Electromagnetic properties for arbitrary spin particles: Natural electromagnetic moments from light-cone arguments

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(Received 4 February 2009; revised manuscript received 15 May 2009; published 24 June 2009)

We revisit the old-standing problem of the electromagnetic interaction for particles of arbitrary spin. Based on the assumption that light-cone helicity at tree level and $Q^2 = 0$ should be conserved nontrivially by the electromagnetic interaction, we are able to derive *all* the natural electromagnetic moments for a pointlike particle of *any* spin. We provide here a transparent decomposition of the electromagnetic current in terms of covariant vertex functions. We also define in a general way the electromagnetic multipole form factors, and show their relation with the electromagnetic moments and covariant vertex functions. The light-cone helicity conservation argument determines *uniquely* the values of all electromagnetic moments, which we refer to as the ''natural'' ones. These specific values are in accordance with the standard model, and the prediction of universal $g = 2$ gyromagnetic factor is naturally recovered. We provide a very simple and compact formula for these natural moments. As an application of our results, we generalize the discussion of quark transverse charge densities to particles with arbitrary spin, giving more physical support to the light-cone helicity conservation argument.

DOI: 10.1103/PhysRevD.79.113011 PACS numbers: 13.40.Em, 13.88.+e

I. INTRODUCTION

High-energy physics involves high-spin particles for many reasons. Among those, let us mention that

(i) experimentally, more than 50 baryons with spins ranging from 3/2 to 15/2 and about the same number of mesons with spins ranging from 2 to 6 have been observed [1];

(ii) these resonances enter, e.g. as intermediate states in the description of the photo- and electro-pion production off protons, which are a main focus of experiments at electron facilities such as Jefferson Lab, ELSA, MAMI [2], or are produced at e^+e^- collider facilities such as DAΦNE, BELLE, *BABAR*, and BES;

(iii) in proposals for physics beyond the standard model (SM) based on supersymmetry, which will be explored soon by the LHC, elementary particles with high spin (e.g. the gravitino) are required.

For further motivations to study high-spin particles, see e.g. [3] and references therein.

Formalisms to study high-spin particles have been proposed a long time ago. Dirac, Fierz, and Pauli were the first to develop a general theory of particles with arbitrary spin [4], but this formalism is quite complicated. Later, Rarita and Schwinger (RS) proposed a more convenient formalism for arbitrary half-integer spin particles [5], which is still the most often used to date in the literature. In the case of integer spin particles, the Klein-Gordon (KG) equation with subsidiary conditions is more suited. Actually, it has been shown by Moldauer and Case that both RS and KG approaches can be derived from the Dirac, Fierz, and Pauli formalism [6]. There exists another formalism due to Bargmann and Wigner [7] from which one can also derive the RS and KG equations together with their subsidiary conditions [8].

Theories with spin >1 are however plagued by arbitrary parameters in both their Lagrangian and propagator [6,9]. This is due to the fact that the corresponding fields contain extra degrees of freedom related to lower spins. To eliminate these extra degrees of freedom, one imposes constraints (the subsidiary conditions in RS and KG formalisms). Unfortunately, interactions usually break these constraints, generating many problems. Among the various difficulties [10], let us mention an important result of Velo and Zwanziger [11] that showed that a massive c-number spin-3/2 field minimally coupled to an external electromagnetic field will lead to the existence of tachyons, and therefore to noncausality. The causality issue has been discussed, e.g. in [3,12].

The gyromagnetic factor of elementary particles has also a long history. Based on the minimal substitution $p^\mu \longrightarrow \pi^\mu \equiv p^\mu - eA^\mu$, it was argued that the gyromagnetic factor at tree level depends on the spin j of the particle [6,13] as

$$g_j = \frac{1}{j}, \tag{1}$$

i.e. the magnetic moment depends only on the mass and electric charge of the particle. This result agrees with the case $j = 1/2$ but seems to disagree with higher-spin cases. Many reasons suggest that the gyromagnetic factor at tree level is in fact *independent* of the spin [14]

$$g_j = 2, \tag{2}$$

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i.e. the magnetic moment is directly proportional to the spin. Among the various reasons, let us mention that

(i) the only higher-spin and charged elementary particle observed is the W boson. At tree level, it has $g_W = 2$ [15] and not $g_W = 1$ as suggested by (1) for spin-1 particles;

(ii) the relativistic equation of motion of the polarization four-vector S_μ in a homogeneous external electromagnetic field is simplified for $g_j = 2$ [16];

(iii) in the supersymmetric sum rules framework, when all magnetic-moment matrix elements are diagonal, the gyromagnetic factor of arbitrary-spin supersymmetric particles must be equal to 2 [17];

(iv) string theory also suggests that $g_j = 2$ [18];

(v) by requiring that forward Compton scattering amplitudes of physical theories possess a good high-energy behavior, Weinberg showed [19] that this implies $g_j \approx 2$ for any nonstrongly interacting spin-j particle, small deviations being due to loop corrections.

The way to reconcile Lagrangian theories with $g_j = 2$ is to allow nonminimal coupling [12,20]. Indeed, using the minimal substitution is ambiguous for spins $j > 1/2$ since one has $[\pi^\mu, \pi^\nu] = -ieF^{\mu\nu}$, while $[p^\mu, p^\nu] = 0$. Note that a nonminimal coupling $F^{\mu\nu}W^+_\mu W^-_\nu$ is already present in the standard model [21].

In order to completely characterize the electromagnetic interaction of an elementary particle with spin $j > 1/2$, the electric charge and the magnetic dipole moment are not sufficient. In general, a spin-j particle has $2j + 1$ electromagnetic multipoles if Lorentz, parity and time-reversal symmetries are respected. The knowledge of these "natural" electromagnetic moments is obviously very important since they

(i) constrain the construction of a consistent higher-spin electromagnetic interaction theory;

(ii) allow one to refute the elementary nature of a particle when the observed electromagnetic moments are significantly different from the expected value (loop corrections included), e.g. proton anomalous magnetic moment $\kappa_p \approx 1.79 \gg 0$;

(iii) allow one to determine the actual shape of composite particles.

In this work, we present our results concerning the electromagnetic interaction for particles with arbitrary spin. Based on the simple assumption that QED conserves nontrivially the light-cone helicity of *any* elementary particle at tree level and real photon point $Q^2 = 0$, we were able to derive *all* natural electromagnetic moments for particles of *any* spin.

The outline of this paper is as follows: In Sec. II, we give the general form of the electromagnetic current in terms of covariant vertex functions. An explicit multipole decomposition is then performed in Sec. III together with a transparent relation to electromagnetic moments. In Sec. IV, we use the Breit frame to obtain a general relation between multipole form factors and covariant vertex functions. In Sec. V, we study the light-cone helicity amplitudes for the electromagnetic vertex of a particle of any spin. Using the assumption that light-cone helicity is conserved nontrivially at tree level and at the real photon point ($Q^2 = 0$) by the electromagnetic interaction for elementary particles, we obtain the natural values of covariant vertex functions, multipole form factors and therefore electromagnetic moments. As an application of our results, we generalize in Sec. VI to arbitrary-spin particles the study of quark transverse charge densities, recently discussed in the literature for spin 1/2 [22], spin 1 [23], and spin 3/2 [24]. Finally, we summarize our conclusions. A number of technical derivations are given in three appendices.

II. ELECTROMAGNETIC CURRENT FOR ARBITRARY SPIN

The interaction of a particle with the electromagnetic field can be described in terms of matrix elements, also known as on-shell vertex functions, of the following form:

$$J^\mu \equiv \langle p', \lambda' | J^\mu_{\rm EM}(0) | p, \lambda \rangle, \tag{3}$$

where p (resp. p') is the initial (resp. final) four-momentum of the particle, and $|p, \lambda\rangle$ is the spin-j single-particle state with four-momentum p and polarization λ. For further convenience, we define the scalar quantity $Q^2 \equiv -q^2$ as *minus* the square of the four-momentum transfer $q = p' - p$ due to the photon.

If the particle respects parity and time-reversal symmetries, its matrix elements can in principle be conveniently written in terms of $2j + 1$ independent covariant vertex functions $F_k(Q^2)$ [25], also often called form factors. Note however that the decomposition is not unique. One can choose different sets of covariant vertex functions. All the sets are equivalent and are related through on-shell relations.[1] In this sense, matrix elements are more fundamental

[1] To illustrate this point, consider the case of a spin-1/2 particle where the current is usually written in the form

$$J^\mu_{(1/2)} = \bar{u}(p', \lambda')\left[\gamma^\mu F_1(Q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(Q^2)\right]u(p, \lambda).$$

It is however also legitimate to write this current as

$$J^\mu_{(1/2)} = \frac{1}{2M}\bar{u}(p', \lambda')[P^\mu G_1(Q^2) + i\sigma^{\mu\nu}q_\nu G_2(Q^2)]u(p, \lambda),$$

where $P = p' + p$ is twice the averaged four-momentum of the particle. Clearly $G_1 = F_1$ and $G_2 = F_1 + F_2$ because of Gordon's identity

$$\bar{u}(p', \lambda')\gamma^\mu u(p, \lambda) = \frac{1}{2M}\bar{u}(p', \lambda')[P^\mu + i\sigma^{\mu\nu}q_\nu]u(p, \lambda).$$

Nevertheless, the set $\{F_1, F_2\}$ is the preferred one because it is in direct connection with Dirac theory, where the current of an elementary spin-1/2 particle is given by $u(p', \lambda')\gamma^\mu u(p, \lambda)$.

than covariant vertex functions. Some decompositions have been proposed long ago. Unfortunately, they are usually either not applicable to any spin or very obscure and thus not suited for direct use. We therefore propose here a complete and transparent decomposition of any on-shell electromagnetic current.

Since any spin representation can be constructed from spin-$1/2$ and spin-1 representations, it is rather straightforward to obtain a decomposition of matrix elements satisfying Lorentz covariance, explicit gauge invariance, together with parity and time-reversal symmetries. For a particle with spin j and mass M we propose to use

$$J^{\mu}_{(j)} = (-1)^j \varepsilon^*_{\alpha'_1\cdots\alpha'_j}(p',\lambda')\Big[P^\mu \sum_{(k,j)} F_{2k+1}(Q^2) + (g^{\mu\alpha_j} q^{\alpha'_j} - g^{\alpha'_j\mu} q^{\alpha_j}) \sum_{(k,j-1)} F_{2k+2}(Q^2)\Big]\varepsilon_{\alpha_1\cdots\alpha_j}(p,\lambda), \tag{4}$$

when j is integer and

$$J^{\mu}_{(j)} = (-1)^n \bar{u}_{\alpha'_1\cdots\alpha'_n}(p',\lambda')\sum_{(k,n)}\Big[\gamma^\mu F_{2k+1}(Q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M} F_{2k+2}(Q^2)\Big] u_{\alpha_1\cdots\alpha_n}(p,\lambda), \tag{5}$$

when $j = n + 1/2$ is half integer. An explicit expression for the standard polarization tensors $\varepsilon_{\alpha_1\cdots\alpha_j}$ and $u_{\alpha_1\cdots\alpha_n}$ is given in Appendix A and the strange sum stands actually for

$$\sum_{(k,j)} \equiv \sum_{k=0}^{j}\Big[\prod_{i=1}^{k}\Big(-\frac{q^{\alpha'_i}q^{\alpha_i}}{2M^2}\Big)\prod_{i=k+1}^{j} g^{\alpha'_i\alpha_i}\Big]. \tag{6}$$

The decompositions (4) and (5) coincide with the standard spin-$1/2$ and spin-1 [26] currents. For spin-$3/2$ particles, we get

$$J^{\mu}_{(3/2)} = -\bar{u}_{\alpha'}(p',\lambda')\Big\{g^{\alpha'\alpha}\Big[\gamma^\mu F_1(Q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(Q^2)\Big] - \frac{q^{\alpha'}q^{\alpha}}{2M^2}\Big[\gamma^\mu F_3(Q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_4(Q^2)\Big]\Big\}u_\alpha(p,\lambda),$$

which is equivalent to the standard decompositions [27] provided that

$$\begin{aligned} F_1(Q^2) &= F_1^*(Q^2) = a_1(q^2) + a_2(q^2),\\ F_2(Q^2) &= F_2^*(Q^2) = -a_2(q^2),\\ F_3(Q^2) &= -\tfrac{1}{2}F_3^*(Q^2) = -\tfrac{1}{2}[c_1(q^2) + c_2(q^2)],\\ F_4(Q^2) &= -\tfrac{1}{2}F_4^*(Q^2) = \tfrac{1}{2}c_2(q^2). \end{aligned}$$

Our decompositions (4) and (5) have the advantage of avoiding spurious $(-1/2)$ factors in subsequent results.

III. MULTIPOLE FORM FACTORS

Our aim is to obtain the natural values for the electromagnetic moments of a particle of arbitrary spin. These moments are related to the so-called multipole form factors $G_{El}(Q^2)$ and $G_{Ml}(Q^2)$ at real photon point $Q^2 = 0$, which are obtained by means of a standard multipole decomposition of a four-current j^μ. In the literature clear definitions are often absent and multipoles differ by a given factor from one paper to another. For this reason, and also for further convenience, we propose here an explicit decomposition with clear definitions.

The zero component $\mu = 0$ of a four-current j^μ corresponds to the charge density, while the spatial components $\mu = i$ are related to some kind of "magnetic density" [28]. Let us refer, for the moment, to these densities by means of the generic density

$$\rho(\vec{q}) = \int d^3r e^{i\vec{q}\cdot\vec{r}}\rho(\vec{r}). \tag{7}$$

The multipole decomposition of this density can be written in the following form ($Q \equiv |\vec{q}|$)

$$\rho(\vec{q}) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l} Q^l F_{lm}(Q^2) Y_{lm}(\Omega_q), \tag{8}$$

where $Y_{lm}(\Omega_q)$ are the usual spherical harmonics, Ω_q is the solid angle giving the direction of $\vec{q}$. Thanks to the identity [29]

$$e^{i\vec{q}\cdot\vec{r}} = 4\pi\sum_{l=0}^{+\infty}\sum_{m=-l}^{l} i^l j_l(Qr) Y_{lm}(\Omega_q) Y^*_{lm}(\Omega_r),$$

where $j_l(x)$ are the spherical Bessel functions, and using orthonormal relations among the spherical harmonics, one gets

$$F_{lm}(Q^2) = \frac{4\pi i^l}{Q^l}\int d^3r j_l(Qr) Y^*_{lm}(\Omega_r)\rho(\vec{r}).$$

Remembering that we are interested in the electromagnetic properties of particles, one can naturally consider that the electric and magnetic densities exhibit an azimuthal (or cylindrical) symmetry with respect to the quantization axis. Let us identify the spin quantization axis with the z axis. Because of this symmetry, the multipole decomposition (8) reduces to $m = 0$ components only

$$\rho(\vec{q}) = \sum_{l=0}^{+\infty} Q^l F_{l0}(Q^2) Y_{l0}(\Omega_q). \tag{9}$$

Cartesian moments M_l are defined by

$$M_l \equiv \int d^3r C_l(\vec{r})\rho(\vec{r}),$$
$$C_l(\vec{r}) = l! r^l \sqrt{\frac{4\pi}{2l+1}} Y_{l0}(\Omega_r) = l! r^l P_{l0}(\cos\theta_r). \quad (10)$$

These moments are therefore associated with the structures $1, z, (3z^2 - r^2), (15z^3 - 9r^2 z), \cdots$ They are clearly just proportional to $F_{l0}(0)$ and the proportionality factor can be obtained using the expansion of spherical Bessel functions $j_l(Qr)$ for small Q

$$j_l(Qr) = \frac{(Qr)^l}{(2l+1)!!} + \mathcal{O}(Q^{l+2}),$$

leading then to

$$M_l = (-i)^l l!(2l-1)!! \sqrt{\frac{2l+1}{4\pi}} F_{l0}(0).$$

The multipole form factors used in the literature are identified to $F_{l0}(Q^2)$ up to factors depending on l. Note also that a spin-j particle respecting parity symmetry has only *even* electric multipoles and *odd* magnetic multipoles [25]. Moreover, the total number of multipole form factors is equal to the total number of covariant vertex functions, namely, $2j + 1$.

A. Electric multipoles

Electric multipoles[2] are obtained from the charge density $\rho(\vec{r}) := j^0(\vec{r})$. We define the electric multipole form factors $G_{El}(Q^2)$ as follows:

$$G_{El}(Q^2) \equiv (-i)^l \frac{(2l-1)!!}{l!} \frac{(2M)^l}{e} \sqrt{\frac{2l+1}{4\pi}} F_{l0}(Q^2), \quad (11)$$

where e is *minus* the electric charge of an electron. The lth electric moment Q_l [28] in (natural) unit of e/M^l is therefore given by

$$Q_l \equiv \int d^3r C_l(\vec{r}) j^0(\vec{r}) = \frac{(l!)^2}{2^l} G_{El}(0). \quad (12)$$

The expansion (9) of the charge density of a spin-j particle in terms of multipole form factors (11) takes the form

$$j^0(\vec{q}) = e \sum_{\substack{l=0 \\ l \text{ even}}}^{2j} i^l \tau^{l/2} \frac{1}{\tilde{C}^{l-1}_{2l-1}} G_{El}(Q^2) \sqrt{\frac{4\pi}{2l+1}} Y_{l0}(\Omega_q), \quad (13)$$

where $\tau \equiv \frac{Q^2}{4M^2}$ and with the definition[3]

$$\tilde{C}^k_n \equiv \begin{cases} \frac{n!!}{k!!(n-k)!!}, & n \geq k \geq -1 \\ 0, & \text{otherwise} \end{cases}. \quad (14)$$

More explicitly, this amounts to the following expansion:

$$j^0(\vec{q}) = e\Big[G_{E0}(Q^2)\sqrt{4\pi} Y_{00}(\Omega_q) - \frac{2}{3}\tau G_{E2}(Q^2) \times \sqrt{\frac{4\pi}{5}} Y_{20}(\Omega_q) + \cdots\Big].$$

B. Magnetic multipoles

Magnetic multipoles are obtained from the magnetic density $\rho(\vec{r}) := \vec{\nabla} \cdot (\vec{j}(\vec{r}) \times \vec{r})$. We define the magnetic multipole form factors $G_{Ml}(Q^2)$ as follows:

$$G_{Ml}(Q^2) \equiv (-i)^l \frac{(2l-1)!!}{(l+1)!} \frac{(2M)^l}{e} \sqrt{\frac{2l+1}{4\pi}} F_{l0}(Q^2). \quad (15)$$

The lth magnetic moment μ_l [28] in (natural) unit of $e/2M^l$ is therefore given by

$$\mu_l \equiv \int d^3r C_l(\vec{r}) \frac{\vec{\nabla} \cdot (\vec{j}(\vec{r}) \times \vec{r})}{l+1} = \frac{(l!)^2}{2^{l-1}} G_{Ml}(0). \quad (16)$$

The expansion (9) of the magnetic density of a spin-j particle in terms of the multipole form factors (15) takes the form

$$\vec{\nabla} \cdot (\vec{j}(\vec{q}) \times \vec{q}) = e \sum_{\substack{l=0 \\ l \text{ odd}}}^{2j} i^l \tau^{l/2} \frac{(l+1)}{\tilde{C}^{l-1}_{2l-1}} G_{Ml}(Q^2) \times \sqrt{\frac{4\pi}{2l+1}} Y_{l0}(\Omega_q), \quad (17)$$

i.e. more explicitly

$$\vec{\nabla} \cdot (\vec{j}(\vec{q}) \times \vec{q}) = e2i\sqrt{\tau}\Big[G_{M1}(Q^2)\sqrt{\frac{4\pi}{3}} Y_{10}(\Omega_q) - \frac{4}{5}\tau G_{M3}(Q^2)\sqrt{\frac{4\pi}{7}} Y_{30}(\Omega_q) + \cdots\Big].$$

As a closing remark for this section, we would like to emphasize that *only* $G_{E0}(0)$, $G_{M1}(0)$, and $G_{E2}(0)$ can directly be interpreted as electromagnetic moments. For $l > 2$, the factor between (Cartesian) electromagnetic moments and multipole form factors at $Q^2 = 0$ differs from unity, as one can see from Eqs. (12) and (16).

IV. BREIT FRAME

The set of covariant vertex functions $\{F_k(Q^2)\}$ and the set of multipole form factors $\{G_{E,Ml}(Q^2)\}$ are not independent. Following the common usage, they can be connected in the so-called *Breit frame* by identifying the classical electromagnetic current j^μ of Sec. III with the quantum-

[2] If we follow more rigorously the literature terminology, they should be called Coulomb multipoles.
[3] We remind that by definition $(-1)!! = 1$.

mechanical one J^μ of Sec. II

$$j^\mu(\vec{q}) \equiv \frac{e}{2M}\langle p', j|J^\mu_{\rm EM}(0)|p, j\rangle|_{\rm Breit} = \frac{e}{2M}J^\mu_B. \qquad (18)$$

The derivation of this connection is straightforward but a bit technical, so details have been relegated to Appendix B.

A. Bosonic case

For an integer spin particle, we found that the multipole form factors are related to our covariant vertex functions at any Q^2 according to (see Appendix B 2):

$$\begin{aligned}
&\sum_{m=t}^{j}(-1)^{m+t}\frac{\tau^{m-t}(C^t_m)^2}{\tilde{C}^{2m+2t-1}_{4m-1}}G_{E2m}(Q^2)\\
&\quad=\sum_{k=0}^{t}(1+\tau)^{k+(1/2)}C^{j-t}_{j-k}\\
&\quad\times\left[F_{2k+1}(Q^2)-\frac{1-\delta_{k,0}}{1+\tau}F_{2k}(Q^2)\right],\\
&\sum_{m=t}^{j-1}(-1)^{m+t}(m+1)\frac{\tau^{m-t}(C^t_m)^2}{\tilde{C}^{2m+2t+1}_{4m+1}}G_{M2m+1}(Q^2)\\
&\quad=(t+1)\sum_{k=0}^{t}(1+\tau)^{k+(1/2)}C^{j-t-1}_{j-k-1}F_{2k+2}(Q^2), \qquad (19)
\end{aligned}$$

where

$$C^k_n = \binom{n}{k}$$

is the standard binomial function. Let us, for example, consider the case $j = 1$

$$\begin{aligned}
G_{E0}(Q^2) - \tfrac{2}{3}\tau G_{E2}(Q^2) &= \sqrt{1+\tau}F_1(Q^2),\\
G_{M1}(Q^2) &= \sqrt{1+\tau}F_2(Q^2),\\
G_{E2}(Q^2) &= \sqrt{1+\tau}[F_1(Q^2) - F_2(Q^2)\\
&\quad + (1+\tau)F_3(Q^2)],
\end{aligned}$$

which coincides with the well-known expression for the electromagnetic interaction of vector particles [26], provided that

$$\begin{aligned}
G_{E0}(Q^2) &= \sqrt{1+\tau}G_C(q^2),\\
G_{M1}(Q^2) &= \sqrt{1+\tau}G_M(q^2),\\
G_{E2}(Q^2) &= \sqrt{1+\tau}G_Q(q^2).
\end{aligned}$$

The factor $\sqrt{1+\tau}$ is present for any integer spin. One is free to define new multipole form factors without this inelegant factor, at the cost of having different multipole decompositions for fermions and bosons. Nevertheless, at $Q^2 = 0$ there is no difference between both definitions.

At real photon point $Q^2 = 0$, the connections (19) reduce to

$$\begin{aligned}
G_{E2m}(0) &= \sum_{k=0}^{m}C^{j-m}_{j-k}[F_{2k+1}(0) - (1-\delta_{k,0})F_{2k}(0)],\\
G_{M2m+1}(0) &= \sum_{k=0}^{m}C^{j-m-1}_{j-k-1}F_{2k+2}(0), \qquad (20)
\end{aligned}$$

which can be inverted

$$\begin{aligned}
F_{2k+1}(0) &= \sum_{l=0}^{k}C^{j-k}_{j-l}(-1)^{k-l}[G_{E2l}(0) + (1-\delta_{l,0})\\
&\quad\times G_{M2l-1}(0)],\\
F_{2k+2}(0) &= \sum_{l=0}^{k}C^{j-k-l}_{j-l-1}(-1)^{k-l}G_{M2l+1}(0), \qquad (21)
\end{aligned}$$

thanks to the following identity

$$\sum_{l=m}^{k}C^{j-k}_{j-l}C^{j-l}_{j-m}(-1)^{k-l} = \delta_{k,m}, \quad \forall\ j \geq k. \qquad (22)$$

B. Fermionic case

We proceed with half-integer spin particles. In this case, we found that the multipole form factors are related to our covariant vertex functions at any Q^2 according to (see Appendix B 3)

$$\begin{aligned}
&\sum_{m=t}^{n}(-1)^{m+t}\frac{\tau^{m-t}(C^t_m)^2}{\tilde{C}^{2m+2t-1}_{4m-1}}G_{E2t}(Q^2)\\
&\quad=\sum_{k=0}^{t}(1+\tau)^k C^{n-t}_{n-k}[F_{2k+1}(Q^2) - \tau F_{2k+2}(Q^2)],\\
&\sum_{m=t}^{n}(-1)^{m+t}(m+1)\frac{\tau^{m-t}(C^t_m)^2}{\tilde{C}^{2m+2t+1}_{4m+1}}G_{M2t+1}(Q^2)\\
&\quad=(t+1)\sum_{k=0}^{t}(1+\tau)^k C^{n-t}_{n-k}[F_{2k+1}(Q^2) + F_{2k+2}(Q^2)].
\end{aligned} \qquad (23)$$

Let us consider two examples. For $j = 1/2$, we get

$$\begin{aligned}
G_{E0}(Q^2) &= F_1(Q^2) - \tau F_2(Q^2),\\
G_{M1}(Q^2) &= F_1(Q^2) + F_2(Q^2),
\end{aligned}$$

which coincides with the well-known expression for the electromagnetic interaction of spin-1/2 particles. The multipole form factors $G_{E0}(Q^2)$ and $G_{M1}(Q^2)$ are nothing else than the Sachs electric and magnetic form factors $G_E(Q^2)$ and $G_M(Q^2)$. For $j = 3/2$, we get

$$G_{E0}(Q^2) - \frac{2}{3}\tau G_{E2}(Q^2) = F_1(Q^2) - \tau F_2(Q^2),$$
$$G_{M1}(Q^2) - \frac{4}{5}\tau G_{M3}(Q^2) = F_1(Q^2) + F_2(Q^2),$$
$$G_{E2}(Q^2) = F_1(Q^2) - \tau F_2(Q^2) + (1+\tau)[F_3(Q^2) - \tau F_4(Q^2)],$$
$$G_{M3}(Q^2) = F_1(Q^2) + F_2(Q^2) + (1+\tau)[F_3(Q^2) + F_4(Q^2)],$$

which also coincides with the well-known expression for the electromagnetic interaction of spin-3/2 particles [27], excepted that the spurious $(-1/2)$ factors are absent. Clearly, our decomposition of the current in terms of covariant vertex function is more economical. We would like also to remind that $G_{M3}(0)$ does *not* correspond to the value of the (Cartesian) magnetic octupole, but represents one-ninth of its value.

At real photon point $Q^2 = 0$, the connections (23) reduce to

$$G_{E2m}(0) = \sum_{k=0}^{m} C_{n-k}^{n-m} F_{2k+1}(0),$$
$$G_{M2m+1}(0) = \sum_{k=0}^{m} C_{n-k}^{n-m}[F_{2k+1}(0) + F_{2k+2}(0)], \tag{24}$$

which can also be inverted thanks to (22)

$$F_{2k+1}(0) = \sum_{l=0}^{k} C_{n-l}^{n-k}(-1)^{k-l} G_{E2l}(0),$$
$$F_{2k+2}(0) = \sum_{l=0}^{k} C_{n-l}^{n-k}(-1)^{k-l}[G_{M2l+1}(0) - G_{E2l}(0)]. \tag{25}$$

V. LIGHT-CONE HELICITY AMPLITUDES AND NATURAL MOMENTS

We discuss in this section the light-cone helicity amplitudes of the $+$ component of the current $J^+_{\text{EM}} = J^0_{\text{EM}} + J^3_{\text{EM}}$. We will work in the usual Drell-Yan-West (DYW) frame $q^+ = 0$ [30], and one can furthermore choose a frame where the transverse momenta of the initial and final particles are opposite. We will write such light-cone helicity amplitude in the form

$$A_{\lambda',\lambda}(Q^2) \equiv \frac{e^{i(\lambda'-\lambda)\phi_q}}{2p^+}\left\langle p^+, \frac{\vec{q}_\perp}{2}, \lambda' \middle| J^+_{\text{EM}}(0) \middle| p^+, -\frac{\vec{q}_\perp}{2}, \lambda\right\rangle, \tag{26}$$

where $Q^2 \equiv -q^2 = \vec{q}^{\,2}_\perp$, with $\vec{q}_\perp = Q(\cos\phi_q \hat{e}_x + \sin\phi_q \hat{e}_y)$, and λ, λ' are the light-cone helicities of the initial and final particles, respectively.

A spin-j particle has $2j+1$ possible polarization states. This means that there are in principle $(2j+1)^2$ helicity amplitudes. However, there are only $2j+1$ covariant vertex functions. This means that out of the $(2j+1)^2$ helicity amplitudes, only $2j+1$ are in fact *independent*. One needs therefore $2j(2j+1)$ constraints. These constraints arise due to, on the one hand, discrete space-time symmetries, and on the other hand, angular momentum conservation.

A. Light-cone discrete symmetries

Obviously, parity and time-reversal discrete symmetries are not compatible with the DYW frame $q^+ = 0$. However, relevant light-cone parity and time-reversal operators can be defined by compounding the usual parity and time-reversal operators with π rotation about the y axis, choosing the x axis so that all momenta lie in the x-z plane [31]. This results in the parity relation for light-cone helicity amplitudes and identical particles, given by

$$A_{-\lambda',-\lambda}(Q^2) = (-1)^{\lambda'-\lambda} A_{\lambda',\lambda}(Q^2), \tag{27}$$

while the time-reversal relation reads

$$A_{\lambda,\lambda'}(Q^2) = (-1)^{\lambda'-\lambda} A_{\lambda',\lambda}(Q^2). \tag{28}$$

From these relations, one easily deduces that the number of pertinent amplitudes is reduced to $(j+1/2)(j+3/2)$ and $(j+1)^2$ for half-integer and integer spin, respectively.

B. Light-cone angular conditions

We know that at the end there should only be $2j+1$ independent light-cone amplitudes. The remaining constraints to be imposed are provided by considerations of angular momentum conservation. Such relations are known in the literature as *angular conditions*. The link between these relations and angular momentum conservation can be made transparent in the Breit frame. This has been discussed in [32,33]. The number of angular conditions must obviously be $j^2 - 1/4$ for half-integer spin, and j^2 for integer spin.

Since we will work with explicit expressions for helicity amplitudes in terms of covariant vertex functions, we automatically satisfy angular momentum conservation. There are therefore two ways of obtaining the angular conditions. One is to directly impose angular momentum conservation at the level of amplitudes, referring to a specific frame where this condition is simple (Breit frame) and then performing a transformation to the light-cone frame. The other possibility is to directly work with the explicit decomposition in terms of covariant vertex functions. Light-cone helicity amplitudes then appear as linear combinations of these covariant vertex functions, and only a subset of them constitutes linearly independent combinations. It turns out that we can choose, for example, the set $\{A_{j,j-m} | m = 0, \cdots, 2j\}$ as the independent amplitudes. The explicit expression we obtained for these amplitudes in terms of covariant vertex functions is

$$A_{j,j-m}(Q^2) = \frac{(4\tau)^{m/2}}{\sqrt{C_{2j}^m}} \sum_{k=0}^{[j]} \sum_{t=0}^{\min\{k,m/2\}} \frac{(-1)^t}{2^{2t}} C_k^t \tau^{k-t} \times \left[C_{[j]-t}^{m-2t} F_{2k+1}(Q^2) - \frac{1-\delta_{k,j}}{2} C_{[j-(1/2)]-t}^{m-2t-1} F_{2k+2}(Q^2) \right], \quad (29)$$

where $[x]$ means the largest integer i such that $i \leq x$. Details of the derivation can be found in Appendix C. Any other light-cone helicity amplitude can be written as a linear combination of the elements of this set, arising either from a discrete light-cone symmetry relation (only one element of the set of independent amplitudes is needed) or from an angular condition (many elements of the set of independent amplitudes are needed).

C. Natural electromagnetic moments

Since we are interested in electromagnetic moments, let us consider the limit $Q^2 \to 0$. Because of the factor $\tau^{m/2}$ in (29), a light-cone amplitude involving m units of helicity flip behaves at least as Q^m in this limit and so the only nonvanishing amplitude is

$$A_{j,j}(0) = F_1(0) \equiv Z, \quad (30)$$

where Z stands for the particle charge in units of e. In other words, the helicity-conserving amplitude at $Q^2 = 0$ just gives the electric charge of the particle. We define new light-cone helicity amplitudes where this trivial Q^2 dependence is removed

$$G_{j,j-m}(Q^2) \equiv Q^{-m} A_{j,j-m}(Q^2). \quad (31)$$

At $Q^2 = 0$, these amplitudes simply read

$$G_{j,j-m}(0) = \frac{1}{M^m \sqrt{C_{2j}^m}} \sum_{k=0}^{m} \frac{(-1)^{[((k+1)/2)]}}{2^k} C_{[j-k/2]}^{m-k} F_{k+1}(0). \quad (32)$$

Since the highest possible covariant vertex function involved in $G_{j,j-m}(0)$ is $F_{m+1}(0)$, these amplitudes are necessarily independent, which in turn implies that the $A_{j,j-m}(Q^2)$ are also independent. Moreover, thanks to the relations (C4)–(C7) of Appendix C, one can see that we have in fact

$$G_{j-k,j-k-m}(0) = G_{j,j-m}(0), \quad \forall\ k \in [0, 2j-m]. \quad (33)$$

In order to derive the natural electromagnetic moments of any particle, we need an assumption concerning the electromagnetic interaction. We propose to *assume* that, at tree level and $Q^2 = 0$, the light-cone helicity of any elementary particle is *nontrivially* conserved. In other words, we assume that

$$G_{j,j-m}(0) = \delta_{m,0} Z. \quad (34)$$

Any violation of this condition will be due to internal structure. The elementary constituents of a composite particle will naturally conserve their helicity, but they are allowed to jump from one orbital to another, leading thus to a nonconservation of the composite particle's helicity. Using (32) and the following identity

$$\sum_{k=0}^{m} (-1)^k C_{[j-k/2]}^{m-k} C_{[j+(k-1)/2]}^k = \delta_{m,0}, \quad (35)$$

the condition (34) imposes specific values to the covariant vertex functions

$$F_{k+1}(0) = (-1)^{[k/2]} 2^k C_{[j+(k-1)/2]}^k Z,$$

which are not of great interest since they depend on the chosen decomposition of the current in terms of covariant vertex functions. Nevertheless, using finally (20) and (24) allows us to obtain the natural values of multipole form factors and (Cartesian) electromagnetic moments

$$\begin{aligned} G_{El}(0) + iG_{Ml}(0) &= i^l C_{2j}^l Z, \\ Q_l(0) + \frac{i}{2}\mu_l(0) &= i^l \frac{(l!)^2}{2^l} C_{2j}^l Z, \end{aligned} \quad (36)$$

which have a direct physical meaning. This particularly simple and elegant formula constitutes the master result of this paper. It is quite surprising to see that the natural values of the multipole form factors turned out to be just given, up to a sign, by a binomial function. These values depend only on the spin j of the particle, the order l of the multipole, and are proportional to the particle electric charge Z. Our result also shows explicitly that the highest nonvanishing moment is of order $l = 2j$.

Let us develop the expression (36) for a particle with unit electric charge $Z = +1$

$$\begin{aligned} Q_0 &= G_{E0}(0) = 1, \\ \mu_1 &= G_{M_1}(0) = 2j, \\ Q_2 &= G_{E2}(0) = -j(2j-1), \\ \mu_3 &= 9G_{M3}(0) = -3j(2j-1)(2j-2), \\ &\vdots \end{aligned}$$

From the second line, one can see that our assumption about helicity conservation agrees with a universal gyromagnetic factor $g = 2$ for any elementary particle [14–19]. Our derivation of this universal factor is interesting in the sense that it explains naturally why $g = 2$ and not any other number. The value 2 has the same origin as the fact that spins can be half integer, i.e. the spin group $SU(2)$ covers *twice* the group of rotations in 3-dimensional Euclidean space $SO(3)$. Note also that the general form of the quadrupole (third line) is also in accordance with a good high-energy behavior [34].

TABLE I. The natural multipoles of a spin-j particle with electric charge $Z = +1$ are organized according to a pseudo Pascal triangle, when expressed in terms of natural units of e/M^l and $e/2M^l$ for $G_{El}(0)$ and $G_{Ml}(0)$, respectively.

j	$G_{E0}(0)$	$G_{M1}(0)$	$G_{E2}(0)$	$G_{M3}(0)$	$G_{E4}(0)$	$G_{M5}(0)$	$G_{E6}(0)$	$G_{M7}(0)$	$G_{E8}(0)$
0	1	0	0	0	0	0	0	0	0
1/2	1	1	0	0	0	0	0	0	0
1	1	2	−1	0	0	0	0	0	0
3/2	1	3	−3	−1	0	0	0	0	0
2	1	4	−6	−4	1	0	0	0	0
5/2	1	5	−10	−10	5	1	0	0	0
3	1	6	−15	−20	15	6	−1	0	0
7/2	1	7	−21	−35	35	21	−7	−1	0
4	1	8	−28	−56	70	56	−28	−8	1

The standard model (SM) contains only elementary particles up to spin 1. It requires that at tree level, the elementary weak bosons W and Z have $G_{M1}(0) = 2$ and $G_{E2}(0) = -1$, in order to satisfy the Gerasimov-Drell-Hearn sum rule to lowest order in perturbation theory [35]. The only known consistent theory for spin-3/2 particles is the extended supergravity [36]. The gravitino is described as a spin-3/2 particle, which couples consistently to electromagnetism, in the framework of $\mathcal{N} = 2$ supergravity. One can therefore consider that the multipoles arising from this theory are the natural ones, namely, $G_{E0}(0) = 1$, $G_{M1}(0) = 3$, $G_{E2}(0) = -3$, and $G_{M3}(0) = -1$ [24]. In Table I, we give the natural values of multipoles we obtain for a particle with unit electric charge $Z = +1$, up to spin $j = 4$. As one can see, we are in complete agreement with both SM and supergravity. The table obtained is just a (pseudo) Pascal triangle because of the binomial function.

The same suggestion as (36) has been obtained in a complementary way by the authors of Ref. [37], which derived model-independent, nonperturbative supersymmetric sum rules for the electromagnetic moments of any theory with $\mathcal{N} = 1$ supersymmetry. They find that in any irreducible $\mathcal{N} = 1$ supermultiplet, the diagonal matrix elements of the lth moments are completely fixed in terms of their off-diagonal matrix elements and the diagonal $(l-1)$th moments. Setting the off-diagonal matrix elements to zero, any given moment has the same structure for all members of the supermultiplet. This specific case is then considered as leading to the "preferred" value of the electromagnetic moments

$$\mathcal{T}_j^{(l)^{(e,m)}} = \mp\frac{1}{M}\mathcal{T}_j^{(l-1)^{(m,e)}}, \tag{37}$$

where the $\mathcal{T}_j^{(l)^{(e,m)}}$ are the generalization[4] to higher multipoles of the gyromagnetic factor $\mathcal{T}_j^{(1)^{(m)}} \equiv g_j \frac{Ze}{2M}$. One can easily see that the value of $\mathcal{T}_j^{(l)^{(e,m)}}$ is at the end uniquely fixed by the electric charge $\mathcal{T}_j^{(0)^{(e)}} \equiv Ze$. The expression for the lth moment $M_j^{(l)^{(e,m)}}$ is given in absolute value by

$$M_j^{(l)^{(e,m)}} = \frac{2j(2j-1)\cdots(2j-l+1)}{C_{2l}^l}\mathcal{T}_j^{(l)^{(e,m)}}. \tag{38}$$

To see actually that (36) and (38) do coincide, one has to remember that our moments are given in natural units and to take into account that our definition of multipoles has an additional factor of $(2l-1)!!$ compared to [37].

Before closing this section, we would like to comment on a previous study concerning light-cone helicity conservation [32]. The conclusion was that one cannot satisfy at the same time both angular momentum conservation and light-cone helicity conservation. This result is not in contradiction with the present study. In [32], the author imposes helicity conservation for *any* Q^2

$$A_{\lambda',\lambda}(Q^2) \propto \delta_{\lambda',\lambda}. \tag{39}$$

This assumption is simply different from ours (34). Up to spin 1, the conclusions are the same. The discrepancies appear when we consider spins higher than 1. This can be easily understood as follows. Let us consider the light-cone helicity-flip amplitudes (29) for spin 1/2

$$A_{(1/2),-(1/2)}(Q^2) = -\sqrt{\tau}F_2(Q^2),$$

and spin 1

$$A_{1,0}(Q^2) = \sqrt{2\tau}\left[F_1(Q^2) - \frac{1}{2}F_2(Q^2) + \tau F_3(Q^3)\right],$$
$$A_{1,-1}(Q^2) = -\tau F_3(Q^2).$$

All the other helicity-flip amplitudes are related to these ones by discrete space-time symmetries (27) and (28). At $Q^2 = 0$, both assumptions (34) and (39) about helicity conservation lead to the same result, namely, $F_2(0) = 0$ for spin 1/2, and $F_2(0) = 2F_1(0)$ and $F_3(0) = 0$ for spin 1.

[4]Rotational invariance allows one to characterize completely each lth moment by means of a single quantity. This quantity $\mathcal{T}_j^{(l)^{(e,m)}}$ is essentially a reduced matrix element according to the Wigner-Eckart theorem.

Our assumption (34) is less restrictive since only the value at $Q^2 = 0$ is fixed. The other one (39) imposes the stronger condition $F_2(Q^2) = 2F_1(Q^2)$. Now, for spin 3/2, let us consider the light-cone helicity-flip amplitudes (29)

$$A_{(3/2),(1/2)}(Q^2) = \frac{2\sqrt{\tau}}{\sqrt{3}}\Big[F_1(Q^2) - \frac{1}{2}F_2(Q^2) + \tau F_3(Q^3) - \frac{\tau}{2}F_4(Q^2)\Big],$$

$$A_{(3/2),-(1/2)}(Q^2) = -\frac{2\tau}{\sqrt{3}}\Big[F_2(Q^2) + \frac{1}{2}F_3(Q^3) + \tau F_4(Q^2)\Big],$$

$$A_{(3/2),-(3/2)}(Q^2) = \tau^{3/2}F_4(Q^3),$$

and the nonindependent one [24]

$$A_{(1/2),-(1/2)}(Q^2) = \frac{\sqrt{\tau}}{3}[4F_1(Q^2) - 2(1-2\tau)F_2(Q^2) + 4\tau F_3(Q^3) - \tau(1-4\tau)F_4(Q^2)].$$

Here also, all the other helicity-flip amplitudes are related to these ones by discrete space-time symmetries (27) and (28). This set of amplitudes with the requirement of helicity conservation for any Q^2 only accepts as a solution the trivial case $F_k(Q^2) = 0$ for all k. Equivalently, one can obtain the same conclusion by considering angular conditions. While for spin 1, the unique angular condition [33]

$$(1+2\tau)A_{1,1}(Q^2) - 2\sqrt{2\tau}A_{1,0}(Q^2) + A_{1,-1}(Q^2) - A_{0,0}(Q^2) = 0$$

is compatible with the requirement of helicity conservation for any Q^2, leading to $A_{0,0}(Q^2) = (1+2\tau)A_{1,1}(Q^2)$, the second of the two angular conditions for spin 3/2 [24,33]

$$(1+4\tau)\sqrt{3}A_{(3/2),(3/2)}(Q^2) - 8\sqrt{\tau}A_{(3/2),(1/2)}(Q^2) + 2A_{(3/2),-(1/2)}(Q^2) - \sqrt{3}A_{(1/2),(1/2)}(Q^2) = 0,$$

$$8\tau^{3/2}A_{(3/2),(3/2)}(Q^2) + 2\sqrt{3}(1-2\tau)A_{(3/2),(1/2)}(Q^2) + A_{(3/2),-(3/2)}(Q^2) - 3A_{(1/2),-(1/2)}(Q^2) = 0$$

would imply that $A_{(3/2),(3/2)}(Q^2) = 0$, i.e. no electric charge. So, in general, starting from spin 3/2 there is at least one angular condition that allows one to write a helicity-conserving amplitude in terms of helicity-flip amplitudes only [32], leading to a vanishing electric charge under the assumption (39). Since with our assumption (34) we found a nontrivial solution for the covariant vertex functions, we are automatically consistent with the angular conditions.

VI. TRANSVERSE SPIN

In this section, we discuss a first application of our helicity conservation assumption (34). We will consider the electromagnetic $h \to h$ transition (Fig. 1) from the viewpoint of a light front moving toward the hadron h.

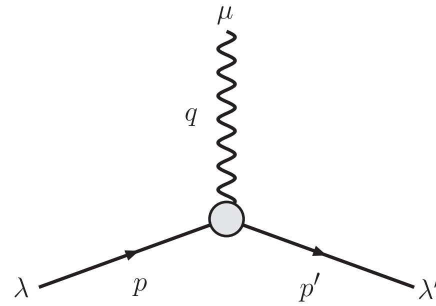

FIG. 1. Electromagnetic vertex function (blob). The initial (resp. final) hadron has four-momentum p (resp. p') and helicity λ (resp. λ'). The photon has four-momentum $q = p' - p$, and we work with the light-cone component $\mu = +$.

This is equivalent to the infinite momentum frame picture where the hadron has a large momentum along the z axis chosen along the direction of $(p' + p)$, where p and p' are as before the initial and final four-momenta of the particle. In the symmetric light-cone frame, the virtual photon couples only to forward-moving partons and the component $J^+_{\rm EM}(0)$ of the electromagnetic current has the interpretation of quark charge density operator. If one considers only the two light quarks u and d, this operator is given by

$$J^+_{\rm EM}(0) = \tfrac{2}{3}\bar{u}(0)\gamma^+ u(0) - \tfrac{1}{3}\bar{d}(0)\gamma^+ d(0).$$

Each term in this expression is a positive operator since $\bar{\psi}\gamma^+\psi \propto |\gamma^+\psi|^2$. One can define a transverse quark charge density in a hadron with definite light-cone helicity λ by the Fourier transform

$$\rho^h_\lambda(\vec{b}) \equiv \int \frac{d^2\vec{q}_\perp}{(2\pi)^2} e^{-i\vec{q}_\perp\cdot\vec{b}} \frac{1}{2p^+} \times \Big\langle p^+, \frac{\vec{q}_\perp}{2}, \lambda' \Big| J^+_{\rm EM}(0) \Big| p^+, -\frac{\vec{q}_\perp}{2}, \lambda \Big\rangle = \int_0^{+\infty} \frac{dQ}{2\pi} Q J_0(Qb) A_{\lambda,\lambda}(Q^2), \qquad (40)$$

which is obviously circularly symmetric. For a spin-j hadron h, there will be $[j+1]$ independent quark charge densities $\rho^h_\lambda(\vec{b})$ for $\lambda = j, j-1, \cdots, j-[j]$. Quark charge densities provide us therefore with $[j+1]$ independent combinations of covariant vertex functions. To get information for the other covariant vertex functions, we consider also charge densities in a hadron state with the spin transversely polarized.

A. Transversely polarized quark charge densities

First, we want to write the transverse spin eigenstates $|j, s_\perp\rangle$ in terms of the helicity eigenstates $|j, \lambda\rangle$. The direction of the transverse polarization is denoted by $\vec{S}_\perp = \cos\phi_S \hat{e}_x + \sin\phi_S \hat{e}_y$. Transverse spin states can be obtained from a rotation of the helicity states

$$|j, s_\perp\rangle = \mathcal{R}(\phi_S, \pi/2, 0)|j, m = s_\perp\rangle, \quad (41)$$

where $\mathcal{R}(\alpha, \beta, \gamma) = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}$ is the rotation operator with α, β, γ the Euler angles. Using the Wigner (small) d matrix $d^j_{mm'}(\theta) = \langle j, m|e^{-i\theta J_y}|j, m'\rangle$, we can write (41) as

$$|j, s_\perp\rangle = \sum_{\lambda=-j}^{j} e^{-i\lambda\phi_S} d^j_{\lambda s_\perp}(\pi/2)|j, \lambda\rangle. \quad (42)$$

The explicit expression for the Wigner (small) d matrix at $\theta = \pi/2$ is

$$d^j_{mm'}(\pi/2) = \sum_k \frac{(-1)^{j-m'+k}\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{2^j k!(j-m-k)!(j-m'-k)!(m+m'+k)!}, \quad (43)$$

where the sum is over the integer values of k such that the factorial arguments are non-negative. Note that for the maximal-spin projection $m' = j$, the expression becomes very simple $d^j_{mj}(\pi/2) = 2^{-j}\sqrt{C^{j-m}_{2j}}$.

Next, we want to write the transverse quark charge densities in terms of light-cone helicity amplitudes. The transverse quark charge densities with definite transverse polarization are defined as

$$\rho^h_{Ts_\perp}(\vec{b}) = \int \frac{d^2\vec{q}_\perp}{(2\pi)^2} e^{-i\vec{q}_\perp\cdot\vec{b}} \rho^h_{Ts_\perp}(\vec{q}_\perp), \quad (44)$$

with

$$\rho^h_{Ts_\perp}(\vec{q}_\perp) \equiv \frac{1}{2p^+}\left\langle p^+, \frac{\vec{q}_\perp}{2}, s_\perp \middle| J^+(0) \middle| p^+, -\frac{\vec{q}_\perp}{2}, s_\perp\right\rangle. \quad (45)$$

Using the change of basis (42), we can write

$$\rho^h_{Ts_\perp}(\vec{q}_\perp) = \sum_{\lambda,\lambda'=-j}^{j} d^j_{\lambda' s_\perp}(\pi/2) d^j_{\lambda s_\perp}(\pi/2) e^{i(\lambda'-\lambda)\phi} A_{\lambda',\lambda}(Q^2),$$

with $\phi \equiv \phi_S - \phi_q$. Then, thanks to both parity (27) and time-reversal (28) relations, this can be reduced to

$$\rho^h_{Ts_\perp}(\vec{q}_\perp) = \sum_{m=0}^{2j} Q^m \mathrm{Ctrig}(m, \phi) B_{s_\perp m}(Q^2), \quad (46)$$

where we have defined the function

$$\mathrm{Ctrig}\,(n, \alpha) \equiv \frac{1+(-1)^n}{2}\cos(n\alpha) + i\frac{1+(-1)^{n+1}}{2} \times \sin(n\alpha),$$

and used the compact notation

$$B_{s_\perp m}(Q^2) = (2-\delta_{m,0}) \sum_{\lambda'=m/2}^{j} (2-\delta_{\lambda',m/2}) d^j_{\lambda' s_\perp}(\pi/2) \times d^j_{(\lambda'-m)s_\perp}(\pi/2) G_{\lambda',\lambda'-m}(Q^2). \quad (47)$$

Finally, using the Bessel function $J_n(x) = \frac{(-i)^n}{2\pi} \times \int_0^{2\pi} d\phi \cos(n\phi) e^{ix\cos\phi}$ in the Fourier transform of (44) and inserting (46), we find that we can write the transversely polarized quark charge densities as

$$\rho^h_{Ts_\perp}(\vec{b}) = \sum_{m=0}^{2j} i^m \mathrm{Ctrig}(m, \phi_b - \phi_S) \times \int \frac{dQ}{2\pi} Q^{m+1} J_m(Qb) B_{s_\perp m}(Q^2). \quad (48)$$

B. Transverse electric moments

The trigonometric functions appearing in (48) show that the transversely polarized quark charge densities are not circularly symmetric. Let us therefore consider a bidimensional multipole decomposition by means of circular harmonics. Namely, any function $f(r, \theta)$ with $r \geq 0$ and period 2π in θ can be decomposed as

$$f(r, \theta) = \sum_{m=-\infty}^{+\infty} f_m(r) e^{im\theta}.$$

Note however that not all circular harmonics are present in (48). For a spin-j particle, there are in fact $2j+1$ multipoles. In general, we found that the lth electric multipole is associated to light-cone amplitudes with $|\lambda' - \lambda| = l$ units of helicity flip. Consequently, it is in principle *sufficient* to know, e.g. the transversely polarized quark charge density $\rho^h_{Tj}(\vec{b})$ only, in order to have an information about all the $2j+1$ covariant vertex functions.

Choosing the x axis to be parallel to the transverse spin, i.e. $\phi_S = 0$, we can define the transverse electric moments Q_{Tl} as follows:

$$Q_{Tl} \equiv e\int d^2\vec{b}\, C_l(\vec{b}) \rho^h_{Ts_\perp}(\vec{b}), \qquad C_l(\vec{b}) \equiv b^l \mathrm{Rtrig}(l, \phi_b), \quad (49)$$

with the circular harmonics

$$\mathrm{Rtrig}\,(n, \alpha) \equiv \frac{1+(-1)^n}{2}\cos(n\alpha) + \frac{1+(-1)^{n+1}}{2} \times \sin(n\alpha).$$

Working out the Fourier transform leads to

$$Q_{Tl} = e\int d^2\vec{q}_\perp \delta^{(2)}(\vec{q}_\perp) C_l(-i\vec{\nabla}_q) \rho^h_{Ts_\perp}(\vec{q}_\perp).$$

Inserting now (46) in this equation gives

$$Q_{Tl} = (-1)^{[((l+1)/2)]} 2^{l-1} l! (1+\delta_{l,0}) e B_{s_\perp l}(0). \quad (50)$$

Let us discuss first the transverse electric charge Q_{T0}. A charge monopole is spherically symmetric and should therefore not depend on the orientation of spin. We therefore expect that $Q_{T0} = Q_0$. From (50) with $l = 0$ and (47), we get

$$Q_{T0} = e \sum_{\lambda'=-j}^{j} [d^j_{\lambda' s_\perp}(\pi/2)]^2 G_{\lambda',\lambda'}(0).$$

Thanks to (33), the expression can be simplified, leading to the expected relation $Q_{T0} = eZ = Q_0$.

The other transverse moments are directly proportional to helicity-flip amplitudes. Assuming that helicity at tree level and $Q^2 = 0$ is nontrivially conserved (34) and using once more Eq. (33), we conclude that for an elementary particle, the higher transverse electric moments are vanishing. This has as immediate consequence that higher transverse electric moments are just functions of the *anomalous* moments. Such an observation has already been reported for spin up to $3/2$ [22–24]. Our assumption allows us to generalize this observation to any value of the spin.

We would like to emphasize that these higher transverse electric moments are not intrinsic, but are in fact *induced* moments due to a light-cone point of view. We do not claim that particles have, e.g. an intrinsic dipole electric moment, which would violate parity and time-reversal invariance. This effect is purely induced and is consonant with the observation [38] that an object with a magnetic dipole moment at rest, will exhibit an electric dipole moment when moving, orthogonal to both magnetic moment and momentum directions. The magnetic moment of a particle is the source of a magnetic dipole field, which is accompanied by an electric field when the particle is moving in a direction different from the magnetic dipole one. Such an electric field will induce electric polarizations in the particle, and thus electric moments, only if the particle has constituents that can migrate. From this point of view, it is clear that the induced polarizations can only be functions of the anomalous moments. We can actually use the argument the other way around. Since, on the light-cone, the particle is subject to induced fields that tend to polarize it electrically, its constituents with electric charge (if any) would migrate leading to the appearance of induced electric moments. An elementary particle, i.e. structureless or pointlike particle, does not have such constituents and cannot therefore, at tree level, present induced electric moments. As we have shown in this work explicitly, this is equivalent to say that, for an elementary particle, the light-cone helicity-flip amplitudes have to vanish nontrivially at $Q^2 = 0$ and tree level. The particular values for the usual electromagnetic moments we were able to derive from this condition can therefore be called natural.

VII. CONCLUSION

In this paper, we addressed the problem of electromagnetic interaction for arbitrary-spin particles. This problem is an old and a very important one, and requires new constraints in order to be solved. The knowledge of natural electromagnetic moments is rightly one kind of constraints that will help in the construction of a physical Lagrangian theory of electromagnetic interaction with high-spin particles, even though this would not be sufficient to solve the causality problem plaguing higher-spin field theories.

Firstly, we proposed a transparent expression for the arbitrary-spin electromagnetic current in terms of covariant vertex functions. Performing an explicit multipole decomposition, we have defined generally the multipole form factors and worked out their relation with the electromagnetic moments. In the Breit frame, we were able to derive the general relation between the covariant vertex functions and multipole form factors. We naturally recover the low-spin cases studied so far.

Besides the fact that the steps explained in this paper are necessary for our aim of obtaining the natural electromagnetic moments, the results presented here will be relevant for other studies. For example, based on our decomposition in terms of covariant vertex functions, it should be in principle possible to determine within lattice QCD the structure of high-spin resonances. Moreover, we draw the attention that multipole form factors at $Q^2 = 0$ are in general *not* equal to the (Cartesian) electromagnetic moments. The identification is valid only up to quadrupoles.

Subsequently, we have derived the explicit expression of light-cone helicity amplitudes in terms of covariant vertex functions. Under the assumption that light-cone helicity is nontrivially conserved by electromagnetism at $Q^2 = 0$ and tree level, we have derived the natural value of all electromagnetic moments for any particle. The result turns out to be surprisingly simple. The natural values of multipole form factors at $Q^2 = 0$ are just given (up to a sign) by a binomial function times the electric charge of the particle.

The result agrees with the values from the standard model for elementary spin-$1/2$ (e.g. electrons) and spin-1 (e.g. $W^\pm$ gauge bosons) particles. It is also in accordance with the prediction from $\mathcal{N} = 2$ supergravity for gravitinos. Moreover, we also reproduce in a simple way the universality of the gyromagnetic factor $g = 2$ and its counterpart for electric quadrupole, as derived from considerations of tree unitarity. Finally, it has also been realized that this result is in fact exactly the same as one obtains from $\mathcal{N} = 1$ supersymmetric sum rules, when one considers that electromagnetic properties do not mix the members of the supermultiplet. All these agreements can hardly be seen as a pure coincidence. Naturally, one still has to completely understand the deep field theoretic implications of nontrivial light-cone helicity conservation. It seems highly probable that this condition can be related to the tree-unitarity argument. This relation is beyond the scope of the present study but will be the subject of further investigations.

As an application of our results, we have generalized the discussion on quark transverse charge densities to particles of arbitrary spin. Our assumption concerning helicity conservation directly leads to the conclusion that the transverse higher electric moments can only be functions of the anomalous electromagnetic moments. Using the argument the other way around, an elementary particle cannot present *induced* electric moments in a light-cone framework. This requirement is equivalent to saying that light-cone helicity is conserved at tree level and $Q^2 = 0$, justifying our assumption *a posteriori*.

As a final comment, let us add that knowing the natural moments allows one to distinguish in a certain limit between composite and elementary particle already at low Q^2 without any reference to a specific field theory and to determine in a fully consistent way the actual shape of hadrons. The natural moments of a pointlike particle being nonzero, any statement concerning the shape of a hadron based on the values of multipole form factors should compare the actual values not with zero but with the natural moments. That is why, even though its electric quadrupole moment is negative according to lattice QCD calculations, the Δ^+ baryon has a prolate shape when viewed from a light front [24].

ACKNOWLEDGMENTS

The author is grateful to M. Vanderhaeghen, V. Pascalutsa, and T. Ledwig for numerous enlightening discussions and comments.

APPENDIX A: POLARIZATION TENSORS

In this appendix we remind the explicit construction of polarization tensors in RS and KG formalisms.

A particle with spin j and mass M can be described in terms of a polarization tensor $\varepsilon_{\alpha_1\cdots\alpha_j}(p,\lambda)$ when j is integer or a polarization spin tensor $u_{\alpha_1\cdots\alpha_n}(p,\lambda)$ when $j = n + 1/2$ is half-integer. These polarization tensors are completely symmetric and satisfy the following subsidiary conditions[5]

$$p^\mu \varepsilon_{\mu\alpha_2\cdots\alpha_j}(p,\lambda) = 0, \qquad \varepsilon^\mu{}_{\mu\alpha_3\cdots\alpha_j}(p,\lambda) = 0, \tag{A1}$$

$$(\not p - M)u_{\alpha_1\cdots\alpha_n}(p,\lambda) = 0, \qquad \gamma^\mu u_{\mu\alpha_2\cdots\alpha_n}(p,\lambda) = 0, \tag{A2}$$

in order to ensure that the number of degrees of freedom is $2j+1$.

An explicit construction of these polarization tensors has been proposed a long time ago by Auvil and Brehm [8,39]. By always coupling the maximum possible spin of two lower-order polarization tensors, the product will satisfy KG or RS equations together with the subsidiary conditions. For a polarization spin tensor, it is convenient to consider the product of a spin-1/2 spinor with a polarization tensor

$$\begin{aligned} u_{\alpha_1\cdots\alpha_n}(p,\lambda) &= \sum_{m,m'}\left\langle\frac{1}{2}\,\frac{m}{2}, nm'\Big|j\lambda\right\rangle u(p,m)\varepsilon_{\alpha_1\cdots\alpha_n}(p,m') \\ &= \sqrt{\frac{j+\lambda}{2j}}u(p,+)\varepsilon_{\alpha_1\cdots\alpha_n}\left(p,\lambda-\frac{1}{2}\right) \\ &\quad + \sqrt{\frac{j-\lambda}{2j}}u(p,-)\varepsilon_{\alpha_1\cdots\alpha_n}\left(p,\lambda+\frac{1}{2}\right), \end{aligned} \tag{A3}$$

where $\langle j_1m_1, j_2m_2|jm\rangle$ represents the Clebsch-Gordan coefficient in the Condon-Shortley phase convention. With such a construction, one can focus on integer spin polarization tensors only. The latter are obtained from the following recursion formula

$$\begin{aligned}\varepsilon_{\alpha_1\cdots\alpha_j}(p,\lambda) = \sum_{m,m'}\langle 1m, (j &\\ - 1)m'|j\lambda\rangle\varepsilon_{\alpha_1}(p,m)\varepsilon_{\alpha_2\cdots\alpha_j}(p,m'),&\end{aligned} \tag{A4}$$

where $\varepsilon_{\alpha_i}(p,\lambda)$ is just the standard polarization four-vector. Iterating this formula, one obtains

$$\begin{aligned}\varepsilon_{\alpha_1\cdots\alpha_j}(p,\lambda) = \sum_{m_l=0,\pm1}\Bigg[\prod_{l=1}^{j-1}\langle 1m_l lm'_l|(l+1)m'_{l+1}\rangle \\ \times\,\varepsilon_{\alpha_l}(p,m_l)\Bigg]\varepsilon_{\alpha_j}(p,m_j),\end{aligned}$$

where the sum is implicitly restricted to configurations such that $\sum_{l=1}^j m_l = \lambda$, and where $m'_l = m_j + \sum_{k=1}^{l-1} m_k$. Since Clebsch-Gordan coefficients can be written as [29]

$$\begin{aligned}\langle 1m_l lm'_l|(l+1)(m_l+m'_l)\rangle &= \sqrt{\frac{C_2^{1+m_l}C_{2l}^{l+m'_l}}{C_{2l+2}^{l+m_l+m'_l+1}}}, \\ C_n^k = \begin{pmatrix} n \\ k\end{pmatrix} &\equiv \begin{cases}\frac{n!}{k!(n-k)!}, & n\geq k\geq 0 \\ 0, & \text{otherwise}\end{cases},\end{aligned} \tag{A5}$$

one obtains the expression

$$\varepsilon_{\alpha_1\cdots\alpha_j}(p,\lambda) = \sum_{m_l=0,\pm1}\frac{\prod_{l=1}^j\sqrt{C_2^{1+m_l}}\,\varepsilon_{\alpha_l}(p,m_l)}{\sqrt{C_{2j}^{j+\lambda}}},$$

which can be rewritten more conveniently as [40]

[5] Note that multiplying (A2) on the left by γ^{α_2} implies that (A1) with $\varepsilon_{\alpha_1\cdots\alpha_j}$ replaced by $u_{\alpha_1\cdots\alpha_n}$ is also satisfied.

$$\varepsilon_{\alpha_1\cdots\alpha_j}(p, j-m) = \sum_{k=0}^{m/2} \frac{\sum_{\mathcal{P}}[\prod_{l=1}^{k}\varepsilon_{\alpha_{\mathcal{P}(l)}}(p,-)][\prod_{l=k+1}^{m-k}\varepsilon_{\alpha_{\mathcal{P}(l)}}(p,0)][\prod_{l=m-k+1}^{j}\varepsilon_{\alpha_{\mathcal{P}(l)}}(p,+)]}{2^{k-m/2}k!(m-2k)!(j-m+k)!\sqrt{C_{2j}^{m}}}, \tag{A6}$$

where $\mathcal{P}$ stands for a permutation of $\{1, \cdots, j\}$. The presence of factorials in the denominator is due to the fact that permuting the indices of two polarization four-vectors with the same polarization does not give a new contribution.

APPENDIX B: LINKING COVARIANT VERTEX FUNCTIONS WITH MULTIPOLE FORM FACTORS

In this appendix, we report details of the derivation in the Breit frame, which led us to the connections (19) and (23) between covariant vertex functions and multipole form factors.

1. Breit frame kinematics and expansion of spherical harmonics

The Breit frame (or brick-wall frame) is the frame where no energy is transferred to the system by the photon[6]

$$q^\mu = (0, \vec{q}), \qquad p^\mu = \left(p_0, -\frac{\vec{q}}{2}\right), \qquad p'^\mu = \left(p_0, \frac{\vec{q}}{2}\right),$$
$$p_0 = \sqrt{M^2 + \frac{|\vec{q}|^2}{4}} = M\sqrt{1+\tau}. \tag{B1}$$

In this particular frame, the current J_B^μ has a nonrelativistic appearance once explicitly expressed in terms of rest-frame polarization vectors $\vec{\varepsilon}_\lambda$ and rest-frame spinors χ_λ [26]. Since we identify the spin quantization axis with the z axis, the rest-frame polarization vectors and spinors are given by

$$\vec{\varepsilon}_{\pm} = \frac{1}{\sqrt{2}}(\mp 1, -i, 0), \qquad \vec{\varepsilon}_0 = (0,0,1),$$
$$\chi_+ = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \chi_- = \begin{pmatrix}0\\1\end{pmatrix}. \tag{B2}$$

In the following, we will use standard polarization four-vectors and spinors in Dirac representation

$$\varepsilon^\mu(p,\lambda) = \left(\frac{\vec{\varepsilon}_\lambda\cdot\vec{p}}{M}, \vec{\varepsilon}_\lambda + \frac{\vec{p}(\vec{\varepsilon}_\lambda\cdot\vec{p})}{M(p_0+M)}\right),$$
$$u(p,\lambda) = \sqrt{p_0+M}\begin{pmatrix}1\\ \frac{\vec{p}\cdot\vec{\sigma}}{p_0+M}\end{pmatrix}\chi_\lambda. \tag{B3}$$

The connection between covariant vertex functions and multipole form factors can conveniently be achieved once the spherical harmonics[7] $Y_{l0}(\Omega)$ involved in (13) and (17) are expressed in terms of $\sin\theta$ and $\cos\theta$ [29]

$$Y_{l0}(\Omega) = \sqrt{\frac{2l+1}{4\pi}}\sum_{\substack{s=0\\ s\ \text{even}}}^{l}(-1)^{s/2} \times \begin{cases}\tilde{C}_l^s\tilde{C}_{l+s-1}^{l-1}\sin^s\theta, & \text{for } l \text{ even},\\ \tilde{C}_{l-1}^{l-s-1}\tilde{C}_{l+s}^{s}\sin^s\theta\cos\theta, & \text{for } l \text{ odd},\end{cases} \tag{B4}$$

leading, after a few algebraic manipulations, to

$$j^0(\vec{q}) = e\sum_{t=0}^{n}\sum_{m=t}^{n}(-1)^{m+t}\tau^m\frac{(C_m^t)^2}{\tilde{C}_{4m-1}^{2m+2t-1}}(\sin^2\theta)^t \times G_{E2m}(Q^2),$$
$$\vec{\nabla}\cdot(\vec{j}(\vec{q})\times\vec{q}) = e2i\sqrt{\tau}\cos\theta\sum_{t=0}^{n}\sum_{m=t}^{n}(-1)^{m+t}(1-\delta_{j,m}) \times \tau^m(m+1)\frac{(C_m^t)^2}{\tilde{C}_{4m+1}^{2m+2t+1}}(\sin^2\theta)^t \times G_{M2m+1}(Q^2), \tag{B5}$$

where $j = n$ for integer spin and $j = n + 1/2$ for half-integer spin.

2. Integer spin case

Considering $\lambda = \lambda' = j$, the polarization tensor (A6) of the particle takes a very simple form

$$\varepsilon_{\alpha_1\cdots\alpha_j}(p,j) = \prod_{l=1}^{j}\varepsilon_{\alpha_l}(p,+).$$

Concerning the charge density of integer spin particles, we find that all the complexity reduces to three structures

[6]The reader might be worried by the fact that the definition of momentum Q is different in Sec. II compared to Sec. III. Covariant vertex functions and multipole form factors are however related in the Breit frame, where both definitions do actually match $Q^2 \equiv -q^2 = \vec{q}^2$.

[7]For the sake of clarity, we omit the index q attached to the angles, since we will not refer anymore to configuration space.

$$\begin{aligned} \varepsilon^*(p', +) \cdot \varepsilon(p, +) &= -1 - \tau\sin^2\theta, \\ \frac{[\varepsilon^*(p', +) \cdot q][\varepsilon(p, +) \cdot q]}{2M^2} &= (1 + \tau)\tau\sin^2\theta, \\ \varepsilon^0(p, +)[\varepsilon^*(p', +) \cdot q] - \varepsilon^{0*}(p', +)[\varepsilon(p, +) \cdot q] &= 2p_0\tau\sin^2\theta, \end{aligned} \tag{B6}$$

allowing us to write the quantum-mechanical electric density in terms of $\sin\theta$

$$J_B^0 = 2M \sum_{t=0}^{j} \sum_{k=0}^{t} (1 + \tau)^{k+(1/2)} C_{j-k}^{j-t} (\tau\sin^2\theta)^t \times \left[F_{2k+1}(Q^2) - \frac{1 - \delta_{k,0}}{1 + \tau} F_{2k}(Q^2) \right]. \tag{B7}$$

Concerning the magnetic density of integer spin particles, the first simplification is $\vec{P} \times \vec{q} = 0$, so that only covariant vertex functions with even value of k will contribute. Moreover, since in the Breit frame

$$\begin{aligned} \vec{\nabla} \cdot [\vec{\varepsilon}(p, \lambda) \times \vec{q}] &= 0, \\ [\vec{\varepsilon}(p, \lambda) \times \vec{q}] \cdot \vec{\nabla}[\varepsilon(p, \lambda) \cdot q] &= 0, \\ [\vec{\varepsilon}(p, \lambda) \times \vec{q}] \cdot \vec{\nabla}[\varepsilon^*(p', \lambda') \cdot q] &= -\frac{p_0}{M} (\vec{\varepsilon}^*_{\lambda'} \times \vec{\varepsilon}_\lambda) \cdot \vec{q}, \end{aligned} \tag{B8}$$

we obtain

$$\begin{aligned} &\vec{\nabla} \cdot \left\{ \{\vec{\varepsilon}(p, +)[\varepsilon^*(p', +) \cdot q] - \vec{\varepsilon}^*(p', +)[\varepsilon(p, +) \cdot q]\} \times \vec{q} \left(\frac{[\varepsilon^*(p', +) \cdot q][\varepsilon(p, +) \cdot q]}{2M^2} \right)^k f(Q^2) \right\} \\ &= -2p_0(k + 1)2i\sqrt{\tau}\cos\theta(\tau\sin^2\theta)^k f(Q^2), \end{aligned}$$

so that we can write the quantum-mechanical magnetic density in terms of $\sin\theta$ and $\cos\theta$

$$\vec{\nabla} \cdot (\vec{J}_B \times \vec{q}) = 4iM\sqrt{\tau}\cos\theta \sum_{t=0}^{j-1} \sum_{k=0}^{t} (1 + \tau)^{k+(1/2)} C_{j-k-1}^{j-t-1} \times (\tau\sin^2\theta)^t (t + 1) F_{2k+2}(Q^2). \tag{B9}$$

Now comparing (B5) with (B7) and (B9), we obtain (19).

3. Half-integer spin case

Considering $\lambda = \lambda' = j = n + 1/2$, the polarization spin-tensor (A3) of the particle takes a very simple form

$$u_{\alpha_1 \cdots \alpha_n}(p, j) = u(p, +) \prod_{l=1}^{n} \varepsilon_{\alpha_l}(p, +).$$

Concerning the charge density of half-integer spin particles, we find that all the complexity reduces to the three structures of the bosonic case (B6) *plus* two new ones

$$\begin{aligned} \bar{u}(p', +)\gamma^0 u(p, +) &= 2M, \\ \bar{u}(p', +)\frac{i\sigma^{0\nu} q_\nu}{2M} u(p, +) &= -2M\tau, \end{aligned} \tag{B10}$$

allowing us to write the quantum-mechanical electric density in terms of $\sin\theta$

$$J_B^0 = 2M \sum_{t=0}^{n} \sum_{k=0}^{t} (1 + \tau)^k C_{n-k}^{n-t} (\tau\sin^2\theta)^t \times [F_{2k+1}(Q^2) - \tau F_{2k+2}(Q^2)]. \tag{B11}$$

Concerning the magnetic density of half-integer spin particles, we have in addition

$$\begin{aligned} \bar{u}(p', +)\gamma^k u(p, +) &= \bar{u}(p', +)\frac{i\sigma^{k\nu} q_\nu}{2M} u(p, +) \\ &= \chi_+^\dagger i(\vec{\sigma} \times \vec{q})^k \chi_+, \end{aligned} \tag{B12}$$

from which we obtain

$$\begin{aligned} &\vec{\nabla} \cdot \left\{ \chi_+^\dagger [i(\vec{\sigma} \times \vec{q}) \times \vec{q}] \chi_+ \left[\frac{(\varepsilon'^*(+1) \cdot q)(\varepsilon(+1) \cdot q)}{2M^2} \right]^k f(Q^2) \right\} \\ &= 4M(k + 1)i\sqrt{\tau}\cos\theta(\tau\sin^2\theta)^k f(Q^2), \end{aligned}$$

allowing us to write the quantum-mechanical magnetic density in terms of $\sin\theta$ and $\cos\theta$

$$\vec{\nabla} \cdot (\vec{J}_B \times \vec{q}) = 4Mi\sqrt{\tau}\cos\theta \sum_{t=0}^{n} \sum_{k=0}^{t} (1 + \tau)^k C_{n-k}^{n-t} (\tau\sin^2\theta)^t \times (t + 1)[F_{2k+1}(Q^2) + F_{2k+2}(Q^2)]. \tag{B13}$$

Now comparing (B5) with (B11) and (B13), we obtain (23).

APPENDIX C: HELICITY AMPLITUDES AND COVARIANT VERTEX FUNCTIONS

In this appendix, we report details of the derivation of the expression of light-cone helicity amplitudes in terms of covariant vertex functions (29).

1. Light-cone kinematics

Light-cone helicity amplitudes are obtained by considering the + component of the current $J^+ = J^0 + J^3$ and the proper expressions for the light-cone spinors ($p_{R,L} \equiv p_x \pm ip_y$)

$$u(p,+)=\frac{1}{\sqrt{2p^+}}\begin{pmatrix}p^++M\\ p_R\\ p^+-M\\ p_R\end{pmatrix},\qquad u(p,-)=\frac{1}{\sqrt{2p^+}}\begin{pmatrix}-p_L\\ p^++M\\ p_L\\ -p^++M\end{pmatrix}, \tag{C1}$$

and the light-cone polarization four-vectors ($\hat{e}_{R,L}\equiv\hat{e}_x\pm i\hat{e}_y$)

$$\begin{aligned}\varepsilon^\mu(p,+)&=-\frac{1}{\sqrt{2}}\left(0,\frac{2p_R}{p^+},\hat{e}_R\right),\\ \varepsilon^\mu(p,0)&=\frac{1}{M}\left(p^+,\frac{p_Rp_L-M^2}{p^+},\frac{p_R\hat{e}_L+p_L\hat{e}_R}{2}\right),\\ \varepsilon^\mu(p,-)&=\frac{1}{\sqrt{2}}\left(0,\frac{2p_L}{p^+},\hat{e}_L\right).\end{aligned} \tag{C2}$$

It is particularly convenient to work in the symmetric light-cone frame, which is the DYW frame $q^+=0$ where the light-cone energy $p^-=p^0-p^3$ is conserved

$$q^\mu=(0,0,\vec{q}_\perp),\qquad p^\mu=\left(p^+,\frac{M^2(1+\tau)}{p^+},-\frac{\vec{q}_\perp}{2}\right),\qquad p'^\mu=\left(p^+,\frac{M^2(1+\tau)}{p^+},\frac{\vec{q}_\perp}{2}\right), \tag{C3}$$

with p and p' the four-momenta of the incoming and outgoing particle, respectively. Let us now consider only the set of amplitudes $\{A_{j,j-m}|m=0,\cdots,2j\}$, separately for integer and half-integer spin particles.

2. Integer spin case

In the DYW frame and for spin-j bosons, thanks to the following relations

$$\begin{aligned}\varepsilon^*(p',+)\cdot\varepsilon(p,0)&=\sqrt{2\tau}e^{-i\phi_q}[\varepsilon^*(p',+)\cdot\varepsilon(p,+)],\\ \varepsilon^*(p',+)\cdot\varepsilon(p,-)&=0,\\ \varepsilon(p,0)\cdot q&=\sqrt{2\tau}e^{-i\phi_q}[\varepsilon(p,+)\cdot q],\\ \varepsilon(p,-)\cdot q&=-e^{-2i\phi_q}[\varepsilon(p,+)\cdot q],\end{aligned} \tag{C4}$$

it is straightforward to see that we can rewrite all the contractions of Lorentz indices in terms of $\varepsilon^*(p',+)\cdot\varepsilon(p,+)$ and $[\varepsilon^*(p',+)\cdot q][\varepsilon(p,+)\cdot q]$ only. Moreover, since we have

$$\begin{aligned}\varepsilon^*(p',+)\cdot\varepsilon(p,+)&=-1,\\ -\frac{[\varepsilon^*(p',+)\cdot q][\varepsilon(p,+)\cdot q]}{2M^2}&=-\tau,\end{aligned} \tag{C5}$$

and using the expression for the polarization tensor (A6) with our current decomposition (4), we finally obtain

$$A_{j,j-m}(Q^2)=\frac{(4\tau)^{m/2}}{\sqrt{C_{2j}^m}}\sum_{k=0}^{j}\sum_{t=0}^{\min\{k,m/2\}}\frac{(-1)^t}{2^{2t}}C_k^t\tau^{k-t}\times\left[C_{j-t}^{m-2t}F_{2k+1}(Q^2)-\frac{1-\delta_{k,j}}{2}C_{j-t-1}^{m-2t-1}F_{2k+2}(Q^2)\right]. \tag{C6}$$

3. Half-integer spin case

For fermions with spin $j=n+1/2$, thanks to (C4) and (C5) and the following relations:

$$\begin{aligned}\bar{u}(p',+)\gamma^+u(p,+)&=2p^+,\\ \bar{u}(p',+)\gamma^+u(p,-)&=0,\\ \bar{u}(p',+)\frac{i\sigma^{+\nu}q_\nu}{2M}u(p,+)&=0,\\ \bar{u}(p',+)\frac{i\sigma^{+\nu}q_\nu}{2M}u(p,-)&=-\sqrt{\tau}e^{-i\phi_q}2p^+,\end{aligned} \tag{C7}$$

we can easily obtain the light-cone helicity amplitudes in terms of covariant vertex functions $F_k(q^2)$, using the expression for the polarization tensor (A6) with our current decomposition (5),

$$A_{j,j-m}(Q^2)=\frac{(4\tau)^{m/2}}{\sqrt{C_{2j}^m}}\sum_{k=0}^{n}\sum_{t=0}^{\min\{k,m/2\}}\frac{(-1)^t}{2^{2t}}C_k^t\tau^{k-t}\times\left[C_{n-t}^{m-2t}F_{2k+1}(Q^2)-\frac{1}{2}C_{n-t}^{m-2t-1}F_{2k+2}(Q^2)\right]. \tag{C8}$$

Now, it is straightforward to see that (C6) and (C8) can in fact be written in a single formula (29).

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