On generalized Hölder-Zygmund spaces

D. Kreit (joint work with S. Nicolay)
Université de Liège, Dpt. Mathematics, Liège, Belgium
(D.Kreit@ulg.ac.be)
(S.Nicolay@ulg.ac.be)

Abstract. The Besov spaces $B_{p,q}^s(\mathbb{R}^d)$ and Hölder spaces $C^\alpha(\mathbb{R}^d)$ ($0 < \alpha < 1$) provide natural ways for measuring the smoothness of a function and are used in multiple areas from the solving of PDE to multiresolution analysis. As such, it appears that generalizations of these spaces would prove themselves very useful in many domains and that is why a generalization of Besov Spaces has been extensively studied by many authors during those last 20 years. On the other hand, it has been proved that those spaces coincide with some kind of generalized Hölder spaces in particular cases ([4]). The purpose of this poster is to introduce those new Hölder spaces and to show that all main properties of the classical case are still true for the generalized ones.

**Definition of admissible sequences**
A sequence $\sigma = (\sigma_j)_{j \in \mathbb{Z}}$ of positive numbers is called admissible if there exist two constants $d_0$ and $d_1$ such that
$$d_0 \sigma_j \leq \sigma_{j+1} \leq d_1 \sigma_j, \quad j \in \mathbb{Z}.$$ 

Let $\sigma_j := \inf_{k \geq 0} \frac{\sigma_{j+k}}{\sigma_k}$ and $\sigma_j := \sup_{k \geq 0} \frac{\sigma_{j+k}}{\sigma_k}, \quad j \in \mathbb{N}.$

The lower and upper Boyd index are respectively defined by
$$\underline{\alpha} := \liminf_{j \to +\infty} \frac{\log \sigma_j}{j} \quad \text{and} \quad \overline{\alpha} := \limsup_{j \to +\infty} \frac{\log \sigma_j}{j}.$$ 

**Definition of generalized Hölder-Zygmund spaces**
Let $\alpha > 0$ and $\sigma$ an admissible sequence. A function $f \in L^p(\mathbb{R}^d)$ belongs to the generalized Hölder space $C^{\alpha,\sigma}(\mathbb{R}^d)$ if there exists $C > 0$ such that
$$\sup_{x \in [0,1]^d} \sum_{j \in \mathbb{Z}} \|\Delta_h^{\alpha+1} f(x)\|_{L^p} \leq C \sigma_j, \quad j \in \mathbb{N}.$$

Remark.
The space $C^{\alpha,\sigma}(\mathbb{R}^d)$ is a Banach space, with$
\|f\|_{C^{\alpha,\sigma}} := \|f\|_{L^p} + \|D^\alpha f\|_{C^{\sigma}}$
where $\|D^\alpha f\|_{C^{\sigma}} := \sup_{j \geq 0} \sup_{k \geq 0} \frac{\|\Delta_h^{\alpha+1} f\|_{L^p}}{\sigma_k}.$

**Link with generalized Besov spaces**
If $\|f\|_{C^{\sigma}} > 0$, it can be shown that generalized Hölder spaces $C^{\alpha,\sigma}(\mathbb{R}^d)$ are indeed generalized Besov spaces $B_{\infty,\infty}^{\alpha,\sigma}(\mathbb{R}^d)$ (see [4]).

**Proposition (link with classical regularity)**
Let $K \subset \mathbb{N}$ and $\sigma = (\sigma_j)_{j \in \mathbb{N}}$ an admissible sequence such that $K < g(\sigma^{-1})$. If $f \in C^{g(\sigma^{-1})}(\mathbb{R}^d)$, then $f$ is $K$-times continuously differentiable.

**A characterization by the convolution**
Let $\sigma = (\sigma_j)_{j \in \mathbb{N}}$ be an admissible sequence such that $g(\sigma^{-1}) > 0$. Then
$$C^{g(\sigma^{-1})}(\mathbb{R}^d) = \left\{ f \in L^p(\mathbb{R}^d) : \exists \varphi \in C^\infty(\mathbb{R}^d) \sup_{j \in \mathbb{N}} \frac{1}{\sigma_j} \left( \|f \ast \varphi_j - f\|_{L^p} \right) < \infty \right\},$$
where $\varphi_j(x) := \frac{\varphi(x/j)}{\sigma_j}.$

Moreover, the norm $\|f\|_{C^{g(\sigma^{-1})}}$ is equivalent to $\|f\|_{L^p} + \sup_{j \geq 0} \sigma_j \|f \ast \varphi_j - f\|_{L^p}$, where the infimum is taken on the set of functions $f \in C^{g(\sigma^{-1})}(\mathbb{R}^d)$ that verify the inequality above and such that $\sup_{j \in \mathbb{N}} \|\Delta_h^{\alpha+1} f\|_{L^p} \leq C.$

**Hölder exponent**
In [1], some sufficient conditions on admissible sequences $\sigma^\alpha$ ($\alpha > 0$) are exposed so that we have $\alpha \times j \Rightarrow C^{\alpha,\sigma} \subset C^{\alpha,\sigma^\alpha}.$ Those inclusions allow to define an Hölder exponent linked to these spaces by $H^\alpha := \sup_{x \in [0,1]^d} \|D^\alpha f\|_{L^p}.$

A characterization by wavelet coefficients
Let $\sigma = (\sigma_j)_{j \in \mathbb{N}}$ be a decreasing admissible sequence such that $N - 1 < g(\sigma^{-1}) < N$. Then we have
$$\sup_{x \in [0,1]^d} \|D^\alpha f\|_{L^p(\mathbb{R}^d)} \leq C \sigma_j \quad \text{iff} \quad f \in C^{g(\sigma^{-1})}(\mathbb{R}^d).$$

**Notation**
- $\|f\|_{C^{\sigma}} := \sup_{j \geq 0} \frac{\|\Delta_h\^{\alpha+1} f\|_{L^p}}{\sigma_j}$
- $\sigma_j := \inf_{k \geq 0} \frac{\sigma_{j+k}}{\sigma_k}$ and $\sigma_j := \sup_{k \geq 0} \frac{\sigma_{j+k}}{\sigma_k}, \quad j \in \mathbb{N}.$

**References**