ABSTRACT NUMERATION SYSTEMS

Michel Rigo

Department of Mathematics
University of Liège

Journées Montoises d'Informatique Théorique
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caucal@irisa.fr

Cher Michel,

dans le cadre des Journées Montoises d'informatique théorique qui auront lieu à Rennes fin août-début septembre : `http://www.irisa.fr/JM06`

Véronique et moi-même avons le plaisir de t'inviter à venir faire un exposé de synthèse (45 minutes) sur tes travaux.

Outline of the talk

Where it comes from

Motivation

Abstract numeration systems

First results

Another formalism

Towards a Cobham's theorem

Real numbers

INTEGER BASE NUMERATION SYSTEM, $k \geq 2$

$$n = \sum_{i=0}^{\ell} c_i\, k^i, \quad \text{with} \quad c_i \in \Sigma_k = \{0, \ldots, k-1\},\ c_\ell \neq 0$$

Any integer n corresponds to a word $\mathrm{rep}_k(n) = c_\ell \cdots c_0$ over Σ_k.

DEFINITION

A set $X \subseteq \mathbb{N}$ is *k-recognizable* if $\mathrm{rep}_k(X) \subseteq \Sigma_k^*$ is a regular language (accepted by a DFA).

THEOREM (COBHAM '69)

Let $p, q \geq 2$ be two multiplicatively independent integers. If $X \subseteq \mathbb{N}$ is both p- and q-recognizable, then X is ultimately periodic.

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Divisibility criteria

If $X \subseteq \mathbb{N}$ is ultimately periodic,
then X is k-recognizable for any $k \geq 2$.

Various proof simplifications and generalizations

G. Hansel, D. Perrin, F. Durand, V. Bruyère, F. Point,
C. Michaux, R. Villemaire, A. Bès, J. Bell, J. Honkala,
S. Fabre, C. Reutenauer, A.L. Semenov, L. Waxweiler, ...

Non standard numeration systems

A strictly increasing sequence $(U_k)_{k\geq 0}$ of integers such that

- $U_0 = 1$
- $\frac{U_{k+1}}{U_k}$ is bounded

$$n = \sum_{i=0}^{\ell} c_i\, U_i, \quad \text{with} \quad c_i \in \Sigma_U,\ c_\ell \neq 0, \quad \mathrm{rep}_U(n) = c_\ell \cdots c_0$$

In a non standard setting, you can play the same game : *which subsets of $\mathbb{N}$ have a simple expression, i.e., are U-recognizable ?*

THE LANGUAGE OF THE NUMERATION

In particular, is $\mathcal{L}_U = \text{rep}_U(\mathbb{N})$ regular ?

THEOREM (SHALLIT '94)

If $\mathcal{L}_U$ is regular, then $(U_k)_{k\geq 0}$ satisfies a linear recurrent equation.

THEOREM (N. LORAUD '95, M. HOLLANDER '98)

They give (technical) sufficient conditions for $\mathcal{L}_U$ to be regular: "the characteristic polynomial of the recurrence has a special form".

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Best known case : "Pisot systems"

If the characteristic polynomial of $(U_k)_{k\geq 0}$ is the minimal polynomial of a Pisot number θ then "everything" is fine: $\mathcal{L}_U$ *is regular, addition preserves recognizability, logical first order characterization of recognizable sets*, ...
"*Just*" like in the integer case : $U_k \simeq \theta^k$.

A. Bertrand '89, C. Frougny, B. Solomyak, D. Berend, J. Sakarovitch, V. Bruyère and G. Hansel '97, ...

Example

Take the golden ratio $\tau = \frac{1+\sqrt{5}}{2}$, $M_\tau(X) = X^2 - X - 1$

$$U_{k+2} = U_{k+1} + U_k, \quad U_0 = 1,\ U_1 = 2$$

Fibonacci numeration system : $\mathcal{L}_U = \{\varepsilon\} \cup 1\{0, 01\}^*$.

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A QUESTION

After G. Hansel's talk during JM'94 in Mons and knowing Shallit's results, P. Lecomte has the following question :

- Everybody takes first a sequence $(U_k)_{k\geq 0}$
- then ask for the language $\mathcal{L}_U$ of the numeration to be regular and play with recognizable sets
- Why not proceed backwards ?

REMARK

Let $x, y \in \mathbb{N}$, $x < y \Leftrightarrow \mathrm{rep}_U(x) <_{gen} \mathrm{rep}_U(y)$.

EXAMPLE (FIBONACCI)

$6 < 7$ and $1001 <_{gen} 1010$ (same length)
$6 < 8$ and $1001 <_{gen} 10000$ (different lengths).

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DEFINITION (P. LECOMTE, M.R. '01)

An *abstract numeration system* is a triple $S = (L, \Sigma, <)$ where L is a regular language over a totally ordered alphabet $(\Sigma, <)$. Enumerating the words of L with respect to the genealogical ordering induced by $<$ gives a one-to-one correspondence

$$\mathrm{rep}_S : \mathbb{N} \to L \qquad \mathrm{val}_S = \mathrm{rep}_S^{-1} : L \to \mathbb{N}.$$

REMARK

This generalizes "classical" Pisot systems like integer base systems or Fibonacci system.

EXAMPLE (POSITIONAL)

$L = \{\varepsilon\} \cup \{1, \ldots, k-1\}\{0, \ldots, k-1\}^*$ or $L = \{\varepsilon\} \cup 1\{0, 01\}^*$

EXAMPLE (NON POSITIONAL)

Non positional numeration system : $L = a^*b^*$ $\Sigma = \{a < b\}$

n	0	1	2	3	4	5	6	$\cdots$
$\mathrm{rep}(n)$	ε	a	b	aa	ab	bb	aaa	$\cdots$

$\mathrm{val}(a^p b^q) = \frac{1}{2}(p+q)(p+q+1) + q$

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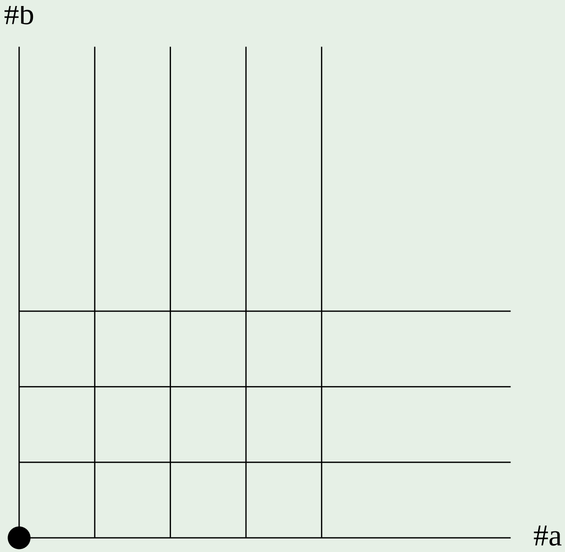
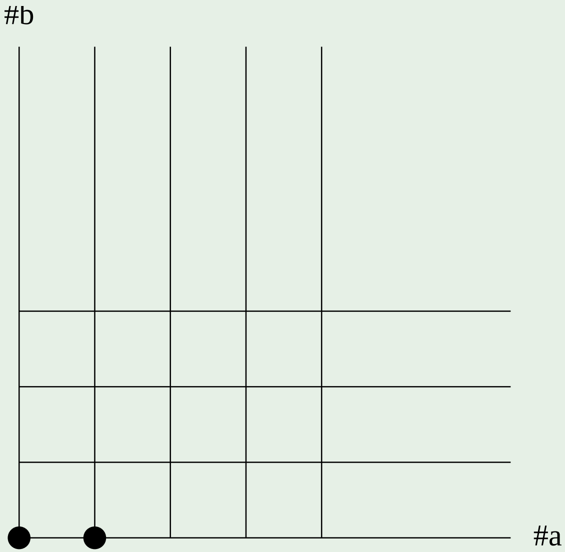


EXAMPLE (CONTINUED...)



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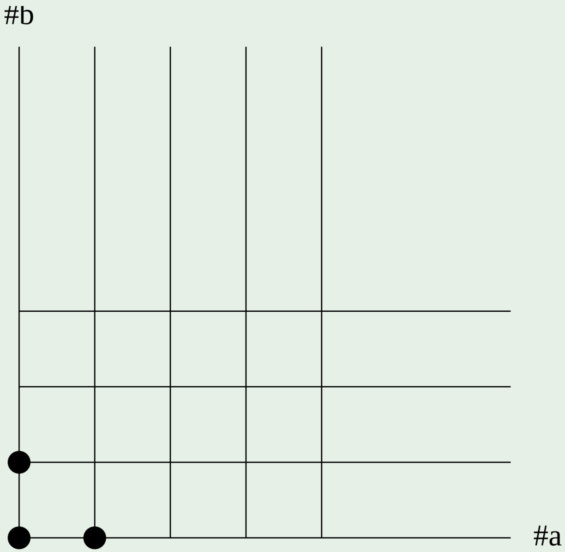

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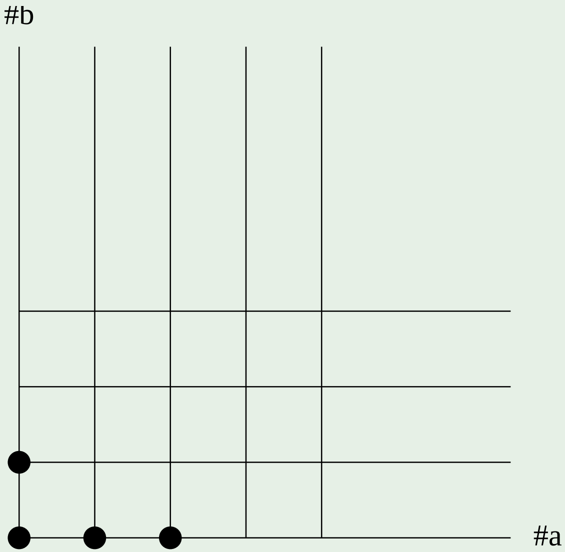

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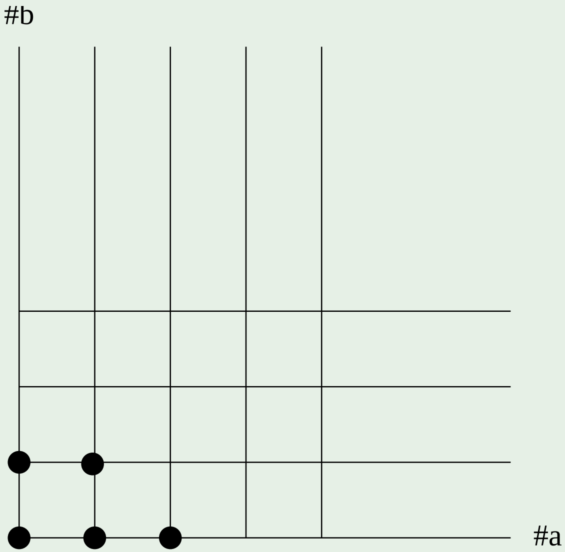

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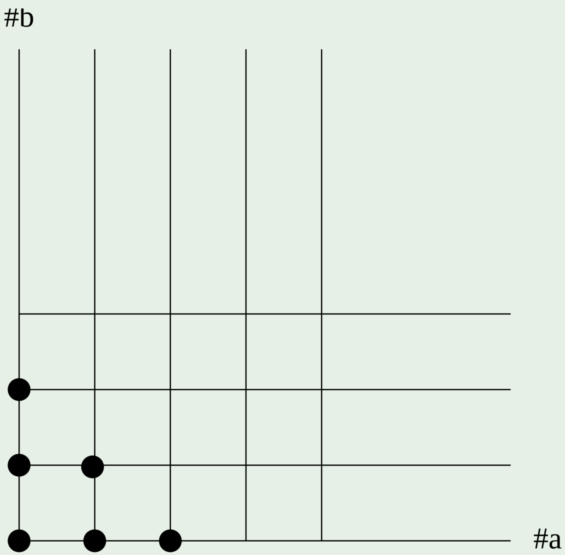

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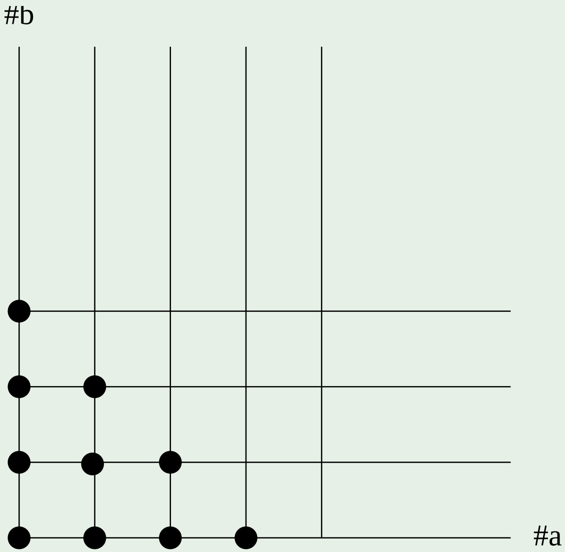

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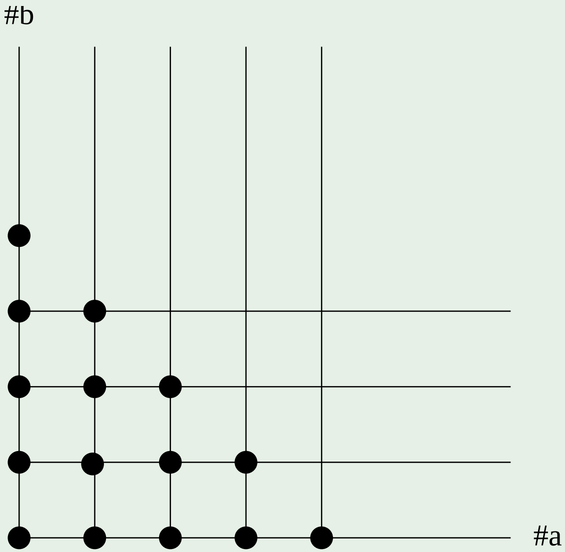

Many natural questions...

- What about S-recognizable sets ?
  - Are ultimately periodic sets S-recognizable for any S ?
  - For a given $X \subseteq \mathbb{N}$, can we find S s.t. X is S-recognizable ?
  - For a given S, what are the S-recognizable sets ?
- Can we compute "easily" in these systems ?
  - Addition, multiplication by a constant, ...
- Are these systems equivalent to something else ?
- Any hope for a Cobham's theorem ?
- Can we also represent real numbers ?
- Number theoretic problems like additive functions ?
- Dynamics, odometer, tilings, logic...

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Theorem

Let $S = (L, \Sigma, <)$ be an abstract numeration system. Any ultimately periodic set is S-recognizable.

Well-known fact (see Eilenberg's book)

The set of squares is never recognizable in any integer base system.

Example

Let $L = a^*b^* \cup a^*c^*$, $a < b < c$.

0	1	2	3	4	5	6	7	8	9	$\cdots$
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\varepsilon & \color{red}{a} & b & c & \color{red}{aa} & ab & ac & bb & cc & \color{red}{aaa} & \cdots
\end{array}
$$

Definition of complexity

Let $\mathcal{A} = (Q, q_0, F, \Sigma, \delta)$, $\mathbf{u}_q(n) = \#\{w \in \Sigma^n \mid \delta(q,w) \in F\}$ i.e., number of words of length n accepted from q in $\mathcal{A}$.

$$\mathbf{u}_{q_0}(n) = \#(L \cap \Sigma^n).$$

Theorem ("Multiplication by a constant")

slender language	$\mathbf{u}_{q_0}(n) \in \mathcal{O}(1)$	*OK*
polynomial language	$\mathbf{u}_{q_0}(n) \in \mathcal{O}(n^k)$	*NOT OK*
exponential language with polynomial complement	$\mathbf{u}_{q_0}(n) \in 2^{\Omega(n)}$	*NOT OK*
exponential language with exponential complement	$\mathbf{u}_{q_0}(n) \in 2^{\Omega(n)}$	*OK ?*

Remember

Émilie Charlier's talk on bounded languages $a^*b^*, \ldots$

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DEFINITION (SUBSTITUTIONS)

Let $f : \Sigma \to \Sigma^*$ and $g : \Sigma \to \Gamma^*$ be to morphisms such that $f(a) \in a\Sigma^+$. We define a *morphic word* over Γ,

$$w = g(\lim_{n\to\infty} f^n(a)) = g(f^\omega(a)).$$

REMARK (COBHAM, ALLOUCHE-SHALLIT)

One can assume f non-erasing and g letter-to-letter.

EXAMPLE (CHARACTERISTIC SEQUENCE OF SQUARES)

$f : a \mapsto abcd,\ b \mapsto b,\ c \mapsto cdd,\ d \mapsto d, g : a, b \mapsto 1,\ c, d \mapsto 0.$

$$f^\omega(a) = abcdbcdddbcdddddbcdddddddbc\cdots$$

$$g(f^\omega(a)) = 110010000100000010000000010\cdots$$

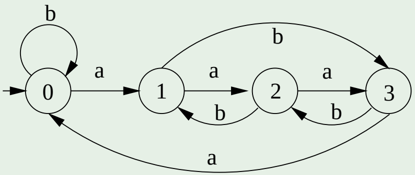
Definition

Let $S = (L, \Sigma, <)$ be an abstract number system and $\mathcal{M} = (Q, q_0, \Sigma, \delta, \Gamma, \tau)$ be a DFAO.
Consider the *S-automatic sequence*

$$x_n = \tau(\delta(q_0, (\mathrm{rep}_S(n))))$$

Example

$S = (a^*, b^*, \{a, b\}, a < b)$



$$0102303120023101012302303120312023100231012 3\cdots$$

Remark

A set $X \subseteq \mathbb{N}$ is S-recognizable iff its characteristic sequence is S-automatic.

Theorem (A. Maes, M.R. '02)

The set of S-automatic sequences (for any S) coincides with the set of morphic words.

F. DURAND '98

Let (f, g, a) (resp. (f', g', a')) be a primitive substitution with a Perron dominating eigenvalue $\alpha > 1$ (resp. $\beta > 1$). If α and β are multiplicatively independent and if the characteristic sequence χ of the set $X \subseteq \mathbb{N}$ is such that

$$g(f^\omega(a)) = \chi = g'(f'^\omega(a'))$$

then X is ultimately periodic.

REMARK 1 DURAND '02 (E. SENETA, D. LIND-B. MARCUS)

Primitiveness assumption can be removed but

$$\forall \sigma \in \Sigma, \lim_{n\to\infty} |f^n(\sigma)| = +\infty \quad \text{i.e., growing substitution}$$

Remark 2

A few cases are problematic, substitutions with no main "sub-substitution" having the same dominating eigenvalue.

Example

$$f : \{a, 0, 1\} \to \{a, 0, 1\}^* : \left\{ \begin{array}{l} a \mapsto aa0 \\ 0 \mapsto 01 \\ 1 \mapsto 0 \end{array} \right. \left| \begin{array}{r} 2 \\ (1+\sqrt{5})/2 \\ \end{array} \right.$$

Lemma

Let $\tau : A \to A^*$ be a substitution on the finite alphabet A. There exists p such that for $\sigma = \tau^p$ and for all $a \in A$, one of the following two situations occurs, either

$$\exists N \in \mathbb{N} : \forall n > N,\ |\sigma^n(a)| = 0,$$

or there exist $d(a) \in \mathbb{N}$ and algebraic numbers $c(a), \alpha(a)$ such that

$$\lim_{n \to +\infty} \frac{|\sigma^n(a)|}{c(a)\, n^{d(a)}\, \alpha(a)^n} = 1.$$

Definition

(d, α) is the growth type of a

$(d, \alpha) < (e, \beta)$ if $\alpha < \beta$ or ($\alpha = \beta$ and $d < e$), i.e., $\frac{n^d\, \alpha^n}{n^e\, \beta^n} \to 0$.

Maximal growth type = growth type of τ.

Link with abstract systems

Let $S = (L, \Sigma, <)$ with $\mathbf{u}_{q_0}(n) = \#(L \cap \Sigma^n) \simeq n^D \Theta^n$ and $E \subseteq \mathbb{N}$ a S-recognizable set of integers

$$\begin{array}{c|l} \Theta > 1 & \chi_E \text{ is } (D, \Theta^\ell)\text{-substitutive} \\ \Theta = 1 & \chi_E \text{ is } (D+1, 1)\text{-substitutive} \end{array}$$

Theorem

Let $d, e \in \mathbb{N} \setminus \{0\}$ and $\alpha, \beta \in [1, +\infty)$ such that $(d, \alpha) \neq (e, \beta)$ and satisfying one of the following three conditions:

1. *α and β are multiplicatively independent;*
2. *$\alpha, \beta > 1$ and $d \neq e$;*
3. *$(\alpha, \beta) \neq (1, 1)$ and, $\beta = 1$ and $e \neq 0$, or, $\alpha = 1$ and $d \neq 0$;*

Let C be a finite alphabet. If $x \in C^{\mathbb{N}}$ is both (d, α)-substitutive and (e, β)-substitutive then the letters of C which have infinitely many occurrences in x appear in x with bounded gaps.

Example

If $X \subseteq \mathbb{N}$ is both S- and T-recognizable where S (resp. T) is built over an exponential (resp. a polynomial) language then X is ultimately periodic.

EXAMPLE (BASE 10)

$$\pi - 3 = .14159265358979323846264338328\cdots$$

$$\frac{1}{10}, \quad \frac{14}{100}, \quad \frac{141}{1000}, \quad \ldots, \quad \frac{val(w_n)}{10^n}, \quad \ldots$$

$$\frac{\mathrm{val}(w)}{\#\{\text{words of length } \leq |w|\}}$$

THIS DESERVES NOTATION

$$\mathbf{v}_{q_0}(n) = \#(L \cap \Sigma^{\leq n}) = \sum_{i=0}^{n} \mathbf{u}_{q_0}(i).$$

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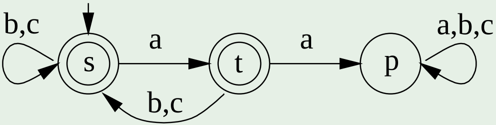
Example (Avoid aa on three letters)



w	$\mathrm{val}(w)$	$\mathbf{v}_{q_0}(\lvert w\rvert)$	$\mathrm{val}(w)/\mathbf{v}_{q_0}(\lvert w\rvert)$
bc	8	12	0.666666666666667
bac	19	34	0.558823529411176
$babc$	52	94	0.553191489361170
$babac$	139	258	0.538759689922248
$bababc$	380	706	0.538243626062322
$bababac$	1035	1930	0.536269430051881
$babababc$	2828	5274	0.53621539628836

$$\lim_{n\to\infty} \frac{\mathrm{val}((ba)^n c)}{\mathbf{v}_{q_0}(2n+1)} = \frac{1}{1+\sqrt{3}} + \frac{3}{9+5\sqrt{3}} \simeq 0.535898.$$

Numerical value of a word $w = w_1 \cdots w_\ell \in L$

$$\mathrm{val}(w) = \sum_{i=1}^{\ell} \sum_{q \in Q} (\theta_{q,i}(w) + \delta_{q,s})\, \mathbf{u}_q(|w| - i)$$

where $\theta_{q,i}(w) = \#\{\sigma < w_i \mid s.w_1 \cdots w_{i-1}\sigma = q\}$

Hypotheses: For all state q of $\mathcal{M}_L$, either

(i) $\exists N_q \in \mathbb{N} : \forall n > N_q,\ \mathbf{u}_q(n) = 0$, or

(ii) $\exists \beta_q \geq 1, P_q(x) \in \mathbb{R}[x]\ , b_q > 0 : \lim_{n\to\infty} \frac{\mathbf{u}_q(n)}{P_q(n)\beta_q^n} = b_q$.

From automata theory, we have

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IF $(w_n)_{n\in\mathbb{N}}$ IS CONVERGING TO $W = W_1 W_2 \cdots$ THEN

$$\lim_{n\to\infty} \frac{\mathrm{val}(w_n)}{\mathbf{v}_{q_0}(|w_n|)} = \frac{\beta-1}{\beta^2} \sum_{j=0}^{\infty} \sum_{q\in Q} \frac{a_q}{a_{q_0}} \left(\theta_{q,j+1}(W) + \delta_{q,s}\right) \beta^{-j}.$$

REMARK [W. STEINER, M.R. '05]

By "normalizing" we can specify the value of a_{q_0}, why not take $a_{q_0} = 1 - \frac{1}{\beta}$?

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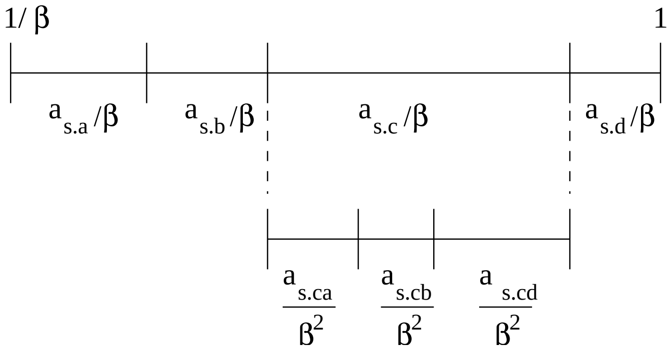
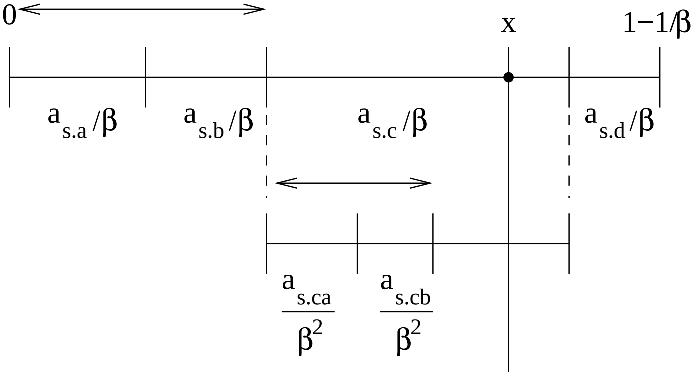
$W = W_1 W_2 \cdots$

$$x = \frac{1}{\beta} + \sum_{j=1}^{\infty} \alpha_{q_0.W_1 \cdots W_{j-1}}(W_j)\ \beta^{-j}$$

where $\alpha_q(\sigma) = \sum_{\tau<\sigma} a_{q.\tau}$.





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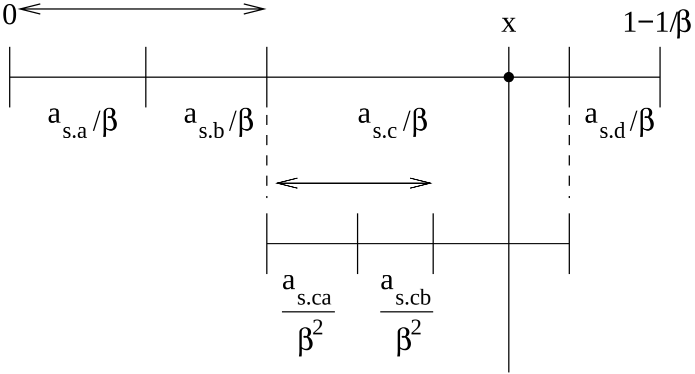

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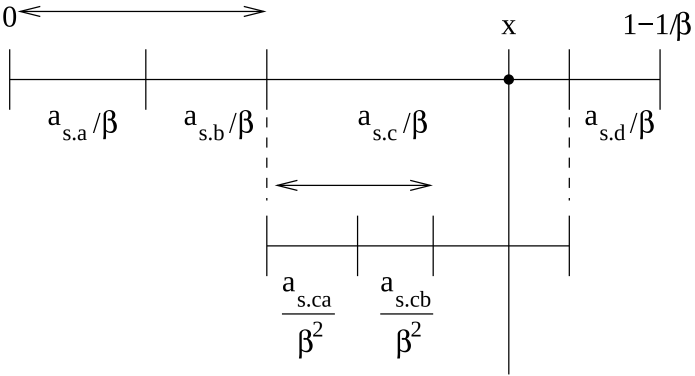

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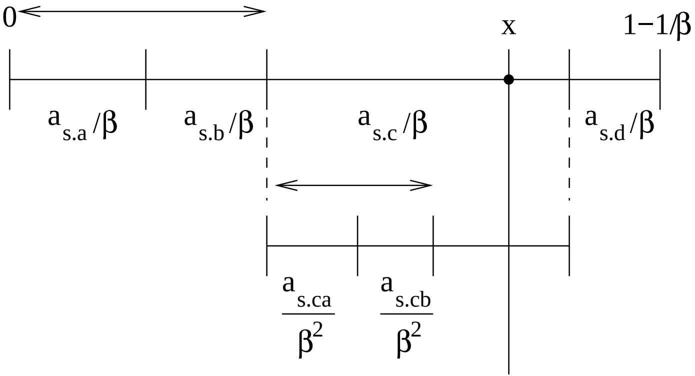

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- Which numbers have several representations
- Determine numbers with (ultimately) periodic expansion
- Conditions to have a ring structure
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