty(\mathbb{R}^2)$, $c \in C^\infty(\mathbb{R})$.
- In this situation, $\mathrm{Obs}(K)^\flat$ exact$\Rightarrow$existence of symmetries.

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

- Conformal Stäckel metric g : g s.t. the Hamilton-Jacobi equation

$$\mathrm{g}^{ij}(\partial_i W)(\partial_j W) = 0$$

admits additive separation in an orthogonal coordinate system.

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- Coordinate x ignorable for g : ∂_x is a conformal Killing vector field for g.
- If g admits one ignorable coordinate x_1, then
$$\mathrm{g} = Q\left(dx_1^2 + (u(x_2) + v(x_3))(dx_2^2 + dx_3^2)\right).$$

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

- ∂_{x_1} is a conformal Killing vector field and

$$K = (u(x_2) + v(x_3))^{-1}(v(x_3)p_2^2 - u(x_2)p_3^2)$$

a conformal Killing 2-tensor.

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

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  a conformal Killing 2-tensor.
- In general, $\mathrm{Obs}(K)^\flat$ not closed$\Rightarrow$no conformal symmetries with principal symbol K.

- Schrödinger equation : $(\Delta_Y + V)\psi = E\psi$, $V \in C^\infty(M)$ is a fixed potential and $E \in \mathbb{R}$ a free parameter

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

- Schrödinger equation : $(\Delta_Y + V)\psi = E\psi$, $V \in C^\infty(M)$ is a fixed potential and $E \in \mathbb{R}$ a free parameter
- Solving Schrödinger equation : finding a solution for all E

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

- Schrödinger equation : $(\Delta_Y + V)\psi = E\psi$, $V \in C^\infty(M)$ is a fixed potential and $E \in \mathbb{R}$ a free parameter
- Solving Schrödinger equation : finding a solution for all E
- Schrödinger equation at zero energy : $(\Delta_Y + V)\psi = 0$, $V \in C^\infty(M)$ is a fixed potential

- Schrödinger equation at zero energy R-separable in an orthogonal coordinates system (x^i) ($\mathrm{g}_{ij} = 0$ if $i \neq j$)

$$\Longleftrightarrow$$

$\exists$ $n+1$ functions $R, h_i \in C^\infty(M)$ and n differential operators $L_i := \partial_i^2 + l_i(x^i)\partial_i + m_i(x^i)$ such that

$$R^{-1}(\Delta_Y + V)R = \sum_{i=1}^{n} h_i L_i.$$

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$\forall\ E \in \mathbb{R}$, $\exists\ n+1$ functions $R, h_i \in C^\infty(M)$ and n differential operators $L_i := \partial_i^2 + l_i(x^i)\partial_i + m_i(x^i)$ such that

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- $R\prod_{i=1}^{n}\phi_i(x^i)$ solution of one of the two previous equations

$$\Longleftrightarrow$$

$$L_i\phi_i = 0 \quad \forall i$$

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  (b) For all $K \in \mathcal{I}$, $\exists$ second order conformal symmetry D, i.e. an operator such that $[\Delta_Y + V, D] \in (\Delta_Y + V)$, with principal symbol $\sigma_2(D) = K$.

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Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

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- Link between the (conformal) symmetries and the R-separating coordinate systems :

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

- Link between the (conformal) symmetries and the R-separating coordinate systems :
- Hyperplans orthogonal to the eigenvectors of the tensors in $\mathcal{I} \longleftrightarrow$ integrable distributions

Second order conformal symmetries of Δ_Y
Conformal Killing tensors
Structure of the conformal symmetries
Examples
DiPirro system
Conformal Stäckel metrics in dimension 3
Application to

- Link between the (conformal) symmetries and the R-separating coordinate systems :
- Hyperplans orthogonal to the eigenvectors of the tensors in $\mathcal{I} \longleftrightarrow$ integrable distributions
- Leaves of the corresponding foliations$\longleftrightarrow$Coordinate hyperplans of the R-separating coordinate systems