Analysis of the speed of convergence

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Neoclassical Production Function

We will assume a production function of the Cobb-Douglas form:

$$F[K(t), L(t), A(t)] = A\, K(t)^{\alpha}\, L(t)^{1-\alpha}$$

where *K(t)* is the physical capital stock at time *t*, *L(t)* is labor and A is the constant level of total factor productivity.

This Cobb-Douglas function is homogeneous of degree 1. Therefore, it is possible to write it in intensive form:

$$f[k(t)] = A\, k(t)^{\alpha}$$

where $k = K/L$ is the capital-labor ratio.

Exogenous Growth Model: The Solow-Swan Model

The fundamental equation of the Solow-Swan model is the equation of the capital accumulation:

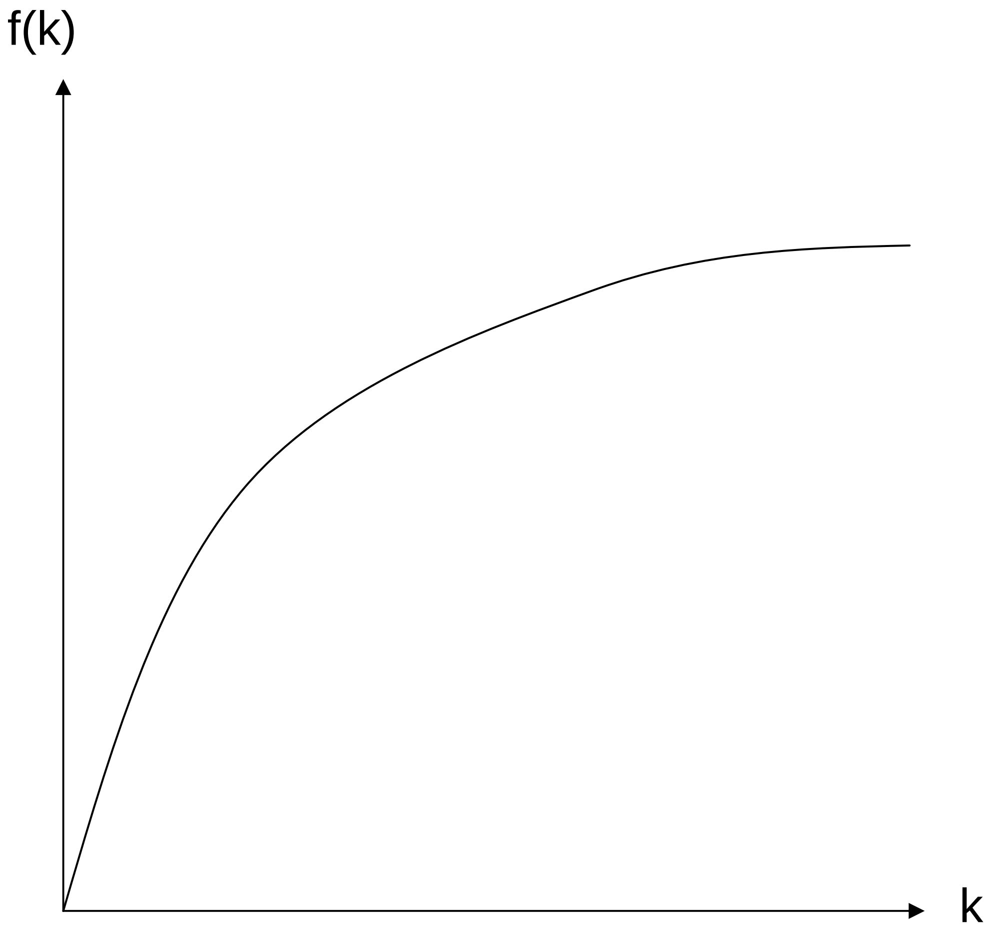
$$\frac{dk}{dt} = sAk^{\alpha} - (n + \delta)\, k$$

The growth rate of the capital-labor ratio is:

$$\frac{dk/dt}{k} = sAk^{\alpha - 1} - (n + \delta)$$

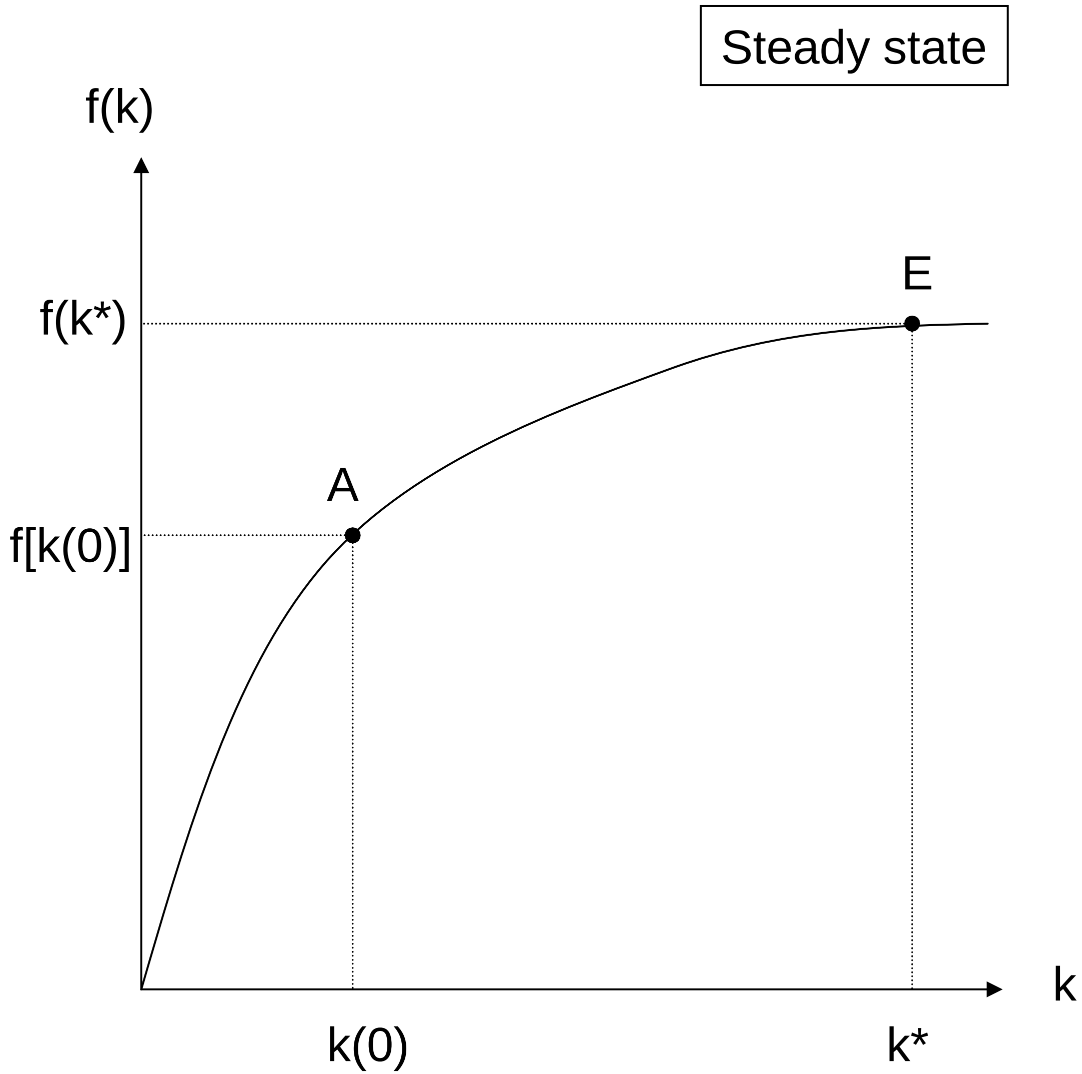
The derivative of this growth rate with respect to *k* is negative.

Production function: graphical representation



This curve is the graphical representation of the production function in intensive form, where k = K/L.

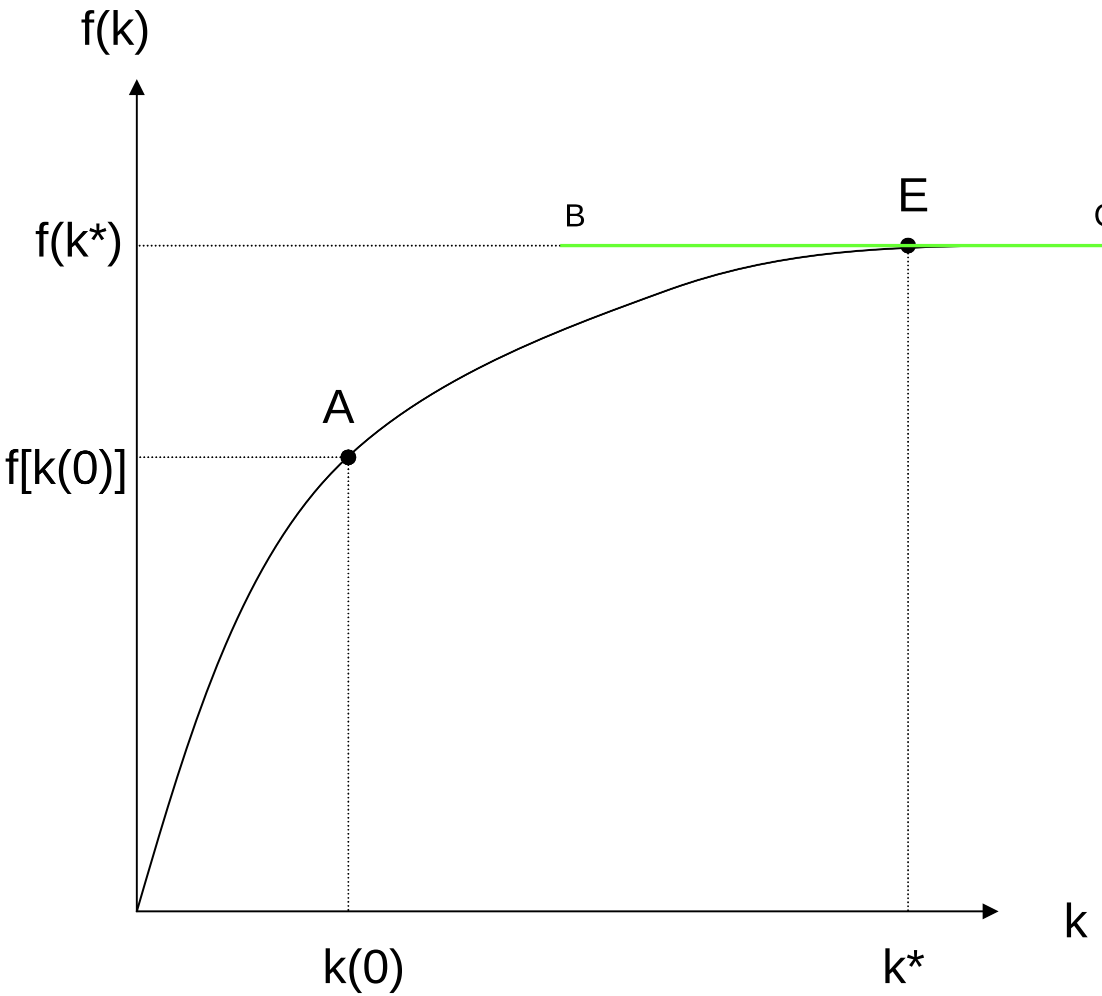
The function f(k) is assumed to concave.



The point *A,* whose coordinates are (k(0);f(k(0)), is the initial level of the economy.

The point *E,* whose coordinates are (k*;f(k*)), is the steady state of the function f(k). This is the long-term level of the economy.

Growth rate at the steady state



The straight line (BC) is the tangent to the point *E*.

On the graph, the slope of the tangent appears to be 0. This slope is the instantaneous rate of variation of the function f(k) at the value k=k*.

Slope of the tangent = f'(k*)

The growth rate of this economy at the steady state (*E*) is equal to 0.

How to calculate the instantaneous rate of variation ?

The instantaneous rate of variation (or derivative at a point) is

$$f'(k^*) = \lim_{k \to k^*} \frac{f(k) - f(k^*)}{k - k^*} = \lim_{h \to 0} \frac{f(k^* + h) - f(k^*)}{h}$$

For our production function, the instantaneous rate of variation at the steady state point is

$$f'(k^*) = 0$$

This means that the production per worker does not grow at the steady state.

Growth rate between two points on the curve (e.g. between A and E)

The level of product per worker is *f [k(0)]* when *k = k(0)* and is *f (k*)* when *k = k**.

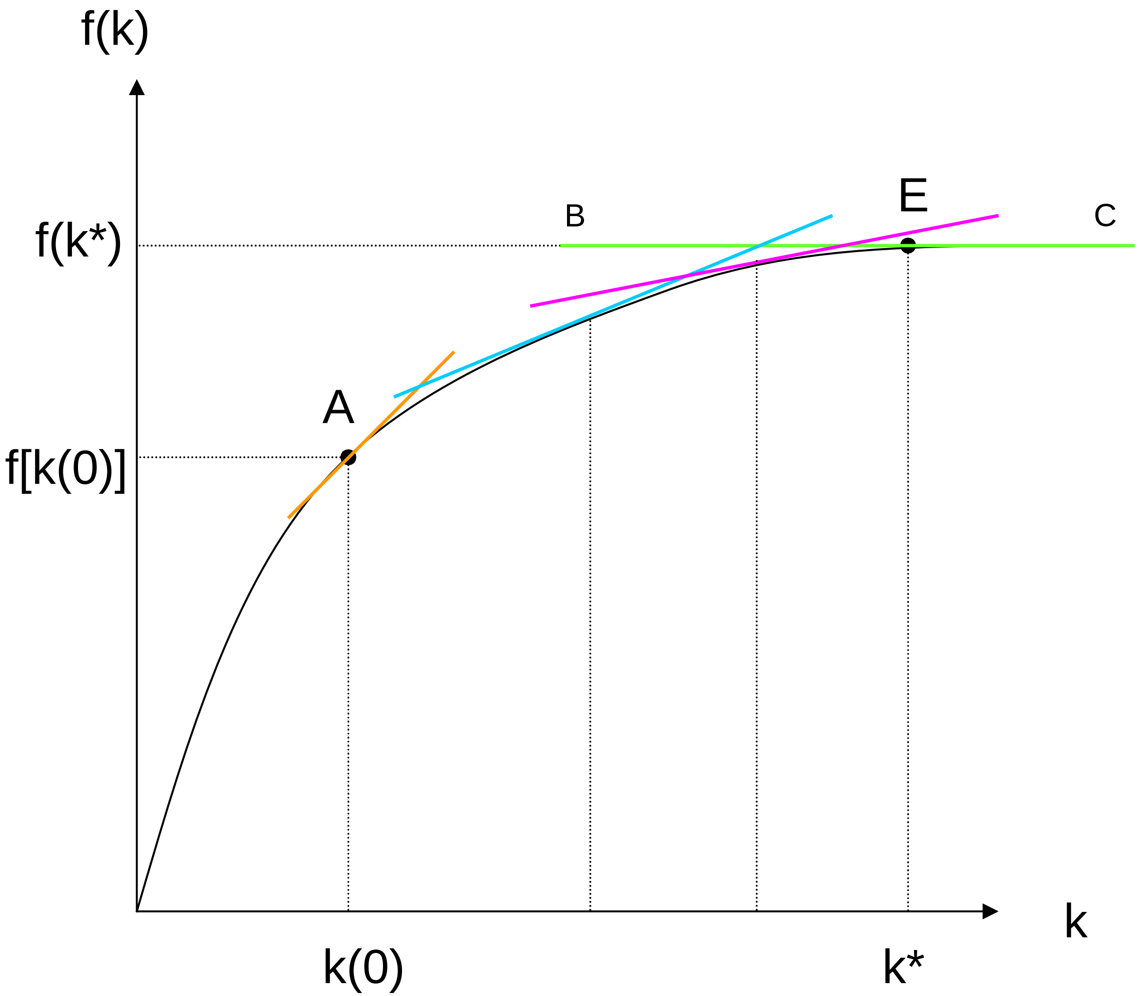
The difference in level *(f (k*) - f [k(0)])* is the increase in product per worker when *k* increases from *k = k(0)* to *k = k**.

We can also calculate the growth rate (*g)* of the product per worker when *k* increases from *k = k(0)* to *k = k**. It is the geometric mean of all the instantaneous rates of variation between *k = k(0)* to *k = k**:

$$g = \frac{dk/dt}{k} = [\underbrace{f'(k(0)) \times \ldots \times f'(k^*)}_{\text{n derivatives}}]^{1/n} \qquad \text{for all } k \in [k(0), k^*] \text{ and } n \rightarrow \infty$$

where n is the number of compounding. When n → ∞ compounding is continuous.

Growth rate between A and E



The instantaneous rates of variation between k(0) and k* are all different since the function is non-linear.

In fact, the instantaneous rates of variation decrease as k increases from k(0) to k*. The slopes are increasingly weaker.

The growth rate between A and E is the geometric mean of all the instantaneous rates of variation

Calculation of average growth rate between A and E

If we know the values for f[k(0)] and f(k*) and the number of periods (e.g. number of years) that elapsed for the economy to go from k(0) to k*, then we can calculate the average growth rate between f[k(0)] and f(k*) :

$$R = \{\ln f(k^*) - \ln f[k(0)]\}^{1/t}$$

where *t* is the number of periods = (number of dates – 1). (e.g. 1991, 1992 and 1993 are 3 dates but 1991-1992 and 1992-1993 are two periods).

This average growth rate *R* is calculated by using the geometric mean where the growth rate compound continuously. If the number of periods is 1, then t = 1 and the growth rate is just the continuous growth rate between f[k(0)] and f(k*).

This continuous growth rate is also called the *speed of convergence*. The name comes from the result that the steady state *E* is stable, hence the economy converges to *E* regardless of its initial start k(0) (except k (0) = 0).

Calculation of average growth rate between A and E (cont.)

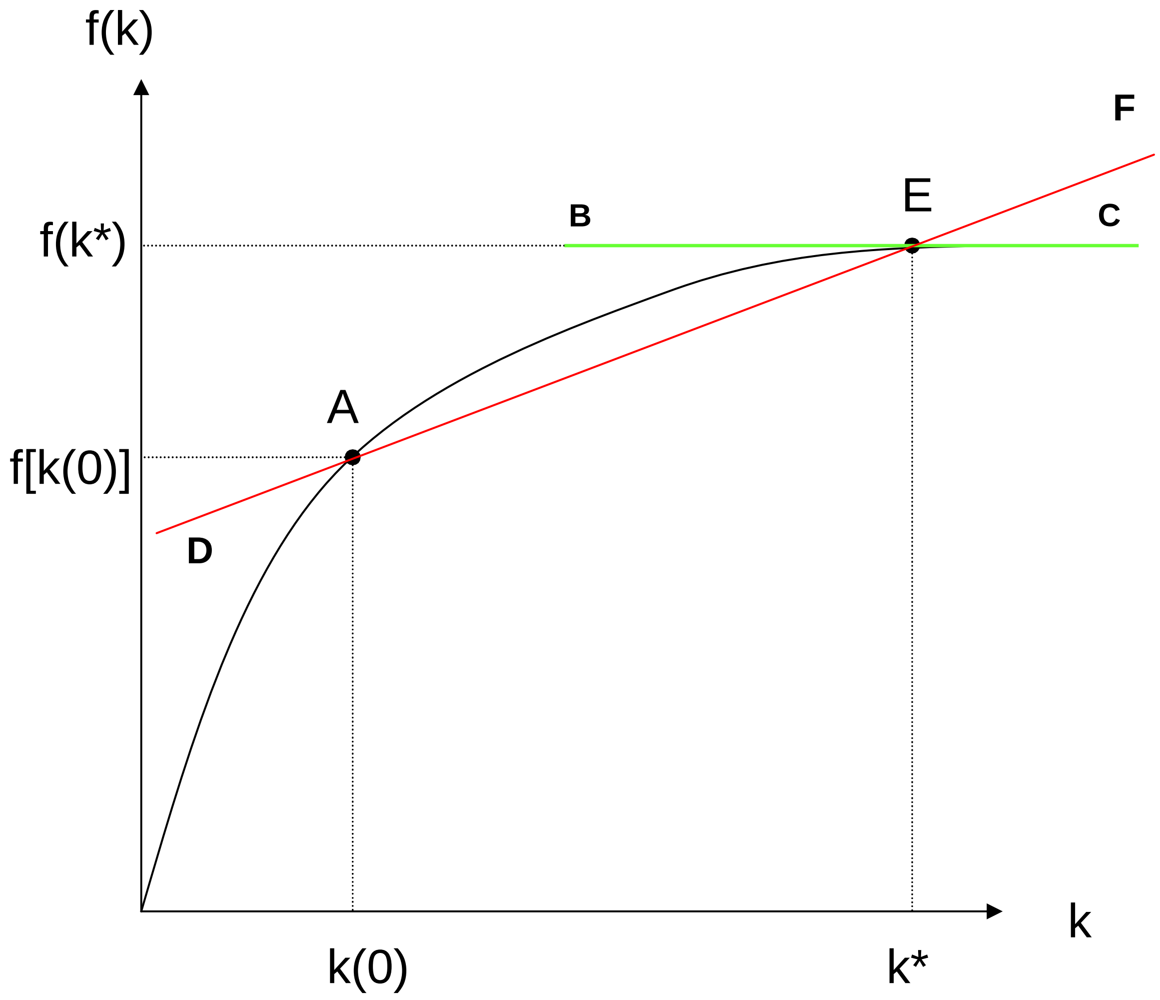
We can obviously use the growth rate to calculate the level of f(k*) if we know f[k(0)]:

$$f(k^*) = \exp(Rt)\, f[k(0)]$$

To sum up, the growth rate of f[k(t)] is a non-linear function of k(t). It decreases as k(t) increases due to the concavity of the production function.

Therefore, for linear estimation purposes, it is necessary to compute a growth rate that is linear in k(t) and could be a reasonable approximation of the true growth rate.

Graphical linear approximation of the growth rate



Graphically, to approximate the growth rate between the points A and E, one has to draw a line (DF) passing through A and E.

The slope of this straight line gives the approximation of the growth rate of the concave function.

The farther A is located from E, the worse is the approximation. In our graph, the approximation is bad because A is too far from E.

Analytical linear approximation of the growth rate

To compute an analytical linear approximation of the growth rate, one has to linearize the growth rate function (dk/dt)/k around its steady state. To do so, we apply to this function a Taylor expansion of order 1 around the steady state k* to obtain a linear function:

$$\frac{dk/dt}{k} = \frac{dk/dt}{k^*} + \left.\frac{\partial \frac{dk/dt}{k}}{\partial k}\right|_{k = k^*} (k - k^*)$$

In the Solow-Swan model the growth rate is

$$\frac{dk/dt}{k} = sAk^{\alpha - 1} - (n + \delta)$$

At the steady state, the growth rate is 0, then :

$$sAk^{\alpha - 1} = (n + \delta)$$

Analytical linear approximation of the growth rate

Let us approximate the nonlinear Solow growth rate function by a Taylor polynomial of the first order:

$$\frac{dk/dt}{k} = sA(k^*)^{\alpha - 1} - (n + \delta) + \left.\frac{\partial\,(sAk(t)^{\alpha - 1} - (n + \delta))}{\partial\, k}\right|_{k(t) = k^*} (k(t) - k^*)$$

$$= \quad 0 \quad + (\alpha - 1)sA(k^*)^{\alpha - 2}\,(k(t) - k^*)$$

Since $sAk^{\alpha - 1} = (n + \delta)$ at the steady state, we can further simplify to:

$$\frac{dk/dt}{dk} = -(1 - \alpha)\,(n + \delta)\,\frac{k(t) - k^*}{k^*}$$

where $[(k(t) - k^*)/k^*]$ is the rate of variation of $k(t)$ around the steady state. This new growth function is linear in $k(t)$. An increase in $k(t)$ yields a decrease in the growth rate of $-(1 - \alpha)\,(n + \delta)/k^*$.

log – linear approximation of the growth rate

A more convenient way for econometric analysis is to log – linearize the original growth rate function. It allows to interpret the result as a percentage deviation from the steady state. The log – linearization consists in applying a first-order Taylor expansion of log(k) around log(k*).

Let us write $\frac{dk/dt}{dk} = sAk^{\alpha - 1} - (n + \delta)$ in log:

$$\frac{d \log k(t)}{dt} = sA\, e^{(\alpha - 1) \log k(t)} - (n + \delta)$$

Let us define $g[\log k(t)] = sA\, e^{(\alpha - 1) \log k(t)} - (n + \delta)$. Let us approximate this function:

$$g[\log k(t)] = g[\log k^*] + \left.\frac{\partial\, g[\log k(t)]}{\partial \log k(t)}\right|_{\log k(t) = \log k^*} (\log k(t) - \log k^*)$$

log – linear approximation of the growth rate

$$g[\log k(t)] = sA\, e^{(\alpha - 1)\log k^*} - (n + \delta) \quad + \quad (\alpha - 1)\, sA\, e^{(\alpha - 1)\log k^*}\,(\log k(t) - \log k^*)$$

$$= \qquad 0 \qquad + \quad (\alpha - 1)(n + \delta)\,(\log k(t) - \log k^*)$$

Therefore, the log – linear form of the growth rate function is:

$$\frac{d \log k(t)}{dt} = -\,(1 - \alpha)(n + \delta)\,(\log k(t) - \log k^*)$$

And $$\frac{d\{d \log k(t)/dt\}}{d \log k(t)} = -\,(1 - \alpha)(n + \delta)$$

where $\boldsymbol{\beta} = -\dfrac{\mathbf{d\{d \log k(t)/dt\}}}{\mathbf{d \log k(t)}}$ is called the **speed of convergence** in the economic growth literature. 1% deviation from *k** yields a percentage change in the growth rate of *k* equal to $-(1 - \alpha)(n + \delta)$ when the production function is Cobb-Douglas.

log – linear approximation of the growth rate

In fact, we are interested in the growth rate of income per capita rather than in the growth rate of the capital –labor ratio. But, they are the same:

$$\frac{dy(t)/dt}{y(t)} = \frac{d\ \ln y(t)}{dt} = \frac{d\ \ln k(t)^{\alpha}}{dt} = \frac{d\ [\alpha \ln k(t)]}{dt} = \frac{d\ [\alpha \ln k(t)]}{dk(t)} \cdot \frac{d\ k(t)}{dt}$$

$$= \alpha \frac{dk/dt}{k}$$

And $y(t) = k(t)^{\alpha}$ => $\frac{y(t)}{y^*} = \frac{k(t)^{\alpha}}{y^*} = \frac{k(t)^{\alpha}}{(k^*)^{\alpha}}$ Taking the log :

$\log \frac{y(t)}{y^*} = \alpha \log \frac{k(t)}{k^*}$ => $\log y(t) - \log y^* = \alpha [\log k(t) - \log k^*]$. Then

$$\frac{d \log k(t)}{dt} = -\beta (\log k(t) - \log k^*) \Rightarrow \frac{1}{\alpha} \frac{d \log y(t)}{dt} = -\beta \frac{1}{\alpha} (\log y(t) - \log y^*)$$

log – linear approximation of the growth rate

As a result:

$$\frac{d \log y(t)}{dt} = - \beta (\log y(t) - \log y^*) \qquad (1)$$

The speed of convergence is the same for the income per capita as for the capital-labor ratio.

Equation (1) is a first-order differential equation of the type:

$$\log y'(t) + \beta \log y(t) = \beta \log y^*$$

where log y'(t) is the time derivative of log y(t). It can be solved in four steps:

Solution of the linear differential growth equation of the first-order

Let us first define: $z(t) = \log y(t)$

First step: Solution of the corresponding homogenous equation $z'(t) + \beta\, z(t) = 0$

$$\frac{z'(t)}{z(t)} = -\beta \quad \Rightarrow \quad \int \frac{z'(t)}{z(t)}\, dt = -\int \beta\, dt$$

$$\Rightarrow \quad \log z(t) + b_1 = -\beta t + b_2$$

$$\Rightarrow \quad \log z(t) = -\beta t + b \qquad \text{where } b = b_1 + b_2$$

$$\Rightarrow \quad e^{\log z(t)} = e^{-\beta t + b}$$

$$\Rightarrow \quad z_1(t) = e^{-\beta t}\, e^{b}$$

$$\Rightarrow \quad z_1(t) = e^{-\beta t}\, \theta \qquad \text{where } \theta = e^{b}$$

Second step: Particular solution of the equation $z'(t) + \beta\, z(t) = \beta\, z^*$

An obvious particular solution is at the steady state where $z'(t) = 0$, then $z_2(t) = z^*$.

Solution of the linear differential growth equation of the first-order

Third step: General solution of the equation $z'(t) + \beta\, z(t) = \beta\, z^*$

This is the sum of the solution of the homogenous equation and the particular solution of our equation:

$$z(t) = z_1(t) + z_2(t) = e^{-\beta t}\,\theta + z^* \qquad (2)$$

Fourth step: Final solution of the equation $z'(t) + \beta\, z(t) = \beta\, z^*$

What is left to do is to give a value for θ. This value can be determined by a value for z(t) at a particular date t. For example, the initial condition is a good candidate: z(0) for t = 0. Then, at t = 0,

$$z(0) = e^0\,\theta + z^* \quad \Rightarrow \quad \theta = z(0) - z^*$$

Substituting in (2) for θ :

$$z(t) = e^{-\beta t}\,[z(0) - z^*] + z^* \quad \Rightarrow \quad z(t) = (1 - e^{-\beta t})\, z^* + e^{-\beta t}\, z(0)$$

Solution of the linear differential growth equation of the first-order

Eventually, as $z(t) = \log y(t)$, the solution of our differential equation is

$$\mathbf{\log y(t) = (1 - e^{-\beta t}) \log y^* + e^{-\beta t} \log y(0)} \qquad (3)$$

where $\beta = (1 - \alpha)(n + \delta)$ and $y^* = (k^*)^\alpha$

If we have data on GDP per capita in an initial date and a terminal date, then we can estimate the speed of convergence β. If we substract log y(0) from both sides of (3) and substitute for y* then

$$\log y(t) - \log y(0) = (1 - e^{-\beta t}) \log \frac{1}{1 - \alpha} [\log sA - \log (n + \delta] + (1 - e^{-\beta t}) \log y(0)$$

In the Solow-Swan model, the growth of income (left-hand side) is a function of the determinants of the steady state and the initial level of income.