Upper bounds for the many-channel Marchenko transformation operator and its derivatives

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We specify the class of perturbative complex matrix potentials for which the corresponding many-channel Marchenko type transformation operators are bounded and integrable. Our reference matrix potential contains Coulomb interactions, different threshold energies, and centrifugal potentials with different angular momenta. Estimates for the transformation operator and its derivatives are obtained; they enable us to improve our recent results and are necessary for the establishment of a unique solution to the "generalized Marchenko fundamental equation." From the existence of an integrable transformation operator, the analyticity of the Jost solution as a function of k_1 is deduced in the upper-half of the physical k_1 plane.

1. INTRODUCTION

The transformation operator plays a prominent part in the theory of the inverse problem of scattering: Indeed, the starting point of the method developed by Agranovich and Marchenko[1] is to specify the class of potentials for which the existence of a bounded transformation operator can be proved. Within this class of potentials, a fundamental equation for the inverse problem is then derived. In the first part of their monograph, however, they only considered coupled channels without either the centrifugal terms or Coulomb interactions and with the same threshold energy in each channel. In the second part of their book, an indirect iterative approach for the singular centrifugal part was then used via the transformations introduced by Crum and Krein.[2] We consider here a system of differential equations containing different (and even not necessarily integer) angular momenta, different threshold energies and Coulomb interactions, by a direct method using the Riemann–Green solution.[3]

Cox[4] considered a system of coupled channels with different threshold energies. He was able to apply the method of Jost and Kohn[5] and to get a generalized Gel'fand–Levitan equation. However, he did not say for which potentials his equation is valid. His Gel'fand transformation operator is not necessarily a bounded function; it could happen that the transformation operator is only defined as a distribution. In that case, it is only useful if we can show that its diagonal part is bounded so that a well-behaved potential can be obtained by differentiation. Therefore, we want to find the class of perturbative potentials for which a transformation operator exists in the sense of functions theory and not in the enlarged sense of distributions. We only consider Marchenko's type transformation operator for the reasons explained in Ref. 6, and upper bounds for this operator and its derivatives are obtained. These bounds are necessary for the establishment of a stable and unique solution to the generalized fundamental Marchenko equation.[7] The recent results obtained in Refs. 3, 6, 8 must be modified according to our new estimate of the Riemann–Green solution. The present paper is divided into five sections: The introduction forms the first section. In Sec. 2, the definition of a transformation operator and the use of the Riemann–Green functions are briefly recalled. New estimates for the Riemann–Green solution are found in Sec. 3. In Sec. 4, the results for the Riemann–Green function enable us to obtain new upper bounds for the transformation operator and to specify the class of perturbative potentials for which a bounded integrable transformation operator exists. From the existence of an integrable transformation operator, the analyticity of the Jost solution in the upper half of the physical k_1 plane is shown in Sec. 5. The paper includes five appendices. In Appendix A, a spectral representation is derived for the complete Riemann–Green solution. Upper bounds for the unperturbated Jost solution, the derivatives of the unperturbated, and the complete Riemann–Green solution are obtained in Appendices B, C, D, respectively. The derivatives of the transformation operator are estimated in Appendix E.

2. THE TRANSFORMATION OPERATOR AND THE USE OF THE RIEMANN–GREEN SOLUTION

Two systems L_0 and L of n coupled differential equations are defined by the following two equations:

$$L_0(x)\phi_0(\Lambda,x) = \left[\frac{d^2}{dx^2} I + \Lambda - V_0(x)\right]\phi_0(\Lambda,x) \quad (1)$$

and

$$L(x)\phi(\Lambda,x) = \left[\frac{d^2}{dx^2} I + \Lambda - V_0(x) - V(x)\right]\phi(\Lambda,x), \quad (2)$$

where I, Λ, V_0, V stand for the unity matrix, the diagonal matrix of different channel wave numbers k_i^2 $(i = 1,n)$, the reference matrix potential and the perturbative matrix potential respectively. The reference diagonal potential contains the usual singularities: The centrifugal potential and the Coulomb interaction, while the perturbative potential is allowed to be complex (non-Hermitian). The Jost matrix solutions ϕ_0 and ϕ satisfy the same boundary conditions:

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$$\lim_{x\to\infty} [\phi(\Lambda,x)]_{ij} = \lim_{x\to\infty} [\phi_0(\Lambda,x)]_{ij} = \delta_{ij}\exp\{i[k_j x - l_j\pi/2 + \sigma_j - (\alpha_j/2k_j)\ln(2k_j x)]\}, \tag{3}$$

where the Coulomb phase σ_j is defined by Eq. (4):

$$e^{2i\sigma_j} = \frac{\Gamma(l_j+1+i\alpha_j/2k_j)}{\Gamma(l_j+1-i\alpha_j/2k_j)}. \tag{4}$$

ϕ_0 is the irregular Coulomb diagonal matrix defined by Eq. (5):

$$[\phi_0(\Lambda,x)]_{ij} = \delta_{ij} W_{-i\alpha_i/2k_i,l_i+1/2}(-2ik_i x) \times\exp[-\tfrac{1}{2}i\pi(l_i+1+i\alpha_i/2k_i)+i\sigma_i], \tag{5}$$

where W denotes the Whittaker's function (see Ref. 9). From Ref. 10, we know that when the Coulomb interaction is attractive, the reference and the perturbated problems have both an infinite number of bound states.

We are looking for a possible integral representation of the form:

$$\phi(\Lambda,x) = \phi_0(\Lambda,x) + \int_x^\infty K(x,t)\phi_0(\Lambda,t)\,dt, \tag{6}$$

when Λ belongs to the spectrum of $L\phi = 0$ and where $K(x,t)$ is the transformation kernel. This kernel $K(x,t)$ is connected with the solution of the inverse problem by the equation:

$$-2\frac{d}{dx}K(x,x) = V(x). \tag{7}$$

We want to specify the class of perturbative potentials V for which such a continuous bounded kernel exists.

It is shown in Ref. 3 that the transformation matrix elements have to satisfy the partial differential system (8):

$$\left[\frac{\partial^2}{\partial x^2} + k_i^2 - \frac{l_i(l_i+1)}{x^2} - \frac{\alpha_i}{x}\right]K_{ij}(x,y) - \sum_l V_{il}(x)K_{lj}(x,y) = \left[\frac{\partial^2}{\partial y^2} + k_j^2 - \frac{l_j(l_j+1)}{y^2} - \frac{\alpha_j}{y}\right]K_{ij}(x,y), \tag{8a}$$

$$\lim_{y\to\infty} K_{ij}(x,y) = \lim_{y\to\infty}\frac{\partial}{\partial y}K_{ij}(x,y) = 0, \tag{8b}$$

$$K_{ij}(x,x) = \tfrac{1}{2}\int_x^\infty V_{ij}(s)\,ds, \quad i,j = 1,n. \tag{8c}$$

The partial differential system (8) is equivalent to the integral system (9):

$$K_{ij}(x,y) = \tfrac{1}{2}\int_{(x+y)/2}^\infty R_{ij}(x,y;s,s)V_{ij}(s)\,ds + \tfrac{1}{2}\int_{\mathscr{D}}\int R_{ij}(x,y;s,u)\sum_l V_{il}(s) \times K_{lj}(s,u)\,du\,ds, \quad i,j = 1,n. \tag{9}$$

In Eq. (9), $\mathscr{D}$ denotes the Marchenko domain represented in Fig. 1 and the $R_{ij}(x,y,s,u)$ are the Riemann–Green solutions, satisfying the partial differential equations (10):

$$\left[\frac{\partial^2}{\partial x^2} + k_i^2 - \frac{l_i(l_i+1)}{x^2} - \frac{\alpha_i}{x}\right]R_{ij}(x,y;s,u) = \left[\frac{\partial^2}{\partial y^2} + k_j^2 - \frac{l_j(l_j+1)}{y^2} - \frac{\alpha_j}{y}\right]R_{ij}(x,y;s,u), \tag{10a}$$

$$R_{ij}(x,y;s,u) = 1 \quad \text{if} \quad |x-s| = |y-u|. \tag{10b}$$

If we use the canonical variables

$$\eta = \frac{x+y}{2}, \quad \xi = \frac{y-x}{2}, \quad \eta_0 = \frac{u+s}{2}, \quad \xi_0 = \frac{u-s}{2}, \tag{11}$$

Eq. (10) reads

$$\left[\frac{\partial}{\partial\eta\,\partial\xi} - \frac{l_j(l_j+1)}{(\eta+\xi)^2} - \frac{\alpha_j}{(\eta+\xi)} + k_j^2 + \frac{l_i(l_i+1)}{(\eta-\xi)^2} + \frac{\alpha_i}{(\eta-\xi)} - k_i^2\right] \times R_{ij}(\eta,\xi;\eta_0,\xi_0) = 0, \tag{12a}$$

$$R_{ij}(\eta,\xi;\eta_0,\xi_0) = 1 \quad \text{if} \quad \eta = \eta_0, \quad \text{or} \quad \xi = \xi_0. \tag{12b}$$

In Appendix A, the spectral representation (A10)–(A11) is obtained for $R_{ij}(x,y;s,u)$, using the techniques of Ref. 11. However, we are not able to write this spectral representation into a close form. Simple expressions for the Riemann functions $\mathscr{R}_{ij}$ and R^0_{ij} corresponding to the cases $l_i = \alpha_i = 0$, $i = 1,n$ and $k_i = \alpha_i = 0$, $i = 1,n$ respectively, can be deduced from this spectral representation:

$$\mathscr{R}_{ij}(x,y;s,u) = \mathscr{J}_0(\{\Delta_{ij}^2[(x-s)^2-(y-u)^2]\}^{1/2}), \tag{13}$$

where $\mathscr{J}_0$ is the Bessel function of zero order and

$$\Delta_{ij}^2 = k_i^2 - k_j^2, \tag{14}$$

and

$$R^0_{ij}(x,y;s,u) = P_{l_i}(1-2x_2) - 2x_1\int_0^1 P_{l_i}(1-2x_2+2x_2 t)P'_{l_i}(1-2x_1 t)\,dt, \tag{15}$$

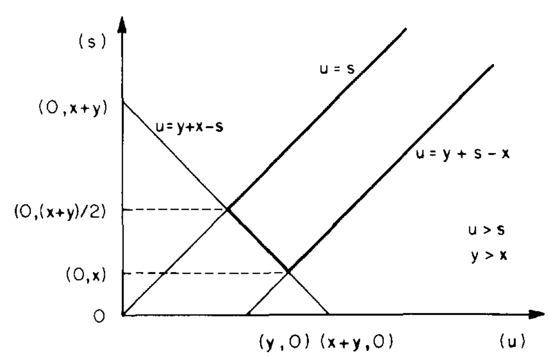

FIG. 1. Marchenko domain.

where

$$x_1 = \frac{(\eta - \eta_0)(\xi - \xi_0)}{(\eta - \xi)(\eta_0 - \xi_0)}, \tag{16}$$

$$x = \frac{(\eta_0 - \eta)(\xi - \xi_0)}{(\eta + \xi)(\eta_0 + \xi_0)}, \tag{17}$$

and P_{l_i} is the Legendre polynomial.

This last function R^0_{ij} will be used in the next section in order to evaluate the complete Riemann solution R_{ij}.

3. UPPER BOUNDS FOR THE FUNCTIONS R^0_{ij} AND R_{ij}

In the domain $\mathscr{D}$ of Marchenko, the following inequalities are satisfied:

$$0 \leqslant \xi_0 \leqslant \xi \leqslant \eta \leqslant \eta_0 \leqslant \infty. \tag{18}$$

From Eqs. (16) and (17), one easily gets

$$x_1 < 0 \tag{19}$$

and

$$0 \leqslant x_2 \leqslant \frac{\eta_0 \xi}{(\eta + \xi)(\eta_0 + \xi_0)} \leqslant \frac{\eta_0 \xi}{2\eta_0 \xi} \leqslant \frac{1}{2}$$

or

$$0 \leqslant 1 - 2x_2 \leqslant 1. \tag{20}$$

Our next step is to prove the inequality (21):

$$1 \leqslant 1 - 2x_1 = 1 + 2\frac{(\eta_0 - \eta)(\xi - \xi_0)}{(\eta - \xi)(\eta_0 - \xi_0)} \leqslant \frac{2(\eta + \xi)(\eta_0 - \xi_0)}{(\eta - \xi)(\eta_0 + \xi_0)}. \tag{21}$$

After multiplication of both sides of Eq. (21) by $(\eta - \xi)(\eta_0 + \xi_0)/(\eta + \xi)(\eta_0 - \xi_0) \geqslant 0$, we must show that

$$E(\eta,\xi;\eta_0,\xi_0) = \frac{(\eta - \xi)(\eta_0 + \xi_0)}{(\eta + \xi)(\eta_0 - \xi_0)} + 2\frac{(\eta_0 + \xi_0)(\eta_0 - \eta)(\xi - \xi_0)}{(\eta + \xi)(\eta_0 - \xi_0)^2} \leqslant 2. \tag{22}$$

The following inequalities are successively obtained:

$$E(\eta,\xi;\eta_0,\xi_0)$$

$$\leqslant \frac{(\eta - \xi)(\eta_0 + \xi_0)}{(\eta + \xi)(\eta_0 - \xi_0)} + 2\frac{(\eta_0 + \xi_0)(\xi - \xi_0)}{(\eta + \xi)(\eta_0 - \xi_0)} \tag{23}$$

$$\leqslant \frac{(\eta + \xi)(\eta_0 + \xi_0) - 2\xi_0(\eta_0 + \xi_0)}{(\eta + \xi)(\eta_0 - \xi_0)} \tag{24}$$

$$\leqslant \frac{(\eta + \xi)(\eta_0 - \xi_0) + 2\xi_0(\eta + \xi - \eta_0 - \xi_0)}{(\eta + \xi)(\eta_0 - \xi_0)} \tag{25}$$

$$\leqslant 1 + 2\frac{\xi_0(\eta + \xi - \eta_0 - \xi_0)}{(\eta + \xi)(\eta_0 - \xi_0)}. \tag{26}$$

As

$$\frac{\partial}{\partial \xi}\left(\frac{\eta + \xi - \eta_0 - \xi_0}{\eta + \xi}\right) = \frac{\eta_0 + \xi_0}{(\eta + \xi)^2} > 0,$$

the right-hand side of Eq. (26) is certainly overestimated, if we replace ξ by $\eta > \xi$. Finally, we get the requested result (21):

$$E(\eta,\xi;\eta_0,\xi_0) \leqslant 1 + \frac{\xi_0}{\eta}\frac{(2\eta - \eta_0 - \xi_0)}{(\eta_0 - \xi_0)}$$

$$\leqslant 1 + \frac{\xi_0}{\eta}\frac{(\eta - \xi_0)}{(\eta_0 - \xi_0)} \leqslant 2. \tag{27}$$

We are looking now for an upper bound for $|R^0_{ij}|$. Starting from its representation (15), we get

$$|R^0_{ij}| \leqslant |P_{l_i}(1 - 2x_2)| + \left|\int_0^1 P_{l_i}(1 - 2x_2 + 2x_2 t)\, dP_{l_i}(1 - 2x_1 t)\right|. \tag{28}$$

Since

$$0 \leqslant 1 - 2x_2(1 - t) \leqslant 1$$

and

$$1 \leqslant 1 - 2x_1 t \leqslant 2\frac{(\eta + \xi)(\eta_0 - \xi_0)}{(\eta - \xi)(\eta_0 + \xi_0)}$$

for $0 \leqslant t \leqslant 1$, we can write

$$|R^0_{ij}(\eta,\xi;\eta_0,\xi_0)| \leqslant 1 + |P_{l_i}(1 - 2x_1) - 1| \leqslant P_{l_i}(1 - 2x_1) \leqslant \left[4\frac{(\eta + \xi)(\eta_0 - \xi_0)}{(\eta - \xi)(\eta_0 + \xi_0)}\right]^{l_i}, \tag{29}$$

where the following property has been used:

$1 \leqslant P_l(x) \leqslant (2x)^l$ for $x > 1$ and l real $\geqslant 0$.

We can construct the complete Riemann function R_{ij} from the function R^0_{ij}, using the composition formula

$$R_{ij}(\eta,\xi;\eta_0,\xi_0) = R^0_{ij}(\eta,\xi;\eta_0,\xi_0) + \int_{\xi_0}^{\xi} \mathrm{d}\xi_1 \int_{\eta}^{\eta_0} \mathrm{d}\eta_1\, R^0_{ij}(\eta,\xi;\eta_1,\xi_1) \times\left(k_i^2 - k_j^2 - \frac{\alpha_i}{\eta_1 - \xi_1} + \frac{\alpha_j}{\eta_1 + \xi_1}\right) R_{ij}(\eta_1,\xi_1;\eta_0,\xi_0). \tag{30}$$

The bound (29) for $|R^0_{ij}|$ is then used to find an upper bound for $|R_{ij}|$. Setting

$$\tilde{R}^0_{ij}(\eta,\xi;\eta_0,\xi_0) = R^0_{ij}(\eta,\xi;\eta_0,\xi_0)\left[\frac{(\eta - \xi)(\eta_0 + \xi_0)}{4(\eta + \xi)(\eta_0 - \xi_0)}\right]^{l_i}, \tag{31}$$

$$\tilde{R}_{ij}(\eta,\xi;\eta_0,\xi_0) = R_{ij}(\eta,\xi;\eta_0,\xi_0)\left[\frac{(\eta - \xi)(\eta_0 + \xi_0)}{4(\eta + \xi)(\eta_0 - \xi_0)}\right]^{l_i}, \tag{32}$$

$$\mu_{ij}^2 = 4^{l_i}|k_i^2 - k_j^2| \tag{33}$$

and

$$a_{ij} = 2^{(2l_i + 1)}\max(\alpha_i,\alpha_j), \tag{34}$$

Eq. (30) can be written as

$$\tilde{R}_{ij}(\eta,\xi;\eta_0,\xi_0) = \tilde{R}^0_{ij}(\eta,\xi;\eta_0,\xi_0) + \int_{\xi_0}^{\xi} d\xi_1 \int_{\eta}^{\eta_0} d\eta_1\, \tilde{R}^0_{ij}(\eta,\xi;\eta_1,\xi_1) \times 4^{l_i}\left[(k_i^2 - k_j^2) + \left(\frac{-\alpha_i}{\eta_1 - \xi_1} + \frac{\alpha_j}{\eta_1 + \xi_1}\right)\right] \times \tilde{R}_{ij}(\eta_1,\xi_1;\eta_0,\xi_0), \tag{35}$$

where

$$|\tilde R^{\,0}_{ij}(\eta,\xi;\eta_0,\xi_0)|\leqslant 1. \qquad (36)$$

The successive approximations applied to the absolute value of Eq. (35) lead to

$$|\tilde R^{(0)}_{ij}(\eta,\xi;\eta_0,\xi_0)|\leqslant|\tilde R^{\,0}_{ij}(\eta,\xi;\eta_0,\xi_0)|\leqslant 1, \qquad (37)$$

$$\begin{aligned}
|\tilde R^{(1)}_{ij}|(\eta,\xi,\eta_0,\xi_0)
&=\Big|\int_{\xi_0}^{\xi}d\xi_1\int_{\eta}^{\eta_0}d\eta_1\,\tilde R^{\,0}_{ij}(\eta,\xi;\eta_1,\xi_1)\\
&\quad\times 4^{l_i}\Big(k_i^2-k_j^2+\frac{\alpha_j}{\eta_1+\xi_1}-\frac{\alpha_i}{\eta_1-\xi_1}\Big)\\
&\quad\times\tilde R^{(0)}_{ij}(\eta_1,\xi_1;\eta_0,\xi_0)\Big|\\
&\leqslant\int_{\xi_0}^{\xi}d\xi_1\int_{\eta}^{\eta_0}d\eta_1\Big(\mu_{ij}^2+\frac{a_{ij}\eta_1}{\eta_1^2-\xi_1^2}\Big)\\
&\leqslant\int_{\xi_0}^{\xi}d\xi_1\int_{\eta}^{\eta_0}d\eta_1\Big(\mu_{ij}^2+\frac{a_{ij}}{\eta_1-\xi_1}\Big)\\
&\leqslant\int_{\xi_0}^{\xi}d\xi_1\int_{\eta}^{\eta_0}d\eta_1\frac{[\mu_{ij}^2(\eta_0-\xi_0)+a_{ij}]}{(\eta-\xi_1)^{1/2}(\eta_1-\xi)^{1/2}},
\end{aligned}\qquad (38)$$

$$\begin{aligned}
|\tilde R^{(p)}_{ij}(\eta,\xi;\eta_0,\xi_0)|
&=\Big|\int_{\xi_0}^{\xi}d\xi_1\int_{\eta}^{\eta_0}d\eta_1\,\tilde R^{\,0}_{ij}(\eta,\xi;\eta_1,\xi_1)\\
&\quad 4^{l_i}\Big(k_i^2-k_j^2+\frac{\alpha_j}{\eta_1+\xi_1}-\frac{\alpha_i}{\eta_1-\xi_1}\Big)\\
&\quad\times\int_{\xi_0}^{\xi_1}d\xi_2\int_{\eta_1}^{\eta_0}d\eta_2\,\tilde R^{\,0}_{ij}(\eta_1,\xi_1;\eta_2,\xi_2)\\
&\quad\times 4^{l_i}\Big(k_i^2-k_j^2+\frac{\alpha_j}{\eta_2+\xi_2}-\frac{\alpha_i}{\eta_2-\xi_2}\Big)\cdots\\
&\quad\times\int_{\xi_0}^{\xi_{p-1}}d\xi_p\int_{\eta_{p-1}}^{\eta_0}d\eta_p\,\tilde R^{\,0}_{ij}(\eta_{p-1},\xi_{p-1};\eta_p,\xi_p)\\
&\quad\times 4^{l_i}\Big(k_i^2-k_j^2+\frac{\alpha_j}{\eta_p+\xi_p}-\frac{\alpha_i}{\eta_p-\xi_p}\Big)\\
&\quad\tilde R^{(0)}_{ij}(\eta_p,\xi_p;\eta_0,\xi_0)\Big|\\
&\leqslant[\mu_{ij}^2(\eta_0-\xi_0)+a_{ij}]^p\int_{\xi_0}^{\xi}d\xi_1\frac{1}{(\eta-\xi_1)^{1/2}}\\
&\quad\times\int_{\xi_0}^{\xi_1}d\xi_1\frac{1}{(\eta-\xi_2)^{1/2}}\cdots\int_{\xi_0}^{\xi_{p-1}}d\xi_p\frac{1}{(\eta-\xi_p)^{1/2}}\\
&\quad\times\int_{\eta}^{\eta_0}d\eta_1\frac{1}{(\eta_1-\xi)^{1/2}}\int_{\eta_1}^{\eta_0}d\eta_2\frac{1}{(\eta_2-\xi)^{1/2}}\cdots\\
&\quad\times\int_{\eta_{p-1}}^{\eta_0}d\eta_p\frac{1}{(\eta_p-\xi)^{1/2}}
\end{aligned}$$

(notice that for $p=0$, $\xi_p=\xi$, and $\eta_p=\eta$).

$$\begin{aligned}
|R^{(p)}_{ij}(\eta,\xi;\eta_0,\xi_0)|
&\leqslant[\mu_{ij}^2(\eta_0-\xi_0)+a_{ij}]^p\\
&\quad\times\frac{[\int_{\xi_0}^{\xi}d\xi_1/(\eta-\xi_1)^{1/2}]^p}{p!}\,\frac{[\int_{\eta}^{\eta_0}d\eta_1/(\eta_1-\xi)^{1/2}]^p}{p!}.
\end{aligned}\qquad (39)$$

Since the modified Bessel function can be defined by

$$\mathscr{T}_0(z)=\sum_{p=0}^{\infty}\frac{(z^2/4)^p}{(p!)^2}=\frac{1}{\pi}\int_0^{\pi}e^{z\cos\theta}\,d\theta\leqslant e^z, \qquad (40)$$

we easily get the following upper bounds:

$$\begin{aligned}
&|R_{ij}(\eta,\xi;\eta_0,\xi_0)|\\
&\leqslant 4^{l_i}\Big[\frac{(\eta+\xi)}{(\eta-\xi)}\frac{(\eta_0-\xi_0)}{(\eta_0+\xi_0)}\Big]^{l_i}\mathscr{T}_0\Big[2\Big([\mu_{ij}^2(\eta_0-\xi_0)+a_{ij}]\\
&\quad\times\int_{\xi_0}^{\xi}d\xi_1\frac{1}{(\eta-\xi_1)^{1/2}}\int_{\eta}^{\eta_0}d\eta_1\frac{1}{(\eta_1-\xi)^{1/2}}\Big)^{1/2}\Big]\\
&\leqslant 4^{l_i}\Big[\frac{(\eta+\xi)(\eta_0-\xi_0)}{(\eta-\xi)(\eta_0+\xi_0)}\Big]^{l_i}\\
&\quad\times\exp(4\{[\mu_{ij}^2(\eta_0-\xi_0)+a_{ij}](\eta_0-\xi_0)\}^{1/2}).
\end{aligned}\qquad (41a)$$

The same method, directly applied to the case $\alpha_i=0$, leads to

$$\begin{aligned}
&|R_{ij}(\eta,\xi;\eta_0,\xi_0)|\\
&\quad\leqslant 4^{l_i}\Big[\frac{(\eta+\xi)(\eta_0-\xi_0)}{(\eta-\xi)(\eta_0+\xi_0)}\Big]^{l_i}\exp[2\mu_{ij}(\eta_0-\xi_0)]
\end{aligned}\qquad (42)$$

The inequality (42) differs from the inequality (41) with $a_{ij}=0$, by a factor 2 in the exponent; this is due to the approximations we have done, in the evaluation of (41), in order to get separable integrands in Eqs. (38), (39). The estimate (41a) of R_{ij} is much more general and much easier to use than the estimate (58) of Ref. 3.

Several further approximations to Eq. (41a) can be performed:

(i) Since $\eta\xi_0\leqslant\eta_0\xi$, we have

$$(\eta+\xi)(\eta_0-\xi_0)/(\eta-\xi)(\eta_0+\xi_0)\geqslant 1$$

and

$$\begin{aligned}
&|R_{ij}(\eta,\xi;\eta_0,\xi_0)|\\
&\quad\leqslant 4^{l_i}\Big(\frac{(\eta+\xi)(\eta_0-\xi_0)}{(\eta_0+\xi_0)(\eta-\xi)}\Big)^{l_{\max}}\\
&\qquad\times\exp(4\{[\mu_i^2(\eta_0-\xi_0)+a_i](\eta_0-\xi_0)\}^{1/2}),
\end{aligned}\qquad (41b)$$

where

$$\mu_i=\max_j\mu_{ij}$$

and

$$a_i=\max_j a_{ij}$$

(ii) Since

$$\frac{\eta+\xi}{\eta_0+\xi_0}\leqslant\frac{\eta+\xi}{\eta_0}\leqslant\frac{\eta+\xi}{\eta}\leqslant 2,$$

we can also get from Eq. (41a)

$$|R_{ij}(\eta,\xi;\eta_0,\xi_0)| \leqslant 8^{l_i}\left(\frac{\eta_0-\xi_0}{\eta_0+\xi_0}\right)^{l_i} \times \exp(4\{[\mu_i^2(\eta_0-\xi_0)+a_i](\eta_0-\xi_0)\}^{1/2}). \tag{41c}$$

This inequality (41c) was the only one obtained in Ref. 2 by another method. In Sec. 4, estimates of K are obtained from the different approximations (41), and we explain why the use of inequality (41b) must be preferred.

4. AN UPPER BOUND FOR THE TRANSFORMATION OPERATOR

Using the canonical variables (11), the integral equation (9) for the transformation kernel reads

$$K_{ij}(\eta,\xi) = \tfrac{1}{2}\int_\eta^\infty d\eta_0\, R_{ij}(\eta,\xi;\eta_0,0)U_{ij}(\eta_0) + \int_\eta^\infty d\eta_0 \int_0^\xi d\xi_0\, R_{ij}(\eta,\xi;\eta_0,\xi_0) \times \sum_l U_{il}((\eta_0-\xi_0)K_{lj}(\eta_0,\xi_0). \tag{43}$$

If we introduce, in Eq. (43), the matrices $\tilde{R}$, $\tilde{K}$, and $\tilde{U}$ defined by the following equations,

$$R_{ij}(\eta,\xi;\eta_0,\xi_0) = \tilde{R}_{ij}(\eta,\xi;\eta_0,\xi_0)\left[\frac{(\eta+\xi)}{(\eta_0+\xi_0)}\,\frac{(\eta_0-\xi_0)}{(\eta-\xi)}\right]^{l_{max}} \times \exp(4\{[\mu_i^2(\eta_0-\xi_0)+a_i](\eta_0-\xi_0)\}^{1/2})4^{l_i}, \tag{44}$$

$$K_{ij}(\eta,\xi) = \tilde{K}_{ij}(\eta,\xi)\left(\frac{\eta+\xi}{\eta-\xi}\right)^{l_{max}}, \tag{45}$$

$$\tilde{U}_{ij}(\eta) = 4^{l_i}\exp(4\{[\mu_i^2(\eta_0-\xi_0)+a_i] \times (\eta_0-\xi_0)\}^{1/2})U_{ij}(\eta), \tag{46}$$

where

$$\mu_i = \max_j(\mu_{ij}) \tag{47}$$

and

$$a_i = \max_j(a_{ij}), \tag{48}$$

we get

$$\tilde{K}_{ij}(\eta,\xi) = \tfrac{1}{2}\int_\eta^\infty d\eta_0\, \tilde{R}_{ij}(\eta,\xi;\eta_0,0)\tilde{U}_{ij}(\eta_0) + \int_\eta^\infty d\eta_0\int_0^\xi d\xi_0\, \tilde{R}_{ij}(\eta,\xi;\eta_0,\xi_0) \times \sum_l \tilde{U}_{il}(\eta_0-\xi_0)\tilde{K}_{lj}(\eta_0,\xi_0). \tag{49}$$

From Eq. (41b), we see that $|\tilde{R}_{ij}| \leqslant 1$ and successive approximations applied to the absolute value of Eq. (49) give

$$|\tilde{K}_{ij}^{(0)}(\eta,\xi)| \leqslant \tfrac{1}{2}\int_\eta^\infty d\eta_0 |\tilde{U}_{ij}(\eta_0)| \leqslant \tfrac{1}{2}\int_\eta^\infty d\eta_0 \|\tilde{U}(\eta_0)\| = \tfrac{1}{2}\tilde{\sigma}^0(\eta), \tag{50}$$

where

$$\|A\| = \sup_i \sum_{j=1}^n |A_{ij}|,$$

$$\{\tilde{K}_{ij}^{(1)}(\eta,\xi)\} \leqslant \int_\eta^\infty \mathrm{d}\eta_0 \int_0^\xi \mathrm{d}\xi_0 \sum_l |\tilde{U}_{il}(\eta_0-\xi_0)|\,|\tilde{K}_{lj}^{(0)}(\eta_0,\xi_0)| \leqslant \tfrac{1}{2}\tilde{\sigma}^0(\eta)\int_\eta^\infty d\eta_0 \int_0^\xi \mathrm{d}\xi_0 \|\tilde{U}(\eta_0-\xi_0)\|, \tag{51}$$

$$|\tilde{K}_{ij}^{(p)}(\eta,\xi)| \leqslant \frac{1}{2}\,\frac{\tilde{\sigma}^0(\eta)}{p!}\left[\int_\eta^\infty d\eta_0\int_0^\xi \mathrm{d}\xi_0 \|\tilde{U}(\eta_0-\xi_0)\|\right]^p, \tag{52}$$

and finally

$$\|\tilde{K}(\eta,\xi)\| \leqslant \frac{n}{2}\tilde{\sigma}^0(\eta)\exp\left[\int_\eta^\infty d\eta_0\int_0^\xi \mathrm{d}\xi_0 \|\tilde{U}(\eta_0-\xi_0)\|\right]. \tag{53}$$

Using the physical variables again, we get

$$\int_\eta^\infty d\eta_0\int_0^\xi \mathrm{d}\xi_0 \|\tilde{U}(\eta_0-\xi_0)\| = \tfrac{1}{2}\int_x^{(x+y)/2} ds\int_{x+y-s}^{y-x+s} du\|\tilde{U}(s)\| + \tfrac{1}{2}\int_{(x+y)/2}^\infty ds\int_s^{y-x+s} du\,\|\tilde{U}(s)\|$$
$$= \int_x^{(x+y)/2} ds\,\|\tilde{U}(s)\|(s-x) + \tfrac{1}{2}\int_{(x+y)/2}^\infty \mathrm{d}s\,\|\tilde{U}(s)\|(y-x) \leqslant \int_x^\infty \|\tilde{U}(s)\|s\,ds = \tilde{\sigma}^1(x)$$

and also

$$\leqslant \tfrac{1}{2}(y-x)\tilde{\sigma}(x). \tag{54}$$

So we have

$$\|K(x,y)\| \leqslant \frac{n}{2}\left(\frac{y}{x}\right)^{l_{max}}\tilde{\sigma}^0\left(\frac{x+y}{2}\right) \times \exp\left[\tilde{\sigma}^0(x)\left(\frac{y-x}{2}\right)\right] \tag{55a}$$

and also

$$\|K(x,y)\| \leqslant \frac{n}{2}\left(\frac{y}{x}\right)^{l_{max}}\tilde{\sigma}^0\left(\frac{x+y}{2}\right)\exp[\tilde{\sigma}^1(x)] \tag{55b}$$

$$\leqslant \frac{2^{l_{\max}-1} n}{x^{l_{\max}}} \tilde{\sigma}^{l_{\max}}\left(\frac{x+y}{2}\right) \exp[\tilde{\sigma}^1(x)] \quad (56)$$

if $\tilde{\sigma}^{l_{\max}}$ exists and where

$$\tilde{\sigma}^l\left(\frac{x+y}{2}\right) = \sup_i \sum_{j=1}^n \int_{(x+y)/2}^{\infty} \eta_0^l 4^{l_i} | U_{ij}(\eta_0) |$$

$$\times \exp\{4[\mu_i^2(\eta_0 - \xi_0)^2$$

$$+ a_i(\eta_0 - \xi_0)]^{1/2}\} d\eta_0, \quad (57)$$

$$\mu_i^2 = \max_j (4^{l_i} | k_i^2 - k_j^2 |), \quad (58)$$

$$a_i = 2^{(2l_i+1)} \max_j (|\alpha_j|). \quad (59)$$

The same method with the use of estimate (41c) leads to the estimate

$$|K_{ij}(x,y)| \leqslant \frac{1}{2}\left(\frac{2}{x}\right)^{l_i} \tilde{\sigma}^{l_i}\left(\frac{x+y}{2}\right) \exp[\tilde{\sigma}^{(l_i - l_i + 1)}(x)], \quad (60)$$

which corresponds to the one obtained in Ref. 3 and used in Refs. 6 and 8. However, from estimate (60), it is not obvious if $\lim_{x\to 0} K(x,x)$ is finite while it is so from estimate (55). From that point of view, Eq. (55) is much better. If we use estimate (41a), without any approximation, the method of successive approximations gives

$$|K_{ij}(x,y)| \leqslant \frac{1}{2}\left(\frac{y}{x}\right)^{l_i} \tilde{\sigma}^0\left(\frac{x+y}{2}\right) \exp[\|\Sigma(x,y)\|], \quad (61)$$

with

$$\Sigma_{ij}(x,y) = \frac{1}{2}\int_x^{(x+y)/2} ds \int_{x+y-s}^{y-x+s} du \, | \tilde{U}_{ij}(s) | \left(\frac{u}{s}\right)^{l_j - l_i}$$
$$+ \frac{1}{2}\int_{(x+y)/2}^{\infty} ds \int_s^{y-x+s} du \, | \tilde{U}_{ij}(s) | \left(\frac{u}{s}\right)^{l_j - l_i}. \quad (62)$$

If $l_i \geqslant l_j$, since $u/s > 1$, we have

$$\|\Sigma(x,y)\| \leqslant \tilde{\sigma}^1(x). \quad (63a)$$

For $l_i < l_j$, we get

$$\Sigma_{ij}(x,y) \leqslant 3^{(l_j - l_i)} \tilde{\sigma}^1\left(\frac{x+y}{2}\right)$$
$$+ \frac{1}{2}\int_x^{(x+y)/2} ds \int_{x+y-s}^{y-x+s} du \, | \tilde{U}_{ij}(s) | \left(\frac{u}{s}\right)^{l_j - l_i}$$
$$\leqslant 3^{(l_j - l_i)} \tilde{\sigma}^1\left(\frac{x+y}{2}\right) + \left(\frac{y}{x}\right)^{l_j - l_i}$$
$$\times \left[\tilde{\sigma}^1(x) - \tilde{\sigma}^1\left(\frac{x+y}{2}\right)\right]. \quad (63b)$$

From estimate (61), it is not obvious whether $\lim_{y\to\infty} K(x,y)$ is finite whereas it is the case from estimate (56). The most general estimate (41a) for R_{ij} will, however, be used in Appendices C, D, E in order to get estimates for the derivatives of R_{ij}^0, R_{ij}, and K_{ij}. The results of Refs. 6 and 8 may be modified according to our new estimates for K.

In order to ensure the convergence of the integral in Eq. (6), since, in Appendix B, an upper bound (B12) for the function $\phi_0(\Lambda,x)$ is found for $\mathrm{Im} k_i = \delta \geqslant 0$, $| k_i | \geqslant s > 0$, and $\alpha_i \geqslant 0$, it is sufficient to impose that

$$\int_x^{\infty} y^{l_{\max}} \tilde{\sigma}^0\left(\frac{x+y}{2}\right) e^{-\delta y} \frac{[1 + 2|k_i|y]^{l_i}}{(2|k_i|y)^{l_i}} < \infty \quad \text{for } x \geqslant 0$$

or

$$\int_x^{\infty} dy \, \tilde{\sigma}^{l_{\max}}\left(\frac{x+y}{2}\right)$$
$$= 4^{l_{\max}} \int_x^{\infty} dy \int_{(x+y)/2}^{\infty} d\eta_0 \, \eta_0^{l_{\max}} \| \tilde{U}(\eta_0) \|$$
$$= 4^{l_{\max}} \int_x^{\infty} d\eta_0 \int_x^{2\eta_0 - x} dy \, \eta_0^{l_{\max}} \| \tilde{U}(\eta_0) \|$$
$$\leqslant 2^{(2l_{\max}+1)} \int_x^{\infty} d\eta_0 \, \eta_0^{l_{\max}+1} \| \tilde{U}(\eta_0) \|$$
$$= 2\tilde{\sigma}^{l_{\max}+1}(x) < \infty. \quad (64)$$

Since we suppose that $\tilde{\sigma}^{l_{\max}+1}(x)$ exists for $x \geqslant 0$, we can justify the interchange of the order of the integrals in Eq. (61) by the Tonelli–Fubini theorem. Equations (55)–(64) show that $x^{l_{\max}} K(x,y)$ is bounded and integrable if $\tilde{\sigma}^{l_{\max}+1}(0)$ remains finite. Nowhere have we made the assumption that U is a real or Hermitian matrix; our results are thus valid for non-Hermitian potentials. We see that an exponential decrease is required for the perturbation potential; the rate of this decrease is measured by $4(\mu^2 x^2 + ax)^{1/2}$. In absence of Coulomb interactions, $U_{ij}(x)$ must decrease faster than $\exp[-2^{l_i+1} | k_i^2 - k_j^2 |^{1/2} x]$. (The exponent obtained in Ref. 3 contains a wrong factor.) In Appendices C and D, bounds for the derivatives of R_{ij}^0 and R_{ij} are obtained. These bounds enable us to get a bound for the derivatives of the transformation operator K in Appendix E. The estimates (50), (51), and (E3) obtained for $K(x,y)$ and its derivatives are necessary if we want to prove the unicity and the stability of a suitable solution to the fundamental equation of the inverse problem the existence of which Coz and the author[7] have generalized for non-Hermitian systems of coupled channels.

5. ANALYTICITY OF THE JOST MATRIX SOLUTION

We first consider the Jost solution as a function of the n variables $(k_1,...,k_n)$. Equation (6) yields

$$[\phi(\Lambda,x)]_{ij} = \delta_{ij}[\phi_0(\Lambda,x)]_{jj}$$
$$+ \int_x^{\infty} K_{ij}(x,y)[\phi_0(\Lambda,y)]_{jj} \, dy. \quad (65)$$

Since $[\phi_0(\Lambda,x)]_{jj}$ depends only on k_j and since the solution

$K_{ij}(x,y)$ of Eq. (8) is an entire function of

$$\Delta_{ji}^2 = k_j^2 - k_i^2 \tag{66}$$

(from a theorem demonstrated by Poincaré[12]), $[\phi(\Lambda,x)]_{ij}$ is a function of k_j and Δ_{ji}^2 only. This shows that $[\phi(\Lambda,x)]_{ij}$ is an even function of all the k_l's, except k_j as is well known from Ref. 12.

Instead of considering $\phi(\Lambda,x)$ as a function of many variables k_l $(l = 1,n)$, the conservation of energy (66) between the channels may be used to eliminate $k_2,...,k_n$ in favor of the largest variable k_1 and the $n-1$ constants Δ_{1l}^2 $(l = 2,n)$. By doing this, we must define, as in Refs. 13, a k_1 Riemann surface, consisting of 2^{n-1} sheets, having branch points at $k_1 = \pm \Delta_{l1}$ $(l = 2,...,n)$. With each sheet, Weidenmüller[13] associates a vector $\boldsymbol{\tau}$ of $n-1$ elements $\tau_l = \pm 1$, defined according the rule

$$\text{sgn Re}k_{l+1} = \tau_l \,\text{sgn Re}k_1, \tag{67}$$

$$\text{sgn Im}k_{l+1} = \tau_l \,\text{sgn Im}k_1. \tag{68}$$

The physical sheet is defined by $\tau_l = +1$ $(l = 1,n-1)$. In order to prove the analyticity of $\Phi(k_1,x)$ with respect to k_1 in the upper half of the physical plane, the existence and the continuity of $\Phi(k_1,x)$ and its first derivative with respect to k_1 must be shown. The existence of a bounded continuous $\Phi_{ij}(k_1,x)$ has already been shown if $\tilde{\sigma}^{l_{\max}+1}(0)$ exists for $\text{Im}k_1 > 0$, hence for $\text{Im}k_j > 0$. Since the integral in Eq. (6) converges absolutely, the differentiation of Φ with respect to k_1 can be performed under the integral sign:

$$\left[\frac{d}{dk_1}\Phi(k_1,x)\right]_{ij} = \left[\frac{d}{dk_1}\Phi_0(k_1,x)\right]_{jj}\delta_{ij} + \int_x^\infty K_{ij}(x,t) \times\left(\frac{d}{dk_1}\Phi_0(k_1,t)\right)_{jj} dt. \tag{69}$$

Since $[(d/dk_1)\Phi_0(k_1,x)]$ exists and can be bounded for $\text{Im}k_1 > 0$ and since $K_{ij}(x,t)$ is absolutely integrable, $[(d/dk_1)\Phi(k_1,x)]_{ij}$ is well defined and bounded. The matrix function $\Phi(k_1,x)$ is thus analytical in the upper half of the physical k_1 plane, for all fixed $x > 0$.

6. CONCLUSION

We have found sufficient conditions that the matrix perturbative potential should satisfy in order to get a bounded and integrable transformation operator with bounded first derivatives. These conditions are, of course, dependent on the reference potential: The centrifugal part imposes that the perturbative potential has certain moments [see Eq. (56)] while the Coulomb interaction or the different threshold energies lead to an exponential decrease of the potential [see Eq. (57)]. For this class of potentials, the analyticity of the Jost solution has been shown in the upper half of the physical k_1 plane (for $\alpha_i > 0$). This property of analyticity is essential if we want to establish a fundamental equation for the inverse problem. The upper bounds (55), (56), (E3) that we have obtained for the transformation operator and its derivatives are of basic importance for two reasons:

(i) They allow us to get an upper bound for the kernel of the fundamental equation and consequently give us the conditions to be imposed to the scattering data in order to have a suitable solution to the inverse problem.

(ii) They are also necessary to ensure the stability of the inverse problem (see Ref. 14). Indeed, the experimental scattering data are only known up to a certain energy and the question naturally arises whether this is sufficient to well define the potential. Of course, this is not sufficient if we do not restrict the class of acceptable potentials (see Ref. 15). On the other hand, if we impose that the solution of the inverse problem should belong to the above-defined class of potentials, then the estimates obtained for K and its first derivatives will enable us to show that the solution is stable with respect to small changes in the phase shifts above a certain energy.

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APPENDIX A: SPECTRAL REPRESENTATION OF THE COMPLETE RIEMANN–GREEN FUNCTION

Since the partial differential Eq. (10a) satisfied by the Riemann–Green matrix element $R_{ij}(x,y;s,u)$ is separable with respect to the variables x,y, a spectral representation can easily be obtained, by a generalization of a method developed by Riemann and described p. 328 of Ref. 11.

Setting $\Delta_{ij}^2 = k_i^2 - k_j^2$, we consider the following two differential equations:

$$\left[\frac{d^2}{dx^2} + \Delta_{ij}^2 + \lambda^2 - \frac{l_i(l_i+1)}{x^2} - \frac{\alpha_i}{x}\right] v(\lambda,x) = 0, \tag{A1}$$

$$\left[\frac{d^2}{dy^2} + \lambda^2 - \frac{l_j(l_j+1)}{y^2} - \frac{\alpha_j}{y}\right] w(\lambda,y) = 0. \tag{A2}$$

The regular $F_j(\lambda,y)$ and irregular $G_j(\lambda,y)$ real Coulomb functions, defined by the equation:

$$[\phi_0(\Lambda,y)]_{jj} = G_j(k_j,y) + iF_j(k_j,y), \tag{A3}$$

form a system of two linearly independent solutions of Eq. (A2), the Wronskian W of which is constant:

$$W = \text{Wr}[F,G] = F\frac{\partial}{\partial y}G - G\frac{\partial}{\partial y}F = -1. \tag{A4}$$

The solution v_{x_1} of Eq. (A1) is chosen in such a way that the boundary conditions (A5) and (A6) are satisfied:

$$v_{x_1}(\lambda,x) = 0 \quad \text{for } x = x_1 \tag{A5}$$

and

$$\frac{d}{dx}v_{x_1}(\lambda,x) = 1 \quad \text{for } x = x_1. \tag{A6}$$

An expression for $v_{x_1}(\lambda,x)$ in terms of F_i and G_i is easily obtained:

$$v_{x_1}(\lambda,x) = [G_i((\lambda^2 + \Delta_{ij}^2)^{1/2},x_1)F_i((\lambda^2 + \Delta_{ij}^2)^{1/2},x)$$

$$- G_i((\lambda^2 + \Delta_{ij}^2)^{1/2},x)F_i((\lambda^2 + \Delta_{ij}^2)^{1/2},x_1)]. \tag{A7}$$

We consider the solution $w(\lambda,y)$ of Eq. (A2) with which a function $\overline{w}(\lambda,y)$ can be associated, such that the following equality is verified for any function $f(x)$ belonging to $L^2(0,\infty)$:

$$f(x) = \left(\int \sum\right) d\lambda\ \overline{W}(\lambda,x) \int_0^\infty dy\, w(\lambda,y)f(y) \tag{A8}$$

where $(\int\Sigma)$ denotes the integration over the continuous spectrum, plus an infinite summation over the point spectrum when α_j is negative. From Refs. 10 and 16, we know that such functions $w(\lambda,y)$ and $\overline{w}(\lambda,y)$ exist:

$$w(\lambda,y) = \tfrac{1}{2}\pi\overline{w}(\lambda,y) = F_j(\lambda,y). \tag{A9}$$

The Riemann–Green matrix element R_{ij} can be written for $|x-s| > |y-u|$, which is always satisfied inside the Marchenko domain (and also inside the Gel'fand domain):

$$R_{ij}(x,y;s,u) = 2\,\mathrm{sgn}(s-x)\left(\int \sum\right) d\lambda \overline{w}(\lambda,y)w(\lambda,u)v_x(\lambda,s), \tag{A10}$$

where

$$\mathrm{sgn}(a) = a/|a|.$$

Of course, when $|x-s| = |y-u|$, we have

$$R_{ij}(x,y;s,u) = 1. \tag{A11}$$

However, we did not manage to rewrite the spectral representation (A10) for the complete Riemann–Green function into a close form. This can only be done in particular cases when $\alpha_i = l_i = 0$ or when $\alpha_i = k_i = 0$ and leads to well-known results.[11,17,18] We must acknowledge that not very much progress has been achieved since 1930: Most of the actually known Riemann-Green solutions were already discovered at that time by Darboux[11] and Chaundy,[18] in spite of the fact that new constructive methods[19,20] have been proposed.

APPENDIX B: BOUND FOR ϕ_0

Taking the absolute value of Eq. (4), we get

$$|[\phi_0(\Lambda,x)]_{ij}| = \delta_{ij}|W_{-i\alpha_i/2k_i,l_i+1/2}(-2ik_ix)|e^{\alpha_i\pi/4k_i}. \tag{B1}$$

Assuming that the Coulomb interaction is repulsive ($\alpha_i \geqslant 0$) and setting $Z = -2ik_ix$, $\kappa = -i\alpha_i/2k_i$, $\nu = l_i + \frac{1}{2}$, the Whittaker's functions W can be expressed in terms of the Kummer function U by the Eq. (B2) [see Eq. (13.1.33) of Ref. 9]

$$W_{\kappa,\nu}(Z) = e^{-Z/2}Z^{\nu+1/2}U(\tfrac{1}{2}+\nu-\kappa,1+2\nu,Z). \tag{B2}$$

If $\mathrm{Re}(1+2\nu) > \mathrm{Re}(\frac{1}{2}+\nu-\kappa) > 0$, the integral representation (13.2.5) of Ref. 9 for U can be used:

$$U(\tfrac{1}{2}+\nu-\kappa,1+2\nu,Z) = \frac{1}{\Gamma(\nu+\frac{1}{2}-\kappa)}\int_0^\infty e^{-Zt}t^{\nu-\kappa-1/2}(1+t)^{\nu+\kappa-1/2}\,dt \tag{B3}$$

if

$$\mathrm{Re}(l_i + 1 + i\alpha_i/2k_i) > 0, \tag{B4}$$

or

$$U(\tfrac{1}{2}+\nu-\kappa,1+2\nu,Z) = \frac{1}{\Gamma(\nu+\frac{1}{2}-\kappa)}\frac{1}{Z} \times \int_0^\infty e^{-t}\left(\frac{t}{Z}\right)^{\nu-\kappa-\frac{1}{2}}\left(1+\frac{t}{Z}\right)^{\nu+\kappa-\frac{1}{2}} dt. \tag{B5}$$

It readily follows from Eqs. (B2) and (B5) that

$$W_{\kappa,\nu}(Z) = \frac{e^{-Z/2}}{\Gamma(\nu+\frac{1}{2}-\kappa)Z^{\nu-1/2}} \times \int_0^\infty e^{-t}t^{2\nu-1}\left(\frac{Z}{t}+1\right)^{\nu+\kappa-1/2} dt, \tag{B6}$$

if

$$\mathrm{Im}k_i/|k_i|^2 \geqslant -(l_i+1)2/\alpha_i. \tag{B7}$$

Our Φ_0 Jost solution behaves like Z^{-l} at the origin and has a discontinuity along the imaginary negative k_i axis [we take $-\pi < \arg Z < \pi$, $\arg(-i) = -\pi/2$, and $3\pi/2 < \arg k_i < -\pi/2$].

Setting $k_i = \gamma + i\delta$, we get for $\delta \geqslant 0$ and $|k_i| \geqslant s > 0$

$$|[\phi_0(\Lambda<x)]_{ij}| \leqslant \frac{\exp(\alpha_i\pi\delta/4|k_i|^2 - \delta x)\exp(\alpha_i\pi/2s)_{\delta_{ij}}}{|\Gamma(l_i+1+i\alpha_i/2k_i)|(2|k_i|x)^{l_i}} \times \int_0^\infty e^{-t}t^{2l_i}\left(1+\frac{2|k_i|}{t}\right)^{l_i} dt, \tag{B8}$$

since

$$\left|\left(\frac{Z}{t}+1\right)^{(l_i - i\alpha_i/2k_i)}\right| = \exp\left\{\mathrm{Re}\left[\left(l_i - i\frac{\alpha_i}{2k_i}\right)\ln\left(\frac{Z}{t}+1\right)\right]\right\} = \left|\frac{Z}{t}+1\right|^{(l_i-\alpha_i\delta/2|k_i|^2)}\exp\left[\frac{\alpha_i\gamma}{2|k_i|^2}\arg\left(1+\frac{Z}{t}\right)\right] = \left|1+\frac{2|k_i|x}{t}\right|^{l_i}e^{\alpha_i\pi/2s}. \tag{B9}$$

The integral in (B8) can be evaluated:

$$\int_0^\infty e^{-t}t^{l_i}(t+2|k_i|x)^{l_i} = \sum_{j=0}^{l_i} C^j_{l_i}(2|k_i|x)^{l_i-j}(l_i+j)! \tag{B10}$$

$$\leqslant (2l_i)![1+2|k_i|x]^{l_i}, \tag{B11}$$

so that the final result can be written as

$$|[\phi_0(\Lambda,x)]_{ii}| \leqslant \frac{\exp(3\alpha_i\pi/4s-\delta x)}{|\Gamma(l_i+1+i\alpha_i/2k_i)|} \times (2l_i)! \frac{[1+2|k_i|x]^{l_i}}{[2|k_i|x]^{l_i}}, \tag{B12}$$

where

$$\left|\Gamma\left(l_i+1+\frac{i\alpha_i}{2k_i}\right)\right| = \left\{\left(l_i^2+\frac{\alpha_i^2}{4k_i^2}\right)\left[(l_i-1)^2+\frac{\alpha_i^2}{4k_i^2}\right]\cdots \times\left(1+\frac{\alpha_i^2}{4k_i^2}\right)\frac{\pi\alpha_i}{2k_i\sinh(\pi\alpha_i/2k_i)}\right\}^{1/2}. \tag{B13}$$

APPENDIX C: BOUNDS FOR $|(\partial/\partial\eta)R_{ij}^0(\eta,\xi;\eta_0,\xi_0)|$, $|(\partial/\partial\xi)R_{ij}^0(\eta,\xi;\eta_0,\xi_0)|$

The derivative of Eq. (15) with respect to η or ξ can be written as

$$R_{ij}'^{0} = -2x_2'P_{l_i}'(1-2x_2) - 2x_1'\int_0^1 P_{l_j}(1-2x_2t) \times P_{l_i}'(1-2x_1+2x_1t)\,dt + 2x_12x_2' \times\int_0^1 t\,P_{l_j}'(1-2x_2t)P_{l_i}'(1-2x_1+2x_1t)\,dt - 2x_12x_1'\int_0^1 (t-1)P_{l_j}(1-2x_2t) \times P_{l_i}''(1-2x_1+2x_1t)\,dt, \tag{C1}$$

where the primes denote derivatives with respect to any of the two variables η or ξ. Before proceeding further, the following bounds are recalled:

$$|P_l(x)|\leqslant 1 \quad \text{for } |x|\leqslant 1, \tag{C2}$$

$$P_l'(x)\leqslant l(l+1)/2 \quad \text{for } |x|\leqslant 1, \tag{C3}$$

$$P_l(x)\leqslant (2x)^l \quad \text{for } x\geqslant 1, \tag{C4}$$

$$P_l'(x)\leqslant l(l+1)(2x)^{l-1} \quad \text{for } x\geqslant 1. \tag{C5}$$

Setting $Z=1-2x_1$ and using Eqs. (20) and (21), we obtain

$$|R_{ij}'^{0}| \leqslant |x_2'|l_j(l_j+1) + 4|x_1'|\,|P_{l_i}'(z)| + |x_2'|l_j(l_j+1)l_i(l_i+1)(2Z)^{l_i} \leqslant 4|x_1'|l_i(l_i+1)(2Z)^{l_i-1} + |x_2'|l_j(l_j+1)[1+l_i(l_i+1)(2Z)^{l_i}], \tag{C6}$$

$$|R_{ij}'^{0}| \leqslant (2Z)^{l_i}(|x_1'|+|x_2'|)l_{\max}(l_{\max}+1) \times 2[1+l_{\max}(l_{\max}+1)] \leqslant \mathrm{A}(2\mathrm{Z})^{l_i}(|x_1'|+|x_2'|). \tag{C7}$$

To pursue, we have to find upper bounds for $[\,|(\partial/\partial\eta)x_1| + |(\partial/\partial\eta)x_2|\,]$ and $[\,|(\partial/\partial\xi)x_1| + |(\partial/\partial\xi)x_2|\,]$, where

$$\frac{1}{x_1}\frac{\partial x_1}{\partial\eta} = \frac{\eta_0-\xi}{(\eta-\eta_0)(\eta-\xi)}\leqslant 0,$$

$$\frac{1}{x_1}\frac{\partial x_1}{\partial\xi} = \frac{\xi_0-\eta}{(\xi-\xi_0)(\xi-\eta)}\geqslant 0,$$

$$\frac{1}{x_2}\frac{\partial x_2}{\partial\eta} = \frac{\xi+\eta_0}{(\eta-\eta_0)(\eta+\xi)}\leqslant 0,$$

$$\frac{1}{x_2}\frac{\partial x_2}{\partial\xi} = \frac{\eta+\xi_0}{(\xi-\xi_0)(\xi+\eta)}\geqslant 0.$$

$$\left|\frac{\partial x_1}{\partial\eta}\right| + \left|\frac{\partial x_2}{\partial\eta}\right| \leqslant \frac{(\xi-\xi_0)(\eta_0-\xi)}{(\eta-\xi)^2(\eta_0-\xi_0)} + \frac{(\eta_0+\xi)(\xi-\xi_0)}{(\eta+\xi)^2(\eta_0+\xi_0)} \leqslant \frac{(\eta_0-\xi)}{(\eta-\xi)^2} + \frac{(\eta_0+\xi)}{(\eta+\xi)^2} \leqslant \frac{2\eta_0}{(\eta-\xi)^2}, \tag{C8}$$

$$\left|\frac{\partial x_1}{\partial\xi}\right| + \left|\frac{\partial x_2}{\partial\xi}\right| \leqslant \frac{(\eta-\xi_0)(\eta_0-\eta)}{(\eta-\xi)^2(\eta_0-\xi_0)} + \frac{(\eta+\xi_0)(\eta_0-\eta)}{(\eta+\xi)^2(\eta_0+\xi_0)} \leqslant \frac{(\eta-\xi_0)}{(\eta-\xi)^2} + \frac{(\eta+\xi_0)}{(\eta+\xi)^2} \leqslant \frac{2\eta}{(\eta-\xi)^2}. \tag{C9}$$

The bounds for the derivatives of R_{ij}^0 read then

$$\left|\frac{\partial}{\partial\eta}R_{ij}^0(\eta,\xi;\eta_0,\xi_0)\right| \leqslant \frac{2A\eta_0}{(\eta-\xi)^2}\left[4\frac{(\eta+\xi)(\eta_0-\xi_0)}{(\eta-\xi)(\eta_0+\xi_0)}\right]^{l_i} \tag{C10}$$

and

$$\left|\frac{\partial}{\partial\xi}R_{ij}^0(\eta,\xi;\eta_0,\xi_0)\right| \leqslant \frac{2A\eta}{(\eta-\xi)^2}\left[4\frac{(\eta+\xi)(\eta_0-\xi_0)}{(\eta-\xi)(\eta_0+\xi_0)}\right]^{l_i}. \tag{C11}$$

APPENDIX D: BOUNDS FOR $|(\partial/\partial\eta)R_{ij}|$ AND $|(\partial/\partial\xi)R_{ij}|$

In this appendix, the estimates (29, (41), (C10), and (C11) for R_{ij}^0, R_{ij}, and the derivatives of R_{ij}^0 are used to find bounds for the absolute values of the complete Riemann function derivatives. Taking the derivatives with respect to η and ξ of Eq. (30) and setting $T_{ij}(\eta,\xi) = k_i^2 - k_j^2 - \alpha_i/(\eta-\xi) + \alpha_j/(\eta+\xi)$, it readily follows that

$$\frac{\partial}{\partial\eta} R_{ij}(\eta,\xi;\eta_0,\xi_0) = \frac{\partial}{\partial\eta} R^0_{ij}(\eta,\xi;\eta_0,\xi_0) - \int_{\xi_0}^{\xi} d\xi_1\, R^0_{ij}(\eta,\xi;\eta,\xi_1) \times T_{ij}(\eta,\xi_1) R_{ij}(\eta,\xi_1;\eta_0,\xi_0) + \int_{\xi_0}^{\xi} d\xi_1 \int_{\eta}^{\eta_0} d\eta_1 \frac{\partial}{\partial\eta} R^0_{ij}(\eta,\xi;\eta_1,\xi_1) \times T_{ij}(\eta_1,\xi_1) R_{ij}(\eta_1,\xi_1;\eta_0,\xi_0), \tag{D1}$$

$$\frac{\partial}{\partial\xi} R_{ij}(\eta,\xi;\eta_0,\xi_0) = \frac{\partial}{\partial\xi} R^0_{ij}(\eta,\xi;\eta_0,\xi_0) + \int_{\eta}^{\eta_0} d\eta_1\, R^0_{ij}(\eta,\xi;\eta_1,\xi) T_{ij}(\eta_1,\xi) \times R_{ij}(\eta_1,\xi;\eta_0,\xi_0) + \int_{\xi_0}^{\xi} d\xi_1 \int_{\eta}^{\eta_0} d\eta_1 \frac{\partial}{\partial\xi} R^0_{ij}(\eta,\xi;\eta_1,\xi_1) \times T_{ij}(\eta_1,\xi_1) R_{ij}(\eta_1,\xi_1;\eta_0,\xi_0). \tag{D2}$$

Defining

$$D_{ij}(\eta,\xi;\eta_0,\xi_0) = \left[\frac{(\eta_0+\xi_0)(\eta-\xi)}{4(\eta_0-\xi_0)(\eta+\xi)}\right]^{1/2} \frac{\partial}{\partial\eta} R_{ij}(\eta,\xi;\eta_0,\xi_0), \tag{D3}$$

$$C_{ij}(\eta,\xi;\eta_0,\xi_0) = \left[\frac{(\eta_0+\xi_0)(\eta-\xi)}{4(\eta_0-\xi_0)(\eta+\xi)}\right]^{1/2} \frac{\partial}{\partial\xi} \mathrm{R_{ij}}(\eta,\xi;\eta_0,\xi_0), \tag{D4}$$

$$\tilde{R}^0_{ij}(\eta,\xi;\eta_0,\xi_0) = \left[\frac{(\eta_0+\xi_0)(\eta-\xi)}{4(\eta_0-\xi_0)(\eta+\xi)}\right]^{1/2} R^0_{ij}(\eta,\xi;\eta_0,\xi_0), \tag{D5}$$

$$\tilde{R}_{ij}(\eta,\xi;\eta_0,\xi_0) = \left[\frac{(\eta_0+\xi_0)(\eta-\xi)}{4(\eta_0-\xi_0)(\eta+\xi)}\right]^{1/2} R_{ij}(\eta,\xi;\eta_0,\xi_0), \tag{D6}$$

$$E(\eta_0-\xi_0) = 4^{1/2} \exp\{4[\mu_{ij}^2(\eta_0-\xi_0)^2 + a_{ij}(\eta_0-\xi_0)]^{1/2}\}, \tag{D7}$$

Eqs. (D1) and (C10) lead to

$$|D_{ij}(\eta,\xi;\eta_0,\xi_0)| \leqslant \frac{2A\eta_0}{(\eta-\xi)^2} + E(\eta_0-\xi_0)\left[\mu_{ij}^2(\xi-\xi_0) + a_{ij}\ln\frac{\eta-\xi_0}{\eta-\xi}\right] + \frac{2A}{(\eta-\xi)^2} E(\eta_0-\xi_0)\left[\mu_{ij}^2 \frac{(\eta_0^2-\eta^2)(\xi-\xi_0)}{2} + a_{ij}\left(\frac{(\eta_0-\eta)(\xi-\xi_0)}{2} + (\eta_0^2-\xi_0^2)\ln\frac{\eta+\xi}{\eta-\xi}\right)\right] \tag{D8}$$

and

$$|C_{ij}(\eta,\xi;\eta_0,\xi_0)| \leqslant \frac{2A\eta}{(\eta-\xi)^2} + E(\eta_0-\xi_0)\left[\mu_{ij}^2(\eta_0-\eta) + a_{ij}\ln\frac{\eta_0-\xi}{\eta-\xi}\right] + \frac{2A\eta}{(\eta-\xi)^2} E(\eta_0-\xi_0) \times \left\{\mu_{ij}^2(\eta_0-\eta)(\xi-\xi_0) + \tfrac{1}{2}a_{ij}\left[\eta_1 \ln\frac{(\eta_1+\xi)(\eta_1-\xi_0)}{(\eta_1-\xi)(\eta_1+\xi_0)}\right]_{\eta_1=\eta_0}^{\eta_1=\eta} + \tfrac{1}{2}a_{ij}\left[\xi_1 \ln\frac{(\eta_0+\xi_1)(\eta_0-\xi_1)}{(\eta+\xi_1)(\eta-\xi_1)}\right]_{\xi_1=\xi_0}^{\xi_1=\xi}\right\}. \tag{D9}$$

If we use the well-known fact that $\ln x < x$ for $x > 1$, Eqs. (D7) and (D8) can be written as

$$\left|\frac{\partial}{\partial\eta} R_{ij}(\eta,\xi;\eta_0,\xi_0)\right| \leqslant \left[\frac{4(\eta_0-\xi_0)(\eta+\xi)}{(\eta_0+\xi_0)(\eta-\xi)}\right]^{1/2} \left\{\frac{2A\eta_0}{(\eta-\xi)^2} + E(\eta_0-\xi_0)\left[\mu_{ij}^2(\xi-\xi_0) + a_{ij}\frac{\eta-\xi_0}{\eta-\xi}\right] + \frac{2A}{(\eta-\xi)^2} E(\eta_0-\xi_0) \times \left[\mu_{ij}^2 \frac{(\eta_0^2-\eta^2)(\xi-\xi_0)}{2} + a_{ij}\frac{(\eta_0-\eta)(\xi-\xi_0)}{2} + a_{ij}\frac{(\eta_0^2-\xi_0^2)(\eta+\xi)}{(\eta-\xi)}\right]\right\} \tag{D10}$$

and

$$\left|\frac{\partial}{\partial\xi} R_{ij}(\eta,\xi;\eta_0,\xi_0)\right| \leqslant \left[4\frac{(\eta_0-\xi_0)(\eta+\xi)}{(\eta_0+\xi_0)(\eta-\xi)}\right]^{1/2} \left\{\frac{2A\eta}{(\eta-\xi)^2} + E(\eta_0-\xi_0)\left[\mu_{ij}^2(\eta_0-\eta) + a_{ij}\frac{\eta_0-\xi}{\eta-\xi}\right] + \frac{2A\eta}{(\eta-\xi)^2} E(\eta_0-\xi_0)\right.$$

$$\times\left[\mu_{ij}^2(\eta_0-\eta)(\xi-\xi_0)+\tfrac{1}{2}a_{ij}\eta\,\frac{(\eta+\xi)}{(\eta-\xi)}+\tfrac{1}{2}a_{ij}\xi\,\frac{\eta_0^2-\xi^2}{\eta^2-\xi^2}\right]\Bigg\}. \tag{D11}$$

APPENDIX E: UPPER BOUNDS FOR THE DERIVATIVES OF THE TRANSFORMATION OPERATOR

Performing the derivative of Eq. (43) with respect to η, we obtain

$$\begin{aligned}\frac{\partial}{\partial\eta}K_{ij}(\eta,\xi)=&-\tfrac{1}{2}R_{ij}(\eta,\xi,\eta,0)U_{ij}(\eta)+\tfrac{1}{2}\int_\eta^\infty d\eta_0\,\frac{\partial}{\partial\eta}R_{ij}(\eta,\xi,\eta_0,0)U_{ij}(\eta_0)\\&-\sum_l\int_0^\xi d\xi_0\,R_{ij}(\eta,\xi;\eta,\xi_0)U_{il}(\eta-\xi_0)K_{lj}(\eta,\xi_0)\\&+\sum_l\int_\eta^\infty d\eta_0\int_0^\xi d\xi_0\,\frac{\partial}{\partial\eta}R_{ij}(\eta,\xi;\eta_0,\xi_0)U_{il}(\eta_0-\xi_0)K_{lj}(\eta_0,\xi_0).\end{aligned}\tag{E1}$$

Setting $\tilde U_{ij}(\eta)=E(\eta)U_{ij}(\eta)$, we get the following upper bound:

$$\begin{aligned}\left|\frac{\partial}{\partial\eta}K_{ij}(\eta,\xi)\right|\left[\frac{\eta-\xi}{(\eta+\xi)4}\right]^{l_{\max}}\leqslant&\tfrac{1}{2}\tilde U_{ij}(\eta)+\tfrac{1}{2}\int_\eta^\infty d\eta_0\bigg[\frac{2A\eta_0}{(\eta-\xi)^2}+E(\eta_0)\left(\mu_{ij}^2\xi+a_{ij}\frac{\eta}{\eta-\xi}\right)+\frac{2A}{(\eta-\xi)^2}E(\eta_0)\\&\times\left(\frac{\mu_{ij}^2\eta_0^2\xi}{2}+a_{ij}\frac{3\eta\eta_0^2}{(\eta-\xi)}\right)\bigg]|U_{ij}(\eta_0)|+\sum_l\int_0^\xi d\xi_0\left(\frac{\eta-\xi_0}{\eta+\xi_0}\right)^{l_{\max}}|\tilde U_{il}(\eta-\xi_0)|\;|K_{lj}(\eta,\xi_0)|\\&+\sum_l\int_\eta^\infty d\eta_0\int_0^\xi d\xi_0\left(\frac{\eta_0-\xi_0}{\eta_0+\xi_0}\right)^{l_{\max}}\bigg\{\frac{2A\eta_0}{(\eta-\xi)^2}+E(\eta_0-\xi_0)\left[\mu_{ij}^2(\xi-\xi_0)+a_{ij}\frac{\eta-\xi_0}{\eta-\xi}\right]\\&+\frac{2A}{(\eta-\xi)^2}E(\eta_0-\xi_0)\bigg[\frac{\mu_{ij}^2(\eta_0^2-\eta^2)(\xi-\xi_0)}{2}+a_{ij}\frac{(\eta_0-\eta)(\xi-\xi_0)}{2}\\&+a_{ij}\frac{(\eta_0^2-\xi_0^2)(\eta+\xi)}{(\eta-\xi)}\bigg]\bigg\}|U_{il}(\eta_0-\xi_0)|\;|K_{1j}(\eta_0,\xi_0)|.\end{aligned}\tag{E2}$$

Since we have

$$|K_{lj}(\eta_0,\xi_0)|\leqslant\frac{n}{2}\left(\frac{\eta_0+\xi_0}{\eta_0-\xi_0}\right)^{l_{\max}}\tilde\sigma^0(\eta_0)\exp[\tilde\sigma^1(\eta_0-\xi_0)],\tag{55b}$$

it readily follows that

$$\begin{aligned}\left|\left|\frac{\partial}{\partial\eta}\mathrm{K}(\eta,\xi)\right|\right|\left[\frac{\eta-\xi}{4(\eta+\xi)}\right]^{l_{\max}}\leqslant&\tfrac{1}{2}\|\tilde U(\eta)\|+\frac{A}{(\eta-\xi)^2}\sigma^1(\eta)+\tfrac{1}{2}\mu_{ij}^2\xi\left[\tilde\sigma^0(\eta)+\frac{A}{(\eta-\xi)^2}\tilde\sigma^2(\eta)\right]\\&+a_{ij}\frac{\eta}{2(\eta-\xi)}\left[\tilde\sigma^0(\eta)+\frac{6A}{(\eta-\xi)}\tilde\sigma^2(\eta)\right]+\frac{n}{2}\tilde\sigma^0(\eta)\exp[\tilde\sigma^1(\eta-\xi)]\int_{\xi-\xi_0}^\eta\|U(s)\|\,ds\\&+n\tilde\sigma^0(\eta)\exp[\tilde\sigma^1(\eta-\xi)]\,\frac{A}{(\eta-\xi)^2}\int_\eta^\infty d\eta_0\int_0^\xi d\xi_0\,\eta_0\|U(\eta_0-\xi_0)\|\\&+\frac{n}{2}\tilde\sigma^0(\eta)\exp[\tilde\sigma^1(\eta-\xi)]\int_\eta^\infty d\eta_0\int_0^\xi d\xi_0\bigg\{\mu_{ij}^2\xi\left[1+\frac{A\eta_0^2}{(\eta-\xi)^2}\right]\\&+\frac{a_{ij}}{(\eta-\xi)}\left[\eta+\frac{A\xi}{(\eta-\xi)}\eta_0+\frac{2A(\eta+\xi)}{(\eta-\xi)^2}\eta_0^2\right]\bigg\}\|\tilde U(\eta_0-\xi_0)\|.\end{aligned}\tag{E3}$$

To proceed further, we must evaluate the following integral:

$$\int_{\eta}^{\infty} d\eta_0 \int_{0}^{\xi} d\xi_0\, \eta_0^l f(\eta_0 - \xi_0)$$

with $l = 0$, 1, or 2 and $f(\eta_0 - \xi_0) = \|U(\eta_0 - \xi_0)\|$ or $\|\tilde{U}(\eta_0 - \xi_0)\|$. We easily obtain

$$\int_{\eta}^{\infty} d\eta_0 \int_{\eta_0 - \xi}^{\eta_0} ds\, \eta_0^l f(s) \leqslant \int_{\eta - \xi}^{\infty} ds \int_{s}^{s+\xi} d\eta_0\, \eta_0^l f(s)$$

$$= \int_{\eta - \xi}^{\infty} ds\, f(s) \frac{(s+\xi)^{l+1} - s^{l+1}}{l+1} = \frac{1}{l+1} \sum_{j=0}^{l} C_{l+1}^{j} \sum^{j} (\eta - \xi) \xi^{l+1-j}, \tag{E4}$$

where

$$\sum^{j} (\eta - \xi) = \begin{cases} \sigma^j(\eta - \xi) & \text{if } f(\eta - \xi) = \|U(\eta - \xi)\|, \\ \tilde{\sigma}^j(\eta - \xi) & \text{if } f(\eta - \xi) = \|\tilde{U}(\eta - \xi)\|. \end{cases} \tag{E5}$$

If U has the moments implied by Eqs. (E3), (E4), and (E5), the transformation operator has a bounded first derivative with respect to η. (If $l_{\max} \geqslant 3$, this does not introduce new conditions.) The same simple but tiresome considerations will prove that the ξ derivative of K is bounded and so are the x and y derivatives.

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