Singularly perturbed phase response curves

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1 Introduction

Oscillator models of spiking neurons involve dynamics on multiple time-scales. Due to the presence of singularities in the slow manifolds, their reduction to unidimensional phase models is only valid in a small neighborhood of the periodic orbit. As a consequence, such a phase reduction is informative only for inputs that are much smaller than the singular perturbation parameter [1] and, therefore, vanishing in the singular limit.

Nevertheless, the (non-infinitesimal) phase response curve which has proven a fundamental input–output characteristic for oscillator models is still meaningful. This curve is usually computed through numerical simulations [2].

In this work, we determine the shape of (non-infinitesimal) phase response curves for fast-slow systems from their singular limit. It is completely determined by the geometry of slow manifolds and the type of singularities that lead to jump between them. We apply this geometric approach to relaxation oscillators and bursters.

2 Fast-slow systems and phase response curves

The canonical form of fast-slow systems is

$$\begin{aligned} \dot{x} &= \quad f(x,z,\mu) + \varepsilon p_x(x,z)u_x, & x \in \mathbb{R}^{n_x}, \\ \dot{z} &= \mu\,[g(x,z,\mu) + \varepsilon p_z(x,z)u_z], & z \in \mathbb{R}^{n_z}, \end{aligned}$$

where $\mu \ll 1$ is a small positive parameter explicitly denoting the separation of time-scales between the fast variables x and the slow variables y. We focus on fast-slow systems whose fast subsystem is bistable (that is, it has two co-existing attractive slow manifold) for a wide range of the slow variable.

Studying the singular limit and using standard persistence results [3], we approximate semi-analytically the phase response curve for finite impulsive inputs. We show that fast-slow oscillators are sensitive to finite impulse input only near a small region of the slow manifold that precede jump points.

3 Applications

We apply this geometric approach to relaxation oscillators and bursters. Figure 1 shows a numerically computed finite phase response curves for the van der Pol oscillator. The shape of this phase response curve shows that the oscillator is sensitive to inputs only in a short range of phases. Our analysis permits to derive such a shape semi-analytically and a phase-based explanation to the fast threshold modulation phenomenon [4]. Similar results are derived for bursters, suggesting a universal synchronization mechanism for different types of singularly perturbed hysteretic oscillators.

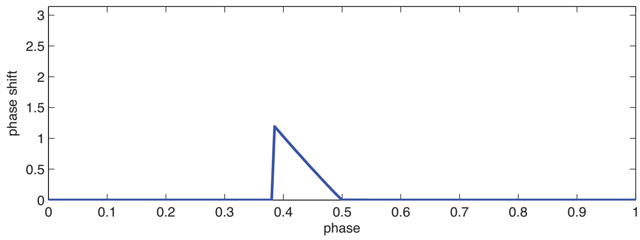

Figure 1: Phase response curve of the van der Pol oscillator for finite impulse input in the singular limit.

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