About the Gurson model and his extensions

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Nomenclature

Abbreviations

GTN	Gurson-Tvergaard-Needleman	

Greek letters

α	Huang [1991] correction constant	
β	Huang [1991] correction constant	
$\boldsymbol{\sigma}$	Stress tensor	[MPa]
ϵ_N	mean effective plastic strain of the matrix at incipient nucleation from Chu and Needleman [1980]	
ϵ_{eq}	equivalent effective plastic strain in the matrix	
$\overline{\sigma}$	Equivalent flow stress in the matrix	[MPa]
σ_Y	Flow stress	[MPa]
σ_{eq}	Macroscopic effective stress	[MPa]
σ_m	Macroscopic mean stress	[MPa]
$\boldsymbol{\epsilon}^p$	Plastic strain tensor	
σ^∞	Stress in the matrix	[MPa]
${\sigma^\infty}_{eq}$	Equivalent stress in the matrix	[MPa]

Roman letters

$\dot{f}_g$	Porosity (growth contribution)	
$\dot{f}_n$	Porosity (nucleation contribution)	
$\mathcal{A}$	Strain contribution from the Needleman and Rice [1978] nucleation law	

$\mathcal{B}$	Strain contribution from the Needleman and Rice [1978] nucleation law	
c	Material parameter from the Needleman and Rice [1978] nucleation law	
f	Microvoid volume fraction (Porosity)	
f_F	Porosity at final failure from Tvergaard and Needleman [1984]	
f_N	Nucleated void volume fraction from Chu and Needleman [1980]	
f_u	Ultimate value of porosity at the occurrence of ductile rupture from Tvergaard and Needleman [1984]	
f_{cr}	Critical void volume fraction at the onset of coalescence from Tvergaard and Needleman [1984]	
q_1	Parameter from the GTN model	
q_2	Parameter from the GTN model	
q_3	Parameter from the GTN model	
R	Void radius	[m]
S_N	Gaussian standard deviation of the normal distribution inclusions from Chu and Needleman [1980]	
T	Triaxiality	
V_A	Volume of the cavity and the matrix	[m]
V_M	Volume of the cavity	[m]
p^{∞}	Hydrostatic pressure in the matrix	[MPa]

1 Introduction

The goal of the damage models is the prediction of fracture during forming processes or structural loading after a progressive deterioration of material properties. From all the types of damage (brittle, ductile, creep, fatigue, ...), we are particularly interested in the ductile damage, which is associated with large plastic deformation in the neighbourhood of crystal defects.

The accurate prediction of the ductile fracture on material is very important because it is associated with a damage mechanism that affects the material properties, after certain level of plastic deformation. There are two common approaches:

1. Uncoupled approach: The calculation of the rupture follows a criteria which has no effect into the classical constitutive behaviour of the material.

2. Coupled approach: The damage development is incorporated into the constitute equation to develop a brand new continuum damage theory.

The coupled approach has some advantages over the uncoupled one, allowing the prediction of different fracture types and a better characterisation of the fracture zone. Nevertheless, the integration into a finite element code is more difficult than the uncoupled approach.

The Gurson model belongs to the coupled approach group, and his strong physical roots has allow a great increase of his use during the last 30 years. In this report a general description of this model, the parameters involved and the finite element implementation in the LAGAMINE FE code are discussed.

2 Historical background

Physical observation regarding ductile fracture phenomena in metals dates backs to the sixties (see early references by Rice and Tracey [1969] and Gurson [1977]), when it became more or less clear[1] that the fracture phenomena in metal involved the generation, growth and coalescence of microscopic voids. Since the early work by Bridgman [1952], when analyzing the effect of the external pressure in the development of plasticity and fracture, the experimental evidence regarding the effect of the stress state on fracture has been studied. One of the main conclusion of this study was that the external pressure has a significant effect on damage leading to failure; eventually,

[1]Considering the obvious technological and theoretical limitations at that time, above all the ignorance of the exact physical mechanisms triggering the damage development.

failure can occur with or without damage development depending on the applied external pressure.

Based on these observations, the first micromechanical studies looked for a relation between the growth of the void and the stress and strain fields. The pioneer work of McClintock [1968] and then Rice and Tracey [1969], both studying the growth of isolated voids, gave the first theoretical framework for ductile failure. In this respect, McClintock [1968] proposed a model for an isolated cylindrical cavity, where the growth rate is an increasing function of the stress triaxiality ratio[2], founding that the void expansion increases exponentially with the transverse stress. This work was later extended by Rice and Tracey [1969], who performed a variational analysis of a single spherical cavity within an infinite perfectly plastic medium under J_2-plasticity. The following evolution equation for the void radius was obtained:

$$\frac{\dot{R}}{R} = 0.283 \exp\left(\frac{1}{2}\frac{{\sigma^\infty}_{kk}}{{\sigma^\infty}_{eq}}\right)\dot{p}^\infty \tag{1}$$

Later, Huang [1991] modified Eq. 1 using additional velocity fields:

$$\frac{\dot{R}}{R} = \begin{cases} \alpha \exp(\beta T)\dot{p} & \text{if } T > 1 \\ \alpha T^{1/4} \exp(\beta T)\dot{p} & \text{if } T \leq 1 \end{cases} \tag{2}$$

With $\alpha = 0.427$ and $\beta = 1.5$. For high triaxiality values, the solutions obtained by McClintock [1968] and Rice and Tracey [1969], Huang [1991] are very similar, mainly because void shape effects are negligible at large values of triaxiality. It is interesting to present these results in their original forms as they, despite the time since they were formulated, present a simple relation for void growth. Moreover, it is notable that these models can used in the post-processing of elastic-plastic calculations as a failure criteria, taken a critical void growth as a material parameter.

[2] In fact, triaxiality also affect nucleation of new voids [Benzerga and Leblond, 2010].

3 General description of the Gurson model

The Gurson model is a mathematical representation of ductile damage based on the michromechanics of the material, using the continuum mechanics approach. It cames as a result of the application of the homogenization theory in the analysis of the plastic stress field in a microscopic medium composed of a dense matrix and cavities. The model is expressed as a macroscopic yield criteria, introducing a micromechnical variable as the damage parameter: the *void volume fraction*. Herein, a brief explanation of the model and his parameters is presented. No intend is for given a detailed description. For a more complete presentation of the model, the reader is encouraged to read the review papers of Tvergaard [1989], Pardoen and Besson [2004], Besson [2009], Benzerga and Leblond [2010] and François et al. [2013].

3.1 Gurson [1977]

The Gurson [1975, 1977] model was born from the experimental evidence regarding the influence of microvoid growth on plastic deformation and the ductile fracture. Hence, the key feature of this model is the void volume fraction (porosity), which acts as an *imperfection* [Li et al., 2011] during the plastic flow. It is defined by:

$$f = \frac{V_A - V_M}{V_A} \tag{3}$$

One void is surrounded by a plastic material matrix, which is incompressible, with no hardening (rigid plastic matrix), isotropic behaviour and no viscosity. The resulting yield locus is shown in Eq. 4.

$$F_p(\boldsymbol{\sigma}, f, \sigma_Y) = \frac{\sigma_{eq}^2}{\overline{\sigma}^2} - 1 + \underbrace{2f \cosh \frac{3}{2}\frac{\sigma_m}{\overline{\sigma}} - f^2}_{\text{Damage}} = 0 \tag{4}$$

It is important to note that when $f = 0$, the Gurson yield locus recovers the classical isotropic von Mises yield locus. The growth of the void is considered including the following equation:

$$\dot{f} = \dot{f}_g = \frac{V_M \dot{V}_A}{V_A^2} = (1 - f)\, tr\dot{\boldsymbol{\epsilon}}^p \tag{5}$$

This equation comes from the apparent volume change, mass conservation and plastic incompressibility of the matrix, derived from Eq. 3.

In his original form, the Gurson model does not take into account the plastic anisotropy, the mixed hardening of the dense matrix, the appearance of new voids, the coalescence leading to the crack and other phenomenas involved in ductile fracture. Moreover, the voids are consider spherical or confined into a infinite cylinder, which is certainly a severe hypothesis. Even if by definition of the yield locus the macroscopic media is compressible (I_1-*dependent*), the fully dense matrix surrounding the voids is in fact incompressible, being governed by a J_2 flow theory. For these and other reasons, numerous extensions had been proposed in the literature.

3.2 Gurson-Tvergaard-Needleman extension

Several extensions had been introduced into original Gurson model and between them, the Gurson-Tvergaard-Needleman (hereafter called the GTN model) was one of the first to compile robustly the three stages of damage development: void nucleation, growth and coalescence. In the following section, a brief descriptions[3] of each of these mechanism and the equations considered into the GTN model are presented.

3.2.1 Void nucleation

Within the nucleation of new microscopic voids two main mechanisms are found:

- Decohesion of matrix-inclusion or matrix-second phase interfaces.
- Hard particle fracture.

They can be observed separately or as combination or both, as there are parameters favouring one or the another. A concise summary of them can be found in Benzerga and Leblond [2010]. As a general rule, nucleation results from inhomogeneity of the plastic deformation between the matrix and the inclusions [Pineau, 2004]. Sometimes nucleation is stress controlled, while in other cases can be strain controlled because the energy condition for the separation of the interfaces can be met both by plastic deformation accumulation or stress within the interface. Nevertheless, the formulation of an adequate condition for void nucleation by interface fracture is difficult to obtain and usually another approach is used [François et al., 2013]. This *continuum* approach will be described below.

[3] No micromechanical models or cell studies are presented for the sake of simplicity. Refer to the reviewing papers for a more complete presentation of the topic.

Assuming that nucleation is a mechanism that is not linked with void growth, like void growth[4], the total porosity can be decomposed in the nucleated and growth part [Chu and Needleman, 1980]:

$$\dot{f} = \dot{f}_g + \dot{f}_n \tag{6}$$

Where f_g is already described in Eq. 5. The nucleation model of Needleman and Rice [1978] assumes that nucleation is a random function of both effective plastic strain or the equivalent stress:

$$\dot{f} = \underbrace{\mathcal{A}\dot{\epsilon}_{eq}^{P}}_{\text{Strain}} + \underbrace{\mathcal{B}\left(\dot{\sigma}_{eq} + c\dot{\sigma}_M\right)}_{\text{Stress}} \tag{7}$$

With $\dot{\sigma}_{eq} + c\dot{\sigma}_M > 0$, and c is a parameter adjusted by cell computations [Needleman, 1987]. Later, Chu and Needleman [1980] proposed a normal distribution for $\mathcal{A}$ and $\mathcal{B}$ to represent the heterogeneity of the nucleation phenomena.

$$\begin{aligned}\mathcal{A}(\epsilon_{eq}) &= \frac{f_N}{S_N\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\epsilon_{eq}-\epsilon_N}{S_N}\right)^2\right] \\ \mathcal{B}(\boldsymbol{\sigma}) &= 0\end{aligned} \tag{8}$$

Where f_n is the nucleated void volume fraction, f_N is the potential nucleated void volume fraction in relation with the inclusion volume fraction, ϵ_N is the mean effective plastic strain of the matrix at incipient nucleation, S_N is the Gaussian standard deviation of the normal distribution inclusions and ϵ_{eq} is the equivalent effective plastic strain in the matrix.

Stress controlled nucleation is not often used because they are harder to implement numerically [Besson, 2009, François et al., 2013]

3.2.2 Void coalescence

In coalescence we can distinguish three different behaviors, sorted in chronological order of observation within the literature [Benzerga and Leblond, 2010]:

- Internal necking: the grow of the voids is large enough to create a neck in the space between cavities. Often observed at high stress triaxialities.

[4] An assumption that has a micromechanical root [Benzerga and Leblond, 2010]

- Void sheeting: due to the formation of a secondary population of voids on small particles, creating shear bands. Dominant at low stress triaxialities.

- Necklace: due to formation of voided columns, prominent in steel containing elongated MnS inclusions and favored at low triaxiality level [Benzerga and Leblond, 2010]

Internal necking was studied by Thomason [1968, 1985] based on limit-load analysis of the ligament between voids. Void sheeting was described by Brown and Embury [1973], where a 45° band can be formed when the distance between voids is approximately equal to their height. With the exception of the third mode, failure by internal necking or void sheeting seems to be mainly dependent on the stress triaxiality and the microstructure, but also on the void distribution [Weck and Wilkinson, 2008].

At the macroscopic level, the coalescence can be easily observed in a load-displacement curve after an abrupt change in the slope at the onset of a (macroscopic) crack, then descending as the crack propagates. According to the Gurson [1977] model, the loss of material stress carrying capacity occurs when the voids have grown so large that the Gurson yield surface becames a point in the stress space i.e., $f = \dfrac{1}{q_1}$. Nevertheless, the experimental evidence shows that the material occurs much before.

Hence, and in order to incorporate coalescence into the Gurson model, Tvergaard and Needleman [1984] proposed to use f_c not as an additive part of the porosity (like Eq. 6) but a specific coalescence function f^*, which replaces the porosity:

$$f^* = \begin{cases} f & \text{if } f < f_{cr} \\ f_{cr} + \dfrac{f_u - f_{cr}}{f_F - f_{cr}}(f - f_{cr}) & \text{if } f > f_{cr} \end{cases} \tag{9}$$

Where f_u is the ultimate value of f^* at the occurrence of ductile rupture, f_{cr} is the critical void volume fraction at the onset of coalescence and f_F is the porosity at final failure. The aim of f^* is to model the complete vanishing of the carrying stress capacity due to void coalescence. Both q_i and f^* allows to recover results from a mesoscopic approach near rupture.

Coalescence is associated to mechanisms of plastic flow localization[5] within the matrix, which is certainly harder to capture compared to diffuse plastic flow during void growth. For a detailed discussion of coalescence

[5] These mechanisms involves, for instance, void growth or void interactions.

models and their physical roots, see Benzerga and Leblond [2010]. It is interesting to note that the main effect of the Lode angle on ductile fracture seems to be through the failure mode.

3.2.3 Correction factors

Tvergaard [1982] introduced the factors q_1 and q_2 (and a third one $q_3 = {q_1}^2$) describing more accurately void growth kinematics in unit cell calculations. A modified version of Eq. 4 is thus obtained:

$$F_p(\boldsymbol{\sigma}, f, \sigma_Y) = \frac{\sigma_{eq}^2}{\sigma_Y^2} - 1 + 2q_1 f \cosh -\frac{3q_2\sigma_m}{2\sigma_Y} - q_3 f^2 = 0 \tag{10}$$

Originally, the Gurson model gives somewhat too large localization strains if q_1 =1.0. Nevertheless, using a value of q_1 =1.5 allows to the continuum model to be in good agreement with the localization strain for the cell analysis carried on by [Tvergaard, 1981].

Some authors claim, wrongly motivated from these previous results, that q_1 and q_2 are parameters accounting for the void shape or the interactions between voids. The evidence, in this aspect, is sparse and is more likely that these parameters reflect the inner imperfections of the model [Benzerga and Leblond, 2010]. For instance, cell analysis by Koplik and Needleman [1988] and Gao et al. [1998] have shown that both parameters vary with the geometry and loading conditions. Faleskog et al. [1998] showed that these parameters also depends on the plastic hardening exponent and the ratio of the yield stress over the Young modulus. Ben Bettaieb et al. [2012] mathematically demonstrated that fixing q_2 lead to bad results, so it is integrated as a state variable within the calculations. This could be related that the void it is assumed to be an empty space where no inclusion is considered inside.

4 Versions available in Lagamine

In its current state, Lagamine has four version of the Gurson model. All of them were programmed by Mohammed Ben Bettaieb with the exception of `GUR3D`. The main difference between the three versions programmed by Ben Bettaieb is the calculation and integration of the damage. All this laws are 3D, so they can only be used with solid elements[6].

4.1 GUR3D

Is the first version implemented in the Lagamine code. It for isotropic materials with mixed isotropic and kinematic hardening. No further information is given.

4.2 GUR3Dclas

Is the classic Gurson model plus plastic anisotropy and mixed isotropic (Swift) and kinematic (Armstrong and Fredrick) hardening. The evolution of cavities is classical and it is integrated implicitly. This law is the recommended for general applications. The yield equation is defined by:

$$F_p(\boldsymbol{\sigma}, \boldsymbol{\alpha}, f, \sigma_Y) = \frac{\widehat{\sigma}_{eq}^2}{\sigma_Y^2} - 1 + 2q_1 f \cosh - \frac{3q_2\widehat{\sigma}_m}{\kappa\sigma_Y} - q_3 f^2 = 0 \tag{11}$$

Where $\widehat{\boldsymbol{\sigma}}$ is the effective stress tensor, defined by:

$$\widehat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{X} \tag{12}$$

And $\widehat{\sigma}_{eq}$:

$$\widehat{\sigma}_{eq} = \sqrt{\left(\frac{1}{2}\right)(\widehat{\boldsymbol{\sigma}} : \mathbb{H} : \widehat{\boldsymbol{\sigma}})} \tag{13}$$

And κ is a coefficient reflecting the plastic anisotropy effect [Benzerga and Besson, 2001, Benzerga et al., 2004a,b]. The porosity evolution is given by:

$$\dot{f} = \dot{f}_g = (1 - f)\mathrm{tr}\left(\dot{\epsilon}^P\right) \tag{14}$$

Equivalence between the rates of macroscopic and matrix plastic work is assumed:

$$\widehat{\boldsymbol{\sigma}} : \dot{\boldsymbol{\epsilon}}^P = (1 - f)\sigma_Y \dot{\epsilon_m}^P \tag{15}$$

[6] A plane strain version of the Gurson model is available, but there is no further information.

Void shape effects are not considered and the initial spherical shape is kept constant. Nucleation is also neglected. A more detailed description can be found in Ben Bettaieb et al. [2011b].

4.3 GUR3Dani

It is a model specific for Dual-Phase (DP) steels. It can be decomposed in two mains parts:

- The modeling of the matrix, extending the original model with plastic anisotropy and mixed isotropic (Swift) and kinematic (Armstrong and Fredrick) hardening.
- The evolution of the porosity, neglecting the morphological distribution and shape evolution of the voids (which are assumed spherical).

The algorithm is explicit respect to the porosity and implicit respect to other variables (equivalent plastic strain, yield stress, etc.). See Ben Bettaieb et al. [2011a] for further details.

4.4 GUR3Darcelor

Few times used in the Ben Bettaieb works.

A Triaxiality

Triaxiality has traditionally been used as a metric to characterize the stress state. It copuld be understand (roughly), a ratio between the hydrostatic (first invariant) and deviatoric (second deviatoric invariant) effects on the stress state.

$$T(I_1, J_2) = \frac{\sigma_m}{\sigma_{eq}} = \frac{1}{3\sqrt{3}} \frac{I_1}{\sqrt{J_2}} \tag{16}$$

The last years has shown an increase interest in other stress state measure,

Table 1: Stress state for different triaxialities.

Triaxiality	Stress state
< 0	Compression
0	Pure shear
$1/3$	Uniaxial tension
$2/3$	Biaxial tension
∞	Pure hydrostatic

the *Lode angle*. A better understanding of the stress state could be reached using this parameter under low triaxialities. For a detailed description of the Lode angle definition and effect on ductile fracture, see the respective internal report.

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