Multidimensional Generalized Automatic Sequences and Shape-Symmetric Morphic Words

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k-ary numeration system, $k \geq 2$
$\Sigma_k = \{0, \ldots, k-1\}$

$$n = \sum_{i=0}^{\ell} d_i\, k^i,\ d_\ell \neq 0, \quad \mathrm{rep}_k(n) = d_\ell \cdots d_0 \in {\Sigma_k}^*$$

An infinite word $x = (x_n)_{n\geq 0}$ is *k-automatic* if there exists a DFAO $\mathcal{A} = (Q, q_0, \Sigma_k, \delta, \Gamma, \tau)$ s.t. for all $n \geq 0$,

$$x_n = \tau(\delta(q_0, \mathrm{rep}_k(n))).$$

THEOREM (COBHAM)

Let $k \geq 2$. An infinite word is k-automatic iif it is the image under a coding of an infinite fixed point of a k-uniform morphism.

Definition

An *abstract numeration system* is a triple $S = (L, \Sigma, <)$ where L is a regular language over a totally ordered alphabet $(\Sigma, <)$.

Enumerating the words of L with respect to the genealogical ordering induced by $<$ gives a one-to-one correspondence

$$\mathrm{rep}_S : \mathbb{N} \to L \qquad \mathrm{val}_S = \mathrm{rep}_S^{-1} : L \to \mathbb{N}.$$

Example

$L = a^*, \ \Sigma = \{a\}$

n	0	1	2	3	4	$\cdots$
rep (n)	ε	a	aa	aaa	$aaaa$	$\cdots$

Example

$L = \{a, b\}^*, \ \Sigma = \{a, b\}, \ a < b$

n	0	1	2	3	4	5	6	7	$\cdots$
$\text{rep}(n)$	ε	a	b	aa	ab	ba	bb	aaa	$\cdots$

Example

$L = a^* b^*, \ \Sigma = \{a, b\}, \ a < b$

n	0	1	2	3	4	5	6	$\cdots$
$\text{rep}(n)$	ε	a	b	aa	ab	bb	aaa	$\cdots$

Definition

Let $S = (L, \Sigma, <)$ be an abstract numeration system.
An infinite word $x = (x_n)_{n\geq 0}$ is *S-automatic* if there exists a DFAO $\mathcal{A} = (Q, q_0, \Sigma, \delta, \Gamma, \tau)$ s.t. for all $n \geq 0$,

$$x_n = \tau(\delta(q_0, \mathrm{rep}_S(n))).$$

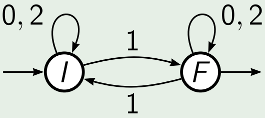
Theorem (Maes, Rigo)

An infinite word is S-automatic for some abstract numeration system S iif it is the image under a coding of an infinite fixed point of a morphism, i.e. a morphic word.

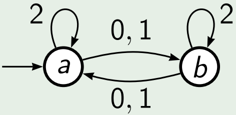
Example (S-Automatic → Morphic)

$S = (L, \{0,1,2\}, 0 < 1 < 2)$ where $L = \{w \in \Sigma^* : |w|_1 \text{ is odd}\}$

minimal automaton of L

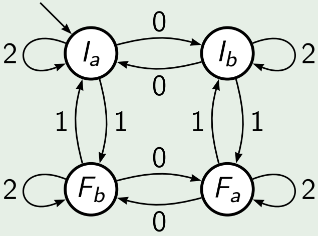


DFAO generating x



n	0	1	2	3	4	5	6	7	8	$\cdots$
$\mathrm{rep}_S(n)$	1	01	10	12	21	001	010	012	021	$\cdots$
x	b	a	a	b	b	b	b	a	a	$\cdots$

Example (Continued)



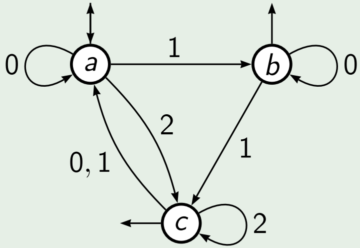
$$
\begin{aligned}
f : \alpha &\mapsto \alpha I_a & F_a &\mapsto F_b I_b F_a & g : \alpha, I_a, I_b &\mapsto \varepsilon \\
I_a &\mapsto I_b F_b I_a & F_b &\mapsto F_a I_a F_b & F_a &\mapsto a \\
I_b &\mapsto I_a F_a I_b & & & F_b &\mapsto b
\end{aligned}
$$

$L \subseteq \Sigma^*$		ε	0	1	2	00	01	02	10	11	12
$f^\omega(\alpha)$	α	I_a	I_b	F_b	I_a	I_a	F_a	I_b	F_a	I_a	F_b
x				b			a		a		b

$$g(f^\omega(\alpha)) = x$$

Example (Morphic → S-Automatic)

Consider the morphism μ defined by $a \mapsto abc$; $b \mapsto bc$; $c \mapsto aac$.
We have $\mu^\omega(a) = abcbcaacbcaacabcabcaacbcaacabcabc\cdots$.
One canonically associates the DFA $\mathcal{A}_{\mu,a}$



$L_{\mu,a} = \{\varepsilon, 1, 2, 10, 11, 20, 21, 22, 100, 101, 110, 111, 112, 200, \ldots\}$

If $S = (L_{\mu,a}, \{0,1,2\}, 0 < 1 < 2)$, then

$$(\mu^\omega(a))_n = \delta_\mu(a, \mathrm{rep}_S(n)) \text{ for all } n \geq 0.$$

A *d-dimensional infinite word* over an alphabet Σ is a map $x : \mathbb{N}^d \to \Sigma$. We use notation like $x_{n_1,\dots,n_d}$ or $x(n_1,\dots,n_d)$ to denote the value of x at $(n_1,\dots,n_d)$.

If $w_1,\dots,w_d$ are finite words over the alphabet Σ,

$$(w_1,\dots,w_d)^{\#} := (\#^{m-|w_1|}w_1,\dots,\#^{m-|w_d|}w_d)$$

where $m = \max\{|w_1|,\dots,|w_d|\}$.

Example

$(ab, bbaa)^{\#} = (\#\#ab, bbaa)$

A d-dimensional infinite word over an alphabet Γ is *k-automatic* if there exists a DFAO

$$\mathcal{A} = (Q, q_0, (\Sigma_k)^d \setminus \{0, \ldots, 0\}, \delta, \Gamma, \tau)$$

s.t. for all $n_1, \ldots, n_d \geq 0$,

$$\tau\left(\delta\left(q_0, (\text{rep}_k(n_1), \ldots, \text{rep}_k(n_d))^0\right)\right) = x_{n_1,\ldots,n_d}.$$

Theorem (Salon)

Let $k \geq 2$ and $d \geq 1$. A d-dimensional infinite word is k-automatic iif it is the image under a coding of a fixed point of a k-uniform d-dimensional morphism.

A *d-dimensional picture* over the alphabet Σ is a map

$$x\colon [\![0, s_1 - 1]\!] \times \cdots \times [\![0, s_d - 1]\!] \to \Sigma.$$

The *shape* of x is $|x| = (s_1, \ldots, s_d)$.

If $s_i < \infty$ for all $i \in [\![1, d]\!]$, then x is said to be *bounded*.

The set of d-dimensional bounded pictures over Σ is $B_d(\Sigma)$.

A bounded picture x is a *square* of *size* c if $s_i = c$ for all $i \in [\![1, d]\!]$.

Example

Consider the two bidimensional pictures

$$x = \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} \quad \text{and} \quad y = \begin{array}{|c|c|c|} \hline a & a & b \\ \hline b & c & d \\ \hline \end{array}$$

of shapes $|x| = (2,2)$ and $|y| = (3,2)$ respectively.
Since $|x|_2 = |y|_2 = 2$, we get

$$x \odot^1 y = \begin{array}{|c|c|c|c|c|} \hline a & b & a & a & b \\ \hline c & d & b & c & d \\ \hline \end{array}.$$

But notice that $x \odot^2 y$ is not defined because $2 = |x|_1 \neq |y|_1 = 3$.

Example

Consider the map μ given by

$$a \mapsto \begin{array}{|c|c|}\hline a & a \\ \hline b & d \\ \hline\end{array}, \quad b \mapsto \begin{array}{|c|}\hline c \\ \hline b \\ \hline\end{array}, \quad c \mapsto \begin{array}{|c|c|}\hline a & a \\ \hline\end{array}, \quad d \mapsto \begin{array}{|c|}\hline d \\ \hline\end{array}.$$

Let

$$x = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}.$$

Since $|\mu(a)|_2 = |\mu(b)|_2 = 2$, $|\mu(c)|_2 = |\mu(d)|_2 = 1$, $|\mu(a)|_1 = |\mu(c)|_1 = 2$ and $|\mu(b)|_1 = |\mu(d)|_1 = 1$, $\mu(x)$ is well defined and given by

$$\mu(x) = \begin{array}{|c|c|c|}\hline a & a & c \\ \hline b & d & b \\ \hline a & a & d \\ \hline\end{array}.$$

Notice that $\mu^2(x)$ is not well defined.

Definition

Let $\mu : \Sigma \to B_d(\Sigma)$ be a map. If for all $a \in \Sigma$ and all $n \geq 0$, $\mu^n(a)$ is well defined from $\mu^{n-1}(a)$, then μ is said to be a *d-dimensional morphism*.

Example

If for all $a \in \Sigma$, $\mu(a)$ has a fixed shape (s_1, s_2), then the map μ is a morphism.

Definition

If for all $a \in \Sigma$, $\mu(a)$ is a square of size k, then μ is said to be a *k-uniform morphism*.

Let μ be a d-dimensional morphism and a be a letter such that $(\mu(a))_{\mathbf{0}} = a$. We say that μ is *prolongable on a* and the limit

$$w = \mu^{\omega}(a) := \lim_{n \to +\infty} \mu^{n}(a)$$

is well defined and $w = \mu(w)$ is a *fixed point* of μ.

A d-dimensional infinite word x over Σ is *purely morphic* if it is a fixed point of a d-dimensional morphism.

It is *morphic* if there exists a coding $\nu : \Gamma \to \Sigma$ such that $x = \nu(y)$ for some purely morphic word y over Γ.

Definition

Let $S = (L, \Sigma, <)$ be an abstract numeration system.
A *d-dimensional infinite word* over the alphabet Γ is *S-automatic* if there exists a DFAO

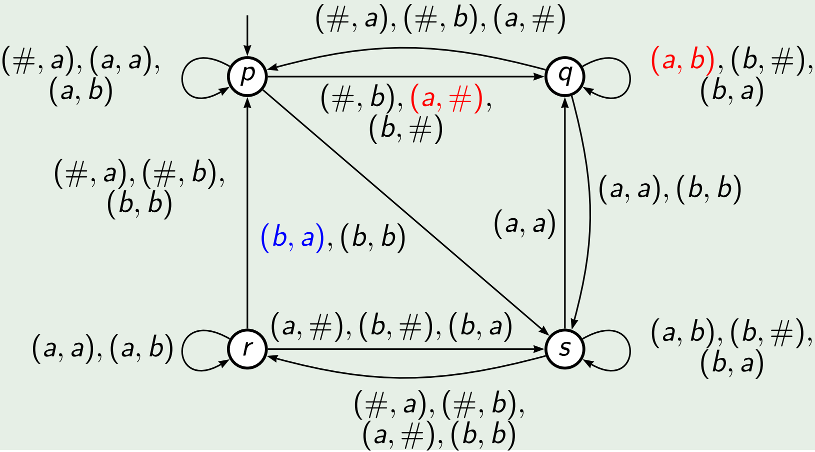
$$\mathcal{A} = (Q, q_0, (\Sigma \cup \{\#\})^d, \delta, \Gamma, \tau)$$

s.t. for all $n_1, \ldots, n_d \geq 0$,

$$\tau\left(\delta\left(q_0, (\text{rep}_S(n_1), \ldots, \text{rep}_S(n_d))^{\#}\right)\right) = x_{n_1,\ldots,n_d}.$$

Example

Consider $S = (\{a, ba\}^*\{\varepsilon, b\}, \{a, b\}, a < b)$ and the DFAO



Example (Continued)

We produce the following bidimensional infinite S-automatic word :

	ε	a	b	aa	ab	ba	aaa	aab	$\cdots$
ε	p	q	q	p	q	p	q	q	$\cdots$
a	p	p	s	s	q	s	p	s	
b	q	p	s	q	s	q	p	s	
aa	p	p	s	p	s	q	q	s	
ab	q	p	s	p	s	s	s	r	
ba	p	s	q	p	s	q	s	q	
aaa	p	p	s	p	s	q	p	s	
aab	q	p	s	p	s	s	p	s	
$\vdots$	$\vdots$								$\ddots$

Definition (Maes)

Let $\mu : \Sigma \to B_d(\Sigma)$ be a d-dimensional morphism having the d-dimensional infinite word x as a fixed point.

If for any permutation f of $\{1, \ldots, d\}$ and for all $n_1, \ldots, n_d > 0$,

$$|\mu(x_{n_1,\ldots,n_d})| = (s_1, \ldots, s_d)$$
$$\Downarrow$$
$$|\mu(x_{n_{f(1)},\ldots,n_{f(d)}})| = (s_{f(1)}, \ldots, s_{f(d)}),$$

then x is said to be *shape-symmetric with respect to μ*.

Example

$$\mu(a)=\mu(f)=\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array},\ \mu(b)=\begin{array}{|c|}\hline e \\ \hline c \\ \hline\end{array},\ \mu(c)=\begin{array}{|c|c|}\hline e & b \\ \hline\end{array},\mu(d)=\begin{array}{|c|}\hline f \\ \hline\end{array},$$

$$\mu(e)=\begin{array}{|c|c|}\hline e & b \\ \hline g & d \\ \hline\end{array},\ \mu(g)=\begin{array}{|c|c|}\hline h & b \\ \hline\end{array},\ \mu(h)=\begin{array}{|c|c|}\hline h & b \\ \hline c & d \\ \hline\end{array}.$$

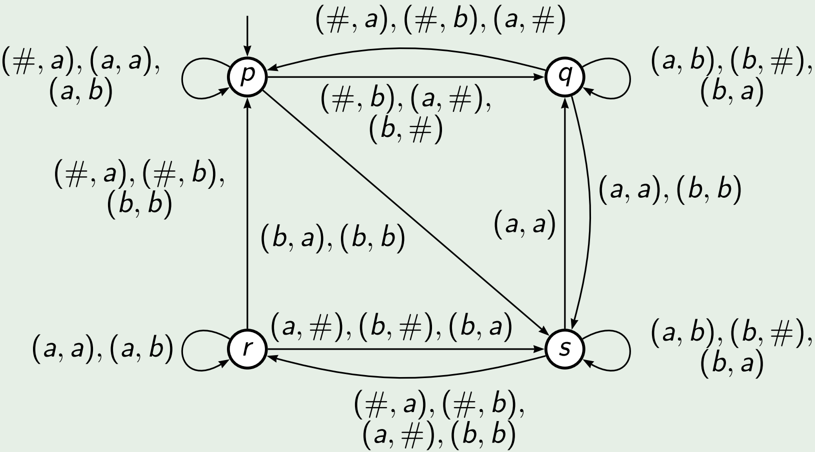
$$\mu^{\omega}(a)=\begin{array}{|cc|c|cc|cc|c|c}
\hline
a & b & e & e & b & e & b & e & \cdots \\
c & d & c & g & d & g & d & c & \\
\hline
e & b & f & e & b & h & b & f & \\
\hline
e & b & e & a & b & e & b & e & \\
g & d & c & c & d & g & d & c & \\
\hline
e & b & e & e & b & a & b & e & \\
g & d & c & g & d & c & d & c & \\
\hline
h & b & f & e & b & e & b & f & \\
\hline
\vdots & & & & & & & & \ddots
\end{array}$$

Theorem (C., Kärki, Rigo)

Let $d \geq 1$. The d-dimensional infinite word x is S-automatic for some abstract numeration system $S = (L, \Sigma, <)$ where $\varepsilon \in L$ iif x is the image under a coding of a shape-symmetric infinite d-dimensional word.

Example (S-Automatic → Shape-Symmetric)

Consider $S = (\{a, ba\}^*\{\varepsilon, b\}, \{a, b\}, a < b)$ and the DFAO $\mathcal{A}$



Example (Continued)

One associates a uniform bidimensional morphism $\mu_{\mathcal{A}}$ to $\mathcal{A}$:
If $\sigma_0 = \#$, $\sigma_1 = a$ and $\sigma_2 = b$, then

$$\mu_{\mathcal{A}}(t) = z \text{ with } z_{m,n} = \delta(t, (\sigma_m, \sigma_n)).$$

$$\mu_{\mathcal{A}}(p) = \begin{array}{|c|c|c|}\hline p & q & q \\ \hline p & p & s \\ \hline q & p & s \\ \hline \end{array}\ ;\ \mu_{\mathcal{A}}(q) = \begin{array}{|c|c|c|}\hline q & p & q \\ \hline p & s & q \\ \hline p & q & s \\ \hline \end{array}\ ;$$

$$\mu_{\mathcal{A}}(r) = \begin{array}{|c|c|c|}\hline r & s & s \\ \hline p & r & s \\ \hline p & r & p \\ \hline \end{array}\ ;\ \mu_{\mathcal{A}}(s) = \begin{array}{|c|c|c|}\hline s & r & s \\ \hline r & q & s \\ \hline r & s & r \\ \hline \end{array}\ .$$

Iterating $\mu_{\mathcal{A}}$ from p, we obtain ...

Example (Continued)

$L \times L$	ε	a	b	##	#a	#b	a#	aa	ab	b#	ba	bb	$\cdots$
ε	p	q	q	q	p	q	q	p	q	q	p	q	$\cdots$
a	p	p	s	p	s	q	p	s	q	p	s	p	
b	q	p	s	p	q	s	p	q	s	p	q	s	
##	p	q	q	p	q	q	s	r	s	p	q	q	
#a	p	s	p	p	s	r	q	s	p	p	p	s	
#b	q	p	s	q	p	s	r	s	r	q	p	s	
a#	q	p	q	p	q	q	s	r	s	p	q	q	
aa	p	s	q	p	p	s	r	q	s	p	p	s	
ab	p	q	s	q	p	s	r	s	r	q	p	s	
b#	p	q	q	q	p	q	q	p	q	p	q	q	
ba	p	p	s	p	s	q	p	s	q	p	p	s	
bb	q	p	s	p	q	s	p	q	s	p	q	s	
	$\vdots$												$\ddots$

The subword in blue is the S-automatic word from the beginning.

Idea of the Proof in Dimension 2

Example (Shape-Symmetric → S-Automatic)

$$\mu(a) = \mu(f) = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array};\ \mu(b) = \begin{array}{|c|}\hline e \\ \hline c \\ \hline\end{array};\ \mu(c) = \begin{array}{|c|c|}\hline e & b \\ \hline\end{array};\ \mu(d) = \begin{array}{|c|}\hline f \\ \hline\end{array}$$

$$\mu(e) = \begin{array}{|c|c|}\hline e & b \\ \hline g & d \\ \hline\end{array};\ \mu(g) = \begin{array}{|c|c|}\hline h & b \\ \hline\end{array};\ \mu(h) = \begin{array}{|c|c|}\hline h & b \\ \hline c & d \\ \hline\end{array}.$$

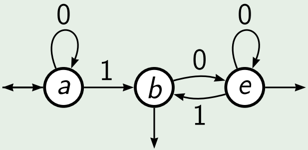
$$\mu^{\omega}(a) = \begin{array}{|cc|c|cc|cc|c|c}
\hline
a & b & e & e & b & e & b & e & \cdots \\
c & d & c & g & d & g & d & c & \\
\hline
e & b & f & e & b & h & b & f & \\
\hline
e & b & e & a & b & e & b & e & \\
g & d & c & c & d & g & d & c & \\
\hline
e & b & e & e & b & a & b & e & \\
g & d & c & g & d & c & d & c & \\
\hline
h & b & f & e & b & e & b & f & \\
\hline
\vdots & & & & & & & & \ddots
\end{array}$$

Example (Continued)

Consider the morphism μ_1 defined by

$$a \mapsto ab\,;\, b \mapsto e\,;\, e \mapsto eb.$$

We have $\mu_1^\omega(a) = abeebebeebeebebeebebeebeeb\cdots$.
One canonically associates the DFA $\mathcal{A}_{\mu_1,a}$



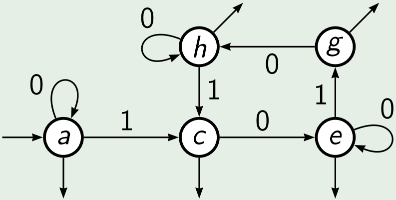
$$L_{\mu_1,a} = \{\varepsilon, 1, 10, 100, 101, 1000, 1001, 1010, 10000, \ldots\}$$

Example (Continued)

Consider the morphism μ_2 defined by

$$a \mapsto ac\ ; c \mapsto e\ ; e \mapsto eg\ ; g \mapsto h\ ; h \mapsto hc.$$

We have $\mu_2^\omega(a) = aceegegheghhceghhchceeghhchce \cdots$.
One canonically associates the DFA $\mathcal{A}_{\mu_2,a}$



$$L_{\mu_2,a} = \{\varepsilon, 1, 10, 100, 101, 1000, 1001, 1010, 10000, \ldots\}$$

$$L_{\mu,a} := L_{\mu_1,a} = L_{\mu_2,a}$$

Definition

Let $\mu : \Sigma \to \Sigma^*$ be a morphism having the infinite word $x = x_0 x_1 x_2 \cdots$ as a fixed point.
The *shape sequence of x with respect to μ* is

$$\mathrm{Shape}_\mu(x) = (|\mu(x_k)|)_{k \geq 0}.$$

Lemma

Let x, y be two infinite (unidimensional) words and λ, μ be two morphisms s.t. there exist letters a, b s.t. $x = \lambda^\omega(a)$ and $y = \mu^\omega(b)$.

$$L_{\lambda,a} = L_{\mu,b} \Leftrightarrow \mathrm{Shape}_\lambda(x) = \mathrm{Shape}_\mu(y).$$

Example (Continued)

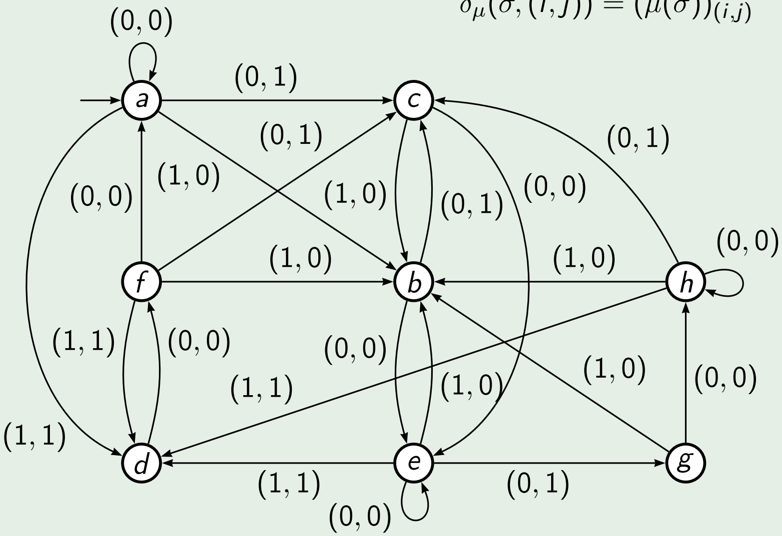
Consider $S = (L_{\mu,a}, \{0,1\}, 0 < 1)$.

	ε	1	10	100	101	1000	1001	1010	$\cdots$
ε	a	b	e	e	b	e	b	e	$\cdots$
1	c	d	c	g	d	g	d	c	
10	e	b	f	e	b	h	b	f	
100	e	b	e	a	b	e	b	e	
101	g	d	c	c	d	g	d	c	
1000	e	b	e	e	b	a	b	e	
1001	g	d	c	g	d	c	d	c	
1010	h	b	f	e	b	e	b	f	
$\vdots$		$\vdots$							$\ddots$

$$\left(\mu(y_{\mathrm{val}_S(101),\mathrm{val}_S(10)})\right)_{0,1} = y_{\mathrm{val}_S(1010),\mathrm{val}_S(101)}$$

Example (Continued)

$$\delta_\mu(\sigma, (i,j)) = (\mu(\sigma))_{(i,j)}$$



$$(\mu^\omega(a))_{m,n} = \delta_\mu(a, (\mathrm{rep}_S(m), \mathrm{rep}_S(n))^0) \text{ for all } m, n \geq 0.$$